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Lattictic non-archimedean random stability of ACQ functional equation

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Abstract

In this paper, we prove the generalized Hyers-Ulam stability of the following additive-cubic-quartic functional equation

$$\begin{aligned} &11f(x+2y)+11f(x-2y)\\ &\quad =44f(x+y)+44f(x-y)+12f(3y)-48f(2y)+60f(y)-66f(x) \end{aligned} \tag{1}$$

in various complete lattictic random normed spaces.

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1. Introduction

Probability theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence and others. The random topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. The usual uncertainty principle of Werner Heisenberg leads to a generalized uncertainty principle, which has been motivated by string theory and non-commutative geometry. In strong quantum gravity, regime space-time points are determined in a random manner. Thus, impossibility of determining the position of particles gives the space-time a random structure. Because of this random structure, position space representation of quantum mechanics breaks down and so a generalized normed space of quasi-position eigenfunction is required. Hence one needs to discuss on a new family of random norms. There are many situations where the norm of a vector is not possible to be found and the concept of random norm seems to be more suitable in such cases, i.e., we can deal with such situations by modeling the inexactness through the random norm.

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4,6-27]).

In [28,29], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (2), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [8], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{3}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (3), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [30,31]*Let* (X, d) *be a complete generalized metric space and* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for any* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [32] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [33-38]).

2. Preliminaries

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied by Alsina [39], Mirmostafaee, Mirzavaziri and Moslehian [40,35], Miheţ, and Radu [41], Miheţ, et al. [42,43], Baktash et. al [44], Najati [45] and Saadati et. al. [24].

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, i.e., a partially ordered set in which every nonempty subset admits supremum and infimum and $0_{\mathcal{L}} = infL$, $1_{\mathcal{L}} = supL$. The space of latticetic random distribution functions, denoted by Δ_L^+, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that F is left continuous, non-decreasing on $\mathbb{R}$ and $F(+\infty) = 1_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

The subspace $D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ_L^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, \text{ if } t \leq 0, \\ 1_{\mathcal{L}}, \text{ if } t > 0. \end{cases}$$

Definition 2.1. [46] A *triangular norm* (t-norm) on L is a mapping $\mathcal{T} : (L)^2 \to L$ satisfying the following conditions:

(1) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (: boundary condition);
(2) $(\forall (x, y) \in (L)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (: commutativity);
(3) $(\forall (x, y, z) \in (L)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (: associativity);
(4) $(\forall (x, x', y, y') \in (L)^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (: monotonicity).

Let $\{x_n\}$ be a sequence in L converges to $x \in L$ (equipped the order topology). The t-norm $\mathcal{T}$ is called a *continuous t-norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y),$$

for any $y \in L$.

A t-norm $\mathcal{T}$ can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, ..., x_n) \in L^n$ the value $\mathcal{T}(x_1, \ldots, x_n)$ defined by

$$\mathcal{T}_{i=1}^0 x_i = 1, \quad \mathcal{T}_{i=1}^n x_i = \mathcal{T}(\mathcal{T}_{i=1}^{n-1} x_i, x_n) = \mathcal{T}(x_1, \ldots, x_n).$$

The t-norm $\mathcal{T}$ can also be extended to a countable operation taking, for any sequence $\{x_n\}$ in L, the value

$$\mathcal{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathcal{T}_{i=1}^n x_i. \tag{4}$$

The limit on the right side of (4) exists since the sequence $(\mathcal{T}_{i=1}^n x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Note that we put $\mathcal{T} = T$ whenever $L = [0, 1]$. If T is a t-norm then, for all $x \in [0, 1]$ and $n \in N \cup \{0\}$, $x_T^{(n)}$ is defined by 1 if $n = 0$ and $T(x_T^{(n-1)}, x)$ if $n \geq 1$. A t-norm T is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$ (see [47]).

Definition 2.2. [46] A continuous t-norm $\mathcal{T}$ on $L = [0, 1]^2$ is said to be continuous *t-representable* if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0, 1]^2$ are continuous t-representable.

Define the mapping $\mathcal{T}_\wedge$ from L^2 to L by

$$\mathcal{T}_\wedge(x, y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases}$$

Recall (see [47,48]) that, if $\{x_n\}$ is a given sequence in L, then $(\mathcal{T}_\wedge)_{i=1}^n x_i$ is defined recurrently by $(\mathcal{T}_\wedge)_{i=1}^1 x_i = x_1$ and $(\mathcal{T}_\wedge)_{i=1}^n x_i = \mathcal{T}_\wedge((\mathcal{T}_\wedge)_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$.

A *negation* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*. In the following, $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

Definition 2.3. *A latticetic random normed space* is a triple $(X, \mu, \mathcal{T}_\wedge)$, where X is a vector space and μ is a mapping from X into D_L^+ satisfying the following conditions:

(LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(LRN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$;

(LRN3) $\mu_{x+y}(t + s) \geq_L \mathcal{T}_\wedge(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note that, from (LPN2), it follows $\mu_{-x}(t) = \mu_x(t)$ for all $x \in X$ and $t \geq 0$.

Example 2.4. Let $L = [0, 1] \times [0, 1]$ and an operation $\leq_L$ be defined by

$$L = \{(a_1, a_2) : (a_1, a_2) \in [0, 1] \times [0, 1] \text{ and } a_1 + a_2 \leq 1\},$$
$$(a_1, a_2) \leq_L (b_1, b_2) \Leftrightarrow a_1 \leq b_1, \quad a_2 \geq b_2, \quad \forall a = (a_1, a_2),\ b = (b_1, b_2) \in L.$$

Then $(L, \leq_L)$ is a complete lattice (see [46]). In this complete lattice, we denote its units by $0_L = (0, 1)$ and $1_L = (1, 0)$. Let $(X, ||\cdot||)$ be a normed space. Let $\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0, 1] \times [0, 1]$ and μ be a mapping defined by

$$\mu_x(t) = \left(\frac{t}{t + ||x||}, \frac{||x||}{t + ||x||}\right), \quad \forall t \in \mathbb{R}^+.$$

Then, $(X, \mu, \mathcal{T})$ is a latticetic random normed spaces.

If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space, then we have

$$\mathcal{V} = \{V(\varepsilon, \lambda) : \varepsilon >_L 0_\mathcal{L}, \lambda \in L\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$$

is a complete system of neighborhoods of null vector for a linear topology on X generated by the norm F, where

$$V(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) >_L \mathcal{N}(\lambda)\}.$$

Definition 2.5. Let $(X, \mu, \mathcal{T}_\wedge)$ be a latticetic random normed spaces.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $t > 0$ and $\varepsilon \in L\backslash\{0_\mathcal{L}\}$, there exists a positive integer N such that $\mu_{x_n - x}(t) >_L \mathcal{N}(\varepsilon)$ for all $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $t > 0$ and $\varepsilon \in L\backslash\{0_\mathcal{L}\}$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) >_L \mathcal{N}(\varepsilon)$ for all $n \geq m \geq N$.

(3) A latticetic random normed space $(X, \mu, \mathcal{T}_\wedge)$ is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.6. *If* $(X, \mu, \mathcal{T}_\wedge)$*is a latticetic random normed space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$, *then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

Proof. The proof is the same as classical random normed spaces (see [49]). □

Lemma 2.7. *Let* $(X, \mu, \mathcal{T}_\wedge)$*be a latticetic random normed space and* $x \in X$. *If*

$$\mu_x(t) = C, \quad \forall t > 0,$$

then $C = 1_\mathcal{L}$*and* $x = 0$.

Proof. Let $\mu_x(t) = C$ for all $t > 0$. Since $Ran(\mu) \subseteq D_L^+$, we have $C = 1_\mathcal{L}$ and, by (LRN1), we conclude that $x = 0$. □

3. Non-Archimedean Lattictic random normed space

By a *non-Archimedean field*, we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$ and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the *trivial valuation* we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

Let $\mathcal{X}$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : \mathcal{X} \to [0, \infty)$ is called a *non-Archimedean norm*, if it satisfies the following conditions:

(1) $||x|| = 0$ if and only if $x = 0$;
(2) for any $r \in \mathcal{K}$, $x \in \mathcal{X}$, $||rx|| = |r|\,||x||$;
(3) the strong triangle inequality (ultrametric), i.e.,

$$||x + y|| \leq \max\{||x||, ||y||\}, \quad \forall x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, ||\cdot||)$ is called a *non-Archimedean normed space*.

Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\}, \quad \forall m, n \in \mathbb{N}(n > m),$$

a sequence $\{x_n\}$ is a Cauchy sequence if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a *complete non-Archimedean normed space*, we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [50] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then, $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*.

Throughout the paper, we assume that $\mathcal{X}$ is a vector space and $\mathcal{Y}$ is a complete non-Archimedean normed space.

Definition 3.1. *A non-Archimedean lattictic random normed space* (briefly, non-Archimedean LRN-space) is a triple $(\mathcal{X}, \mu, \mathcal{T})$, where X is a linear space over a non-Archimedean field $\mathcal{K}$, $\mathcal{T}$ is a continuous t-norm and is μ is a mapping from $\mathcal{X}$ into D_L^+ satisfying the following conditions hold:

(NA-LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(NA-LRN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in \mathcal{X}$, $t > 0$, $\alpha \neq 0$;

(NA-LRN3) $\mu_{x+y}(\max\{t, s\}) \geq_L \mathcal{T}(\mu_x(t), \mu_y(s))$ for all $x, y, z \in \mathcal{X}$ and $t, s \geq 0$.

It is easy to see that, if (NA-LRN3) holds, then we have

(RN3) $\mu_{x+y}(t + s) \geq_L \mathcal{T}(\mu_x(t), \mu_y(s))$.

As a classical example, if $(\mathcal{X}, ||.||)$ is a non-Archimedean normed linear space, then the triple $(\mathcal{X}, \mu, \mathcal{T})$, where $L = [0, 1]$, $\mathcal{T} = \min$ and

$$\mu_x(t) = \begin{cases} 0, & t \leq ||x||, \\ 1, & t > ||x||, \end{cases}$$

is a non-Archimedean LRN-space.

Example 3.2. Let $(\mathcal{X}, ||.||)$ be is a non-Archimedean normed linear space in which $L = [0, 1]$. Define

$$\mu_x(t) = \frac{t}{t + ||x||}, \quad \forall x \in \mathcal{X}, \ t > 0.$$

Then $(\mathcal{X}, \mu, \min)$ is a non-Archimedean RN-space.

Definition 3.3. Let $(\mathcal{X}, \mu, \mathcal{T})$ be a non-Archimedean LRN-space and $\{x_n\}$ be a sequence in $\mathcal{X}$.

(1) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \mu_{x_n - x}(t) = 1_{\mathcal{L}}$$

for all $t > 0$. In that case, x is called the *limit* of the sequence $\{x_n\}$.

(2) The sequence $\{x_n\}$ in $\mathcal{X}$ is called a *Cauchy sequence* if, for any $\varepsilon \in L \backslash \{0_{\mathcal{L}}\}$ and $t > 0$, there exists a poisitve integer n_0 such that, for all $n \geq n_0$ and $p > 0$, $\mu_{x_{n+p} - x_n}(t) >_L \mathcal{N}(\varepsilon)$.

(3) If every Cauchy sequence is convergent, then the random norm is said to be *complete* and the non-Archimedean RN-space is called a *non-Archimedean random Banach space*.

Remark 3.4. [51] Let $(\mathcal{X}, \mu, \mathcal{T}_\wedge)$ be a non-Archimedean LRN-space. Then, we have

$$\mu_{x_{n+p}-x_n}(t) \geq_L \mathcal{T}_\wedge\{\mu_{x_{n+j+1}-x_{n+j}}(t) : j = 0, 1, 2, ..., p-1\}.$$

Thus the sequence $\{x_n\}$ is Cauchy sequence if, for any $\varepsilon \in L\backslash\{0_\mathcal{L}\}$ and $t > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$\mu_{x_{n+1}-x_n}(t) >_L \mathcal{N}(\varepsilon).$$

4. Generalized Ulam-Hyers stability for functional equation (1): an odd case in non-Archimedean LRN-spaces

Let $\mathcal{K}$ be a non-Archimedean field, $\mathcal{X}$ be a vector space over $\mathcal{K}$ and $(\mathcal{Y}, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$ In this section, we investigate the stability of the functional equation (1): an odd case where f is a mapping from $\mathcal{K}$ to $\mathcal{Y}$.

Let Ψ be a distribution function on $\mathcal{X} \times \mathcal{X}$ to D_L^+ ($\Psi(x, y, t)$ denoted by $\Psi_{x,y}(t)$ such that

$$\Psi_{cx,cx}(t) \geq_L \Psi_{x,x}\left(\frac{t}{|c|}\right), \quad \forall x \in \mathcal{X}, \ c \neq 0.$$

Definition 4.1. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is said to be Ψ-*approximately mixed ACQ* if

$$\mu_{Df(x,y)}(t) \geq_L \Psi_{x,y}(t), \quad \forall x, y \in \mathcal{X}, \ t > 0. \tag{5}$$

We assume that $2 \neq 0$ in $\mathcal{K}$ (i.e., the characteristic of $\mathcal{K}$ is not 2). Our main result, in this section, is as follows:

Theorem 4.2. *Let $\mathcal{K}$be a non-Archimedean field, $\mathcal{X}$be a vector space over $\mathcal{K}$and $(\mathcal{Y}, \mu, T)$be a non-Archimedean complete* LRN*-space over $\mathcal{K}$Let $f : \mathcal{X} \to \mathcal{Y}$be an odd and Ψ-approximately mixed ACQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$ with $|2^k| < \alpha$,*

$$\Psi_{2^{-k}x, 2^{-k}y}(t) \geq \Psi_{x,y}(\alpha t), \quad \forall x \in \mathcal{X}, \ t > 0, \tag{6}$$

and

$$\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_\mathcal{L}, \quad \forall x \in \mathcal{X}, \ t > 0, \tag{7}$$

then there exists a unique cubic mapping $C : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\mu_{f(x)-C(x)}(t) \geq_L \mathcal{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \quad \forall x \in \mathcal{X}, \ t > 0, \tag{8}$$

where

$$M(x, t) := T(\Psi_{x,0}(t), \Psi_{2x,0}(t), \ldots, \Psi_{2^{k-1}x,0}(t)), \quad \forall x \in \mathcal{X}, \ t > 0.$$

Proof. First, by induction on j, we show that for any $x \in \mathcal{X}$, $t > 0$ and $j \geq 2$,

$$\mu_{f(4^j x)-256^j f(x)}(t) \geq M_j(x, t) := T(\Psi(x, 0, t), \ldots, \Psi(4^{j-1}x, 0, t)). \tag{9}$$

Putting $y = 0$ in (5), we obtain

$$\mu_{f(4x)-256f(x)}(t) \geq \Psi(x, 0, t), \quad \forall x \in \mathcal{X}, \ t > 0.$$

This proves (9) for $j = 2$. Assume that (9) holds for some $j \geq 2$. Replacing y by 0 and x by $4^j x$ in (5), we get

$$\mu_{f(4^{j+1}x)-256f(4^jx)}(t) \geq \Psi(4^j x, 0, t), \quad \forall x \in \mathcal{X},\ t > 0.$$

Since $|256| \leq 1$, we have

$$\begin{aligned}\mu_{f(4^{j+1}x)-256^{j+1}f(x)}(t) &\geq T(\mu_{f(4^{j+1}x)-256f(4^jx)}(t), \mu_{256f(4^jx)-256^{j+1}f(x)}(t)) \\ &= T\left(\mu_{f(4^{j+1}x)-256f(4^jx)}(t), \mu_{f(4^jx)-256^jf(x)}\left(\frac{t}{|256|}\right)\right) \\ &\geq T(\mu_{f(4^{j+1}x)-256f(4^jx)}(t), \mu_{f(4^jx)-256^jf(x)}(t)) \\ &\geq T(\Psi(4^j x, 0, t), M_j(x, t)) \\ &= M_{j+1}(x, t), \quad \forall x \in \mathcal{X}.\end{aligned}$$

Thus (9) holds for all $j \geq 2$. In particular,

$$\mu_{f(4^kx)-256^kf(x)}(t) \geq M(x, t), \quad \forall x \in \mathcal{X},\ t > 0. \tag{10}$$

Replacing x by $4^{-(kn+k)}x$ in (10) and using inequality (6), we obtain

$$\begin{aligned}\mu_{f\left(\frac{x}{4^{kn}}\right)-256^kf\left(\frac{x}{4^{kn+k}}\right)}(t) &\geq M\left(\frac{x}{4^{kn+k}}, t\right) \\ &\geq M(x, \alpha^{n+1}t), \quad \forall x \in \mathcal{X},\ t > 0,\ n \geq 0.\end{aligned} \tag{11}$$

Then, we have

$$\mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-(4^{4k})^{n+1}f\left(\frac{x}{(4^k)^{n+1}}\right)}(t) \geq M\left(x, \frac{\alpha^{n+1}}{|(4^{4k})^n|}t\right), \quad \forall x \in \mathcal{X},\ t > 0,\ n \geq 0,$$

and so

$$\begin{aligned}&\mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-(4^{4k})^{n+p}f\left(\frac{x}{(4^k)^{n+p}}\right)}(t) \\ &\geq T_{j=n}^{n+p}\left(\mu_{(4^{4k})^j f\left(\frac{x}{(4^k)^j}\right)-(4^{4k})^{j+p}f\left(\frac{x}{(4^k)^{j+p}}\right)}(t)\right) \\ &\geq T_{j=n}^{n+p} M\left(x, \frac{\alpha^{j+1}}{|(4^{4k})^j|}t\right) \\ &\geq T_{j=n}^{n+p} M\left(x, \frac{\alpha^{j+1}}{|(4^k)^j|}t\right), \quad \forall x \in \mathcal{X},\ t > 0,\ n \geq 0.\end{aligned}$$

Since $\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^{j+1}}{|(4^k)^j|}t\right) = 1$ for all $x \in \mathcal{X}$ and $t > 0$, $\left\{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)\right\}$ is a Cauchy sequence in the non-Archimedean random Banach space $(\mathcal{Y}, \mu, T)$. Hence we can define a mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{n\to\infty} \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-Q(x)}(t) = 1, \quad \forall x \in \mathcal{X},\ t > 0. \tag{12}$$

Next, for all $n \geq 1$, $x \in \mathcal{X}$ and $t > 0$, we have

$$\begin{aligned}\mu_{f(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t) &= \mu_{\sum_{i=0}^{n-1}(4^{4k})^i f\left(\frac{x}{(4^k)^i}\right)-(4^{4k})^{i+1} f\left(\frac{x}{(4^k)^{i+1}}\right)}(t)\\ &\geq \mathrm{T}_{i=0}^{n-1}\left(\mu_{(4^{4k})^i f\left(\frac{x}{(4^k)^i}\right)-(4^{4k})^{i+1} f\left(\frac{x}{(4^k)^{i+1}}\right)}(t)\right)\\ &\geq \mathrm{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|4^{4k}|^i}\right).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\mu_{f(x)-Q(x)}(t) &\geq T\left(\mu_{f(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-Q(x)}(t)\right)\\ &\geq T\left(\mathrm{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|4^{4k}|^i}\right), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-Q(x)}(t)\right).\end{aligned}$$

By letting $n \to \infty$, we obtain

$$\mu_{f(x)-Q(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|4^k|^i}\right),$$

which proves (8). Since T is continuous, from a well-known result in probabilistic metric space (see [49], Chapter 12), it follows that

$$\lim_{n\to\infty} \mu_{\Phi_1(x,y,k)}(t) = \mu_{\Phi_2(x,y)}(t), \quad \forall x, y \in \mathcal{X},\ t > 0,$$

for almost all $t > 0$., where

$$\begin{aligned}\Phi_1(x,y,k) =&(4^k)^n \cdot 16 f(4^{-kn}(x+4y)) + (4^k)^n f(4^{-kn}(4x-y))\\ &- 306[(4^k)^n \cdot 9 f(4^{-kn}(x+\frac{y}{3})) + (4^k)^n f(4^{-kn}(x+2y))]\\ &- 136(4^k)^n f(4^{-kn}(x-y)) + 1394(4^k)^n f(4^{-kn}(x+y))\\ &- 425(4^k)^n f(4^{-kn}y) + 1530(4^k)^n f(4^{-kn}x)\end{aligned}$$

and

$$\begin{aligned}\Phi_2(x,y) =&16Q(x+4y) + Q(4x-y) - 306\left[9Q\left(x+\frac{y}{3}\right) + Q(x+2y)\right]\\ &- 136Q(x-y) + 1394Q(x+y) - 425Q(y) + 1530Q(x).\end{aligned}$$

On the other hand, replacing x, y by $4^{-kn}x$, $4^{-kn}y$, respectively, in (5) and using (NA-RN2) and (6), we get

$$\mu_{\Phi_1(x,y,k)}(t) \geq \Psi\left(4^{-kn}x, 4^{-kn}y, \frac{t}{|4^k|^n}\right) \geq \Psi\left(x, y, \frac{\alpha^n t}{|4^k|^n}\right), \quad \forall x, y \in \mathcal{X},\ t > 0.$$

Since $\lim_{n\to\infty}\Psi\left(x, y, \frac{\alpha^n t}{|4^k|^n}\right) = 1$, it follows that Q is a quartic mapping.

If $Q' : \mathcal{X} \to \mathcal{Y}$ is another quartic mapping such that $\mu_{Q'(x)-f(x)}(t) \geq M(x, t)$ for all $x \in \mathcal{X}$ and $t > 0$, then, for all $n \in N$, $x \in \mathcal{X}$ and $t > 0$,

$$\mu_{Q(x)-Q'(x)}(t) \geq T\left(\mu_{Q(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-Q'(x)}(t), t)\right).$$

Therefore, by (12), we conclude that $Q = Q'$. This completes the proof. □

Corollary 4.3. *Let $\mathcal{K}$be a non-Archimedean field, $\mathcal{X}$be a vector space over $\mathcal{K}$and $(\mathcal{Y}, \mu, T)$be a non-Archimedean random Banach space over $\mathcal{K}$under a t-norm $T \in \mathcal{H}$. Let $f : \mathcal{X} \to \mathcal{Y}$be a Ψ-approximately quartic mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$, with $|4^k| < \alpha$*

$$\Psi(4^{-k}x, 4^{-k}y, t) \geq \Psi(x, y, \alpha t), \quad \forall x \in \mathcal{X}, \ t > 0,$$

then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$such that

$$\mu_{f(x)-Q(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|4|^{ki}}\right), \quad \forall x \in \mathcal{X}, \ t > 0,$$

where

$$M(x, t) := T(\Psi(x, 0, t), \Psi(4x, 0, t), \ldots, \Psi(4^{k-1}x, 0, t)), \quad \forall x \in \mathcal{X}, \ t > 0.$$

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1, \quad \forall x \in \mathcal{X}, \ t > 0,$$

and T is of Hadžić type, it follows that

$$\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1, \quad \forall x \in \mathcal{X}, \ t > 0.$$

Now, if we apply Theorem 4.2, we get the conclusion. □

Example 4.4. Let $(\mathcal{X}, \mu, T_M)$ non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + ||x||}, \quad \forall x \in \mathcal{X}, \ t > 0,$$

and $(\mathcal{Y}, \mu, T_M)$ a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1 + t}.$$

It is easy to see that (6) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1 + t},$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{T}_{M,j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathrm{T}_{M,j=n}^{m} M\left(x, \frac{t}{|4|^{kj}}\right)\right) \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t + |4^k|^n}\right) \\
&= 1, \quad \forall x \in \mathcal{X}, \ t > 0.
\end{aligned}$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a Ψ-approximately quartic mapping. Thus, all the conditions of Theorem 4.2 hold and so there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{t}{t + |4^k|}.$$

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Authors' contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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