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Multiple solutions for a fourth-order nonlinear elliptic problem

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Abstract

The existence of multiple solutions for a class of fourth-order elliptic equation with respect to the generalized asymptotically linear conditions is established by using the minimax method and Morse theory.

Keywords: fourth-order elliptic boundary value problems, multiple solutions, mountain pass theorem, Morse theory.

1 Introduction

Consider the following Navier boundary value problem

$$\begin{cases} \Delta^2 u(x) + c\Delta u = f(x,u), & \text{in } \Omega; \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ^2 is the biharmonic operator, and Ω is a bounded smooth domain in $\mathbb{R}^N$ (N > 4), and $c < \lambda_1^*$ the first eigenvalue of -Δ in $H_0^1(\Omega)$.

The conditions imposed on $f(x, t)$ are as follows:

(H_1') $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$, $f(x,t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$;

(H_2') $f_1(t) \leq f(x, t) \leq f_2(t)$ uniformly in $x \in \Omega$, where $f_1, f_2 \in C(\mathbb{R})$ and we denote

$$f_0^- = \liminf_{|t|\to 0} \frac{f_1(t)}{t}, f_0^+ = \limsup_{|t|\to 0} \frac{f_2(t)}{t},$$

$$f_\infty^- = \liminf_{|t|\to\infty} \frac{f_1(t)}{t}, f_\infty^+ = \limsup_{|t|\to\infty} \frac{f_2(t)}{t}.$$

In view of the condition (H_2'), problem (1) is called generalized asymptotically linear at both zero and infinity. Clearly, $u = 0$ is a trivial solution of problem (1). It follows from (H_1') and (H_2') that the functional

$$I(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega F(x,u)dx \tag{2}$$

is of C^2 on the space $H_0^1(\Omega) \cap H^2(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega (|\Delta u|^2 - c|\nabla u|^2) dx \right)^{\frac{1}{2}},$$

where $F(x, t) = \int_0^t f(x, s) ds$. Under the condition (H_2'), the critical points of I are solutions of problem (1). Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the eigenvalues of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ and $\varphi_1(x) > 0$ be the eigenfunction corresponding to λ_1. In fact, $\lambda_1 = \lambda_1^*(\lambda_1^* - c)$. Let E_{λ_k} denote the eigenspace associated to λ_k. Throughout this article, we denoted by $|\cdot|_p$ the L^p (Ω) norm.

If $f_\infty^- = f_\infty^+$ in the above condition (H_2') is an eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$, then the problem (1) is called resonance at infinity. Otherwise, we call it non-resonance. A main tool of seeking the critical points of functional I is the mountain pass theorem (see [1-3]). To apply this theorem to the functional I in (2), usually we need the following condition [1], i.e., for some $\theta > 2$ and $M > 0$,

(AR) $\quad 0 < \theta F(x, s) \le f(x, s)s$ for a.e. $x \in \Omega$ and $|s| > M$.

It is well known that the condition (AR) plays an important role in verifying that the functional I has a "mountain-pass" geometry and a related $(PS)_c$ sequence is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$ when one uses the mountain pass theorem.

If $f(x, t)$ admits subcritical growth and satisfies (AR) condition by the standard argument of applying mountain pass theorem, we know that problem (1) has nontrivial solutions. Similarly, lase $f(x, t)$ is of critical growth (see, for example, [4-7] and their references).

It follows from the condition (AR) that $\lim_{|t| \to \infty} \dfrac{F(x, t)}{t^2} = +\infty$ after a simple computation. That is, $f(x, t)$ must be superlinear with respect to t at infinity. Noticing our condition (H_2'), the nonlinear term $f(x, t)$ is generalized asymptotically linear, not superlinear, with respect to t at infinity; which means that the usual condition (AR) cannot be assumed in our case. If the mountain pass theorem is used to seek the critical points of I, it is difficult to verify that the functional I has a "mountain pass" structure and the $(PS)c$ sequence is bounded.

In [8], Zhou studied the following elliptic problem

$$-\Delta u = f(x, u) \quad u \in H_0^1(\Omega),$$

where the conditions on $f(x, t)$ are similar to (H_1') and (H_2'). He provided a valid method to verify the (PS) sequence of the variational functional, for above problem is bounded in $H_0^1(\Omega)$ (see also [9,10]).

To the authors' knowledge, there seems few results on problem (1) when $f(x, t)$ is generalized asymptotically linear at infinity. However, the method in [8] cannot be applied directly to the biharmonic problems. For example, for the Laplacian problem, $u \in H_0^1(\Omega)$ implies $|u|, u_+, u_- \in H_0^1(\Omega)$, where $u_+ = \max(u, 0)$, $u_- = \max(-u, 0)$. We can use u_+ or u_- as a test function, which is helpful in proving a solution nonnegative. While for the biharmonic problems, this trick fails completely since $u \in H_0^2(\Omega)$ does not imply u_+, $u_- \in H_0^2(\Omega)$ (see [[11], Remark 2.1.10] and [12,13]). As far as this point is concerned, we will make use of the new methods to discuss in the following Lemma 2.2.

This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second-order problems which have been studied by many authors. In [14], there was a survey of results obtained in this direction. In [15], Micheletti and Pistoia showed that (P_1) admits at least two solutions by a variation of linking if $f(x, u)$ is sublinear. Chipot [16] proved that the problem (P_1) has at least three solutions by a variational reduction method and a degree argument. In [17], Zhang and Li showed that (P_1) admits at least two nontrivial solutions by Morse theory and local linking if $f(x, u)$ is superlinear and subcritical on u.

In this article, we consider multiple solutions of problem (1) in the non-resonance by using the mountain pass theorem and Morse theory. At first, we use the truncated skill and mountain pass theorem to obtain a positive solution and a negative solution of problem (1) under our more general conditions (H_1') and (H_2') with respect to the conditions (H_1) and (H_3) in [8]. In the course of proving existence of positive solution and negative solution, our conditions are general, but the proof of our compact condition is more simple than that in [8]. Furthermore, we can obtain a nontrivial solution when the nonlinear term f is non-resonance at the infinity by using Morse theory.

2 Main result and auxiliary lemmas

Let us now state the main result.

Theorem 2.1. *Assume conditions (H_1') and (H_2') hold, $f_0^+ < \lambda_1$ and $\lambda_k < f_\infty^- \le f_\infty^+ < \lambda_{k+1}$ for some $k \ge 2$, then problem (1) has at least three nontrivial solutions.*

Consider the following problem

$$\begin{cases} \Delta^2 u + c\Delta u = f_+(x, u), \ x \in \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Define a functional $I_+ : H^2(\Omega) \cap H_0^1(\Omega) \to \mathbb{R}$ by

$$I_+(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega F_+(x,u)dx,$$

where $F_+(x,t) = \int_0^t f_+(x,s)ds$, then $I_+ \in C^2(H^2(\Omega) \cap H_0^1(\Omega), \mathbb{R})$.

Lemma 2.2. *I_+ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a sequence such that $|I_+'(u_n)| \le c$, $< I_+'(u_n), \phi > \to 0$ as $n \to \infty$. Note that

$$< I_+'(u_n), \phi > = \int_\Omega (\Delta u_n \Delta \phi - c \nabla u_n \nabla \phi) dx - \int_\Omega f_+(x, u_n) \phi dx = \circ(\|\phi\|) \tag{3}$$

for all $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that $|u_n|_2$ is bounded, taking $\varphi = u_n$ in (3). By (H_2'), there exists $c > 0$ such that $|f_+(x, u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So (u_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2 \to +\infty$, as $n \to \infty$, set $v_n = \dfrac{u_n}{|u_n|_2}$, then $|v_n|_2 = 1$. Taking $\varphi = v_n$ in (3), it follows that $||v_n||$ is bounded. Without loss of generality, we assume that $v_n \rightharpoonup v$ in $H^2(\Omega) \cap H_0^1(\Omega)$, then $v_n \to v$ in $L^2(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (3) by $|u_n|_2$, we get

$$\int_\Omega (\Delta v_n \Delta \phi - c \nabla v_n \nabla \phi) dx - \int_\Omega \frac{f_+(x, u_n)}{|u_n|_2} \phi dx = \circ\left(\frac{\|\phi\|}{|u_n|_2}\right), \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4}$$

Let $\epsilon > 0$ and $f_\infty^- - \varepsilon > \lambda_1$, and choose a constant $M > 0$ such that $f(x, t) \geq (f_\infty^- - \epsilon)t,\ t > M$.

Let $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varphi \geq 0$. From (4), we have

$$\begin{aligned}
\int_\Omega (\Delta v_n \Delta \phi - c \nabla v_n \nabla \phi) dx &= \int_\Omega \frac{f_+(x, u_n)}{|u_n|_2} \phi dx + \circ(1) \\
&= \int_{|u_n^+|_2 v_n < M} \frac{f(x, u_n)}{|u_n|_2} \phi dx + \int_{|u_n^+|_2 v_n \geq M} \frac{f(x, u_n)}{|u_n^+|_2} \phi dx + \circ(1) \\
&\geq (f_\infty^- - \epsilon) \int_\Omega v_n \phi dx - c \int_{0 \leq |u_n^+|_2 v_n \leq M} v_n \phi dx + \circ(1).
\end{aligned}$$

Letting $n \to \infty$, for all $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varphi \geq 0$ we have

$$\int_\Omega (\Delta v \Delta \phi - c \nabla v \nabla \phi) dx \geq (f_\infty^- - \epsilon) \int_\Omega v^+ \phi dx. \tag{5}$$

Then we have

$$\begin{cases} \Delta^2 v + c \Delta v \geq (f_\infty^- - \epsilon) v^+,\ x \in \Omega, \\ v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0. \end{cases}$$

While let $-\Delta v = u$, by the comparison maximum principle $v \geq 0$. Since the definition of v_n, we have $v \not\equiv 0$ and we arrive at a contradiction by choosing $\varphi = \varphi_1$ in (5).

Since $|u_n|_2$ is bounded, from (3) we get the boundedness of $||u_n||$. A standard argument shows that $\{u_n\}$ has a convergent subsequence in $H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, I_+ satisfies (PS) condition.

Lemma 2.3. *Let φ_1 be the eigenfunction corresponding to λ_1 with $||\varphi_1|| = 1$. If $f_0^+ < \lambda_1 < f_\infty^-$, then*

(a) There exist ρ, $\beta > 0$ such that $I_+(u) \geq \beta$ for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ with $||u|| = \rho$;

(b) $I_+(t\varphi_1) = -\infty$ as $t \to +\infty$.

Proof. By (H_1') and (H_2'), if $f_\infty^- \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ and $B = B(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F_+(x,s) \leq \frac{1}{2}(f_0^+ + \varepsilon)s^2 + As^{p+1}, \tag{6}$$

$$F_+(x,s) \geq \frac{1}{2}(f_\infty^- - \varepsilon)s^2 - B, \tag{7}$$

where $p \in \left(1, \ \dfrac{N+4}{N-4}\right)$ if $N > 4$.

Choose $\varepsilon > 0$ such that $f_0^+ + \varepsilon < \lambda_1$. By (6), the Poincar*é* inequality and the Sobolev inequality, we get

$$\begin{aligned} I_+(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega F_+(x,u)dx \\ &\geq \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \frac{1}{2}\int_\Omega [(f_0^+ + \varepsilon)u^2 + A|u|^{p+1}]dx \\ &\geq \frac{1}{2}\left(1 - \frac{f_0^+ + \varepsilon}{\lambda_1}\right)\|u\|^2 - c\|u\|^{p+1}. \end{aligned}$$

So, part (a) holds if we choose $||u|| = \rho > 0$ small enough.

On the other hand, if $f_\infty^- \in (\lambda_1, +\infty)$, take $\varepsilon > 0$ such that $f_\infty^- - \varepsilon > \lambda_1$. By (7), we have

$$I_+(u) \leq \frac{1}{2}\|u\|^2 - \frac{f_\infty^- - \varepsilon}{2}|u|_2^2 + B\,|\Omega|\,.$$

Since $f_\infty^- - \varepsilon > \lambda_1$ and $||\varphi_1|| = 1$, it is easy to see that

$$I_+(t\phi_1) \leq \frac{1}{2}\left(1 - \frac{f_\infty^- - \varepsilon}{\lambda_1}\right)t^2 + B\,|\Omega| \to -\infty \ \text{ as } \ t \to +\infty$$

and part (b) is proved.

Lemma 2.4. *Let* $H^2(\Omega) \cap H_0^1(\Omega) = V \oplus W$, *where* $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$. *If* f *satisfies* (H_1')-(H_3'), *then*

(i) the functional I is coercive on W, that is

$$I(u) \to +\infty \quad \text{as } \|u\| \to +\infty, \ u \in W$$

and bounded from below on W,

(ii) the functional I is anti-coercive on V.

Proof. For $u \in W$, by (H_2'), for any $\varepsilon > 0$, there exists $B_1 = B_1(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{2}(f_\infty^+ + \varepsilon)s^2 + B_1. \tag{8}$$

So we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\Omega} (|\Delta u|^2 - c|\nabla u|^2)dx - \int_{\Omega} F(x,u)dx \\ &\geq \frac{1}{2}\int_{\Omega} (|\Delta u|^2 - c|\nabla u|^2)dx - \frac{1}{2}(f_\infty^+ + \varepsilon)\,|u|_2^2 - B_1\,|\Omega| \\ &\geq \frac{1}{2}\left(1 - \frac{f_\infty^+ + \varepsilon}{\lambda_{k+1}}\right)\|u\|^2 - B_1\,|\Omega|\,. \end{aligned}$$

Choose $\varepsilon > 0$ such that $f_\infty^+ + \varepsilon < \lambda_{k+1}$. This proves (*i*).

(*ii*) When $\lambda_k < f_\infty^- \leq f_\infty^+ < \lambda_{k+1}$, it is easy to see that the conclusion holds.

Lemma 2.5. *If* $\lambda_k < f_\infty^- \leq f_\infty^+ < \lambda_{k+1}$, *then I satisfies the (PS) condition*.

Proof. Let $\{u_n\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a sequence such that $|I(u_n)| \leq c$, $< I'(u_n), \varphi > \to 0$.

Since

$$< I'(u_n), \phi > = \int_{\Omega} (\Delta u_n \Delta \phi - c\nabla u_n \nabla \phi)dx - \int_{\Omega} f(x,u_n)\phi dx = \circ(\|\phi\|) \tag{9}$$

for all $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2$ is bounded, we can take $\varphi = u_n$. By (H_2'), there exists a constant $c > 0$ such that $|f(x, u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So (u_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2 \to +\infty$, as $n \to \infty$, set $v_n = \frac{u_n}{|u_n|_2}$, then $|v_n|_2 = 1$. Taking $\varphi = v_n$ in (9), it follows that $||vn||$ is bounded. Without loss of generality, we assume $vn \rightharpoonup v$ in $H^2(\Omega) \cap H_0^1(\Omega)$, then $vn \to v$ in $L^2(\Omega)$. Hence, $vn \to v$ a.e. in Ω. Dividing both sides of (9) by $|u_n|_2$, for any $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, we get

$$\int_{\Omega} (\Delta v_n \Delta \phi - c\nabla v_n \nabla \phi)dx - \int_{\Omega} \frac{f(x,u_n)}{|u_n|_2}\phi dx = \circ\left(\frac{\|\phi\|}{|u_n|_2}\right). \tag{10}$$

Then for a.e. $x \in \Omega$ and suitable ϵ, we have

$$(f_\infty^- - \epsilon)v^2 \leq \frac{f(x,u_n)}{|u_n|_2}v \leq (f_\infty^+ + \epsilon)v^2$$

as $n \to \infty$. In fact, if $v(x) \neq 0$, by (H_2'), we have

$$|u_n(x)| = |v_n(x)|\,|u_n|_2 \to +\infty$$

and

$$(f_\infty^- - \epsilon)v^2 \leq \frac{f(x,u_n)}{|u_n|_2}v = \frac{f(x,u_n)}{u_n}v_n v \leq (f_\infty^+ + \epsilon)v^2.$$

as $n \to \infty$. If $v(x) = 0$, we have

$$\frac{|f(x,u_n)|}{|u_n|_2} \leq c\,|v_n| \to 0.$$

Since $\frac{|f(x,u_n)|}{|u_n|_2} \le c\,|v_n|$, by choosing $\varphi = v$ in (10) and the Lebesgue dominated convergence theorem, we arrive at

$$\lambda_k\,|v|_2^2 \le \int_\Omega (\Delta v \Delta v - c\nabla v \nabla v)dx \le \lambda_{k+1}\,|v|_2^2\,.$$

From Fourier series theory, it is easy to see that $v \equiv 0$. It contradicts to $\lim_{n\to\infty} |v_n|_2 = |v|_2 = 1$.

It is well known that critical groups and Morse theory are the main tools in solving elliptic partial differential equation. Let us recall some results which will be used later. We refer the readers to the book [18] for more information on Morse theory.

Let H be a Hilbert space and $I \in C^1(H, \mathbb{R})$ be a functional satisfying the (PS) condition or (C) condition, and $H_q\ (X,\ Y)$ be the qth singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of I with $I(u_0) = c$, $c \in \mathbb{R}$, and U be a neighborhood of u_0. The group

$$C_q(I, u_0) := H_q(I^c \cap U, I^c \cap U \backslash \{u_0\}),\ q \in Z$$

is said to be the qth critical group of I at u_0, where $I^c = \{u \in H\colon I(u) \le c\}$.

Let $K\colon = \{u \in H\colon I'(u) = 0\}$ be the set of critical points of I and $a < \inf I(K)$, the critical groups of I at infinity are formally defined by (see [19])

$$C_q(I, \infty) := H_q(H, I^a),\ q \in Z.$$

The following result comes from [18,19] and will be used to prove the result in this article.

Proposition 2.6 [19]. *Assume that* $H = H_\infty^+ \oplus H_\infty^-$, *$I$ is bounded from below on* H_∞^+ *and* $I(u) \to -\infty$ *as* $||u|| \to \infty$ *with* $u \in H_\infty^-$. *Then*

$$C_k(I, \infty) \ncong 0,\ \ if\, k = \dim H_\infty^- < \infty. \qquad (11)$$

3 Proof of the main result

Proof of Theorem 2.1. By Lemmas 2.2 and 2.3 and the mountain pass theorem, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \ge \beta$. Since $I_+(0) = 0$, $u_1 \ne 0$ and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1) and satisfies

$$C_1(I_+, u_1) \neq 0,\ u_1 > 0. \qquad (12)$$

Using the results in [18], we obtain

$$C_q(I, u_1) = C_q(I_{C_0^1(\Omega)}, u_1) = C_q(I_+|_{C_0^1(\Omega)}, u_1) = C_q(I_+, u_1) = \delta_{q1}Z. \qquad (13)$$

Similarly, we can obtain another negative critical point u_2 of I satisfying

$$C_q(I, u_2) = \delta_{q,1}Z. \qquad (14)$$

Since $f_0^+ < \lambda_1$, the zero function is a local minimizer of I, then

$$C_q(I, 0) = \delta_{q,0}Z. \tag{15}$$

On the other hand, by Lemmas 2.4, 2.5 and the Proposition 2.6, we have

$$C_k(I, \infty) \ncong 0. \tag{16}$$

Hence I has a critical point u_3 satisfying

$$C_k(I, u_3) \ncong 0. \tag{17}$$

Since $k \geq 2$, it follows from (13)-(17) that u_1, u_2, and u_3 are three different nontrivial solutions of the problem (1).

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Competing interests
The author declares that he has no competing interests.

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References

1. Ambrosetti, A, Rabinowitz, PH: Dual variational methods in critical point theory and applications. J Funct Anal. **14**, 349–381 (1973). doi:10.1016/0022-1236(73)90051-7
2. Brezis, H, Nirenberg, L: Positive solutions of nonlinear elliptic equation involving critical Sobolev exponents. Commun Pure Appl Math. **36**, 437–477 (1983). doi:10.1002/cpa.3160360405
3. Rabinowitz, PH: Minimax methods in critical point theory with applications to differential equations. In CBMs Regional Conference Series in Mathematics, vol. 65,American Mathematical Society, Providence (1986)
4. Bernis, F, Azorero, JG, Peral, I: Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth-order. Adv Diff Equ. **1**, 219–240 (1996)
5. Deng, YB, Wang, GS: On inhomogeneous biharmonic equations involving critical exponents. Proc Royal Soc Edinburg. **129**, 925–946 (1999). doi:10.1017/S0308210500031012
6. Gazzola, F, Grunau, HC, Squassina, M: Existence and nonexistence results for critical growth biharmonic elliptic equations. Calc Var PDE. **18**, 117–143 (2003). doi:10.1007/s00526-002-0182-9
7. Noussair, ES, Swanson, CA, Yang, JF: Critical semilinear biharmonic equations in $\mathbb{R}^N$. Proc Royal Soc Edinburg. **121**, 139–148 (1992). doi:10.1017/S0308210500014189
8. Zhou, HS: Existence of asymptotically linear Dirichlet problem. Nonlinear Anal TMA. **44**, 909–918 (2001). doi:10.1016/S0362-546X(99)00314-4
9. Stuart, CA, Zhou, HS: Applying the mountain pass theorem to asymptotically linear elliptic equation on $\mathbb{R}^N$. Commun Part Diff Equ. **24**, 1731–1758 (1999). doi:10.1080/03605309908821481
10. Li, GB, Zhou, HS: Multiple solutions to *p*-Laplacian problems with asymptotic nonlinearity as u^{p-1} at infinity. J Lond Math Soc. **65**(2):123–138 (2002)
11. Ziemer, WP: Weakly Differentiable Functions. In Grad Texts in Math, vol. 120,Springer- Verlag, Berlin (1989)
12. Liu, Y, Wang, ZP: Biharmonic equations with asymptotically linear nonlinearities. Acta Math Sci. **27**, 549–560 (2007). doi:10.1016/S0252-9602(07)60055-1
13. Pei, RC: Multiple solutions for biharmonic equations with asymptotically linear nonlinearities. Boundary value problems **2010** (2010). Article ID 241518, 11
14. Lazer, AC, Mckenna, PJ: Large amplitude periodic oscillation in suspension bridges: some new connections with nonlinear analysis. ShAM Rev. **32**, 537–578 (1990)
15. Micheletti, AM, Pistoia, A: Multiplicity solutions for a fourth order semilinear elliptic problems. Nonlinear Anal TMA. **31**, 895–908 (1998). doi:10.1016/S0362-546X(97)00446-X
16. Chipot, M: Variational Inequalities and Flow in Porous Media. Springer-lag, New York (1984)
17. Zhang, JH, Li, SJ: Multiple nontrivial solutions for some fourth-order semilinear elliptic problems. Nonlinear Anal TMA. **60**, 221–230 (2005)
18. Chang, KC: Infinite Dimensional Morse Theory and Multiple Solutions Problems. Birkhäuser, Boston. (1993)
19. Bartsch, T, Li, SJ: Critical point theory for asymptotically quadratic functionals and applications to problems with resonance. Nonlinear Anal TMA. **28**, 419–441 (1997). doi:10.1016/0362-546X(95)00167-T

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