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Research Article

Numerical Solution of Seventh-Order Boundary Value Problems by a Novel Method

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We demonstrate the efficiency of reproducing kernel Hilbert space method on the seventh-order boundary value problems satisfying boundary conditions. These results have been compared with the results that are obtained by variational iteration method (VIM), homotopy perturbation method (HPM), Adomian decomposition method (ADM), variation of parameters method (VPM), and homotopy analysis method (HAM). Obtained results show that our method is very effective.

1. Introduction

Consider the seventh-order boundary value problem [1–5]:

$$u^{(7)}(x) = N(x, u(x)), \quad 0 \le x \le 1, \tag{1}$$

with boundary conditions

$$\begin{aligned} u^{(i)}(0) &= A_i, \quad i = 0, 1, 2, 3, \\ u^{(j)}(1) &= B_j, \quad j = 0, 1, 2. \end{aligned} \tag{2}$$

The analytical solution of seventh-order differential equations are rarely exists in literature. However, there are various numerical methods for the solution of (1)-(2). The aim of this work is to apply reproducing kernel Hilbert space method (RKHSM) [6–28] to solve the seventh-order boundary value problems. Numerical results of the seventh-order boundary value problems have been obtained by this method in our work. This study shows that the proposed method can be considered as an alternative technique for solving linear and nonlinear problems in science and engineering [29–31].

The paper is organized as follows. Section 2 introduces several reproducing kernel spaces. We provide the main results and the exact and approximate solutions of (1)-(2) in Section 3. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 4. There are some conclusions in the last section.

2. Reproducing Kernel Spaces

In this section, we define some useful reproducing kernel spaces.

Definition 1. We define the space $H_2^1[0,1]$ by

$$\begin{aligned} H_2^1[0,1] = \{ f \mid f \text{ is absolutely continuous in } [0,1], \\ f'(x) \in L^2[0,1], x \in [0,1] \}. \end{aligned} \tag{3}$$

The inner product and the norm in $H_2^1[0,1]$ are defined, respectively, by

$$\begin{aligned} \langle f, g \rangle_{H_2^1} &= f(0) g(0) + \int_0^1 f'(x) g'(x) \, dx, \\ &\qquad u, f \in H_2^1[0,1], \\ \|f\|_{H_2^1} &= \sqrt{\langle f, f \rangle_{H_2^1}}, \quad f \in H_2^1[0,1]. \end{aligned} \tag{4}$$

The space $H_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel function r_x is given by

$$r_x(y) = \begin{cases} 1+x, & x \le y, \\ 1+y, & x > y. \end{cases} \quad (5)$$

Definition 2. We define the space $T_2^8[0,1]$ by the following:

$$\begin{aligned} T_2^8[0,1] = \{ f \mid\ & f, f', f'', f^{(3)}, f^{(4)}, f^{(5)}, f^{(6)}, f^{(7)} \\ & \text{are absolutely continuous in } [0,1], \\ & f^{(8)} \in L^2[0,1], x \in [0,1], \\ & f(0) = f'(0) = f''(0) = f^{(3)}(0) = f(1) \\ & = f'(1) = f''(1) = 0 \}. \end{aligned} \quad (6)$$

The inner product and the norm in $T_2^8[0,1]$ are defined, respectively, by

$$\begin{aligned} \langle f, g\rangle_{T_2^8} &= \sum_{i=0}^{7} f^{(i)}(0)\, g^{(i)}(0) \\ &\quad + \int_0^1 f^{(8)}(x)\, g^{(8)}(x)\, dx, \quad f, g \in T_2^8[0,1], \\ \|f\|_{T_2^8} &= \sqrt{\langle f, f\rangle_{T_2^8}}, \quad f \in T_2^8[0,1]. \end{aligned} \quad (7)$$

The space $T_2^8[0,1]$ is a reproducing kernel space; that is, for each fixed $y \in [0,1]$ and any $f \in T_2^8[0,1]$, there exists a function R_y such that

$$f = \langle f, R_y\rangle_{T_2^8}. \quad (8)$$

Theorem 3. *The space $T_2^8[0,1]$ is a reproducing kernel Hilbert space whose reproducing kernel function R_y is given by*

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{16} c_i(y)\, x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{16} d_i(y)\, x^{i-1}, & x > y, \end{cases} \quad (9)$$

where $c_i(y)$ and $d_i(y)$ can be obtained by Maple 16 and proof of Theorem 3 is given in Appendix.

3. Exact and Approximate Solutions of (1)-(2) in $T_2^8[0,1]$

The solution of (1)-(2) is given in the reproducing kernel space $T_2^8[0,1]$. The linear operator

$$L : T_2^8[0,1] \longrightarrow H_2^1[0,1] \quad (10)$$

is bounded. After homogenizing the boundary conditions, we obtain

$$\begin{aligned} Lv &= M(x, v(x)), \quad 0 \le x \le 1, \\ v^{(i)}(0) &= 0, \quad i = 0, 1, 2, 3, \\ v^{(j)}(1) &= 0, \quad j = 0, 1, 2. \end{aligned} \quad (11)$$

We choose a countable dense subset $P = \{x_i\}_{i=1}^{\infty}$ in $[0,1]$ and let

$$\Psi_x(y) = L^* r_x(y), \quad (12)$$

where L^* is conjugate operator of L and r_x is given by (5). Furthermore, for simplicity let $\Psi_i(x) = \Psi_{x_i}(x)$; namely,

$$\Psi_i(x) \overset{\text{def}}{=} \Psi_{x_i}(x) = L^* r_{x_i}(x). \quad (13)$$

Now one can deduce the following lemmas.

Lemma 4. *$\{\Psi_i(x)\}_{i=1}^{\infty}$ is complete system of $T_2^8[0,1]$.*

Proof. For $f \in T_2^8[0,1]$, let $\langle f, \Psi_i\rangle = 0$ $(i = 1, 2, \ldots)$; that is,

$$\langle f, L^* r_{x_i}\rangle = (Lf)(x_i) = 0. \quad (14)$$

Note that $\{x_i\}_{i=1}^{\infty}$ is the dense set in $[0,1]$; therefore, $(Lf)(x) = 0$. It follows that $f(x) = 0$ from the existence of L^{-1}. □

Lemma 5. *The following formula holds:*

$$\Psi_i(x) = (L_\nu R_x(\nu))(x_i), \quad (15)$$

where the subscript ν of operator L_ν indicates that the operator L applies to function of ν.

Proof. Consider the following:

$$\begin{aligned} \Psi_i(x) &= \langle \Psi_i(\xi), R_x(\xi)\rangle_{T_2^8} = \langle L^* r_{x_i}(\xi), R_x(\xi)\rangle_{T_2^8} \\ &= \langle (r_{x_i})(\xi), (L_\nu R_x(\nu))(\xi)\rangle_{H_2^1} \\ &= (L_\nu R_x(\nu))(x_i). \end{aligned} \quad (16)$$

This completes the proof. □

Remark 6. The orthonormal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of $T_2^8[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \Psi_k(x), \quad (\beta_{ii} > 0, i = 1, 2, \ldots), \quad (17)$$

where β_{ik} are orthogonal coefficients.

In the following, we will give the representation of the exact solution of (11) in the reproducing kernel space $T_2^8[0,1]$.

Theorem 7. *If u is the exact solution of* (11), *then*

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i\left(x\right), \quad (18)$$

where $\{x_i\}_{i=1}^{\infty}$ is a dense set in $[0, 1]$.

Proof. From the (17) and uniqueness of solution of (11), we have

$$\begin{aligned} u &= \sum_{i=1}^{\infty}\left\langle u, \overline{\Psi}_i\right\rangle_{T_2^8}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u, L^* r_{x_k}\right\rangle_{T_2^8}\overline{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu, r_{x_k}\right\rangle_{H_2^1}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle M, r_{x_k}\right\rangle_{H_2^1}\overline{\Psi}_i \quad (19) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i\left(x\right). \end{aligned}$$

This completes the proof. □

Now the approximate solution u_n can be obtained by truncating the n-term of the exact solution u as

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i\left(x\right). \quad (20)$$

Lemma 8. *Assume u is the solution of* (11) *and r_n is the error between the approximate solution u_n and the exact solution u. Then the error sequence r_n is monotone decreasing in the sense of $\|\cdot\|_{T_2^8}$ and $\|r_n(x)\|_{T_2^8} \to 0$.*

Proof. From (18) and (20), we obtain

$$\left\|u - u_n\right\|_{T_2^8} = \left\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i\left(x\right)\right\|_{T_2^8}. \quad (21)$$

Thus

$$\left\|u - u_n\right\|_{T_2^8} \longrightarrow 0, \quad n \longrightarrow \infty. \quad (22)$$

In addition

$$\begin{aligned} \left\|u - u_n\right\|_{T_2^8}^2 &= \left\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i\left(x\right)\right\|_{T_2^8}^2 \\ &= \sum_{i=n+1}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}M\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x)_{T_2^8}\right)^2. \end{aligned} \quad (23)$$

Then, $\|u - u_n\|_{T_2^8}$ is monotonically decreasing in n. □

Remark 9. The seventh-order boundary value problems have come out in construction engineering, beam column theory, and chemical reactions. Therefore solutions of the seventh-order boundary value problems are very important in the literature. The reproducing kernel function for seventh-order boundary value problem has not been calculated till now.

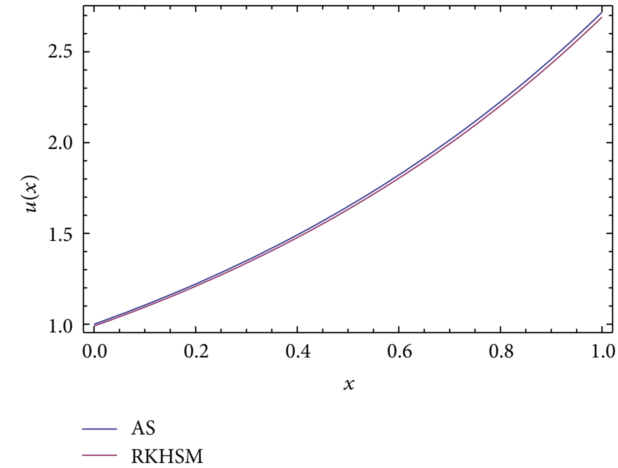

Figure 1: Comparison of analytical solution and RKHSM solution for Example 10.

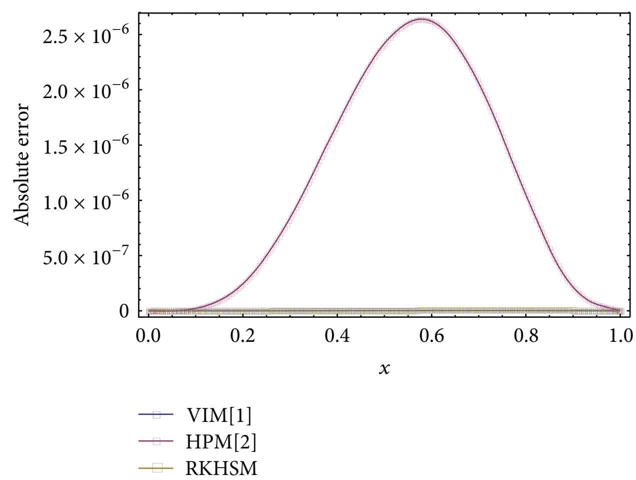

Figure 2: Comparison of absolute error of VIM, HPM, and RKHSM for Example 10.

All computations are performed by Maple 16. The RKHSM does not require discretization of the variables, that is, time and space, and it is not affected by computational round-off errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKHSM for the seventh-order boundary value problems is controllable and absolute errors are small with present choice of x (see Tables 1–6 and Figures 1–6). The obtained numerical results justify the advantage of this methodology. We gave transformations to homogenize the boundary conditions for all examples. Additionally, we improved our programme to find numerical results. As shown in Tables 1, 3, and 5 all the numerical results have been found in very short time.

4. Numerical Results

In this section, three numerical examples are provided to show the accuracy of the present method.

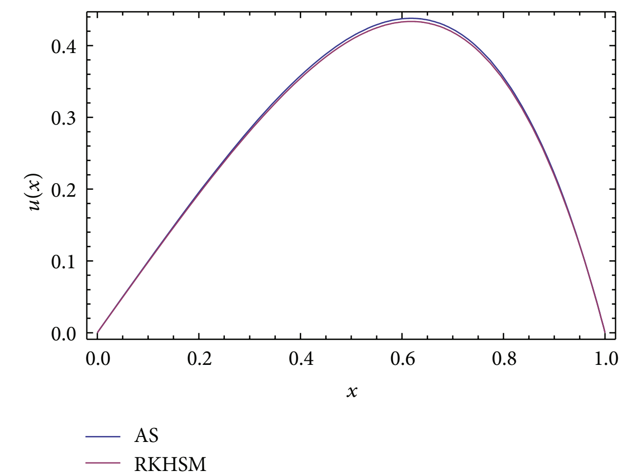

FIGURE 3: Comparison of analytical solution and RKHSM solution for Example 11.

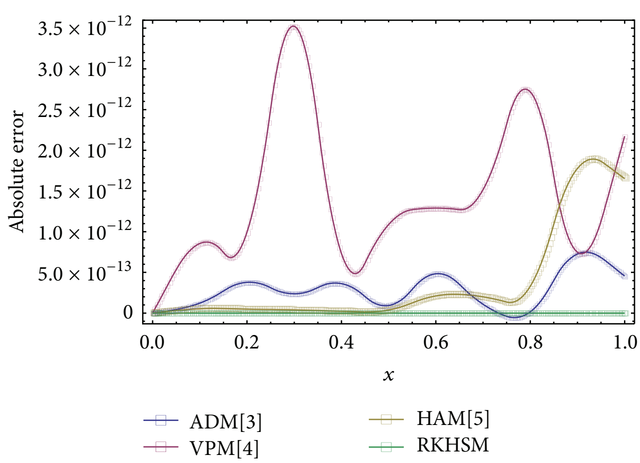

FIGURE 4: Comparison of absolute error of ADM, VPM, HAM, and RKHSM for Example 11.

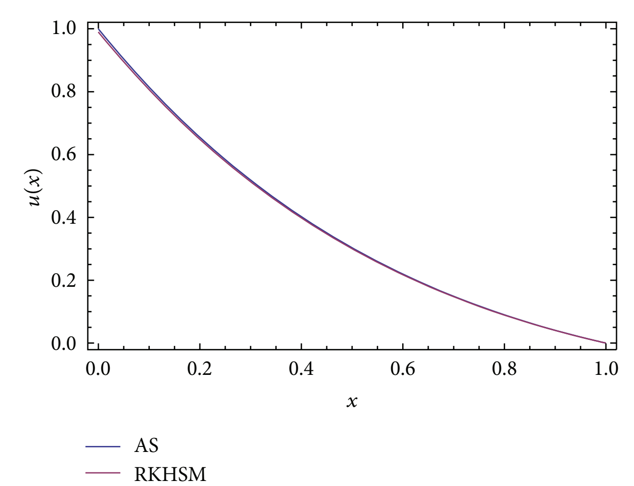

FIGURE 5: Comparison of analytical solution and RKHSM solution for Example 12.

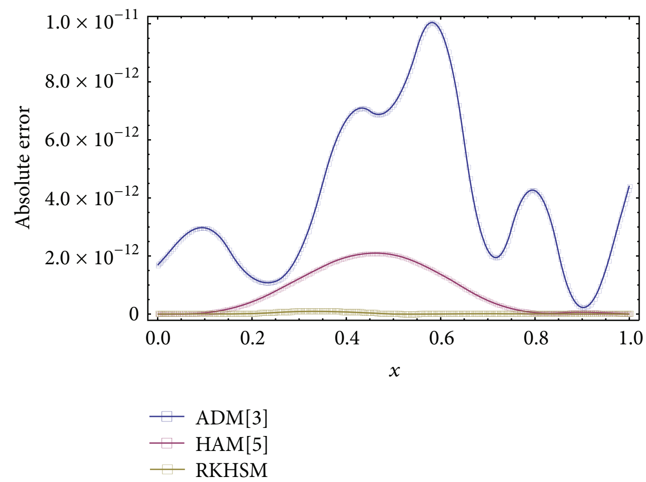

FIGURE 6: Comparison of absolute error of ADM, HAM, and RKHSM for Example 12.

Example 10. We first consider the seventh-order nonlinear boundary value problem:

$$\begin{gathered} u^{(7)}(x) = e^{-x}u^2(x), \quad 0 < x < 1, \\ u(0) = u'(0) = u''(0) = u^{(3)}(0) = 1, \\ u(1) = u'(1) = u''(1) = e. \end{gathered} \tag{24}$$

The exact solution of (24) is given as [1]

$$u(x) = e^x. \tag{25}$$

After homogenizing the boundary conditions of (24), we obtain

$$\begin{gathered} v^{(7)}(x) = e^{-x}\left[v(x) + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^4\left(\frac{21}{2}e - \frac{57}{2}\right)\right. \\ \left. + x^5\left(\frac{87}{2} - 16e\right) + x^6\left(\frac{13}{2}e - \frac{53}{3}\right)\right]^2, \\ 0 \le x \le 1, \\ v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0, \\ v(1) = v'(1) = v''(1) = 0, \end{gathered} \tag{26}$$

where we used the following transformation:

$$\begin{aligned} v(x) = u(x) - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} - x^4\left(\frac{21}{2}e - \frac{57}{2}\right) \\ - x^5\left(\frac{87}{2} - 16e\right) - x^6\left(\frac{13}{2}e - \frac{53}{3}\right). \end{aligned} \tag{27}$$

Using the RKHSM for this example we obtain Tables 1-2 and Figures 1-2.

Example 11. We now consider the seventh-order linear BVP

$$
\begin{gathered}
u^{(7)}(x) = -u(x) - e^{x}\left(35 + 12x + 2x^{2}\right), \quad 0 \le x \le 1, \\
u(0) = 0, \qquad u'(0) = 1, \\
u''(0) = 0, \qquad u^{(3)}(0) = -3, \\
u(1) = 0, \qquad u'(1) = -e, \qquad u''(1) = -4e.
\end{gathered} \tag{28}
$$

The exact solution of (28) is given as [3]

$$
u(x) = x(1-x)e^{x}. \tag{29}
$$

After homogenizing the boundary conditions of (28), we get

$$
\begin{gathered}
v^{(7)}(x) + v(x) = -e^{x}\left(35 + 12x + 2x^{2}\right) - x \\
+ \frac{x^{3}}{2} - x^{4}\left(3e - \frac{17}{2}\right) - x^{5}\left(\frac{27}{2} - 5e\right) \\
- x^{6}\left(2e - \frac{11}{2}\right), \quad 0 \le x \le 1, \\
v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0, \\
v(1) = v'(1) = v''(1) = 0,
\end{gathered} \tag{30}
$$

where, we used the following transformation

$$
\begin{aligned}
v(x) = u(x) - x + x\frac{3}{2} - x^{4}\left(3e - \frac{17}{2}\right) \\
- x^{5}\left(\frac{27}{2} - 5e\right) - x^{6}\left(2e - \frac{11}{2}\right).
\end{aligned} \tag{31}
$$

Using RKHSM for this example we obtain Tables 3-4 and Figures 3-4.

Example 12. Consider the following seventh-order nonlinear BVP

$$
\begin{gathered}
u^{(7)}(x) = u(x)u'(x) + e^{-2x}\left(2 + e^{x}(x-8) - 3x + x^{2}\right), \\
0 \le x \le 1, \\
u(0) = 1, \qquad u'(0) = -2, \\
u''(0) = 3, \qquad u^{(3)}(0) = -4, \\
u(1) = 0, \qquad u'(1) = \frac{-1}{e}, \qquad u''(1) = \frac{2}{e}.
\end{gathered} \tag{32}
$$

The exact solution of (32) is given as [3]

$$
u(x) = (1-x)e^{-x}. \tag{33}
$$

After homogenizing the boundary conditions of (32), we have

$$
\begin{gathered}
v^{(7)}(x) - \left(-2 + 3x - 2x^{2} + 4x^{3}\left(\frac{6}{e} - 2\right) + 5x^{4}\left(4 - \frac{11}{e}\right) + 6x^{5}\left(\frac{5}{e} - \frac{11}{6}\right)\right)v(x) \\
- \left(1 - 2x + \frac{3}{2}x^{2} - \frac{2}{3}x^{3} + x^{4}\left(\frac{6}{e} - 2\right) + x^{5}\left(4 - \frac{11}{e}\right) + x^{6}\left(\frac{5}{e} - \frac{11}{6}\right)\right)v'(x) \\
= \left[\begin{array}{c}\left(1 - 2x + \frac{3}{2}x^{2} - \frac{2}{3}x^{3} + x^{4}\left(\frac{6}{e} - 2\right) + x^{5}\left(4 - \frac{11}{e}\right) + x^{6}\left(\frac{5}{e} - \frac{11}{6}\right)\right) \\ \left(-2 + 3x - 2x^{2} + 4x^{3}\left(\frac{6}{e} - 2\right) + 5x^{4}\left(4 - \frac{11}{e}\right) + 6x^{5}\left(\frac{5}{e} - \frac{11}{6}\right)\right)\end{array}\right] \\
+ v(x)v'(x) + e^{-2x}\left(2 + e^{x}(x-8) - 3x + x^{2}\right), \quad 0 \le x \le 1, \\
v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0, \\
v(1) = v'(1) = v''(1) = 0,
\end{gathered} \tag{34}
$$

where we used the following transformation:

$$
\begin{aligned}
u(x) = v(x) + 1 - 2x + \frac{3}{2}x^{2} - \frac{2}{3}x^{3} \\
+ x^{4}\left(\frac{6}{e} - 2\right) + x^{5}\left(4 - \frac{11}{e}\right) + x^{6}\left(\frac{5}{e} - \frac{11}{6}\right).
\end{aligned} \tag{35}
$$

Using RKHSM for this example we obtain Tables 5-6 and Figures 1–6.

Remark 13. Using our method we chose 36 points on [0, 1]. In Tables 1–6, we computed the absolute errors $|u(x,t) - u_n(x,t)|$ at the points $\{(x_i) : x_i = i,\ i = 0.0, 0.1, \ldots, 1.0\}$. The RKHSM tested on three problems, one linear and two nonlinear. A comparison with VIM [1], HPM [2], ADM [3], VPM [4], and

Table 1: Numerical results for Example 10 (time (s): 1.645).

x	Exact solution	Approximate solution	Absolute error	Relative error
0.0	1.0	1.0	0.0	0.0
0.1	1.1051709180756476248	1.1051709180727538232	2.893×10^{-12}	2.618×10^{-12}
0.2	1.2214027581601698339	1.2214027581330885422	2.708×10^{-11}	2.217×10^{-11}
0.3	1.3498588075760031040	1.3498588075089984856	6.700×10^{-11}	4.963×10^{-11}
0.4	1.4918246976412703178	1.4918246975765762858	6.469×10^{-11}	4.336×10^{-11}
0.5	1.6487212707001281468	1.6487212707474912552	4.736×10^{-11}	2.872×10^{-11}
0.6	1.8221188003905089749	1.8221188006600701498	2.695×10^{-10}	1.479×10^{-10}
0.7	2.0137527074704765216	2.0137527079512967073	4.808×10^{-10}	2.387×10^{-10}
0.8	2.2255409284924676046	2.2255409289689961077	4.765×10^{-10}	2.141×10^{-10}
0.9	2.4596031111569496638	2.4596031113394646558	1.825×10^{-10}	7.420×10^{-11}
1.0	2.7182818284590452354	2.7182818284590452354	0.0	0.0

Table 2: Comparison of absolute error of HPM, VIM, and RKHSM for Example 10.

x	Absolute error [2]	Absolute error [1]	Absolute error [RKHSM]
0.0	0.0	0.0	0.0
0.1	2.15×10^{-8}	3.8478×10^{-12}	$2.8938016 \times 10^{-12}$
0.2	2.45×10^{-7}	1.2366×10^{-10}	$2.70812917 \times 10^{-11}$
0.3	8.42×10^{-7}	2.7788×10^{-10}	$6.70046184 \times 10^{-11}$
0.4	1.69×10^{-6}	7.5864×10^{-10}	$6.46940320 \times 10^{-11}$
0.5	2.42×10^{-6}	1.1571×10^{-9}	$4.73631084 \times 10^{-11}$
0.6	2.62×10^{-6}	1.3132×10^{-9}	$2.695611749 \times 10^{-10}$
0.7	2.06×10^{-6}	1.2228×10^{-9}	$4.808201857 \times 10^{-10}$
0.8	1.05×10^{-6}	6.6023×10^{-10}	$4.765285031 \times 10^{-10}$
0.9	2.14×10^{-7}	1.6534×10^{-10}	$1.825149920 \times 10^{-10}$
1.0	4.44×10^{-16}	1.3265×10^{-11}	0.0

HAM [5] was made and it was seen that the present method yields good results (see Tables 1–6 and Figures 1–6).

5. Conclusion

In this paper, we introduced an algorithm for solving the seventh-order problem with boundary conditions. For illustration purposes, we chose three examples which were selected to show the computational accuracy. It may be concluded that the RKHSM is very powerful and efficient in finding exact solution for a wide class of boundary value problems. The approximate solution obtained by the present method is uniformly convergent. Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKHSM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for the seventh-order boundary value problem.

Appendix

Proof of Theorem 3. Let $f \in T_2^8[0,1]$. By Definition 2 we have

$$\left\langle f, R_y \right\rangle_{T_2^8} = \sum_{i=0}^{7} f^{(i)}(0)\, R_y^{(i)}(0) + \int_0^1 f^{(8)}(x)\, R_y^{(8)}(x)\, dx. \tag{A.1}$$

Through several integrations by parts for (A.1) we have

$$\begin{aligned}\left\langle f, R_y \right\rangle_{T_2^8} &= \sum_{i=0}^{7} u^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(7-i)} R_y^{(15-i)}(0)\right] \\ &\quad + \sum_{i=0}^{7} (-1)^{(7-i)} u^{(i)}(1)\, R_y^{(15-i)}(1) \\ &\quad + \int_0^1 u(x)\, R_y^{(16)}(x)\, dx.\end{aligned} \tag{A.2}$$

Note that property of the reproducing kernel

$$\left\langle u, R_y \right\rangle_{T_2^8} = u(y). \tag{A.3}$$

Table 3: Numerical results for Example 11 (time (s): 3.123).

x	Exact solution	Approximate solution	Absolute error
0.0	0.0	0.0	0.0
0.1	0.09946538262680828 6232	0.09946538262680828 5898	$3.34E-19$
0.2	0.19542444130562717342	0.19542444130562716451	$8.91E-18$
0.3	0.28347034959096065184	0.28347034959096048526	$1.6658E-16$
0.4	0.35803792743390487627	0.35803792743390483786	$3.841E-17$
0.5	0.41218031767503203670	0.41218031767503200826	$2.844E-17$
0.6	0.43730851209372215398	0.43730851209372205668	$9.730E-17$
0.7	0.42288806856880006954	0.42288806856879998460	$8.494E-17$
0.8	0.35608654855879481674	0.35608654855879479280	$2.394E-17$
0.9	0.22136428000412546974	0.22136428000412540564	$6.410E-17$
1.0	0.0	$1.24053E-20$	$1.24053E-20$

Table 4: Comparison of absolute error of VPM, ADM, HAM, and RKHSM.

x	Absolute error [4]	Absolute error [3]	Absolute error [5]	Absolute error [RKHSM]
0.0	0.0	0.0	0.0	0.0
0.1	$8.55607E-13$	$1.23082E-13$	$5.39291E-14$	$3.34E-19$
0.2	$9.94041E-12$	$3.7792E-13$	$4.85167E-14$	$8.91E-18$
0.3	$3.52244E-11$	$2.37421E-13$	$3.92464E-14$	$1.6658E-16$
0.4	$7.3224E-10$	$3.62099E-13$	$2.21489E-14$	$3.841E-17$
0.5	$1.08769E-10$	$9.39249E-14$	$3.84137E-14$	$2.844E-17$
0.6	$1.29035E-10$	$4.82947E-13$	$2.10831E-13$	$9.730E-17$
0.7	$1.51466E-10$	$1.09135E-13$	$1.99785E-13$	$8.494E-17$
0.8	$2.717974E-10$	$1.64868E-14$	$3.29736E-13$	$2.394E-17$
0.9	$7.48179E-10$	$7.25975E-13$	$1.77622E-12$	$6.410E-17$
1.0	$2.1729E-09$	$4.54747E-13$	$1.65159E-12$	$1.24053E-20$

Now, if

$$\begin{gathered} R_y^{(4)}(0)+R_y^{(11)}(0)=0, \\ R_y^{(5)}(0)-R_y^{(10)}(0)=0, \\ R_y^{(6)}(0)+R_y^{(9)}(0)=0, \\ R_y^{(7)}(0)-R_y^{(8)}(0)=0, \\ R_y^{(8)}(1)=0, \\ R_y^{(9)}(1)=0, \\ R_y^{(10)}(1)=0, \\ R_y^{(11)}(1)=0, \\ R_y^{(12)}(1)=0, \end{gathered} \tag{A.4}$$

then (A.2) implies that

$$R_y^{(16)}(x)=\delta(x-y), \tag{A.5}$$

when $x \neq y$, then

$$R_y^{(16)}(x)=0, \tag{A.6}$$

and therefore

$$R_y(x)=\begin{cases} \sum_{i=1}^{16} c_i(y)\, x^{i-1}, & x \le y, \\ \sum_{i=1}^{16} d_i(y)\, x^{i-1}, & x > y. \end{cases} \tag{A.7}$$

Since

$$R_y^{(16)}(x)=\delta(x-y), \tag{A.8}$$

we have

$$\begin{gathered} R_{y^+}^{(k)}(y)=R_{y^-}^{(k)}(y), \quad k=0,\ldots,14, \\ R_{y^+}^{(15)}(y)-R_{y^-}^{(15)}(y)=1. \end{gathered} \tag{A.9}$$

Since $R_y \in T_2^8[0,1]$, it follows that

$$\begin{aligned} R_y(0)&=R_y'(0)=R_y''(0)=R_y^{(3)}(0)=R_y(1) \\ &=R_y'(1)=R_y''(1)=0. \end{aligned} \tag{A.10}$$

From (A.4)–(A.10), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i=1,2,\ldots,16)$ can be obtained. This completes the proof. □

Table 5: Numerical results for Example 12 (time (s): 5.234).

x	Exact solution	Approximate solution	Absolute error
0.0	1.0	1.0	0.0
0.1	0.81435367623236361584	0.81435367623236697064	3.35480×10^{-15}
0.2	0.65498460246238548694	0.65498460246237032242	1.516452×10^{-14}
0.3	0.51857275447720250625	0.51857275447711724998	8.525627×10^{-14}
0.4	0.40219202762138358044	0.40219202762145521926	7.163882×10^{-14}
0.5	0.30326532985631671180	0.30326532985631655134	1.6046×10^{-16}
0.6	0.21952465443761057305	0.21952465443760682026	3.75279×10^{-15}
0.7	0.14897559113742285441	0.14897559113743255406	9.69965×10^{-15}
0.8	0.089865792823444318286	0.089865792823449635588	5.317302×10^{-15}
0.9	0.040656965974059911188	0.040656965974059021714	8.89474×10^{-16}
1.0	0.0	2.708848×10^{-22}	2.708848×10^{-22}

Table 6: Comparison of absolute error of ADM, HAM, and RKHSM.

x	Absolute error [3]	Absolute error [5]	Absolute error [RKHSM]
0.0	$1.67932E-12$	0.0	0.0
0.1	$2.96696E-12$	$4.15223E-14$	$3.3548E-15$
0.2	$1.26055E-12$	$4.18332E-13$	$1.516452E-14$
0.3	$2.10898E-12$	$1.21736E-12$	$8.525627E-14$
0.4	$6.68926E-12$	$1.95471E-12$	$7.163882E-14$
0.5	$7.21923E-12$	$2.03731E-12$	$1.6046E-16$
0.6	$9.75339E-12$	$1.37063E-12$	$3.75279E-15$
0.7	$2.19552E-12$	$4.66988E-13$	$9.69965E-15$
0.8	$4.24917E-12$	$4.8378E-14$	$5.317302E-15$
0.9	$2.27311E-13$	$6.00561E-14$	$8.89474E-16$
1.0	$4.42298E-12$	$1.29172E-15$	$2.708848E-22$

Disclosure

This paper is a part of the Ph.D. thesis of Ali Akgül.

Conflict of Interests

The authors declare that they do not have any competing or conflict of interests.

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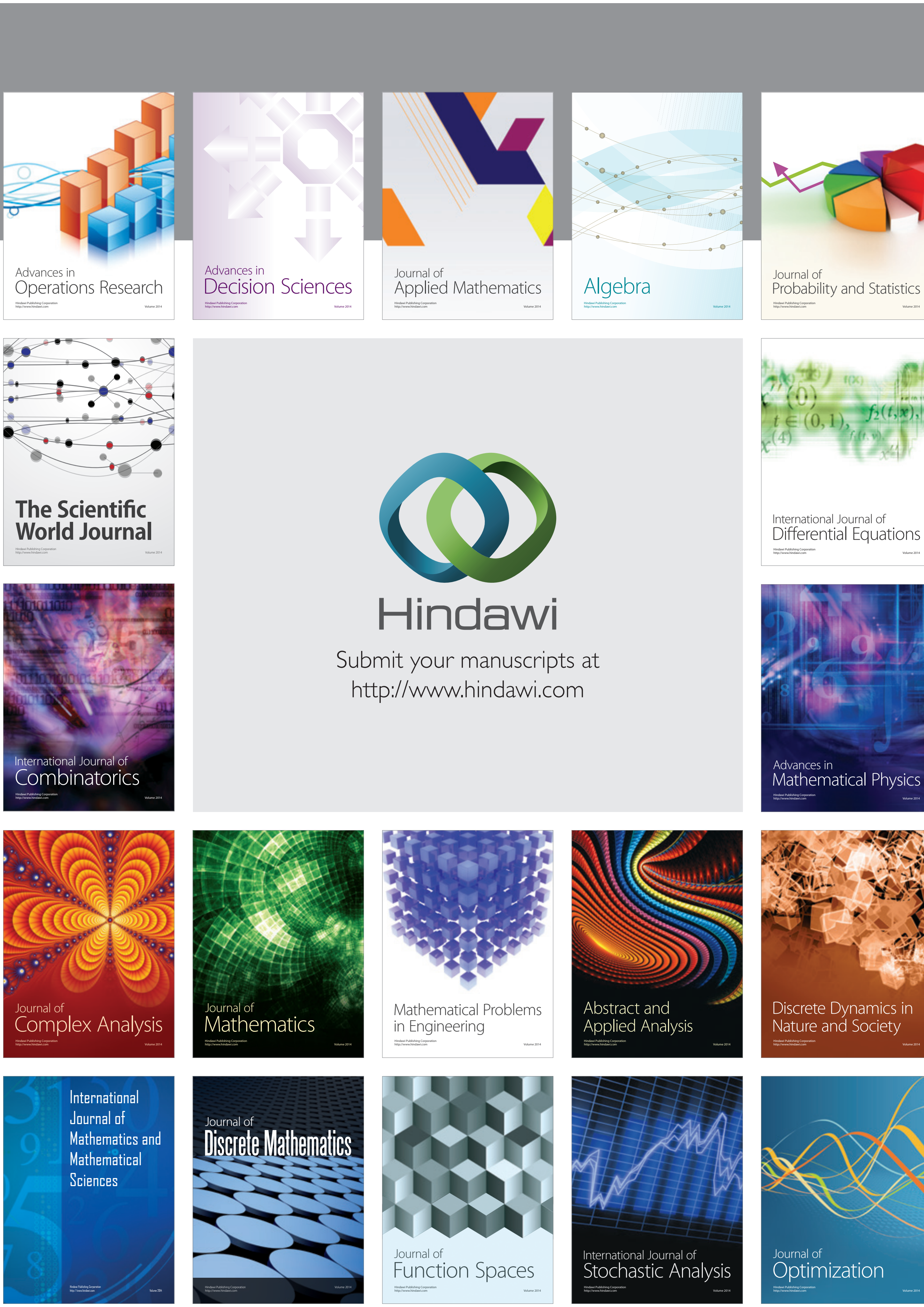

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