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Research Article

Solving the Telegraph and Oscillatory Differential Equations by a Block Hybrid Trigonometrically Fitted Algorithm

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We propose a block hybrid trigonometrically fitted (BHT) method, whose coefficients are functions of the frequency and the step-size for directly solving general second-order initial value problems (IVPs), including systems arising from the semidiscretization of hyperbolic Partial Differential Equations (PDEs), such as the Telegraph equation. The BHT is formulated from eight discrete hybrid formulas which are provided by a continuous two-step hybrid trigonometrically fitted method with two off-grid points. The BHT is implemented in a block-by-block fashion; in this way, the method does not suffer from the disadvantages of requiring starting values and predictors which are inherent in predictor-corrector methods. The stability property of the BHT is discussed and the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency advantages.

1. Introduction

In what follows, we consider the numerical solution of the general second order IVPs of the form

$$\begin{aligned} y'' &= f\left(x, y, y'\right), \\ y\left(x_0\right) &= y_0, \\ y'\left(x_0\right) &= y'_0, \\ & x \in \left[x_0, x_N\right], \end{aligned} \quad (1)$$

where $f : \mathbb{R} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$, $N > 0$ is an integer, and m is the dimension of the system. Problems of the form of (1) frequently arise in several areas of science and engineering such as classical mechanics, celestial mechanics, quantum mechanics, control theory, circuit theory, astrophysics, and biological sciences. Equation (1) is traditionally solved by reducing it into a system of first-order IVPs of double dimension and then solved using the various methods that are available for solving systems of first-order IVPs (see Lambert [1, 2], Hairer and Wanner in [3], Hairer [4], and Brugnano and Trigiante [5, 6]).

Nevertheless, there are numerous methods for directly solving the special second-order IVPs in which the first derivative does not appear explicitly and it has been shown that these methods have the advantages of requiring less storage space and fewer number of function evaluations (see Hairer [4], Hairer et al. [7], Simos [8], Lambert and Watson [9], and Twizell and Khaliq [10]). Fewer methods have been proposed for directly solving second-order IVPs in which the first derivative appears explicitly (see Vigo-Aguiar and Ramos [11], Awoyemi [12], Chawla and Sharma [13], Mahmoud and Osman [14], Franco [15], and Jator [16]). It is also the case that some of these IVPs possess solutions with special properties that may be known in advance and taken advantage of when designing numerical methods. In this light, several methods have been presented in the literature which take advantage of the special properties of the solution that may be known in advance (see Coleman and Duxbury [17], Coleman and Ixaru [18], Simos [19], Vanden Berghe et al. [20], Vigo-Aguiar and Ramos [11], Fang et al. [21], Nguyen et al. [22], Ramos and

Vigo-Aguiar [23], Franco and Gómez [24], and Ozawa [25]). However, most of these methods are restricted to solving special second-order IVPs in a predictor-corrector mode.

Our objective is to present a BHT that is implemented in a block-by-block fashion; in this way, the method does not suffer from the disadvantages of requiring starting values and predictors which are inherent in predictor-corrector methods (see Jator et al. [26], Jator [27], and Ngwane and Jator [28]). We note that multiderivative trigonometrically-fitted block methods for $y'' = f(x, y, y')$ have been proposed in Jator [29] and Jator [27]. However, the BHT proposed in this paper avoids the computation of higher order derivatives which have the potential to increase computational cost, especially when applied to nonlinear systems. We note that Ramos et al. [30] recently proposed a trigonometrically fitted optimized two-step hybrid block method for solving the general second-order IVPs with oscillatory solutions. However, the method given in [30] is of an order 2; hence, in this paper, we propose a BHT which is of order 5 and its application is extended to solving PDEs such as the Telegraph equation.

The organization of this paper is as follows. In Section 2, we derive the BHT for solving (1). The analysis and implementation of the BHT are discussed in Section 3. Numerical examples are given in Section 4 to show the accuracy and efficiency of the BHT. Finally, the conclusion of the paper is given in Section 5.

2. Development of Method

Consider

$$\begin{aligned} y_{n+2} &= \alpha_{2,0} y_n + \alpha_{2,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta_{2,j} f_{n+j} + \beta_{2,n+v} f_{n+v} + \beta_{2,n+\epsilon} f_{n+\epsilon}, \\ y_{n+v} &= \alpha_{v,0} y_n + \alpha_{v,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta_{v,j} f_{n+j} + \beta_{v,n+v} f_{n+v} + \beta_{v,n+\epsilon} f_{n+\epsilon}, \\ y_{n+\epsilon} &= \alpha_{\epsilon,0} y_n + \alpha_{\epsilon,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta_{\epsilon,j} f_{n+j} + \beta_{\epsilon,n+v} f_{n+v} + \beta_{\epsilon,n+\epsilon} f_{n+\epsilon} \end{aligned} \tag{2}$$

which are used together with additional methods given as

$$\begin{aligned} hy'_{n+2} &= \alpha'_{2,0} y_n + \alpha'_{2,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta'_{2,j} f_{n+j} + \beta'_{2,n+v} f_{n+v} + \beta'_{2,n+\epsilon} f_{n+\epsilon}, \\ hy'_{n+v} &= \alpha'_{v,0} y_n + \alpha'_{v,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta'_{v,j} f_{n+j} + \beta'_{v,n+v} f_{n+v} + \beta'_{v,n+\epsilon} f_{n+\epsilon}, \\ hy'_{n+1} &= \alpha'_{1,0} y_n + \alpha'_{1,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta'_{1,j} f_{n+j} + \beta'_{1,n+v} f_{n+v} + \beta'_{1,n+\epsilon} f_{n+\epsilon}, \\ hy'_{n+\epsilon} &= \alpha'_{\epsilon,0} y_n + \alpha'_{\epsilon,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta'_{\epsilon,j} f_{n+j} + \beta'_{\epsilon,n+v} f_{n+v} + \beta'_{\epsilon,n+\epsilon} f_{n+\epsilon}, \\ hy'_{n} &= \alpha'_{0,0} y_n + \alpha'_{0,1} y_{n+1} \\ &\quad + h^2 \sum_{j=0}^{2} \beta'_{0,j} f_{n+j} + \beta'_{0,n+v} f_{n+v} + \beta'_{0,n+\epsilon} f_{n+\epsilon}, \end{aligned} \tag{3}$$

where $\alpha_{j,0}$, $\alpha_{j,1}$, and $\beta_{j,j}$, $j = 0, v, 1, \epsilon, 2$, are coefficients that depend on the step-length h and frequency w. The coefficients of the method are chosen so that the method integrates IVP (1) exactly where the solutions are members of the linear space $\langle 1, x, x^2, x^3, x^4, \sin(wx), \cos(wx) \rangle$.

The main method has the form

$$y_{n+2} = \sum_{i=0}^{1} \alpha_i y_{n+i} + h^2 \sum_{j=0}^{2} \beta_j f_{n+j} + \beta_{n+v} f_{n+v} + \beta_{n+\epsilon} f_{n+\epsilon}, \tag{4}$$

where α_i, $i = 0, 1$, and β_j, $j = 0, v, 1, \epsilon, 2$, are to be determined coefficient functions of the frequency and step-size. To derive the main method and additional methods, we initially seek a continuous local approximation $\Pi(x)$ on the interval $[x_n, x_{n+2}]$ of the form

$$\Pi(x) = \alpha_0(x)\, y_n + \alpha_1(x)\, y_{n+1} + h^2 \sum_j \beta_j(x)\, f_{n+j}, \qquad j = 0, v, 1, \epsilon, 2 \tag{5}$$

which represents our CHT and where $\alpha_0(x)$, $\alpha_1(x)$, and $\beta_j(x)$, $j = 0, v, 1, \epsilon, 2$, are continuous coefficients. The first derivative of (5) is given by

$$\Pi'(x) = \frac{d}{dx} \Pi(x). \tag{6}$$

We assume that $y_{n+j} = \Pi(x_{n+j})$ is the numerical approximation to the analytical solution $y(x_{n+j})$, $y'_{n+j} = \Pi'(x_{n+j})$ is the numerical approximation to $y'(x_{n+j})$, and $f_{n+j} = \Pi''(x_{n+j})$ is an approximation to $y'(x_{n+j})$, $j = 0, v, 1, \epsilon, 2$.

The following theorem shows how continuous method (5) is constructed. This is done by requiring that on the interval from x_n to $x_{n+2} = x_n + 2h$ the exact solution is locally approximated by function (5) with (6) obtained as a consequence.

Theorem 1. *Let $F_i(x) = x^i$, $i = 0, 1, 2, 3, 4$, $F_5(x) = \sin wx$, and $F_6(x) = \cos wx$ be basis functions and $K =$*

$(y_n, y_{n+1}, f_n, f_{n+v}, f_{n+1}, f_{n+\epsilon}, f_{n+2})^T$ a vector, where T is the transpose. Define the matrix G by

$$G = \begin{pmatrix} F_0(x_n) & \cdots & F_6(x_n) \\ F_0(x_{n+1}) & \cdots & F_6(x_{n+1}) \\ F_0''(x_n) & \cdots & F_6''(x_n) \\ F_0''(x_{n+v}) & \cdots & F_6''(x_{n+v}) \\ F_0''(x_{n+1}) & \cdots & F_6''(x_{n+1}) \\ F_0''(x_{n+\epsilon}) & \cdots & F_6''(x_{n+\epsilon}) \\ F_0''(x_{n+2}) & \cdots & F_6''(x_{n+2}) \end{pmatrix} \quad (7)$$

and G_i is obtained by replacing the ith column of G by the vector K. Let the following conditions be satisfied:

$$\begin{aligned} \Pi(x_n) &= y_n, \\ \Pi'(x_n) &= y_n', \\ \Pi''(x_n + j) &= f_{n+j}, \quad j = 0, v, 1, \epsilon, 2; \end{aligned} \quad (8)$$

then continuous representations (5) and (6) are equivalent to the following:

$$\Pi(x) = \sum_{i=0}^{6} \frac{\det(G_i)}{\det(G)} F_i(x), \quad (9)$$

$$\Pi'(x) = \frac{d}{dx}\left(\sum_{i=0}^{6} \frac{\det(G_i)}{\det(G)} F_i(x)\right). \quad (10)$$

Proof. We use the approach given in Jator [16] with appropriate notational modification. Let method (5) be defined by the assumed basis functions:

$$\begin{aligned} \alpha_0(x) &= \sum_{i=0}^{6} \alpha_{i+1,0} F_i(x), \\ \alpha_1(x) &= \sum_{i=0}^{6} \alpha_{i+1,1} F_i(x), \\ h^2\beta_j(x) &= \sum_{i=0}^{6} h^2\beta_{i+1,j} F_i(x), \end{aligned} \quad (11)$$

where $\alpha_{i+1,0}$, $\alpha_{i+1,1}$, and $h^2\beta_{i+1,j}$, are coefficients to be determined. Substituting (11) into (5), we get

$$\begin{aligned} \Pi(x) = {} & \sum_{i=0}^{6} \alpha_{i+1,0} F_i(x) y_n + \sum_{i=0}^{6} \alpha_{i+1,1} F_i(x) y_{n+1} \\ & + \sum_j \sum_{i=0}^{6} h^2 \beta_{i+1,j} F_i(x) f_{n+j} \end{aligned} \quad (12)$$

which is simplified to

$$\begin{aligned} \Pi(x) = \sum_{i=0}^{6} \Bigg(& \alpha_{i+1,0} F_i(x) y_n + \alpha_{i+1,1} F_i(x) y_{n+1} \\ & + \sum_j h^2 \beta_{i+1,j} F_i(x) f_{n+j} \Bigg) \end{aligned} \quad (13)$$

and expressed as

$$\Pi(x) = \sum_{i=0}^{6} \ell_i F_i(x), \quad (14)$$

where

$$\ell_i = \alpha_{i+1,0} y_n + \alpha_{i+1,1} y_{n+1} + \sum_j h^2 \beta_{i+1,j} f_{n+j}. \quad (15)$$

By imposing conditions (8) on (14), we obtain a system of six equations which can be expressed as

$$GL = K, \quad (16)$$

where $L = (\ell_0, \ell_1, \ldots, \ell_6)^T$ is a vector whose coefficients are determined via Cramer's rule as

$$\ell_i = \frac{\det(G_i)}{\det(G)}, \quad i = 0, 1, \ldots, 6, \quad (17)$$

where G_i is obtained by replacing the ith column of G by K. In order to obtain the continuous approximation, we use the elements of L to rewrite (14) as

$$\Pi(x) = \sum_{i=0}^{6} \frac{\det(G_i)}{\det(G)} F_i(x), \quad (18)$$

whose first derivative is given by

$$\Pi'(x) = \frac{d}{dx}\left(\sum_{i=0}^{6} \frac{\det(G_i)}{\det(G)} F_i(x)\right). \quad (19)$$

□

Remark 2. In the derivation of the BHT, the basis functions $F_i(x) = x^i$, $i = 0, 1, 2, 3, 4$, $F_5(x) = \sin wx$, and $F_6(x) = \cos wx$ are chosen because they are simple to analyze. Nevertheless, other possible bases are possible (see Nguyen et al. [22]).

2.1. Specification of the Method. We note that continuous methods (9) and (10) which are equivalent to forms (5) and (6) are used to generate three discrete methods and five additional methods. The discrete and additional methods are then applied as a BHT for solving (1). We choose $v = 1/2$, $\epsilon = 3/2$ and evaluating (9) at $x = x_{n+2}$, $x = x_{n+v}$, and $x = x_{n+\epsilon}$, respectively, gives the three discrete methods $y_{n+2} = \Pi(x_n + 2h)$, $y_{n+v} = \Pi(x_n + vh)$, and $y_{n+\epsilon} = \Pi(x_n + \epsilon h)$ which take the form of the main method. Evaluating (10) at $x = x_n$,

$x = x_{n+v}$, $x = x_{n+1}$, $x = x_{n+\epsilon}$, and $x = x_{n+2}$, respectively, gives the additional methods $y'_n = \Pi'(x_n)$, $y'_{n+v} = \Pi'(x_n + vh)$, $y'_{n+1} = \Pi'(x_n + h)$, $y'_{n+\epsilon} = \Pi'(x_n + \epsilon h)$, and $y'_{n+2} = \Pi'(x_n + 2h)$. The coefficients and their corresponding Taylor series equivalence of y_{n+v}, $y_{n+\epsilon}$, y_{n+2}, hy'_n, hy'_{n+v}, hy'_{n+1}, $hy'_{n+\epsilon}$, and hy'_{n+2} are, respectively, given as follows:

$$\begin{aligned}
\alpha_{v,0} &= \frac{1}{2},\\
\alpha_{v,1} &= \frac{1}{2},\\
\beta_{v,0} &= \frac{\left(-7u^2 + \left(13u^2 - 144\right)\cos(u/2) + 48\cos(u) + 96\right)\csc^4(u/4)}{768u^2}\\
&= -\frac{19}{1920} - \frac{221u^2}{1935360} - \frac{233u^4}{232243200} - \frac{199u^6}{27249868800} - \frac{1543u^8}{34780741632000} - \frac{26281u^{10}}{128549621071872000} + \cdots,\\
\beta_{v,v} &= -\frac{h^2\left(23u^2 - 12\left(u^2 + 32\right)\cos(u/2) + \left(13u^2 + 144\right)\cos(u) + 240\right)\csc^4(u/4)}{768u^2}\\
&= -\frac{17h^2}{160} + \frac{79h^2u^2}{241920} + \frac{79h^2u^4}{29030400} + \frac{61h^2u^6}{3406233600} + \frac{4099h^2u^8}{47823519744000} + \frac{1739h^2u^{10}}{16068702633984000} + \cdots,\\
\beta_{v,1} &= \frac{\left(\left(17u^2 - 144\right)\cos(u/2) + \left(u^2 + 72\right)\cos(u) + 72\right)\csc^4(u/4)}{384u^2}\\
&= -\frac{7}{960} - \frac{19u^2}{64512} - \frac{83u^4}{38707200} - \frac{u^6}{100925440} + \frac{577u^8}{63764692992000} + \frac{773u^{10}}{856997473812480} + \cdots,\\
\beta_{v,\epsilon} &= -\frac{\left(5u^2 + \left(u^2 + 48\right)\cos(u/2) - 48\right)\cot^2(u/4)\csc^2(u/4)}{192u^2}\\
&= -\frac{1}{480} + \frac{u^2}{15120} + \frac{u^4}{7257600} - \frac{u^6}{212889600} - \frac{167u^8}{1707982848000} - \frac{2633u^{10}}{2008587829248000} + \cdots,\\
\beta_{v,2} &= \frac{\left(5u^2 + \left(u^2 + 48\right)\cos(u/2) - 48\right)\csc^4(u/4)}{768u^2}\\
&= \frac{1}{1920} + \frac{31u^2}{1935360} + \frac{67u^4}{232243200} + \frac{109u^6}{27249868800} + \frac{18127u^8}{382588157952000} + \frac{64931u^{10}}{128549621071872000} + \cdots,\\
\alpha_{\epsilon,0} &= -\frac{1}{2},\\
\alpha_{\epsilon,1} &= \frac{3}{2},\\
\beta_{\epsilon,0} &= -\frac{\left(u^2 + \left(5u^2 - 16\right)\cos(u/2) + 16\cos(u)\right)\csc^4(u/4)}{256u^2}\\
&= \frac{17}{1920} + \frac{53u^2}{645120} + \frac{11u^4}{25804800} - \frac{19u^6}{27249868800} - \frac{6427u^8}{127529385984000} - \frac{11509u^{10}}{14283291230208000} + \cdots,\\
\beta_{\epsilon,v} &= \frac{\left(23u^2 - 4u^2\cos(u/2) + \left(5u^2 + 48\right)\cos(u) - 48\right)\csc^4(u/4)}{256u^2}\\
&= \frac{21}{160} - \frac{u^2}{5040} - \frac{u^4}{2419200} + \frac{u^6}{70963200} + \frac{167u^8}{569327616000} + \frac{2633u^{10}}{669529276416000} + \cdots,\\
\beta_{\epsilon,1} &= \frac{3\left(-\left(7u^2 + 16\right)\cos(u/2) + \left(u^2 - 8\right)\cos(u) + 24\right)\csc^4(u/4)}{128u^2}\\
&= \frac{67}{320} + \frac{11u^2}{107520} - \frac{17u^4}{12902400} - \frac{173u^6}{4541644800} - \frac{12277u^8}{21254897664000} - \frac{49729u^{10}}{7141645615104000} + \cdots,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\beta_{\epsilon,\epsilon} &= \frac{\left(19u^2 + 4\left(u^2+32\right)\cos(u/2) + \left(u^2+16\right)\cos(u) - 144\right)\csc^4(u/4)}{256u^2} \\
&= \frac{13}{480} + \frac{u^2}{16128} + \frac{u^4}{460800} + \frac{5u^6}{136249344} + \frac{691u^8}{1449197568000} + \frac{u^{10}}{186856243200} + \cdots, \\
\beta_{\epsilon,2} &= -\frac{\left(5u^2 + \left(u^2+48\right)\cos(u/2) - 48\right)\csc^4(u/4)}{256u^2} \\
&= -\frac{1}{640} - \frac{31u^2}{645120} - \frac{67u^4}{77414400} - \frac{109u^6}{9083289600} - \frac{18127u^8}{127529385984000} - \frac{64931u^{10}}{42849873690624000} + \cdots,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\alpha_{2,0} &= 1, \\
\alpha_{2,1} &= 2, \\
\beta_{2,0} &= -\frac{\left(u^2 + 2u^2\cos(u/2) + 6\cos(u) - 6\right)\csc^4(u/4)}{48u^2} \\
&= \frac{1}{60} + \frac{u^2}{7560} + \frac{u^4}{3628800} - \frac{u^6}{106444800} - \frac{167u^8}{853991424000} - \frac{2633u^{10}}{1004293914624000} + \cdots, \\
\beta_{2,\nu} &= \frac{\left(5u^2 + \left(u^2+12\right)\cos(u) - 12\right)\csc^4(u/4)}{24u^2} = \frac{4}{15} - \frac{u^2}{1890} - \frac{u^4}{907200} + \frac{u^6}{26611200} + \frac{167u^8}{213497856000} + \frac{2633u^{10}}{251073478656000} + \cdots, \\
\beta_{2,1} &= \frac{\left(-10u^2\cos(u/2) + \left(u^2-18\right)\cos(u) + 18\right)\csc^4(u/4)}{24u^2} \\
&= \frac{13}{30} + \frac{u^2}{1260} + \frac{u^4}{604800} - \frac{u^6}{17740800} - \frac{167u^8}{142331904000} - \frac{2633u^{10}}{167382319104000} + \cdots, \\
\beta_{2,\epsilon} &= \frac{\left(5u^2 + \left(u^2+12\right)\cos(u) - 12\right)\csc^4(u/4)}{24u^2} = \frac{4}{15} - \frac{u^2}{1890} - \frac{u^4}{907200} + \frac{u^6}{26611200} + \frac{167u^8}{213497856000} + \frac{2633u^{10}}{251073478656000} + \cdots, \\
\beta_{2,2} &= -\frac{\left(u^2 + 2u^2\cos(u/2) + 6\cos(u) - 6\right)\csc^4(u/4)}{48u^2} \\
&= \frac{1}{60} + \frac{u^2}{7560} + \frac{u^4}{3628800} - \frac{u^6}{106444800} - \frac{167u^8}{853991424000} - \frac{2633u^{10}}{1004293914624000} + \cdots, \\
\alpha'_{0,0} &= -1, \\
\alpha'_{0,1} &= 1, \\
\beta'_{0,0} &= \frac{\csc^4(u/4)\left(-7u^2 + 10u^2\cos(u/2) - 6\cos(5u/4)\sec(u/4) - 6u\sin(5u/4)\sec(u/4) + 6\right)}{96u^2} \\
&= -\frac{53}{360} - \frac{19u^2}{15120} - \frac{37u^4}{2419200} - \frac{163u^6}{638668800} - \frac{461287u^8}{83691159552000} - \frac{12773u^{10}}{95647039488000} + \cdots, \\
\beta'_{0,\nu} & \\
&= -\frac{\csc^4(u/4)\sec(u/4)\left(5u^2\cos(5u/4) + 6\left(u^2+4\right)\cos(u/4) + \left(u^2-6\right)\cos(3u/4) - 6u\sin(3u/4) - 18u\sin(5u/4) - 18\cos(5u/4)\right)}{96u^2} \\
&= -\frac{2}{5} + \frac{13u^2}{3780} + \frac{29u^6}{53222400} + \frac{67u^4}{1814400} + \frac{46559u^8}{4184557977600} + \frac{26753u^{10}}{100429391462400} + \cdots, \\
\beta'_{0,1} &= \frac{\csc^4(u/4)\left(2u^2\cos(u/2) + \left(u^2-6\right)\cos(u) - 6u\sin(u) + 6\right)}{16u^2} \\
&= \frac{1}{12} - \frac{u^2}{360} - \frac{23u^4}{1209600} - \frac{331u^8}{1072963584000} - \frac{u^6}{9676800} + \frac{703u^{10}}{334764638208000} + \cdots,
\end{aligned} \tag{22}$$

$$\beta'_{0,\epsilon} = -\frac{\csc^4(u/4)\sec(u/4)\left(u^2\cos(5u/4)+6\left(u^2+4\right)\cos(u/4)+\left(5u^2-18\right)\cos(3u/4)-18u\sin(3u/4)-6u\sin(5u/4)-6\cos(5u/4)\right)}{96u^2}$$
$$= -\frac{2}{45}-\frac{u^4}{86400}+\frac{u^2}{3780}-\frac{13u^6}{31933440}-\frac{32027u^8}{2988969984000}-\frac{15019u^{10}}{55794106368000}+\cdots,$$
$$\beta'_{0,2} = \frac{\csc^4(u/4)\left(u^2+2\left(u^2-6\right)\cos(u/2)-6u\sin(3u/4)\sec(u/4)+12\right)}{96u^2}$$
$$= \frac{1}{120}+\frac{u^2}{3024}+\frac{13u^4}{1451520}+\frac{47u^6}{212889600}+\frac{452681u^8}{83691159552000}+\frac{269639u^{10}}{2008587829248000}+\cdots, \tag{23}$$

$$\alpha'_{v,0} = -1,$$
$$\alpha'_{v,1} = 1,$$
$$\beta'_{v,0} = \frac{\csc^3(u/4)\sec(u/4)\left(\left(u^2+8\right)\sin(u/2)-4u+8\sin(u)-8u\cos(u/2)\right)}{64u^2}$$
$$= \frac{13}{480}+\frac{17u^2}{32256}+\frac{67u^4}{6451200}+\frac{1549u^6}{6812467200}+\frac{1205803u^8}{223176425472000}+\frac{9739u^{10}}{72873934848000}+\cdots,$$
$$\beta'_{v,v} = -\frac{\left(\left(u^2+24\right)\sin(u/2)-12u\right)(3\cos(u/2)+2)\csc^3(u/4)\sec(u/4)}{96u^2}$$
$$= \frac{7}{144}-\frac{109u^2}{80640}-\frac{139u^4}{5806080}-\frac{4951u^6}{10218700800}-\frac{246931u^8}{22317642547200}-\frac{4329211u^{10}}{16068702633984000}+\cdots,$$
$$\beta'_{v,1} = \frac{\csc^4(u/4)\left(u^2\cos(u/2)-\left(u^2+18\right)\cos(u)-18u\sin(u/2)+18\right)}{48u^2} \tag{24}$$
$$= -\frac{1}{10}+\frac{u^2}{1120}+\frac{23u^4}{2419200}+\frac{19u^6}{212889600}+\frac{7213u^8}{9299017728000}+\frac{4289u^{10}}{669529276416000}+\cdots,$$
$$\beta'_{v,\epsilon} = -\frac{\left(\left(u^2+24\right)\sin(u/2)-12u\right)(\cos(u/2)+2)\csc^3(u/4)\sec(u/4)}{96u^2}$$
$$= \frac{7}{240}+\frac{13u^2}{80640}+\frac{109u^4}{9676800}+\frac{83u^6}{227082240}+\frac{1119247u^8}{111588212736000}+\frac{1397321u^{10}}{5356234211328000}+\cdots,$$
$$\beta'_{v,2} = \frac{\csc^2(u/4)\left(u^2-12u\csc(u/2)+24\right)}{96u^2}$$
$$= -\frac{7}{1440}-\frac{37u^2}{161280}-\frac{419u^4}{58060800}-\frac{577u^6}{2919628800}-\frac{577u^6}{2919628800}-\frac{1148099u^8}{223176425472000}-\frac{169051u^{10}}{1285496210718720}+\cdots,$$
$$\alpha'_{1,0} = -1,$$
$$\alpha'_{1,1} = 1,$$
$$\beta'_{1,0} = -\frac{\csc^4(u/4)\sec(u/4)\left(2u^2\cos(3u/4)+\left(u^2-6\right)\cos(u/4)+6(u\sin(u/4)+\cos(5u/4))\right)}{96u^2}$$
$$= \frac{1}{72}-\frac{u^2}{30240}-\frac{u^4}{172800}-\frac{61u^6}{319334400}-\frac{215521u^8}{41845579776000}-\frac{1769u^{10}}{13390585528320}+\cdots,$$
$$\beta'_{1,v} = \frac{\csc^4(u/4)\sec(u/4)\left(u^2\cos(5u/4)+\left(5u^2-12\right)\cos(u/4)+6u\sin(u/4)+3\cos(3u/4)+9\cos(5u/4)\right)}{48u^2}$$
$$= \frac{13}{45}-\frac{u^2}{15120}+\frac{41u^4}{3628800}+\frac{5u^6}{12773376}+\frac{17569u^8}{1673823191040}+\frac{267983u^{10}}{1004293914624000}+\cdots,$$

$$\beta'_{1,1} = \frac{\left(-10u^2\cos(u/2) + \left(u^2 - 18\right)\cos(u) + 18\right)\csc^4(u/4)}{48u^2}$$
$$= \frac{13}{60} + \frac{u^2}{2520} + \frac{u^4}{1209600} - \frac{u^6}{35481600} - \frac{167u^8}{284663808000} - \frac{2633u^{10}}{334764638208000} + \cdots,$$
$$\beta'_{1,\epsilon} = \frac{\csc^4(u/4)\sec(u/4)\left(\left(5u^2 - 12\right)\cos(u/4) + \left(u^2 + 9\right)\cos(3u/4) + 3\left(\cos(5u/4) - 2u\sin(u/4)\right)\right)}{48u^2}$$
$$= -\frac{1}{45} - \frac{u^2}{2160} - \frac{u^4}{80640} - \frac{113u^6}{319334400} - \frac{406493u^8}{41845579776000} - \frac{85817u^{10}}{334764638208000} + \cdots,$$
$$\beta'_{1,2} = -\frac{\csc^4(u/4)\sec(u/4)\left(\left(u^2 - 2\right)\cos(u/4) + 2\left(\cos(3u/4) - u\sin(u/4)\right)\right)}{32u^2}$$
$$= \frac{1}{360} + \frac{u^2}{6048} + \frac{11u^4}{1814400} + \frac{29u^6}{159667200} + \frac{103669u^8}{20922789888000} + \frac{5911u^{10}}{45649723392000} + \cdots, \tag{25}$$

$$\alpha'_{\epsilon,0} = -1,$$
$$\alpha'_{\epsilon,1} = 1,$$
$$\beta'_{\epsilon,0} = -\frac{\csc^4(u/4)\sec(u/4)\left(7u^2\cos(3u/4) + \left(17u^2 - 24\right)\cos(u/4) + 24\left(\cos(5u/4) - u\sin(u/4)\right)\right)}{384u^2}$$
$$= \frac{31}{1440} + \frac{5u^2}{13824} + \frac{29u^4}{3870720} + \frac{3847u^6}{20437401600} + \frac{3313369u^8}{669529276416000} + \frac{28177u^{10}}{218621804544000} + \cdots,$$
$$\beta'_{\epsilon,\nu}$$
$$= \frac{\csc^4(u/4)\sec(u/4)\left(7u^2\cos(5u/4) + 12\left(7u^2 - 8\right)\cos(u/4) + \left(5u^2 + 24\right)\cos(3u/4) - 72u\sin(u/4) - 24u\sin(3u/4) + 72\cos(5u/4)\right)}{384u^2}$$
$$= \frac{19}{80} - \frac{167u^2}{241920} - \frac{359u^4}{29030400} - \frac{1117u^6}{3406233600} - \frac{47629u^8}{5150225203200} - \frac{4023451u^{10}}{16068702633984000} + \cdots,$$
$$\beta'_{\epsilon,1} = \frac{\csc^4(u/4)\left(-7u^2\cos(u/2) + \left(u^2 - 6\right)\cos(u) + 6u\sin(u/2) + 6\right)}{16u^2} \tag{26}$$
$$= \frac{8}{15} - \frac{u^2}{10080} - \frac{19u^4}{2419200} - \frac{31u^6}{212889600} - \frac{54371u^8}{27897053184000} - \frac{14821u^{10}}{669529276416000} + \cdots,$$
$$\beta'_{\epsilon,\epsilon}$$
$$= \frac{\csc^4(u/4)\sec(u/4)\left(5u^2\cos(5u/4) + 12\left(7u^2 - 8\right)\cos(u/4) + \left(7u^2 + 72\right)\cos(3u/4) - 24u\sin(u/4) - 72u\sin(3u/4) + 24\cos(5u/4)\right)}{384u^2}$$
$$= \frac{157}{720} + \frac{199u^2}{241920} + \frac{221u^4}{9676800} + \frac{97u^6}{185794560} + \frac{3965821u^8}{334764638208000} + \frac{499747u^{10}}{1785411403776000} + \cdots,$$
$$\beta'_{\epsilon,2} = -\frac{\csc^4(u/4)\left(7u^2 + \left(5u^2 + 24\right)\cos(u/2) - 12u\sin(3u/4)\sec(u/4) - 24\right)}{192u^2}$$
$$= -\frac{1}{96} - \frac{191u^2}{483840} - \frac{587u^4}{58060800} - \frac{1613u^6}{6812467200} - \frac{3748337u^8}{669529276416000} - \frac{79621u^{10}}{584316459417600} + \cdots,$$
$$\alpha'_{2,0} = -1,$$
$$\alpha'_{2,1} = 1,$$
$$\beta'_{2,0} = -\frac{\left(2\cos(u/2) + 1\right)\csc^4(u/4)\sec(u/4)\left(\left(u^2 - 2\right)\cos(u/4) + 2\left(\cos(3u/4) - u\sin(u/4)\right)\right)}{32u^2}$$
$$= \frac{1}{120} - \frac{u^2}{5040} - \frac{u^4}{115200} - \frac{7u^6}{30412800} - \frac{156349u^8}{27897053184000} - \frac{6109u^{10}}{44635285094400} + \cdots,$$

$$\beta'_{2,v} = \frac{\csc^4(u/4)\sec(u/4)\left(\left(26u^2-24\right)\cos(u/4)+\left(7u^2+6\right)\cos(3u/4)+3\left(\left(u^2+6\right)\cos(5u/4)-2u\left(3\sin(3u/4)+\sin(5u/4)\right)\right)\right)}{96u^2}$$
$$= \frac{14}{45}-\frac{u^2}{1260}+\frac{19u^4}{1814400}+\frac{71u^6}{159667200}+\frac{2291u^8}{199264665600}+\frac{12767u^{10}}{45649723392000}+\cdots,$$
$$\beta'_{2,1} = -\frac{\csc^4(u/4)\left(26u^2\cos(u/2)+\left(u^2+18\right)\cos(u)-18\left(u\sin(u)+1\right)\right)}{48u^2}$$
$$= \frac{7}{20}+\frac{u^2}{280}+\frac{u^4}{48384}+\frac{u^6}{21288960}-\frac{4021u^8}{4649508864000}-\frac{5969u^{10}}{334764638208000}+\cdots,$$
$$\beta'_{2,\epsilon}$$
$$= \frac{\csc^4(u/4)\sec(u/4)\left(7u^2\cos(5u/4)+\left(26u^2-24\right)\cos(u/4)+3\left(u^2+6\right)\cos(3u/4)-6u\sin(3u/4)-18u\sin(5u/4)+6\cos(5u/4)\right)}{96u^2}$$
$$= \frac{2}{3}-\frac{u^2}{252}-\frac{23u^4}{604800}-\frac{u^6}{1971200}-\frac{72143u^8}{6974263296000}-\frac{6119u^{10}}{23911759872000}+\cdots,$$
$$\beta'_{2,2} = -\frac{\csc^4(u/4)\left(-5u^2+2\left(7u^2+6\right)\cos(u/2)-6u\sin(5u/4)\sec(u/4)-12\right)}{96u^2}$$
$$= \frac{59}{360}+\frac{u^2}{720}+\frac{113u^4}{7257600}+\frac{157u^6}{638668800}+\frac{148307u^8}{27897053184000}+\frac{262967u^{10}}{2008587829248000}+\cdots. \quad (27)$$

Remark 3. We note that the Taylor series expansions in (20) through (27) must be used when $u \to 0$ because the corresponding trigonometric coefficients given in these equations are vulnerable to heavy cancelations (see [8]).

2.2. Block Form. In this subsection, the BHT method is formulated from the eight discrete hybrid formulas stated in (2) and (3). We emphasize that these eight formulas are provided by the continuous two-step hybrid trigonometrically fitted method with two off-grid points given by (5) and its first derivative (6). First, we define the following vectors:

$$\begin{aligned}
&Y_{\mu+1} \\
&\quad = \left[y_{n+v}, y_{n+1}, y_{n+\epsilon}, y_{n+2}, hy'_{n+v}, hy'_{n+1}, hy'_{n+\epsilon}, hy'_{n+2}\right]^T, \\
&Y_\mu = \left[y_{n-\epsilon}, y_{n-1}, y_{n-v}, y_n, hy'_{n-v}, hy'_{n-1}, hy'_{n-\epsilon}, hy'_n\right]^T, \\
&F_{\mu+1} \\
&\quad = \left[f_{n+v}, f_{n+1}, f_{n+\epsilon}, f_{n+2}, hf'_{n+v}, hf'_{n+1}, hf'_{n+\epsilon}, hf'_{n+2}\right]^T, \\
&F_\mu = \left[f_{n-\epsilon}, f_{n-1}, f_{n-v}, f_n, hf'_{n-v}, hf'_{n-1}, hf'_{n-\epsilon}, hf'_n\right]^T,
\end{aligned} \quad (28)$$

where $\mu = 0,\dots,N$; $n = 0,\dots,N$. The three discrete methods whose coefficients are specified by (2) and the five additional methods in (3) whose coefficients are specified by (20) to (27) are combined to give the BHT method, which is expressed as

$$A_1 Y_{\mu+1} = A_0 Y_\mu + h^2\left(B_0 F_\mu + B_1 F_{\mu+1}\right), \quad (29)$$

where A_0, A_1, B_0, and B_1 are matrices of dimension eight whose elements characterize the method and are given by the coefficients of (2) and (3).

3. Error Analysis and Stability

3.1. Local Truncation Error (LTE). Define the local truncation error of (29) as

$$L[Z(x);h] = Z_{\mu+1} - \left[AZ_\mu + h^2 B\overline{F}_\mu + h^2 C\overline{F}_{\mu+1}\right], \quad (30)$$

where

$$\begin{aligned}
Z_{\mu+1} &= \left[y(x_{n+v}), y(x_{n+1}), y(x_{n+\epsilon}), y(x_{n+2}),\right. \\
&\quad \left. hy'(x_{n+v}), hy'(x_{n+1}), hy'(x_{n+\epsilon}), hy'(x_{n+2})\right]^T, \\
Z_\mu &= \left[y(x_{n-\epsilon}), y(x_{n-1}), y(x_{n-v}), y(x_n), hy'(x_{n-\epsilon}),\right. \\
&\quad \left. hy'(x_{n-1}), hy'(x_{n-v}), hy'(x_n)\right]^T, \\
\overline{F}_{\mu+1} &= \left[f(x_{n+\epsilon}, y_{n+\epsilon}), f(x_{n+1}, y_{n+1}), f(x_{n+v}, y_{n+v}),\right. \\
&\quad f(x_{n+2}, y_{n+2}), hf'(x_{n+v}, y_{n+v}), hf'(x_{n+1}, y_{n+1}), \\
&\quad \left. hf'(x_{n+\epsilon}, y_{n+\epsilon}), hf'(x_{n+2}, y_{n+2})\right]^T, \\
\overline{F}_\mu &= \left[f(x_{n-\epsilon}, y_{n-\epsilon}), f(x_{n-1}, y_{n-1}), f(x_{n-v}, y_{n-v}),\right. \\
&\quad f(x_n, y_n), hf'(x_{n-\epsilon}, y_{n-\epsilon}), hf'(x_{n-1}, y_{n-1}), \\
&\quad \left. hf'(x_{n-v}, y_{n-v}), hf'(x_n, y_n)\right]^T, \\
L[Z(x);h] &= \left[L_1[z(x);h], L_2[z(x);h], \dots,\right. \\
&\quad \left. L_8[z(x);h]\right]^T \text{ is linear different operator.}
\end{aligned} \quad (31)$$

Suppose that $Z(x)$ is sufficiently differentiable. Then, a Taylor series expansion of the terms in (30) about the point x gives the following expression for local truncation error:

$$L[Z(x);h] = C_0 Z(x) + C_1 h Z'(x) + \cdots + C_q h^q Z^q(x) + \cdots, \tag{32}$$

where C_i, $i = 0, 1, \ldots$, are constant coefficients (see [16]).

Definition 4. Block method (29) has algebraic order of at least $p \geq 1$ provided there exists a constant $C_{p+2} \neq 0$ such that the local truncation error E_μ satisfies $\|E_\mu\| = C_{p+2} h^{p+2} + O(h^{p+3})$, where $\|\cdot\|$ is the maximum norm.

Remark 5. (i) The local truncation error constants $(\overline{C}_{p+2})$ of $(y_{n+v}, y_{n+1}, y_{n+\epsilon}, y_{n+2}, hy'_{n+v}, hy'_{n+1}, hy'_{n+\epsilon}, hy'_{n+2})^T$ of block method (29) are given, respectively, by $\overline{C}_7 = (-1/61440, -1/5040, 1/61440, 0, -1/5040, 61/645120, -1/40320, 61/645120, -1/5040)^T$, where $\overline{C}_0 = \overline{C}_1 = \overline{C}_2 = \overline{C}_3 = \overline{C}_4 = \overline{C}_5 = \overline{C}_6 = 0$.

(ii) From the local truncation error constant computation, it follows that method (29) has order p of at least five.

3.2. Stability. The linear stability of the BHT is discussed by applying the method to the test equation $y'' = -\lambda^2 y$, where λ is a real constant (see [17]). Letting $\Upsilon = \lambda h$, it is easily shown as in [18] that the application of (29) to the test equation yields

$$Y_{\mu+1} = M(\Upsilon^2; u) Y_\mu, \quad M(\Upsilon^2; u) := (A_1 - \Upsilon^2 B_1)^{-1} (A_0 + \Upsilon^2 B_0), \tag{33}$$

where the matrix $M(\Upsilon^2; u)$ is the amplification matrix which determines the stability of the method. In the spirit of [21], the spectral radius of $\rho(M(\Upsilon^2; u))$ can be obtained from the characteristics equation

$$\rho^2 - 2\Gamma(\Upsilon^2; u)\rho + \Theta(\Upsilon^2; u) = 0, \tag{34}$$

where $\Gamma(\Upsilon^2; u) = \operatorname{trace} M(\Upsilon^2; u)$ and $\Theta(\Upsilon^2; u) = \det M(\Upsilon^2; u)$ are rational functions.

Definition 6. A region of stability is a region in the q-u plane, throughout which $\rho(M(\Upsilon^2; u)) \leq 1$ and any closed curve given by $\rho(M(\Upsilon^2; u)) = 1$ defines the stability boundary of the method (see [21]). We note that the plot for the stability region of the BHT method is given in Figure 1.

3.3. Implementation. We emphasize that the main method and the additional methods specified by (20)–(27) are combined to form block method BHT (29), which is used to solve (1) without requiring starting values and predictors. BHT is implemented in a block-by-block fashion using a Mathematica 10.0 code, enhanced by the feature *NSolve*[] for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature *FindRoot*[] (see Keiper and Gear [35]). Mathematica can symbolically compute derivatives and so the entries of the Jacobian matrix which involve partial derivatives are automatically generated. In what follows, we summarize how BHT is applied.

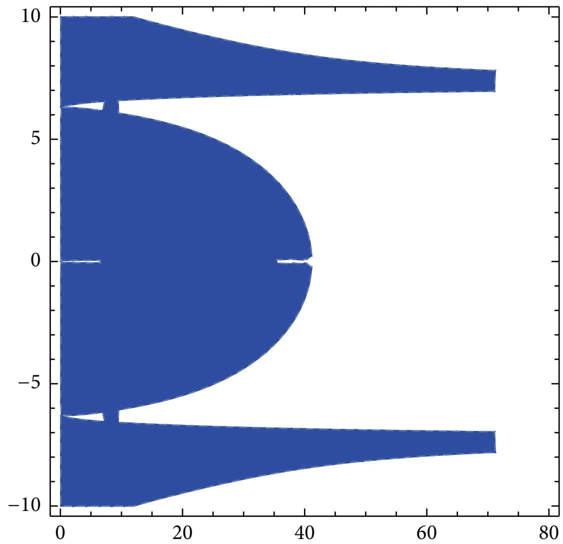


FIGURE 1: The stability region for the BHT plotted in the (q, u)-plane.

Step 1. Choose N, $h = (b - a)/N$, and the number of blocks $\Gamma = N/2$. Using (29), $n = 0$, $\mu = 0$, the values of $(y_{1/2}, y_1, y_{3/2}, y_2)^T$ and $(y'_{1/2}, y'_1, y'_{3/2}, y'_2)^T$ are simultaneously obtained over the subinterval $[x_0, x_2]$, as y_0 and y'_0 are known from IVP (1).

Step 2. For $n = 2$, $\mu = 1$, the values of $(y_{5/2}, y_3, y_{7/2}, y_4)^T$ and $(y'_{5/2}, y'_3, y'_{7/2}, y'_4)^T$ are simultaneously obtained over the subinterval $[x_2, x_4]$, as y_2 and y'_2 are known from the previous block.

Step 3. The process is continued for $n = 4, \ldots, N-2$ and $\mu = 2, \ldots, \Gamma$ to obtain the numerical solution to (1) on the subintervals $[x_0, x_2], [x_2, x_4], \ldots, [x_{N-2}, x_N]$.

4. Numerical Examples

In this section, numerical experiments are performed using a code in Mathematica 10.0 to illustrate the accuracy and efficiency of the method.

Example 1. We consider the following inhomogeneous IVP by Simos [8]:

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11, \quad x \in [0, 1000], \tag{35}$$

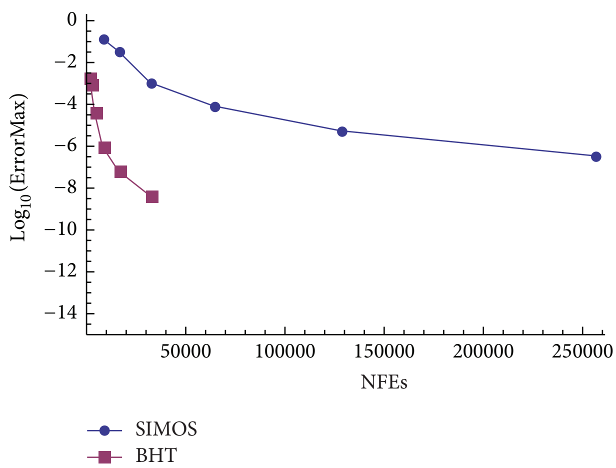


Figure 2: Efficiency curve for Example 1.

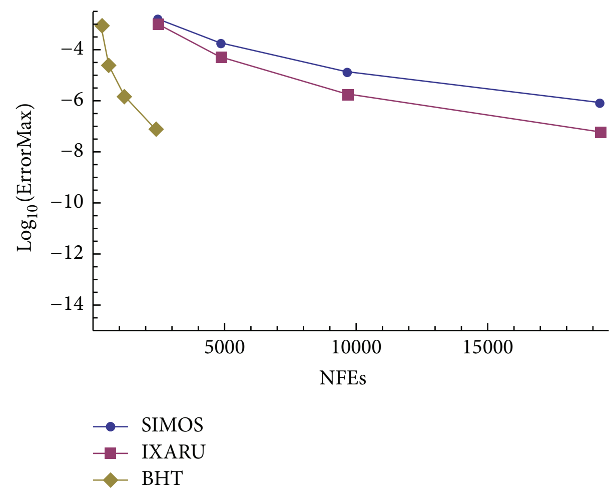


Figure 3: Efficiency curves for Example 2.

Table 1: Results, with $\omega = 10$, for Example 1.

Our method		Simos [8]	
N	Err	N	Err
1000	1.9×10^{-3}	1000	1.4×10^{-1}
2000	8.9×10^{-6}	2000	3.5×10^{-2}
4000	4.2×10^{-8}	4000	1.1×10^{-3}
8000	9.7×10^{-11}	8000	8.4×10^{-5}
16000	6.7×10^{-11}	16000	5.5×10^{-6}
32000	4.3×10^{-13}	32000	3.5×10^{-7}

Table 2: Results, with $\omega = 1.01$, for Example 2.

Our method		Simos [8]		Ixaru and Berghe [31]	
N	Err	N	Err	N	Err
300	7.7×10^{-5}	300	1.7×10^{-3}	300	1.1×10^{-3}
600	1.7×10^{-6}	600	1.9×10^{-4}	600	5.4×10^{-5}
1200	1.4×10^{-8}	1200	1.4×10^{-5}	1200	1.9×10^{-6}
2400	1.9×10^{-10}	2400	8.7×10^{-7}	2400	6.2×10^{-8}

where the analytical solution is given by

$$\text{Exact: } y(x) = \cos(10x) + \sin(10x) + \sin(x). \quad (36)$$

This example was solved using the order 5 BHT and the end-point errors (Err $= |y(x_N) - y_N|$) obtained were compared to the order 4 exponentially fitted method given in Simos [8]. In Table 1, it is shown that BHT is more efficient than the method in Simos [8]. We also compare the computational efficiency of the two methods in Figure 2 by considering the FNEs (number of function evaluations) over N integration steps for each method. Our method (BHT) requires fewer number of function evaluations. Hence, for this example, BHT performs better.

Example 2. We consider the nonlinear Duffing equation which was also solved by Simos [8] and Ixaru and Berghe [31]:

$$\begin{aligned} y'' + y + y^3 &= B\cos(\Omega x), \\ y(0) &= C_0, \\ y'(0) &= 0. \end{aligned} \quad (37)$$

The analytical solution is given by

$$\begin{aligned} \text{Exact: } y(x) = {} & C_1\cos(\Omega x) + C_2\cos(3\Omega x) \\ & + C_3\cos(5\Omega x) + C_4\cos(7\Omega x), \end{aligned} \quad (38)$$

where $\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$, $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, and $C_4 = 0.374 \times 10^{-9}$. We choose $\omega = 1.01$.

We compare the end-point global errors for our method with those of Simos [8] and Ixaru and Berghe [31]. We see from Table 2 that the results produced by our method are competitive to those given in Simos [8] and Ixaru and Berghe [31]. Hence our method is more accurate and efficient as demonstrated in Figure 3.

Example 3. We consider the nonlinear perturbed system on the range $[0, 10]$, with $\varepsilon = 10^{-3}$ that was also solved in Fang et al. [21]:

$$\begin{aligned} y_1'' + 25y_1 + \varepsilon\left(y_1^2 + y_2^2\right) &= \varepsilon\varphi_1(x), \\ y_1(0) &= 1, \\ y_1'(0) &= 0, \\ y_2'' + 25y_2 + \varepsilon\left(y_1^2 + y_2^2\right) &= \varepsilon\varphi_2(x), \\ y_2(0) &= \varepsilon, \\ y_2'(0) &= 5, \end{aligned} \quad (39)$$

Table 3: A comparison of methods for Example 3.

ARKN5(3)		TFARKN5(3)		BHT	
N (rejected)	$-\log_{10}(\text{Err})$	N (rejected)	$-\log_{10}(\text{Err})$	N	$-\log_{10}(\text{Err})$
42 (15)	2.82	29 (6)	2.78	50	3.42
86 (7)	4.96	88 (9)	5.33	100	4.61
260 (5)	7.16	262 (8)	7.85	260	7.52
812 (3)	9.37	811 (4)	10.38	810	10.43

where

$$\begin{aligned}\varphi_1(x) &= 1+\varepsilon^2+2\varepsilon\sin\left(5x+x^2\right)+2\cos\left(x^2\right)\\ &\quad+\left(25-4x^2\right)\sin\left(x^2\right),\\ \varphi_2(x) &= 1+\varepsilon^2+2\varepsilon\sin\left(5x+x^2\right)-2\sin\left(x^2\right)\\ &\quad+\left(25-4x^2\right)\cos\left(x^2\right),\end{aligned} \tag{40}$$

and the exact solution is given by $y_1(x)=\cos(5x)+\varepsilon\sin(x^2)$, $y_2(x)=\sin(5x)+\varepsilon\cos(x^2)$, representing a periodic motion of constant frequency with small perturbation of variable frequency.

This problem was solved using the BHT and the maximum global errors ($\text{Err}=\text{Max}|y(x)-y|$) obtained were compared to the variable step-size trigonometrically fitted Runge-Kutta-Nyström method (TFARKN5(3)) given in Fang et al. [21] and a Runge-Kutta-Nyström method (ARKN5(3)) which was constructed by Franco [15]. In Table 3, the maximum global errors for the three methods are compared. In general, the TFARKN5(3) and ARKN5(3) are expected to perform better because of their variable-step implementation advantage. Nevertheless, the BHT which is implemented in fixed step-size mode is highly competitive to these methods.

Example 4. We consider the following two-body problem which was solved by Ozawa [25] on $[0,50\pi]$:

$$\begin{gathered} y_1''=-\frac{y_1}{r^3},\\ y_2''=-\frac{y_2}{r^3},\\ r=\sqrt{y_1^2+y_2^2},\\ y_1(0)=1-e,\\ y_1'(0)=0,\\ y_2(0)=0,\\ y_2'(0)=\sqrt{\frac{1+e}{1-e}}, \end{gathered} \tag{41}$$

Table 4: Steps and absolute errors, with $\omega = 1$, for Example 4 $[0,50\pi]$.

Our method		FESDIRK4(3) [25]		ESDIRK4(3) [25]	
Steps	Err	Steps	Err	Steps	Err
220	3.52×10^{-5}	170	2.866×10^{-1}	277	2.153×10^{0}
300	1.13×10^{-6}	225	7.846×10^{-3}	496	1.494×10^{-1}
400	1.03×10^{-7}	381	1.399×10^{-3}	884	9.359×10^{-3}
600	3.49×10^{-10}	680	1.690×10^{-4}	1573	6.200×10^{-4}
800	1.14×10^{-11}	1207	1.846×10^{-5}	2796	4.416×10^{-5}
1000	7.68×10^{-13}	2144	1.938×10^{-6}	4970	3.412×10^{-6}
1200	2.8×10^{-14}	3806	1.993×10^{-7}	8833	2.848×10^{-7}
2400	1.02×10^{-13}	6762	2.021×10^{-8}	15706	2.530×10^{-8}

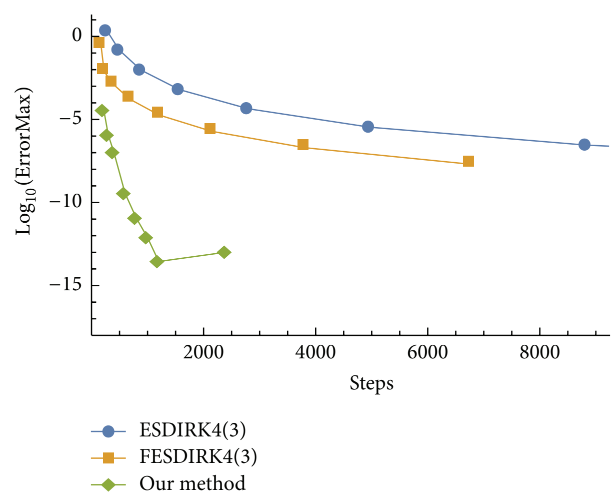


Figure 4: Efficiency curve for Example 4.

where e $(0\le e<1)$ is an eccentricity. The exact solution of this problem is

$$\begin{aligned}\text{Exact: } y_1(x) &= \cos(k)-e,\\ y_2(x) &= \sqrt{1-e^2}\sin(k),\end{aligned} \tag{42}$$

where k is the solution of Kepler's equation $k=x+e\sin(k)$. We choose $\omega=1$.

We show in Table 4 that the results obtained using the BHT method are more accurate than the explicit singly diagonally implicit Runge-Kutta (ESDIRK) and the functionally fitted ESDIRK (FESDIRK) methods given in Ozawa [25]. In Figure 4, we also illustrate the efficiency advantage of the BHT method over those in Ozawa [25].

Table 5: Results, with $\omega = 1$, for Example 5.

Our method		Vigo-Aguiar and Ramos [32]	
N	Err	N	Err
67	1.14×10^{-9}	67	7.11×10^{-7}
82	3.5×10^{-10}	82	9.26×10^{-8}
97	1.3×10^{-10}	97	87.8×10^{-9}
112	5.5×10^{-11}	112	1.12×10^{-10}
125	2.9×10^{-11}	125	2.71×10^{-11}

4.1. Problems Where y' Appears Explicitly

Example 5 (Bessel's IVP). We consider the Bessel differential equation that was also solved by Vigo-Aguiar and Ramos [11]:

$$\begin{aligned} &x^2 y'' + xy' + \left(x^2 - 0.25\right) y = 0, \\ &y(1) = \sqrt{\frac{2}{\pi}} \sin 1 \approx 0.671397071418031, \\ &y'(1) = \frac{(2\cos 1 - \sin 1)}{2\pi} \approx 0.0954005144474746, \\ &\qquad x \in [1, 8], \end{aligned} \tag{43}$$

where the exact (analytical) solution is given by

$$\text{Exact: } y(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x). \tag{44}$$

This problem was chosen to demonstrate the performance of our method on the general second-order IVP with variable coefficients. We compare our results with the variable-step Falkner method of order eight that was implemented in predictor-corrector mode by Vigo-Aguiar and Ramos [32]. The results displayed in Table 5 show that the BHT method performs better.

Example 6. We consider the harmonic oscillator with frequency Ω and small perturbation δ that was solved in Franco [15] and Guo and Yan [36]:

$$\begin{aligned} &y'' + \delta y' + \Omega^2 y = 0, \\ &y(0) = 0, \\ &y'(0) = -\frac{\delta}{2}, \\ &\qquad x \in [0, 1000], \end{aligned} \tag{45}$$

where the analytical solution is given by

$$\text{Exact: } y(x) = e^{(\delta/2)x} \cos\left(\Omega^2 - \frac{\delta^2}{4}\right), \tag{46}$$

where $\Omega = 1$, $\delta = 10^{-6}$, and $\delta = 10^{-10}$. The problem was solved in Guo and Yan [36] using ARKN method. In Table 6, the errors are compared at $x = 1000$. We observed that the BHT is competitive with the order 5 Runge-Kutta-Nyström method.

4.2. Hyperbolic PDEs

Example 7. We consider the given Telegraph equation (see Ding et al. [33]):

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + 2\pi \frac{\partial u}{\partial t} + \pi^2 u \\ &\quad = \frac{\partial^2 u}{\partial x^2} + \pi^2 \sin(\pi x)(\sin(\pi t) + \cos(\pi t)) \\ &\qquad 0 \le x \le 1,\ 0 \le t \le 1. \end{aligned} \tag{47}$$

The exact solution is given by $u(x, y) = \sin(\pi x)\sin(\pi t)$.

In order to solve this PDE using the BHT, we carry out the semidiscretization of the spatial variable x using the second-order finite difference method to obtain the following second-order system in the second variable t:

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} + 2\pi \frac{\partial u_m}{\partial t} + \pi^2 u_m - \frac{(u_{m+1} - 2u_m + u_{m-1})}{(\Delta x)^2} \\ &\quad = g_m, \quad 0 < t < 1,\ m = 1, \ldots, M-1, \\ &u(x_m, 0) = u_m, \\ &u_t(x_m, 0) = u'_m, \end{aligned} \tag{48}$$

where $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0, 1, \ldots, M$, $\mathbf{u} = [u_1(t), \ldots, u_M(t)]^T$, $\mathbf{g} = [g_1(t), \ldots, g_m(t)]^T$, $u_m(t) \approx u(x_m, t)$, and $g_m(t) \approx g(x_m, t) = \pi^2 \sin(\pi x_m)(\sin(\pi t) + \cos(\pi t))$, which can be written in the form

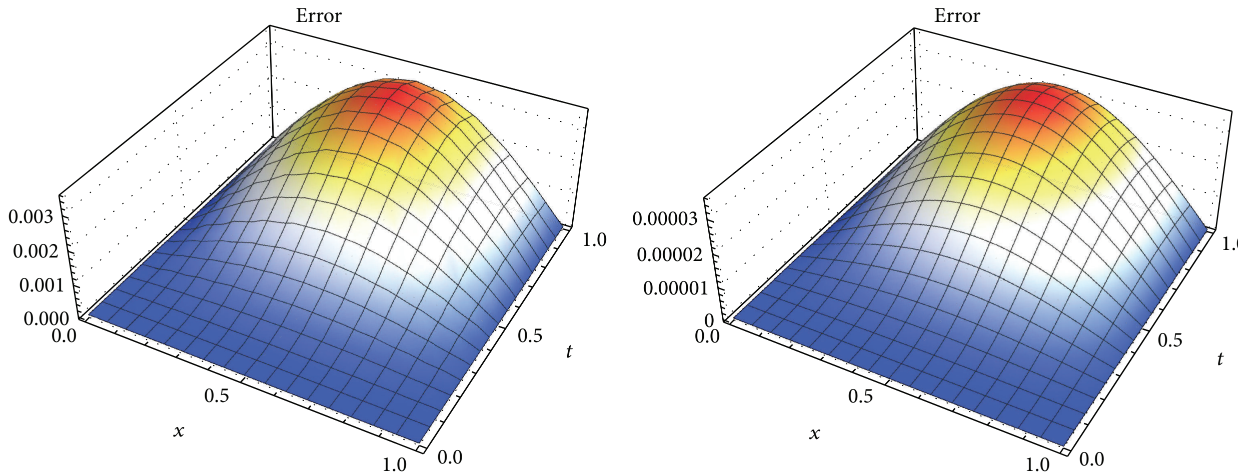
$$\mathbf{u}'' = \mathbf{f}\left(t, \mathbf{u}, \mathbf{u}'\right), \tag{49}$$

subject to the boundary conditions $\mathbf{u}(t_0) = \mathbf{u}_0$, $\mathbf{u}'(t_0) = \mathbf{u}'_0$, where $\mathbf{f}(t, \mathbf{u}, u') = \mathbf{A}\mathbf{u} + \mathbf{g}$, and $\mathbf{A}$ is $M - 1 \times M - 1$, matrix arising from the semidiscretized system, and $\mathbf{g}$ is a vector of constants.

The boundary conditions are chosen accordingly. This example was chosen to demonstrate that the BHT can be used to solve the Telegraph equation. In Table 7, the results produced by the BHT using $\Delta t = 1/100$ and space step $\Delta x = 1/100$ are compared to scheme (3.12) ($\lambda_1 = 1/12$, and $\lambda_2 = 5/6$), time step $\Delta t = 1/200$, and space step $\Delta x = 1/100$, given in Ding et al. [33]. It is obvious from Table 5 that the BHT is more accurate than the method given in [33]. Moreover, the errors produced by BHT method using $\Delta t = 1/100$ and space step $\Delta x = 1/100$ are given in Figure 5.

Table 6: Errors at $x = 1000$, for Example 6.

	BHT			ARKN	
h	Error ($\delta = 10^{-6}$)	Error ($\delta = 10^{-10}$)	h	Error ($\delta = 10^{-6}$)	Error ($\delta = 10^{-10}$)
1	4.12×10^{-8}	1.11×10^{-11}	1/2	9.05×10^{-8}	9.00×10^{-12}
1/2	7.06×10^{-10}	2.19×10^{-13}	1/4	5.43×10^{-9}	7.06×10^{-13}
1/4	1.23×10^{-11}	3.12×10^{-13}	1/8	2.03×10^{-10}	2.87×10^{-13}
1/8	5.23×10^{-12}	5.44×10^{-12}	1/16	7.25×10^{-12}	3.56×10^{-13}
1/16	5.62×10^{-12}	2.94×10^{-12}	1/32	3.45×10^{-13}	5.91×10^{-13}



(a) $N = M = 10$ (b) $N = M = 100$

Figure 5: Absolute errors for Example 7.

Table 7: Results, with $\omega = \pi$, for Example 7.

	Our method	Ding et al. [33]
x	Err	Err
0.2	2.06×10^{-10}	9.62×10^{-10}
0.4	3.33×10^{-10}	1.56×10^{-9}
0.6	3.33×10^{-10}	1.56×10^{-9}
0.8	2.06×10^{-10}	9.62×10^{-10}

Example 8. We consider the wave equation given in Franco [15]. A problem representing a vibrating string with speed ω is given by

$$\frac{\partial^2 u}{\partial t^2} - x(1-x)\frac{\partial^2 u}{\partial x^2} + (\omega^2 - 2)u = 0$$
$$0 < x < 1,\quad 0 < t \le 5,$$
$$u(0,t) = 0,$$
$$u(1,t) = 0, \tag{50}$$
$$u(x,0) = x(1-x),$$
$$u_t(x,0) = 0,$$

where the initial and Dirichlet boundary conditions have been chosen such that the solution is given by $u(x,t) = x(1-x)\cos\omega t$. In order to solve this PDE using the BHT, we carry out the semidiscretization of the spatial variable x using the second-order finite difference method to obtain the following second-order system in the second variable t:

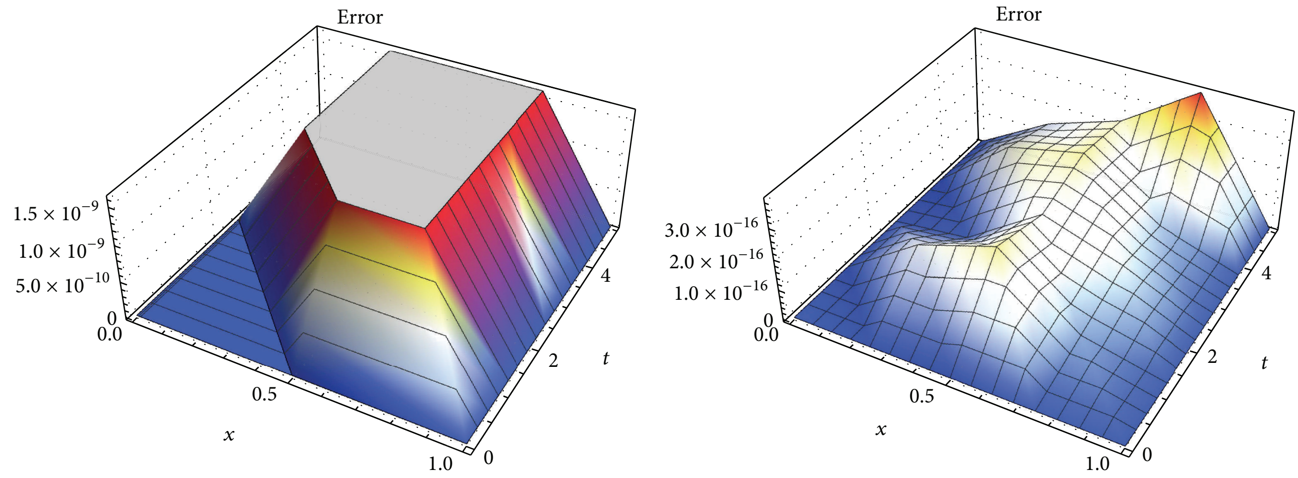
$$\frac{\partial^2 u_m}{\partial t^2} - x_m(1-x_m) + \frac{(u_{m+1} - 2u_m + u_{m-1})}{(\Delta x)^2}$$
$$+ (\omega^2 - 2)u_m = g_m,\quad m = 1,\ldots,M-1, \tag{51}$$
$$u(x_m,0) = x_m(1-x_m),$$
$$u_t(x_m,0) = 0,\quad 0 < t \le 5,$$

where $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0,1,\ldots,M$, $\mathbf{u} = [u_1(t),\ldots,u_M(t)]^T$, $\mathbf{g} = [g_1(t),\ldots,g_m(t)]^T$, $u_m(t) \approx u(x_m,t)$, and $g_m(t) \approx g(x_m,t) = 0$, which can be written in the form

$$\mathbf{u}'' = \mathbf{f}(t,\mathbf{u}), \tag{52}$$

subject to the boundary conditions $\mathbf{u}(t_0) = \mathbf{u}_0$, $\mathbf{u}'(t_0) = \mathbf{u}'_0$, where $\mathbf{f}(t,\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{g}$, and $\mathbf{A}$ is $M-1 \times M-1$, matrix arising from the semidiscretized system, and $\mathbf{g}$ is a vector of constants.

In Figure 6, we give the errors produced by the BHT which show that the method performs very well on this problem.



(a) $N = M = 2$ (b) $N = M = 4$

Figure 6: Absolute errors for Example 8.

Table 8: Comparison of methods for Example 9.

N	Method in [34]	Our method	N	RK4
10	1.12×10^{-4}	2.23×10^{-4}	250	*Overflow*
20	1.69×10^{-6}	3.36×10^{-6}	500	*Overflow*
40	1.22×10^{-8}	2.44×10^{-8}	1000	*Overflow*
80	9.79×10^{-11}	1.96×10^{-10}	2000	*Overflow*
160	1.06×10^{-12}	2.13×10^{-12}	4000	1.20×10^{-13}
320	1.54×10^{-14}	2.48×10^{-14}	8000	7.94×10^{-15}

Example 9. We consider the following mildly stiff IVP which was also solved in [34]:

$$\begin{aligned} y'' &= -1001y' - 1000y, \\ y(0) &= 1, \\ y'(0) &= -1, \qquad x \in [0, 10] \end{aligned} \tag{53}$$

$$\text{Exact: } y(x) = e^{-x}.$$

This example is given to show that the method still performs well on problems with nontrigonometric solutions. The problem was solved using the BHT and the results obtained were compared with the polynomial based method given in [34] and the standard fourth-order Runge-Kutta method (RK4). The results given in Table 8 show that the BHT is competitive with the method in [34] and is superior to RK4 which are designed for problems with nontrigonometric solutions.

5. Conclusion

We have presented a BHT method whose coefficients are functions of the frequency and the step-size for directly solving general second-order initial value problems (IVPs), including systems arising from the semidiscretization of hyperbolic PDEs, such as the Telegraph equation. The BHT is implemented in a block-by-block fashion; in this way, the method does not suffer from the disadvantages of requiring starting values and predictors which are inherent in predictor-corrector methods. We have also shown that the BHT method has a reasonably wide stability region and enjoys accuracy and efficiency advantages when compared to existing methods in the literature. Our future research will be to incorporate a technique for accurately estimating the frequency as suggested in [37, 38] as well as implementing the method in a variable-step mode.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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