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ORBIFOLDS AND FINITE GROUP REPRESENTATIONS

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Abstract. We present our recent understanding on resolutions of Gorenstein orbifolds, which involves the finite group representation theory. We concern only the quotient singularity of hypersurface type. The abelian group $A_r(n)$ for A-type hypersurface quotient singularity of dimension n is introduced. For $n = 4$, the structure of Hilbert scheme of group orbits and crepant resolutions of $A_r(4)$-singularity are obtained. The flop procedure of 4-folds is explicitly constructed through the process.

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1. Introduction. It is well known that the theory of “minimal” resolutions of singularity of algebraic (or analytical) varieties differs significantly when the (complex) dimension of the variety is larger than two. As the prime achievement in algebraic geometry of the 1980s, the minimal model program in the 3-dimensional birational geometry carried out by Mori and others, has provided an effective tool for the study of algebraic 3-folds (see [17] and the references therein). Meanwhile, Gorenstein quotient singularities in dimension 3 has attracted considerable interests among geometers due to the development of string theory, by which the orbifold Euler characteristic of an orbifold was proposed as the vacuum description of models built upon the quotient of a manifold [5]. The consistency of physical theory then demanded the existence of crepant resolutions which are compatible with the orbifold Euler characteristic. The complete mathematical justification of the conjecture was obtained in the mid-90s (see [26] and the references therein). However, due to the computational nature of methods in the proof, the qualitative understanding of these crepant resolutions has still been lacking on certain aspects from a mathematical viewpoint. Until very recently, by the development of Hilbert scheme of a finite group G-orbits, initiated by Nakamura and others with the result obtained in [1, 6, 10, 11, 12, 18, 19] it strongly indicates a promising role of the finite group in problems of resolutions of quotient singularities. In particular, a plausible method has been suggested on the study of geometry of orbifolds through the group representation theory. It has been known that McKay correspondence [16] between representations of Kleinian groups and affine A-D-E root diagrams has revealed a profound geometrical structure on the minimal resolution of the quotient surface singularity (cf. [7]). A similar connection between the finite group and general quotient singularity theories would be expected. Yet, the interest of this interplay of geometry and group representations would not only aim at the research of crepant resolutions, but also at its own right, due to possible implications on understanding some certain special kinds of group representations by engaging the rich algebraic geometry techniques.

In this article, we study problems related to the crepant resolutions of quotient singularities of higher dimension n (mainly for $n \geq 4$). Due to the many complicated exceptional cases of the problem, we restrict ourselves here only to those of the hypersurface singularity type. The purpose of this paper is to present certain primitive results of our first attempt on the study of the higher-dimensional hypersurface orbifolds under the principle of "geometrization" of finite group representations. We give a brief account of the progress recently made. The main issue we deal with in this work is the higher-dimensional generalization of the A-type Kleinian surface singularity, the $A_r(n)$-hypersurface singularity of dimension n (see (3.5) below). For $n = 4$, we are able to determine the detailed structure of $A_r(4)$-Hilbert scheme and its relation with crepant resolutions of $\mathbb{C}^4/A_r(4)$. In the process, an explicit "flop" construction of 4-folds among different crepant resolutions is found. In this article, we only sketch the main ideas behind the proof of these results, referring the reader to our forthcoming paper [2] for a more complete description of the methods and arguments used.

This paper is organized as follows. In Section 2, we give a brief introduction of the general scheme of engaging finite group representations in the birational geometry of orbifolds. Its connection with the Hilbert scheme of G-orbits for a finite linear group G on $\mathbb{C}^n$, $\text{Hilb}^G(\mathbb{C}^n)$, introduced in [11, 12, 18], will be explained in Section 3. In Section 4, we first review certain basic facts in toric geometry, which will be presented in the most suitable form for our goal, then focusing the case on $A_r(n)$-singularity. For $n = 3$, we give a thorough discussion on the explicit toric structure of $\text{Hilb}^{A_r(3)}(\mathbb{C}^3)$ as an illustration of the general result obtained by Nakamura on abelian group G in [19]. For a finite group in $\text{SL}_3(\mathbb{C})$, a recent result in [1] has shown that $\text{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. In Section 5, we deal with a special case of 4-dimensional orbifold with $G = A_1(4)$, and derive the detailed structure of $\text{Hilb}^G(\mathbb{C}^4)$. Its relation with the crepant resolutions of $\mathbb{C}^4/G$ is given, so is the "flop" relation among crepant resolutions. In Section 6, we describe the result of $G = A_r(4)$ for the arbitrary r , then end with some concluding remarks.

NOTATIONS. To present our work, we prepare some notations. By an orbifold we always mean the orbit space for a finite group action on a smooth complex manifold. For a finite group G, we denote

$$\text{Irr}(G) = \{\rho : G \longrightarrow \text{GL}(V_\rho) \text{ an irreducible representative of } G\}. \tag{1.1}$$

The trivial representation of G will be denoted by 1. For a G-module W, that is, a G-linear representation on a vector space W, one has the canonical irreducible decomposition

$$W = \bigoplus_{\rho \in \text{Irr}(G)} W_\rho, \tag{1.2}$$

where W_ρ is a G-submodule of W, G-isomorphic to $V_\rho \otimes W_\rho^0$ for some trivial G-module W_ρ^0. The vector space W_ρ will be called the ρ-factor of the G-module W.

For an analytic variety X, we do not distinct the notions of vector bundle and locally free $\mathbb{O}_X$-sheaf over X. For a vector bundle V over X, an automorphisms of V means a linear automorphism with the identity on X. If the bundle V is acted on by a group G as bundle automorphisms, we call V a G-bundle.

2. Representation theory in algebraic geometry of orbifolds. In this paper, G always denotes a finite (nontrivial) subgroup of $\mathrm{GL}_n(\mathbb{C})$ for $n \geq 2$, and $S_G := \mathbb{C}^n/G$ with the canonical projection,

$$\pi_G : \mathbb{C}^n \longrightarrow S_G, \tag{2.1}$$

and $o := \pi_G(0) \in S_G$. When G is a subgroup of $\mathrm{SL}_n(\mathbb{C})$, which will be our main concern later in this paper, G acts on $\mathbb{C}^n$ freely outside a finite collection of linear subspaces with codimension greater than or equal to 2. Then the orbifold S_G has a nonempty singular set, $\mathrm{Sing}(S_G)$, of codimension greater than or equal to 2, in fact, $o \in \mathrm{Sing}(S_G)$.

For G in $\mathrm{GL}_n(\mathbb{C})$, S_G is a singular variety in general. By a birational morphism of a variety over S_G, we always mean a proper birational morphism σ from variety X to S_G which defines a biregular map between $X \setminus \sigma^{-1}(\mathrm{Sing}(S_G))$ and $S_G \setminus \mathrm{Sing}(S_G)$,

$$\sigma : X \longrightarrow S_G. \tag{2.2}$$

One has the commutative diagram

$$\begin{tikzcd}
X \times_{S_G} \mathbb{C}^n \arrow[r] \arrow[d, "\pi"'] & \mathbb{C}^n \arrow[d, "\pi_G"] \\
X \arrow[r, "\sigma"] & S_G
\end{tikzcd} \tag{2.3}$$

Denote by $\mathcal{F}_X$ the coherent $\mathcal{O}_X$-sheaf over X obtained by the push-forward of the structure sheaf of $X \times_{S_G} \mathbb{C}^n$,

$$\mathcal{F}_X := \pi_* \mathcal{O}_{X \times_{S_G} \mathbb{C}^n}. \tag{2.4}$$

The sheaf $\mathcal{F}_X$ has the following functorial property, namely, for X, X' birational over S_G with the commutative diagram,

$$\begin{tikzcd}
X' \arrow[r, "\sigma'"] \arrow[d, "\mu"'] & S_G \arrow[d, equal] \\
X \arrow[r, "\sigma"] & S_G
\end{tikzcd} \tag{2.5}$$

one has a canonical morphism, $\mu^* \mathcal{F}_X \to \mathcal{F}_{X'}$. In particular, with the morphism (2.2) we have the $\mathcal{O}_X$-morphism

$$\sigma^*(\pi_{G*} \mathcal{O}_{\mathbb{C}^n}) \longrightarrow \mathcal{F}_X. \tag{2.6}$$

Furthermore, all the morphisms in (2.3) and (2.5) are compatible with the natural G-structure on $\mathcal{F}_X$ induced from the G-action on $\mathbb{C}^n$ via (2.3). One has the canonical G-decomposition of $\mathcal{F}_X$

$$\mathcal{F}_X = \bigoplus_{\rho \in \mathrm{Irr}(G)} (\mathcal{F}_X)_\rho, \tag{2.7}$$

where $(\mathcal{F}_X)_\rho$ is the ρ-factor of $\mathcal{F}_X$, and it is a coherent $\mathcal{O}_X$-sheaf over X. The geometrical fiber of $\mathcal{F}_X, (\mathcal{F}_X)_\rho$ over an element x of X are defined by

$$\mathcal{F}_{X,x} = k(x) \bigotimes_{\mathcal{O}_X} \mathcal{F}_X, \qquad (\mathcal{F}_X)_{\rho,x} = k(x) \bigotimes_{\mathcal{O}_X} (\mathcal{F}_X)_\rho, \tag{2.8}$$

where $k(x)(:= \mathcal{O}_{X,x}/\mathcal{M}_x)$ is the residue field at x. Over $X - \sigma^{-1}(\mathrm{Sing}(S_G))$, $\mathcal{F}_X$ is a vector bundle of rank $|G|$ with the regular G-representation on each geometric fiber. Hence $(\mathcal{F}_X)_\rho$ is a vector bundle over $X - \sigma^{-1}(\mathrm{Sing}(S_G))$ of rank equal to $\dim V_\rho$. For $x \in X$, there exists a G-invariant ideal $I(x)$ in $\mathbb{C}[Z](:= \mathbb{C}[Z_1, \dots, Z_n])$ such that the following relation holds:

$$\mathcal{F}_{X,x} = k(x) \bigotimes_{\mathcal{O}_{S_G}} \mathcal{O}_{\mathbb{C}^n}(x) \simeq \frac{\mathbb{C}[Z]}{I(x)}. \tag{2.9}$$

We have $(\mathcal{F}_X)_{\rho,x} \simeq (\mathbb{C}[Z]/I(x))_\rho$. The vector spaces $\mathbb{C}[Z]/I(x)$ form a family of finite-dimensional G-modules parametrized by $x \in X$, which are equivalent to the regular representation for elements outside $\sigma^{-1}(\mathrm{Sing}(S_G))$.

Set $X = S_G$ in (2.9). For $s \in S_G$, there is a G-invariant ideal $I(s)$ of $\mathbb{C}[Z]$; in fact, $I(s)$ is the ideal in $\mathbb{C}[Z]$ generated by the G-invariant polynomials vanishing at $\sigma^{-1}(s)$. Let $\tilde{I}(s)$ be the ideal of $\mathbb{C}[Z]$ consisting of all polynomials in $\mathbb{C}[Z]$ vanishing at $\sigma^{-1}(s)$. Then $\tilde{I}(s)$ is a G-invariant ideal, and we have

$$\tilde{I}(s) \supset I(s). \tag{2.10}$$

In particular, for $s = o$, we have

$$\tilde{I}(o) = \mathbb{C}[Z]_0, \qquad I(o) = \mathbb{C}[Z]_0^G \mathbb{C}[Z], \tag{2.11}$$

where the subscript 0 means the maximal ideal with polynomials vanishing at the origin. For a birational variety X over S_G via σ in (2.2), the following relations of G-invariant ideals of $\mathbb{C}[Z]$ hold:

$$\tilde{I}(s) \supset I(x) \supset I(s), \quad x \in X, \ s = \sigma(x). \tag{2.12}$$

A certain connection exists between algebraic geometry and G-modules through the variety X. For $x \in X$, there is a direct sum G-decomposition of $\mathbb{C}[Z]$,

$$\mathbb{C}[Z] = I(x)^\perp \oplus I(x). \tag{2.13}$$

Here $I(x)^\perp$ is a finite-dimensional subspace of $\mathbb{C}[Z]$ which is G-isomorphic to $\mathbb{C}[Z]/I(x)$. Similarly, we have the G-decomposition of $\mathbb{C}[Z]$ for $s = \sigma(x) \in S_G$,

$$\mathbb{C}[Z] = I(s)^\perp \oplus I(s), \qquad \mathbb{C}[Z] = \tilde{I}(s)^\perp \oplus \tilde{I}(s), \tag{2.14}$$

such that the following relations hold for the finite-dimensional G-modules:

$$\tilde{I}(s)^\perp \subset I(x)^\perp \subset I(s)^\perp. \tag{2.15}$$

Consider the canonical G-decomposition of $I(x)^\perp$,

$$I(x)^\perp = \bigoplus_{\rho \in \mathrm{Irr}(G)} I(x)^\perp_\rho. \tag{2.16}$$

Note that $I(x)_\rho^\perp$ is isomorphic to a positive finite copies of V_ρ. Then the affine structure of X near x is determined by the $\mathbb{C}$-algebra generated by all the G-invariant rational functions $f(Z)$ such that $f(Z)I(x)_\rho^\perp \subset I(x)$ for some ρ.

3. Hilbert scheme of finite group orbits. Among the varieties X that are birational over S_G with $\mathcal{F}_X$ a vector bundle, there exists a minimal object, called the G-Hilbert scheme in [11, 12, 18],

$$\sigma_{\text{Hilb}} : \text{Hilb}^G(\mathbb{C}^n) \longrightarrow S_G. \tag{3.1}$$

For another X, the map (2.2) can be factored through a birational morphism λ from X onto $\text{Hilb}^G(\mathbb{C}^n)$ via σ_{Hilb},

$$\lambda : X \longrightarrow \text{Hilb}^G(\mathbb{C}^n). \tag{3.2}$$

In fact, the ideal $I(x)$, $x \in X$, of (2.9) are with the co-length $|G|$, which gives rise to the above map λ of X to $\text{Hilb}^G(\mathbb{C}^n)$. We denote X_G as the normal variety over S_G defined by

$$X_G := \text{ normalization of } \text{Hilb}^G(\mathbb{C}^n), \quad \sigma_G : X_G \longrightarrow S_G. \tag{3.3}$$

By the fact that every biregular automorphism of S_G can always be lifted to one on $\text{Hilb}^G(\mathbb{C}^n)$, hence on X_G, one has the following result.

Lemma 3.1. *Denote by* $\text{Aut}(S_G)$ *the group of biregular automorphisms of* S_G. *Then* $\text{Hilb}^G(\mathbb{C}^n)$ *and* X_G *are varieties over* S_G *with the* $\text{Aut}(S_G)$*-equivariant covering morphisms.*

By the definition of $\text{Hilb}^G(\mathbb{C}^n)$, an element p of $\text{Hilb}^G(\mathbb{C}^n)$ represents a G-invariant ideal $I(p)$ of $\mathbb{C}[Z]$ of co-length $|G|$. The fiber of the vector bundle $\mathcal{F}_{\text{Hilb}^G(\mathbb{C}^n)}$ over p can be identified with the regular G-representation space $\mathbb{C}[Z]/I(p)$. Our study mainly concentrates on the relation of crepant resolutions of S_G and $\text{Hilb}^G(\mathbb{C}^n)$. For this purpose we assume for the rest of the paper that the group G is a subgroup of $\text{SL}_n(\mathbb{C})$

$$G \subset \text{SL}_n(\mathbb{C}), \tag{3.4}$$

which is the same to say that S_G has the Gorenstein quotient singularity. For $n = 2$, these groups were classified by Klein [14] into A-D-E types, the singularities are called Kleinian singularities. The minimal resolution $\hat{S}_G$ of S_G has the trivial canonical bundle (i.e., crepant), by [9]. In [11, 12, 18], Ito and Nakamura showed that $\text{Hilb}^G(\mathbb{C}^2)$ is equal to the minimal resolution $\hat{S}_G$. For $n = 3$, it has been known that there exist crepant resolutions for a 3-dimensional Gorenstein orbifold (see [26] and the references therein). Two different crepant resolutions of the same orbifolds are connected by a sequence of flop processes (cf. [22]). It was expected that $\text{Hilb}^G(\mathbb{C}^3)$ is one of those crepant resolutions. The assertion has been confirmed in the abelian group case in [19], and in general by [1].

For the motivation of our later study on the higher-dimensional singularities, we now illustrate the relation between G-Hilbert scheme and the minimal resolution in dimension 2, that is, surface singularities. For the rest of this section, we are going to describe the structure of $\mathrm{Hilb}^G(\mathbb{C}^2)$ for the A-type Kleinian group,

$$G = A_r := \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \epsilon^{r+1} = 1 \right\}, \quad r \geq 1. \tag{3.5}$$

The affine ring of $\mathbb{C}^2$ is $\mathbb{C}[Z](=\mathbb{C}[Z_1,Z_2])$ and G-invariant polynomials is the algebra generated by

$$Y_1 := Z_1^{r+1}, \qquad Y_2 := Z_2^{r+1}, \qquad X := Z_1Z_2. \tag{3.6}$$

Hence the ideal $I(o)$ in $\mathbb{C}[Z]$ for $o \in S_G$ is equal to $\langle Z_1^{r+1}, Z_1^{r+1}, Z_1Z_2\rangle$, and Z_1^k, Z_2^k $(0 \leq k \leq r)$, form a basis of the G-module $I(o)^\perp$. For a nontrivial character ρ, $I(o)_\rho^\perp$ is of dimension 2, in fact, it has a basis consisting of a pair, Z_1^k, Z_2^{r+1-k}, for some k. With the method of continued fraction [9], it is known that the minimal resolution $\hat{S}_G$ of S_G has the trivial canonical bundle with an open affine cover $\{\mathcal{U}_k\}_{k=0}^r$, where the coordinates (u_k, v_k) of $\mathcal{U}_k$ is expressed by

$$\mathcal{U}_k \simeq \mathbb{C}^2 \ni (u_k, v_k) = (Z_1^{k+1}Z_2^{-r+k}, Z_1^{-k}Z_2^{r+1-k}). \tag{3.7}$$

Denote by $\hat{o}_k$ the element in $\hat{S}_G$ with the coordinate $u_k = v_k = 0$. The exceptional divisor in $\hat{S}_G$ is $E_1 + \cdots + E_r$ where E_j is a rational (-2)-curve joining $\hat{o}_{j-1}$ and $\hat{o}_j$. The configuration can be realized in the following tree diagram:

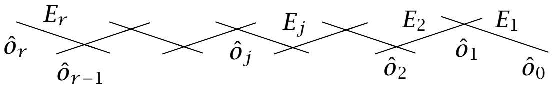

FIGURE 3.1. Exceptional curve configuration in the minimal resolution of $\mathbb{C}^2/A_r$.

It is easy to see that the ideal $I(\hat{o}_k)$ is given by

$$I(\hat{o}_k) = \langle Z_1^{k+1}, Z_2^{r+1-k}, Z_1Z_2\rangle, \tag{3.8}$$

hence the G-module $\mathbb{C}[Z]/I(\hat{o}_k)$ is the regular representation isomorphic to the following one,

$$I(\hat{o}_k)^\perp = \mathbb{C} + \sum_{i=1}^{k} \mathbb{C}Z_1^i + \sum_{j=1}^{r-k} \mathbb{C}Z_2^j. \tag{3.9}$$

One can represent monomials in (3.9) as the ones with • in Figure 3.2.

For $x \in \mathcal{U}_k$, the ideal $I(x)$ has the expression

$$I(x) = \langle Z_1^{k+1} - \alpha Z_2^{r-k}, Z_2^{r+1-k} - \beta Z_1^k, Z_1Z_2 - \alpha\beta\rangle, \quad \alpha, \beta \in \mathbb{C}. \tag{3.10}$$

The classes in $\mathbb{C}[Z]/I(x)$ represented by monomials in (3.9) still form a basis, hence give rise to a local frame of the vector bundle $\mathcal{F}_{\hat{S}_G}$ over $\mathcal{U}_k$. The divisor E_{k+1} is defined by $\beta = 0$, and its element approaches to $\hat{o}_{k+1}$ as α tends to infinity.

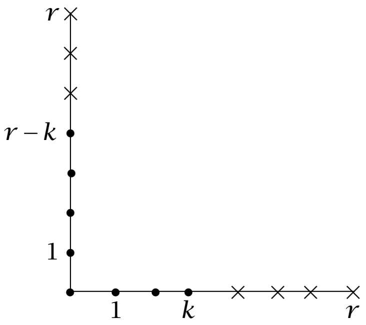

FIGURE 3.2. Representatives of $I(\hat{o}_k)^{\perp}(\simeq \mathbb{C}[Z]/I(\hat{o}_k))$ for the minimal resolution of $\mathbb{C}^2/A_r$.

4. Abelian orbifolds and toric geometry. In this section, we discuss the abelian group case of G in Section 3 using methods in toric geometry. We consider G as a subgroup of the diagonal group of $\mathrm{GL}_n(\mathbb{C})$, denoted by $T_0 := \mathbb{C}^{*n}$, and regard $\mathbb{C}^n$ as the partial compactification of T_0,

$$G \subset T_0 \subset \mathbb{C}^n. \tag{4.1}$$

Define the n-torus T with the toric embedding in $S_G(=\mathbb{C}^n/G)$ by

$$T := \frac{T_0}{G}, \quad T \subset S_G. \tag{4.2}$$

Techniques in toric geometry rely on lattices of one-parameter subgroups, characters of T_0, T,

$$\begin{aligned} N(:=\mathrm{Hom}(\mathbb{C}^*, T)) &\supset N_0(:=\mathrm{Hom}(\mathbb{C}^*, T_0)), \\ M(:=\mathrm{Hom}(T, \mathbb{C}^*)) &\subset M_0(:=\mathrm{Hom}(T_0, \mathbb{C}^*)). \end{aligned} \tag{4.3}$$

For our convenience, we make the following identification of N_0, N with lattices in $\mathbb{R}^n$. An element x in $\mathbb{R}^n$ has the coordinates x_i with respect to the standard basis $(e^1, \dots, e^n)$

$$x = \sum_{i=1}^{n} x_i e^i \in \mathbb{R}^n. \tag{4.4}$$

Then

$$N_0 = \mathbb{Z}^n(:=\exp^{-1}(1)), \qquad N = \exp^{-1}(G), \tag{4.5}$$

where $\exp : \mathbb{R}^n \to T_0$ is defined by $\exp(x) = \sum_i e^{2\pi\sqrt{-1}x_i} e^i$. Note that $G \simeq N/N_0$. The dual lattice M_0 of N_0 is the standard one in the dual space $\mathbb{R}^{n*}$, and we identify it with the group of monomials of $Z_1, \dots, Z_n$ via the correspondence

$$I = \sum_{s=1}^{n} i^s e_s \in M_0 \longleftrightarrow Z^I = \prod_{s=1}^{n} Z_s^{i_s}. \tag{4.6}$$

The dual lattice M of N is the sublattice of M_0 corresponding to the set of G-invariant monomials.

Over the T-space S_G, we now consider only those varieties X which are normal and birational over S_G with a T-structure, hence as it has been known, are presented by certain combinatorial data by the toric method [4, 13, 20]. Note that by Lemma 3.1, X_G is a toric variety over S_G. In general, a toric variety over S_G is described by a fan $\Sigma = \{\sigma_\alpha \mid \sigma \in I\}$ whose support equals to the first quadrant of $\mathbb{R}^n$, that is, a rational convex cone decomposition of the first quadrant of $\mathbb{R}^n$. Equivalently, it is determined by the intersection of the fan and the simplex $\triangle$ where

$$\triangle := \Big\{ x \in \mathbb{R}^n \mid \sum_i x_i = 1,\ x_j \geq 0\ \forall j \Big\}. \tag{4.7}$$

The data in $\triangle$ is given by $\Lambda = \{\triangle_\alpha \mid \alpha \in I\}$, where $\triangle_\alpha := \sigma_\alpha \bigcap \triangle$. The $\triangle_\alpha$s form a decomposition of $\triangle$ by convex subsets, having the vertices in $\triangle \cap \mathbb{Q}^n$. Note that for $\sigma_\alpha = \{\vec{0}\}$, we have $\triangle_\alpha = \emptyset$. We call Λ a rational polytope decomposition of $\triangle$, and denote the corresponding toric variety by X_Λ. We call Λ an integral polytope decomposition of $\triangle$ if all the vertices of Λ are in N. For a rational polytope decomposition Λ of $\triangle$, we define $\Lambda(i) := \{\triangle_\alpha \in \Lambda \mid \dim(\triangle_\alpha) = i\}$ for $-1 \leq i \leq n-1$, (here $\dim(\emptyset) := -1$). Then T-orbits in X_Λ are parametrized by $\bigsqcup_{i=-1}^{n-1} \Lambda(i)$. In fact, for each $\triangle_\alpha \in \Lambda(i)$, there associates a T-orbit of the dimension $n-1-i$, denoted by $\mathrm{orb}(\triangle_\alpha)$. A toric divisor in X_Λ is the closure of an $(n-1)$-dimensional orbit, denoted by $D_v = \overline{\mathrm{orb}(v)}$ for $v \in \Lambda(0)$. The canonical sheaf of X_Λ has the expression in terms of toric divisors (cf. [13]),

$$\omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda}\Bigg(\sum_{v \in \Lambda(0)} (m_v - 1) D_v \Bigg), \tag{4.8}$$

where m_v is the positive integer such that $m_v v$ is a primitive element of N. In particular, the crepant property of X_Λ, that is, $\omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda}$, is given by the integral condition of Λ. The nonsingular criterion of X_Λ is the simplicial decomposition of Λ together with the multiplicity one property, that is, for each $\Lambda_\alpha \in \triangle(n-1)$, the elements $m_v v$, $v \in \Lambda_\alpha \cap \Lambda(0)$, form a $\mathbb{Z}$-basis of N. The following results are known for toric variety over S_G (cf. [21] and the references therein).

(1) The Euler number of X_Λ is given by $\chi(X_\Lambda) = |\Lambda(n-1)|$.

(2) For a rational polytope decomposition Λ of $\triangle$, any two of the following three conditions implies the third one:

$$X_\Lambda \text{ is nonsingular}, \qquad \omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda}, \qquad \chi(X_\Lambda) = |G|. \tag{4.9}$$

It is easy to see that the following result holds for the sheaf $\mathcal{F}_{X_\Lambda}$.

Lemma 4.1. *Let Λ be a rational polytope decomposition of $\triangle$, and let x_0 be the zero-dimensional toric orbit in X_Λ corresponding to an element $\triangle_{\alpha_0}$ in $\Lambda(n-1)$. Let $Z^{I^{(j)}}$, $1 \leq j \leq N$, be a finite collection of monomials whose classes generate the G-module $\mathbb{C}[Z]/I(x_0)$. Then the classes of $Z^{I^{(j)}}$s also generate $\mathbb{C}[Z]/I(y)$ for $y \in \mathrm{orb}(\triangle_\beta)$ with $\triangle_\beta \subseteq \triangle_{\alpha_0}$.*

Define

$$A_r(n) := \{ g \in \mathrm{SL}_n(\mathbb{C}) \mid g \text{ is diagonal and } g^{r+1} = 1 \}, \quad r \geq 1. \tag{4.10}$$

Note that the above group for $n = 2$ is the same as A_r in (3.5). For a general n, $A_r(n)$-invariant polynomials in $\mathbb{C}[Z]$ are generated by the following $(n+1)$ ones:

$$X := \prod_{i=1}^{n} Z_i, \qquad Y_j := Z_j^{r+1} \quad (j = 1,\dots,n). \tag{4.11}$$

This implies that $S_{A_r(n)}$ is the singular hypersurface in $\mathbb{C}^{n+1}$,

$$S_{A_r(n)} = \{(x, y_1,\dots,y_n) \in \mathbb{C}^{n+1} \mid x^{r+1} = y_1 \cdots y_n\}. \tag{4.12}$$

For the rest of this paper, we conduct the discussion of abelian orbifolds mainly on the group $A_r(n)$. The ideal $I(o)$ of $\mathbb{C}[Z]$ associated to the element $o \in S_{A_r(n)}$ is given by

$$I(o) = \langle Z_1^{r+1},\dots,Z_n^{r+1}, Z_1 \cdots Z_n\rangle, \tag{4.13}$$

hence

$$I(o)^{\perp} = \bigoplus \left\{ \mathbb{C}Z^I \mid I = (i^1,\dots,i^n),\ 0 \le i^j \le r,\ \prod_{j=1}^{n} i^j = 0 \right\}. \tag{4.14}$$

For $\mathbf{1} \ne \rho \in \operatorname{Irr}(A_n(r))$, the dimension of $I(o)^{\perp}_{\rho}$ is always greater than one. In fact, one can describe explicitly a set of monomial generators of $I(o)^{\perp}_{\rho}$. For example, say $I(o)^{\perp}_{\rho}$ containing an element Z^I with $I = (i^1,\dots,i^n)$, $i^1 = 0$ and $i^s \le i^{s+1}$, then $I(o)^{\perp}_{\rho}$ is generated by $Z^K s$ with $K = (k^1,\dots,k^n)$ given by

$$k^s = \begin{cases} r+1-i^j+i^s & \text{if } i^s < i^j, \\ i^s - i^j & \text{otherwise}, \end{cases} \tag{4.15}$$

here j runs through 1 to n. In particular for $r = 1$, the dimension of $I(o)^{\perp}_{\rho}$ is equal to 2 for $\rho \ne \mathbf{1}$, with a basis consisting of Z^I, $Z^{I'}$ whose indices satisfy the relations, $0 \le i^s, i^{s'} \le 1$, $i^s + i^{s'} = 1$ for $1 \le s \le n$.

For $n = 3$, by the general result of Nakamura on an abelian group G (see [19, Theorem 4.2]), $\operatorname{Hilb}^{A_r(3)}(\mathbb{C}^3)$ is a crepant toric variety. To illustrate this fact, we give here a direct derivation of the result by working on the explicitly described toric variety.

Example 4.2. It is easy to see that $\triangle \cap N$ consists of the following elements:

$$v^{m_1,m_2,m_3} := \frac{1}{r+1} \sum_{j=1}^{3} m_j e^j, \quad m_j \in \mathbb{Z}_{\ge 0},\ \sum_{j=1}^{3} m_j = r+1. \tag{4.16}$$

Denote by Ξ the simplicial decomposition of $\triangle$ obtained by drawing the three lines parallel to the edges of $\triangle$ through each element of $\triangle \cap N$. The 2-simplexes in $\Xi(2)$ consists of two types of triangles, $\triangle_u^{m_1,m_2,m_3}$, $\triangle_d^{m_1,m_2,m_3}$ (see Figure 4.1):

$$\begin{aligned} \triangle_u^{m_1,m_2,m_3} &= \langle v^{m_1,m_2,m_3}, v^{m_1-1,m_2+1,m_3}, v^{m_1-1,m_2,m_3+1}\rangle, \\ \triangle_d^{m_1,m_2,m_3} &= \langle v^{m_1,m_2,m_3}, v^{m_1-1,m_2+1,m_3}, v^{m_1,m_2+1,m_3-1}\rangle, \end{aligned} \tag{4.17}$$

the vertices of each one form a $\mathbb{Z}$-basis of N.

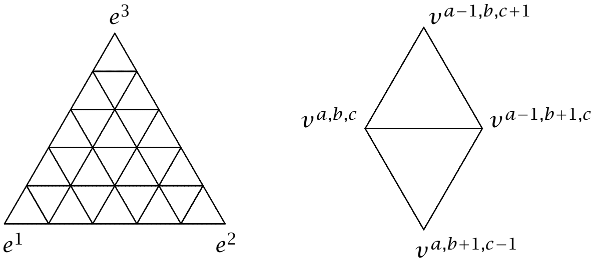

FIGURE 4.1. The right one is the simplicial decomposition Ξ of $\triangle$ for $r = 4$. The left one is the union of $\triangle_u^{a,b,c}$ and $\triangle_d^{a,b,c}$.

Therefore the X_Ξ is a smooth toric variety. The 0-dimensional (toric) orbits corresponding to $\triangle_u^{m_1,m_2,m_3}$, $\triangle_d^{m_1,m_2,m_3}$ are denoted by $x_u^{m_1,m_2,m_3}$, $x_d^{m_1,m_2,m_3}$, respectively. The affine coordinate system centered at $x_u^{m_1,m_2,m_3}$ is given by

$$U_1 = \frac{Z_1^{r+2-m_1}}{(Z_2Z_3)^{m_1-1}}, \qquad U_2 = \frac{Z_2^{r+1-m_2}}{(Z_1Z_3)^{m_2}}, \qquad U_3 = \frac{Z_3^{r+1-m_3}}{(Z_1Z_2)^{m_3}}. \tag{4.18}$$

For y in X_Ξ with the value U_j equal to α_j, by computation, the ideal $I(y)$ has the following generators:

$$\begin{aligned}&Z_1^{r+2-m_1} - \alpha_1(Z_2Z_3)^{m_1-1}, \quad Z_2^{r+1-m_2} - \alpha_2(Z_3Z_1)^{m_2}, \quad Z_3^{r+1-m_3} - \alpha_3(Z_1Z_2)^{m_3},\\ &(Z_1Z_2)^{m_3+1} - \alpha_1\alpha_2 Z_3^{r-m_3}, \quad (Z_2Z_3)^{m_1} - \alpha_2\alpha_3 Z_1^{r+1-m_1}, (Z_3Z_1)^{m_2+1} - \alpha_3\alpha_1 Z_2^{r-m_2},\\ &Z_1Z_2Z_3 - \alpha_1\alpha_2\alpha_3.\end{aligned} \tag{4.19}$$

For the element $x_u^{m_1,m_2,m_3}$, that is, $\alpha_i = 0$ for all i, it is easy to see that there are $(r+1)^2$ monomials not in $I(x_u^{m_1,m_2,m_3})$, which form a basis of $I(x_u^{m_1,m_2,m_3})^\perp$, and give rise to a basis of $\mathbb{C}[Z]/I(y)$ for y in the affine neighborhood of $x_u^{m_1,m_2,m_3}$. The same argument applies to the affine neighborhood near $x_u^{m_1,m_2,m_3}$ with the coordinate functions

$$V_1 = \frac{(Z_2Z_3)^{m_1}}{Z_1^{r+1-m_1}}, \qquad V_2 = \frac{(Z_1Z_3)^{m_2+1}}{Z_2^{r-m_2}}, \qquad V_3 = \frac{(Z_1Z_2)^{m_3}}{Z_3^{r+1-m_3}}, \tag{4.20}$$

hence the description of ideals for y in X_Ξ with $V_j = \beta_j$,

$$\begin{aligned}&Z_1^{r+2-m_1} - \beta_2\beta_3(Z_2Z_3)^{m_1-1}, \quad Z_2^{r+1-m_2} - \beta_3\beta_1(Z_3Z_1)^{m_2}, \quad Z_3^{r+2-m_3} - \beta_1\beta_2(Z_1Z_2)^{m_3-1},\\ &(Z_1Z_2)^{m_3} - \beta_3 Z_3^{r+1-m_3}, \quad (Z_2Z_3)^{m_1} - \beta_1 Z_1^{r+1-m_1}, \quad (Z_3Z_1)^{m_2+1} - \beta_2 Z_2^{r-m_2},\\ &Z_1Z_2Z_3 - \beta_1\beta_2\beta_3.\end{aligned} \tag{4.21}$$

Therefore, we have shown that X_Ξ is birational over $\mathrm{Hilb}^G(\mathbb{C}^3)$. Now we are going to show that they are in fact the same. Let x be an element in $\mathrm{Hilb}^G(\mathbb{C}^3)$ represented by a monomial ideal $J = I(x)$ (i.e., with a set of generators composed of monomials). Then the regular G-module $J^\perp$ is generated by $|G|$ monomials, and x lies over the element o

of S_G, equivalently, J contains the ideal $\mathbb{C}[Z]_0^G$. Denote by l_i the smallest nonnegative integer such that $Z_i^{l_i} \in J$, by $l_{i,j}$ the smallest nonnegative integer with $(Z_iZ_j)^{l_{i,j}} \in J$ for $i \neq j$. Hence $1 \leq l_i \leq r+1$, and $Z_i^{l_i-1} \in J^{\perp}$, which implies $(Z_1Z_2Z_3/Z_i)^{r+2-l_i} \in J$. In particular, $Z_1^{l_1-1} \in J^{\perp}$ and $(Z_2Z_3)^{r+2-l_1} \in J$. By the description (4.15) for $I(o)^{\perp}$, $Z_1^{l_1}$ is the only monomial in the basis of $I(o)^{\perp}$ for the corresponding character of G, the same for $(Z_2Z_3)^{r+1-l_1}$. Hence $(Z_2Z_3)^{r+2-l_1} \in J^{\perp}$, which implies $l_1 + l_{2,3} = r+2$. Similarly, we have $l_2 + l_{1,3} = l_3 + l_{1,2} = r+2$. Again by (4.15), $Z_1^{l_1-1}Z_2^{l_{1,2}-1}$, $Z_2^{r+1-l_1+l_{1,2}}Z_3^{r+2-l_1}$, $Z_1^{l_1-l_{1,2}}Z_3^{l_3}$ are the generators of an eigenspace in $I(o)^{\perp}$. The latter two are elements in J by $(Z_2Z_3)^{r+2-l_1}$, $Z_3^{l_3} \in J$. Therefore $Z_1^{l_1-1}Z_2^{l_{1,2}-1} \in J^{\perp}$. Similarly, one has

$$Z_2^{l_2-1}Z_1^{l_{1,2}-1},\ Z_1^{l_1-1}Z_3^{l_{1,3}-1},\ Z_3^{l_3-1}Z_1^{l_{1,3}-1},\ Z_2^{l_2-1}Z_3^{l_{2,3}-1},\ Z_3^{l_3-1}Z_2^{l_{2,3}-1} \in J^{\perp}. \tag{4.22}$$

This implies that J is the ideal in $\mathbb{C}[Z]$ generated by $Z_i^{l_i}$, $(Z_jZ_k)^{l_{j,k}}$, $Z_1Z_2Z_3$, where $\{i,j,k\} = \{1,2,3\}$ with $j < k$, and l_i, $l_{j,k}$ are integers satisfying the relations $l_i + l_{j,k} = r+2$, $1 \leq l_i \leq r+1$. Therefore the number of monomials in $J^{\perp}$ is given by

$$\begin{aligned} 1 + \sum_{j=1}^{3}(l_j - 1) + \sum_{j<k}(l_j-1)(l_k-1) - (l_j - l_{j,k})(l_k - l_{j,k}) \\ = -\Big(\sum_j l_j\Big)^2 + (4r+7)\sum_j l_j - 4(r+1)^2 - 6(r+1) - 2, \end{aligned} \tag{4.23}$$

which is equal to $(r+1)^2$ by $J \in \mathrm{Hilb}^G(\mathbb{C}^3)$. The only solutions for $\sum_j l_j$ are $2(r+1)+1$, $2(r+1)+2$. Hence the ideal J is characterized by integers l_i between 1 and $r+1$ with the relation

$$\sum_{j=1}^{3} l_j = 2(r+1)+1, \qquad \sum_{j=1}^{3} l_j = 2(r+1)+2. \tag{4.24}$$

If $\sum_{j=1}^{3} l_j = 2(r+1)+1$, the ideal J corresponds to the ideal of $x_u^{m_1,m_2,m_3}$ in X_{Ξ} with

$$l_1 = r+2-m_1, \qquad l_2 = r+1-m_2, \qquad l_3 = r+1-m_3. \tag{4.25}$$

For $\sum_{j=1}^{3} l_j = 2(r+1)+2$, the ideal J corresponds to the ideal for the element $x_d^{m_1,m_2,m_3}$ in X_{Ξ} with

$$l_1 = r+2-m_1, \qquad l_2 = r+1-m_2, \qquad l_3 = r+2-m_3. \tag{4.26}$$

Now by techniques of Groebner basis for ideals in $\mathbb{C}[Z]$ (cf. [3]), given a monomial order, one has a monomial ideal $\mathrm{lt}(J)$ (generated by the leading monomial of the Groebner basis of J) such that all monic monomials outside $\mathrm{lt}(J)$ form a base of $\mathbb{C}[Z]/J$. Thus, one sees that for an element in $\mathrm{Hilb}^G(\mathbb{C}^3)$ that is a G-invariant ideal J' with the regular G-module $\mathbb{C}[Z]/J'$, there is a basis of $\mathbb{C}[Z]/J'$ represented by monomials in $J^{\perp}$ for some J previously described. This shows that $\mathrm{Hilb}^G(\mathbb{C}^3) = X_{\Xi}$.

Remark 4.3. As in the discussion of the case for $n = 2$ in Section 3, one can represent the monomial basis elements of $I(x_u^{m_1,m_2,m_3})^{\perp}$, $I(x_d^{m_1,m_2,m_3})^{\perp}$ in a pictorial way. For example, the ones in Figure 4.2 are for $r = 4$, and $m_1 = 2$, $m_2 = 1$, $m_3 = 2$.

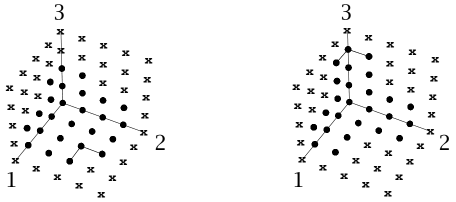

FIGURE 4.2. Graph representation of $I(x_u^{m_1,m_2,m_3})^\perp$, $I(x_d^{m_1,m_2,m_3})^\perp$ for $r = 4$, $(m_1,m_2,m_3) = (2,1,2)$. A dot point • indicates a monomial in $I^\perp$ while an "**x**" means one in I. The difference between two graphs are marked by broken segments.

5. $A_1(4)$-singularity and flop of 4-folds. We now study the $A_r(n)$-singularity with $n \geq 4$. For simplicity, we consider the case $r = 1$, that is, $G = A_1(n)$ (indeed, no conceptual difficulties arise for higher values of r). The N-integral elements in $\triangle$ are as follows:

$$\triangle \bigcap N = \{e^j \mid 1 \leq j \leq n\} \bigcup \{v^{i,j} \mid 1 \leq i < j \leq n\}, \tag{5.1}$$

where $v^{i,j} := (1/2)(e^i + e^j)$ for $i \neq j$. Other than the whole simplex $\triangle$, there is only one integral polytope decomposition of $\triangle$ invariant under permutations of coordinates, denoted by Ξ, which we now describe as follows. There are $n+1$ elements in $\Xi(n-1)$ such that $\triangle_i$, $1 \leq i \leq n$, together with $\diamond$, where $\triangle_i$ is the simplex generated by e^i and $v^{i,j}$ for $j \neq i$, and $\diamond$ = the closure of $\triangle \setminus \bigcup_{i=1}^n \triangle_i$. In fact, $\diamond$ is the convex hull spanned by all the $v^{i,j}$ for $i \neq j$. The lower-dimensional polytopes of Ξ are given by the faces of those in $\Xi(n-1)$. Then X_Ξ has the trivial canonical sheaf. However, only for $n = 2,3$, X_Ξ is a crepant resolution of $S_{A_1(n)}$ (cf. [21]). For $n = 4$, one has the following result.

LEMMA 5.1. *For $n = 4$, the toric variety X_Ξ is smooth except one isolated singularity, which is the 0-dimensional T-orbit corresponding to $\diamond$.*

PROOF. In general, for $n \geq 4$, it is easy to see that for each i, the vertices of $\triangle_i$ form a $\mathbb{Z}$-basis of N, for example, say $i = 1$, it follows from $|A_1(n)| = 2^{n-1}$, and

$$\det(e^1, v^{1,2}, \ldots, v^{1,n}) = \frac{1}{2^{n-1}}. \tag{5.2}$$

Hence X_Ξ is nonsingular near the T-orbits associated to simplices in $\triangle_i$. As $\diamond$ is not a simplex, $\operatorname{orb}(\diamond)$ is always a singular point of X_Ξ. For $n = 4$, the statement of smoothness of X_Ξ except $\operatorname{orb}(\diamond)$ follows from the fact that for $1 \leq i \leq 4$, the vertices $v^{i,j}$ $(j \neq i)$ of X_Ξ, together with $(1/2)\sum_{j=1}^4 e^j$, form a N-basis. □

REMARK 5.2. For $n \geq 4$, the following properties hold for 0-dimensional T-orbits of X_Ξ.

(1) Let $x_j := \operatorname{orb}(\triangle_j) \in X_\Xi$ for $1 \leq j \leq n$. The inverse of the matrix of vertices of $\triangle_j$,

$$(v^{1,j}, \ldots, v^{j-1,j}, e^j, v^{j+1,j}, \ldots, v^{n,j})^{-1}, \tag{5.3}$$

gives rise to affine coordinates $(U_1,\dots,U_n)$ around x_j

$$U_i = Z_i^2 \quad (i \neq j), \qquad U_j = \frac{Z_j}{Z_1 \cdots \check{Z}_j \cdots Z_n}. \tag{5.4}$$

Hence

$$I(x_j) = \langle Z_j, Z_i^2,\ i \neq j\rangle + I_{A_1(n)} \tag{5.5}$$

with the regular $A_1(n)$-module isomorphism,

$$\frac{\mathbb{C}[Z]}{I(x_j)} \simeq \bigoplus \{\mathbb{C}Z^I \mid I = (i_1,\dots,i_n),\ i_j = 0,\ i_k = 0,1 \text{ for } k \neq j\}. \tag{5.6}$$

(2) We denote by $x_\diamond$ the element $\mathrm{orb}(\diamond)$ in X_Ξ, $x_\diamond := \mathrm{orb}(\diamond)$. The singular structure of $x_\diamond$ is determined by those $A_1(n)$-invariant polynomials, corresponding to M-integral elements in the cone dual to the one generated by $\diamond$ in $N_{\mathbb{R}}$. It is easy to see that these polynomials are generated by the following ones:

$$X_j := Z_j^2, \qquad Y_j := \frac{Z_1 \cdots \check{Z}_j \cdots Z_n}{Z_j}. \tag{5.7}$$

Hence we have

$$I(x_\diamond) = \langle Z_1 \cdots \check{Z}_j \cdots Z_n\rangle_{1\le j\le n} + I_{A_1(n)}. \tag{5.8}$$

Note that for $n = 3$, Y_j's form the minimal generators for the invariant polynomials, which implies the smoothness of X_Ξ. For $n \geq 4$, $x_\diamond$ is an isolated singularity, but not of the hypersurface type. For $n = 4$, the X_j, Y_j $(1 \le j \le 4)$ form a minimal set of generators of invariant polynomials, hence the structure near $x_\diamond$ in X_Ξ is the 4-dimensional affine variety in $\mathbb{C}^8$ defined by the relations

$$(x_i, y_i)_{1\le i\le 4} \in \mathbb{C}^8, \quad x_i y_i = x_j y_j, \quad x_i x_j = y_{i'} y_{j'}, \tag{5.9}$$

where $i \neq j$ and $\{i', j'\}$ is the complementary pair of $\{i, j\}$.

For the rest of this section, we consider only the case $n = 4$. We discuss the crepant resolutions of $S_{A_1(4)}$, and its relation with $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$. Now the simplex $\triangle$ is a tetrahedron, and $\diamond$ is an octahedron, on which the symmetric group $\mathfrak{S}_4$ acts as the standard representation. The dual polygon of $\diamond$ is the cubic. Faces of the octahedron $\diamond$ are labelled by F_j, F_j' for $1 \le j \le 4$, where

$$F_j = \diamond \cap \triangle_j, \qquad F_j' = \left\{ \sum_{i=1}^{4} x_j e^j \in \diamond \mid x_j = 0 \right\}. \tag{5.10}$$

The dual of F_j, F_j' in the cubic are vertices, denoted by j, j' as in Figure 5.1

Consider the rational simplicial decomposition Ξ^* of $\triangle$, which is a refinement of Ξ by adding the center

$$c := \frac{1}{4}\sum_{j=1}^{4} e^j \tag{5.11}$$

as a vertex with the barycentric decomposition of $\diamond$ in Ξ. Note that $c \notin N$ and $2c \in N$. See Figure 5.2.

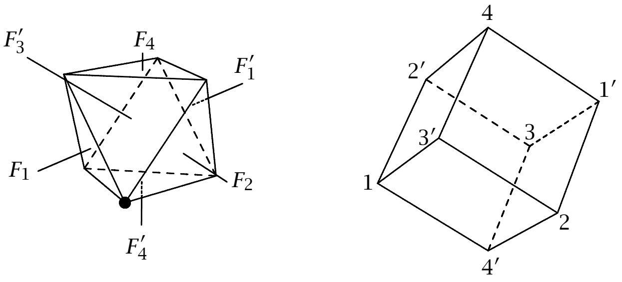

FIGURE 5.1. Dual pair of octahedron and cube. Faces F_j, F_j' of the octahedron on the left dual to vertices j, j' of the cube on the right. The face of the cube with vertices $1, 4', 2, 3'$ corresponds to the dot "•" in the octahedron.

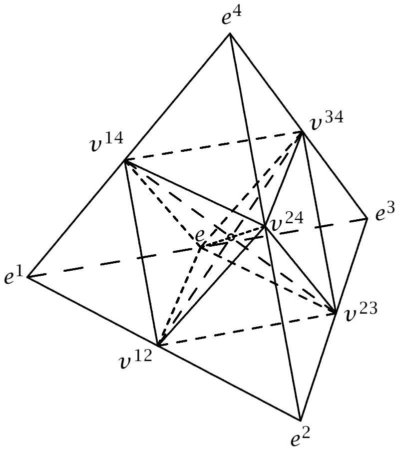

FIGURE 5.2. The rational simplicial decomposition Ξ^* of $\triangle$ for $n = 4$, $r = 1$.

THEOREM 5.3. *For $G = A_1(4)$, there is*

$$\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4) = X_{A_1(4)} \simeq X_{\Xi^*}, \tag{5.12}$$

which is nonsingular with the canonical bundle $\omega = \mathcal{O}_{X_{\Xi^*}}(E)$, where E is an irreducible divisor isomorphic to the triple product of $\mathbb{P}^1$,

$$E = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1. \tag{5.13}$$

Furthermore for $\{i,j,k\} = \{1,2,3\}$, the normal bundle of E when restricted on the $\mathbb{P}^1$-fiber, $\mathbb{P}^1_k$, for the projection on $\mathbb{P}^1 \times \mathbb{P}^1$ via the (i,j)th factor,

$$p_k : E \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \tag{5.14}$$

is the (-1)-hyperplane bundle

$$\mathcal{O}_{X_{\Xi^*}}(E) \otimes \mathcal{O}_{\mathbb{P}^1_k} \simeq \mathcal{O}_{\mathbb{P}^1}(-1). \tag{5.15}$$

PROOF. By Lemma 5.1 and Remark 5.2(1) after that, one can see the smoothness of X_{Ξ^*} on the affine chart corresponding to $\triangle_j$, also its relation with $\mathrm{Hilb}^G(\mathbb{C}^4)$. For the rest of simplexes, the octahedron $\Diamond$ of Ξ is decomposed into eight simplexes corresponding to the faces F_j, F'_j of $\Diamond$. Denote by C_j (C'_j) the simplex of Ξ^* spanned by c and F_j (resp., F'_j), and $x_{C_j}, x_{C'_j}$ the elements in X_{Ξ^*} of the corresponding T-orbit. First we show that for $x = x_{C_j}, x_{C'_j}$, $\mathcal{F}_{X_{\Xi^*},x}$ is a regular G-module. It is easy to see that the vertices of F_j together with $2c$ form an integral basis of N, the same for the vertices of F'_j. For the convenience of notation, we can set $j = 1$, without loss of generality. Then we have the integral basis of M for the cones, dual to C_1, C'_1 as follows:

$$\begin{aligned}
\operatorname{cone}(C_1)^* : (2c, v^{1,2}, v^{1,3}, v^{1,4})^{-1} &= \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \\
\operatorname{cone}(C'_1)^* : (2c, v^{2,3}, v^{2,4}, v^{3,4})^{-1} &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.
\end{aligned} \tag{5.16}$$

Therefore, the following 4 functions form a smooth coordinate of X_{Ξ^*} near x_{C_j} for $j = 1$,

$$U_1 = \frac{Z_2Z_3Z_4}{Z_1}, \qquad U_2 = \frac{Z_1Z_2}{Z_3Z_4}, \qquad U_3 = \frac{Z_1Z_3}{Z_2Z_4}, \qquad U_4 = \frac{Z_1Z_4}{Z_2Z_3}, \tag{5.17}$$

and one has

$$I(x_{C_1}) = \langle Z_2Z_3Z_4, Z_1Z_2, Z_1Z_3, Z_1Z_4\rangle + I_G. \tag{5.18}$$

Similarly, the coordinates near $x_{C'_j}$ for $j = 1$ are given by

$$U'_1 = Z_1^2, \qquad U'_2 = \frac{Z_2Z_3}{Z_1Z_4}, \qquad U'_3 = \frac{Z_2Z_4}{Z_1Z_3}, \qquad U'_4 = \frac{Z_3Z_4}{Z_1Z_2}, \tag{5.19}$$

and we have

$$I(x_{C'_1}) = \langle Z_2Z_3, Z_2Z_4, Z_3Z_4\rangle + I_G. \tag{5.20}$$

It is easy to see that the G-modules, $\mathbb{C}[Z]/I(x_{C_1})$, $\mathbb{C}[Z]/I(x_{C_1})$, are both equivalent to the regular representation. Therefore, the ideals $I(x)$ for $x = x_{\triangle_j}, x_{C_j}, x_{C'_j}$ $(1 \le j \le 4)$, give rise to distinct elements in $\operatorname{Hilb}^{A_1(4)}(\mathbb{C}^4)$. In fact, one can show that $X_{\Xi^*} = \operatorname{Hilb}^{A_1(4)}(\mathbb{C}^4)$ (for the details, see [2]). By (4.8), the canonical bundle of X_{Ξ^*} is given by

$$\omega_{X_{\Xi^*}} = \mathcal{O}_{X_{\Xi^*}}(E), \tag{5.21}$$

where E is the toric divisor D_c. It is known that E is a 3-dimensional complete toric variety arised from the star of c in Ξ^*, which is given by the octahedron in Figure 5.1; in fact, the cube in Figure 5.1 represents the toric orbits' structure. Therefore, E is isomorphic to the triple product of $\mathbb{P}^1$ as in (5.13). The conclusion of the normal

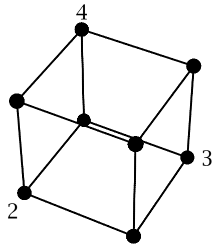

FIGURE 5.3. The monomial basis of the G-module $I(x_{\triangle_1})^\perp(\simeq \mathbb{C}[Z]/I(x_{\triangle_1}))$ in the Z_2-Z_3-Z_4 space.

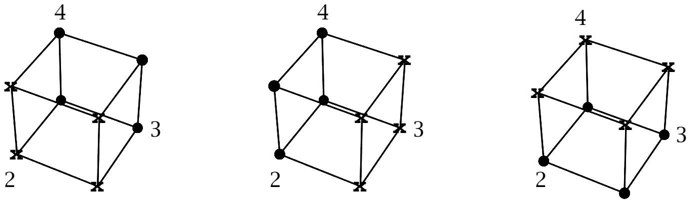

FIGURE 5.4. The corresponding $I^\perp$-graph for the simplex Δ_2, Δ_3, and Δ_4 in X_{Ξ^*}s. A dot point • means a monomial in $I(x_{\triangle_i})^\perp$, while an "**x**" means one in $I(x_{\triangle_i})$.

bundle of E restricting on each $\mathbb{P}^1$-fiber will follow from techniques in toric geometry. For example, for fibers over the projection of E onto the $(\mathbb{P}^1)^2$ corresponding to the 2-convex set spanned by $v^{1,2}$, $v^{1,3}$, $v^{3,4}$, $v^{2,4}$, one can perform the computation as follows. Let (U_1,U_2,U_3,U_4) be the local coordinates near $x_{C_4'}$ dual to the N-basis $(2c,v^{1,2},v^{1,3},v^{2,3})$, similarly the local coordinate (W_1,W_2,W_3,W_4) near x_{C_1} dual to $(2c,v^{1,2},v^{1,3},v^{1,4})$. By $2c = v^{1,4}+v^{2,3}$, one has the relations

$$U_1 = W_1W_4, \qquad U_4 = W_4^{-1}, \qquad U_j = W_j\,(j=2,3). \tag{5.22}$$

This shows that the restriction of the normal bundle of E on a fiber $\mathbb{P}^1$ over (U_2,U_3)-plane is the (-1)-hyperplane bundle. □

The sheaf $\mathcal{F}_{X_{\Xi^*}}$ for X_{Ξ^*} in Theorem 5.3 is a vector bundle with the regular G-module on each fiber. The local frame of the vector bundle is provided by the structure of $\mathbb{C}[Z]/I(x)$ for x being the zero-dimensional toric orbit of X_{Ξ^*}. One can have a pictorial realization of monomial basis of these G-representations as follows. We start with the element $x_{\triangle_1}$, and the identification, $\mathbb{C}[Z]/I(x_{\triangle_1}) = I(x_{\triangle_1})^\perp$. The eigenbasis of the G-module $I(x_{\triangle_1})^\perp$ is given by monomials in the diagram of Figure 5.3.

By the fact that the ρ-eigenspace of $I(o)^\perp$ for the element $o \in S_G$ has the dimension 2 for a nontrivial character ρ, one can present the data of the regular G-module $\mathbb{C}[Z]/I(x)$ for another element x by indicating the ones to be replaced in the Figure 5.3, which will be marked by **x**. All of the replacement are listed in Figures 5.4, 5.5, and 5.6.

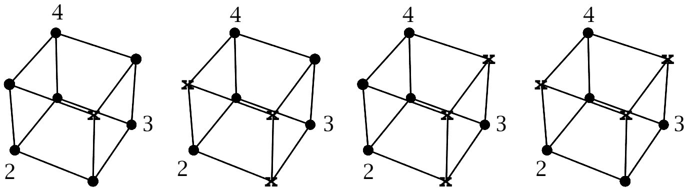

FIGURE 5.5. The corresponding $I^{\perp}$-graph for the simplex C_1, C_2, C_3, and C_4.

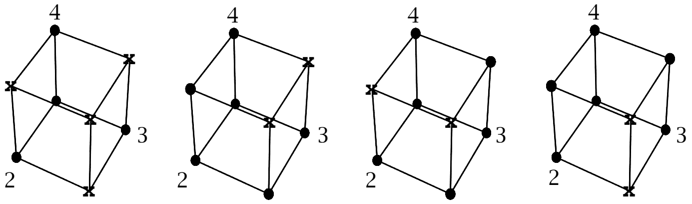

FIGURE 5.6. The corresponding $I^{\perp}$-graph for the simplex C_1, C_2, C_3, and C_4.

By the standard blowing-down criterion of an exceptional divisor, the property (5.15), ensures the existence of a smooth 4-fold $(X_{\Xi^*})_k$ by blowing down the family of $\mathbb{P}^1$s along the projection p_k (5.14) for each k. In fact, $(X_{\Xi^*})_k$ is also a toric variety X_{Ξ_k} where Ξ_k is the refinement of Ξ by adding the segment connecting $v^{k,4}$ and $v^{i,j}$ to divide the central polygon $\diamondsuit$ into 4 simplexes, where $\{i,j,k\} = \{1,2,3\}$. Each X_{Ξ_k} is a crepant resolution of $X_{\Xi}(= S_{A_1(4)})$. We have the relation of refinements: $\Xi \prec \Xi_k \prec \Xi^*$ for $k = 1,2,3$. The polyhedral decomposition in the central part $\diamondsuit$ appeared in the refinement relation are denoted by

$$\diamondsuit \prec \diamondsuit_k \prec \diamondsuit^*, \quad k = 1,2,3, \tag{5.23}$$

of which the pictorial realization is given in Figure 5.7. The connection of smooth 4-folds for different $\diamondsuit_k$ can be regarded as a “flop” of 4-folds suggested by the similar procedure in the theory of 3-dimensional birational geometry. Each one is a “small” resolution of a 4-dimensional isolated singularity defined by (5.9). Here the smallness for a resolution means one with the exceptional locus of codimension greater than or equal to 2. Hence we have shown the following result.

THEOREM 5.4. *For $G = A_1(4)$, there are crepant resolutions of S_G obtained by blowing down the divisor E of $\mathrm{Hilb}^G(\mathbb{C}^4)$ along (5.14) in Theorem 5.3. Any two such resolutions differ by a “flop” procedure of 4-folds.*

6. $A_r(4)$-singularity and conclusion remarks. For $G = A_r(n)$, $n \geq 4$, the structure of $\mathrm{Hilb}^G(\mathbb{C}^n)$ and its relation with possible crepant resolutions of S_G has been an ongoing program under investigation. We have discussed the simplest case $A_1(4)$ in Theorem 5.3. A similar conclusion holds also for $n = 4$, but an arbitrary r, whose proof relies on more complicated techniques. The details will be given in [2].

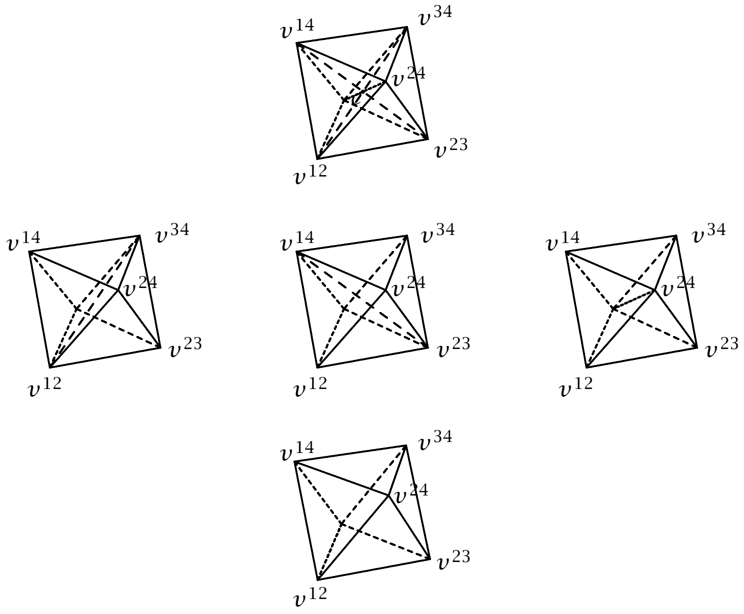

FIGURE 5.7. Toric representation of 4-dimensional flops over a common singular base and dominated by the same 4-fold.

THEOREM 6.1. *The G-Hilbert scheme* $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$ *is the nonsingular toric variety* $X_{A_r(4)}$ *with the canonical bundle*

$$\omega_{X_{A_r(4)}} = \mathbb{O}\left(\sum_{k=1}^{m} E_k\right), \quad m = \frac{r(r+1)(r+2)}{6}, \tag{6.1}$$

where E_k*'s are disjoint smooth exceptional divisors in* $X_{A_r(4)}$*, each of which satisfies conditions (5.13) and (5.15). Associated to a projection (5.14) for each* E_k*, there corresponds a toric crepant resolution* $\widehat{S}_{A_r(4)}$ *of* $S_{A_r(4)}$ *with*

$$\chi(\widehat{S}_{A_r(4)}) = |A_r(4)| = (r+1)^3. \tag{6.2}$$

Furthermore, any two such $\widehat{S}_{A_r(4)}$ *differ by flops of* 4*-fold.*

One can also describe a monomial basis of the $A_r(4)$-module $\mathbb{C}[Z]/I(x)$ for $x \in \mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$, similar to the one we have given in Section 5 for the case $r = 1$.

Another type of hypersurface orbifolds are those from the quotient singularities of simple groups. For $n = 3$, the well-known examples are S_G for $G =$ the icosahedral group I_{60}, Klein group H_{168}, in which cases a crepant resolution of S_G was explicitly constructed in [15, 25], respectively. The structure of $\mathrm{Hilb}^G(\mathbb{C}^3)$ has recently been discussed in [6]. Even though the crepant and smooth property of the group orbit Hilbert schemes for dimension 3 is known by [1], a clear quantitative and qualitative relation would still be interesting for the possible study of some other simple groups

G in higher dimensions. Such program is under consideration with initial progress being made.

Even for the abelian group G in the dimension $n = 3$, the conclusion on the trivial canonical bundle of $\mathrm{Hilb}^G(\mathbb{C}^3)$ would raise a subtle question in the Mirror problem of Calabi-Yau 3-folds in string theory. As an example, a standard well-known one is the Fermat quintic in $\mathbb{P}^4$ with the special marginal deformation family

$$X : \sum_{j=1}^{5} z_i^5 + \lambda z_1 z_2 z_3 z_4 z_5 = 0, \quad \lambda \in \mathbb{C}. \tag{6.3}$$

With the maximal diagonal group SD of z_i's preserving the family X, the mirror X^* is constructed by "the" crepant resolution of X/SD, $X^* = \widehat{X/SD}$ (cf. [8, 23]), by which the roles of $H^{1,1}$, $H^{2,1}$ are interchangeable in the "quantum" sense. When working on the one-dimensional space $H^{1,1}(X) \sim H^{2,1}(X^*)$, the choice of crepant resolution $\widehat{X/SD}$ makes no difference on the conclusion. While on the part of $H^{2,1}(X) \sim H^{1,1}(X^*)$, it has been known that many topological invariants, like Euler characteristic, Hodge numbers, elliptic genus, are independent of the choices of crepant resolutions, hence one obtains the same invariants for different choices of crepant resolutions as the model for X^*. However, the topological triple intersection of cohomologies does differ for two crepant resolutions (cf. [24]), hence the choice of crepant resolution as the mirror $X^* = \widehat{X/SD}$ will lead to the different effect on the topological cubic form of $H^{1,1}(X^*)$, upon which as the "classical" level, the quantum triple product of the physical mirror theory would be built (cf. [27]). The question of the "good" model for X^* has rarely been raised in the past, partly due to the lack of mathematical knowledge on the issue. However, with the G-Hilbert scheme now given in Sections 3 and 4 as the mirror X^*, it seems to have left some fundamental open problems on its formalism of mirror Calabi-Yau spaces and the question of the arbitrariness of the choice of crepant resolutions remains a mathematical question to be completely understood concerning its applicable physical theory.

For the role of G-Hilbert scheme in the study of crepant resolution of S_G, the conclusion we have obtained for $G = A_r(4)$ has indicated that $\mathrm{Hilb}^G(\mathbb{C}^n)$ could not be a crepant resolution of S_G in general when the dimension n is greater than 3. Nevertheless the structure of $\mathrm{Hilb}^G(\mathbb{C}^n)$ is worthwhile for further study on its own right due to the interplay of geometry and group representations. Its understanding could still lead to the construction of crepant resolutions of S_G in case such one does exist. It would be a promising direction of the geometrical study of orbifolds.

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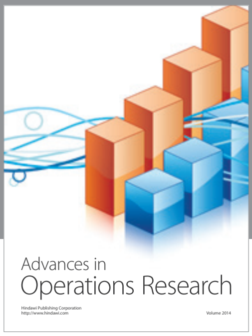
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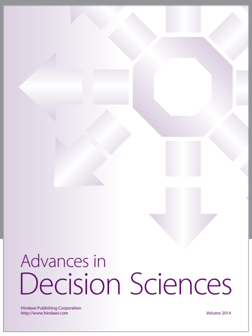
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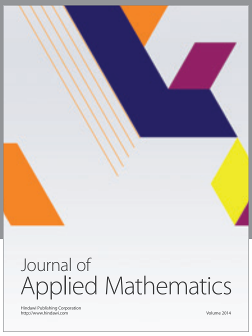
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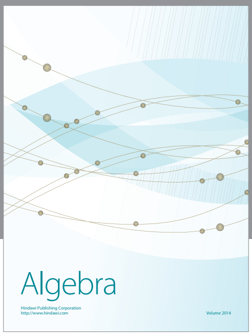
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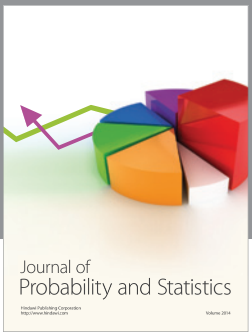
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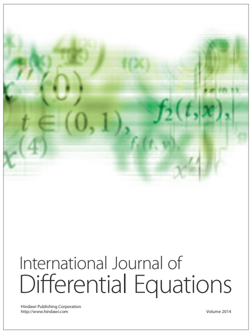
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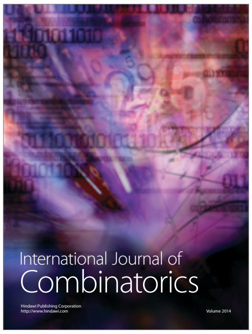
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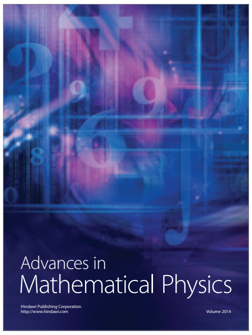
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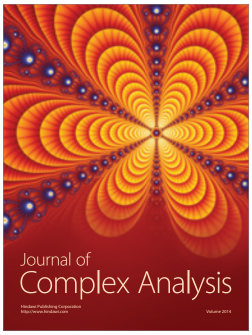
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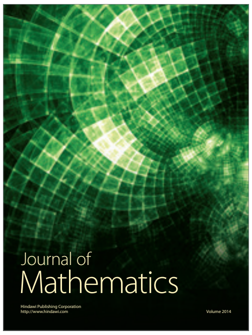
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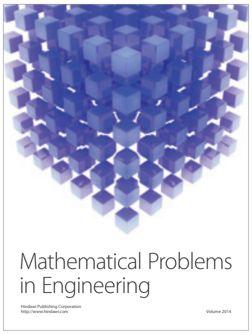
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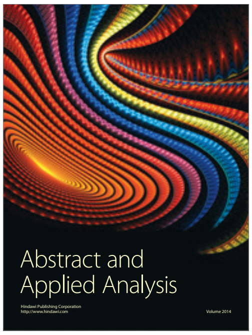
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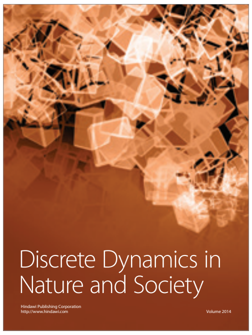
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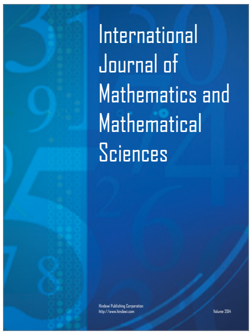
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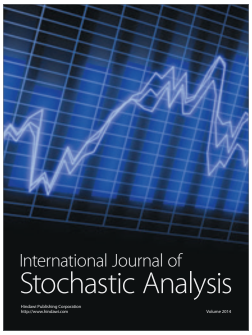
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