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Research Article

Approximate Analytical Solutions for Mathematical Model of Tumour Invasion and Metastasis Using Modified Adomian Decomposition and Homotopy Perturbation Methods

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The modified decomposition method (MDM) and homotopy perturbation method (HPM) are applied to obtain the approximate solution of the nonlinear model of tumour invasion and metastasis. The study highlights the significant features of the employed methods and their ability to handle nonlinear partial differential equations. The methods do not need linearization and weak nonlinearity assumptions. Although the main difference between MDM and Adomian decomposition method (ADM) is a slight variation in the definition of the initial condition, modification eliminates massive computation work. The approximate analytical solution obtained by MDM logically contains the solution obtained by HPM. It shows that HPM does not involve the Adomian polynomials when dealing with nonlinear problems.

1. Introduction

Over the years, many mathematical models of tumour growth have appeared in literature [1–3]. These problems and phenomena are modeled by partial differential equations (PDE) such as deterministic reaction-diffusion equations which are used to model the spatial spread of tumours both at early growth and later invasive stages [4, 5]. In most cases, these problems do not admit analytical solution. So these equations should be solved using some particular techniques. Chaplain [6] used numerical solution (finite difference method) to solve the above problem. However, this method involved linearization, discretization, and assumption. Therefore, the real problem has to undergo simplification before it can be solved. In recent years, much attention has been devoted to the newly developed methods to construct an analytical solution of equation such as Adomian decomposition method (ADM) [7] and homotopy perturbation method (HPM) [8]. Both methods yield rapidly convergent series solutions for linear and nonlinear equations. The advantages of these methods are that they provide direct scheme for solving the problem, that is, without the need for linearization and discretization. The accuracy of the ADM method was studied extensively by Hashim et al. [9] and compared with other methods [10, 11]. Anderson et al. [12] proposed a modification of the ADM by a slight variation from the standard ADM. The modified method (MDM) was established based on the assumption that the initial function can be divided into two parts and the success of the MDM depends mainly on the proper choice of the parts. In this paper, we present approximate analytical solution of tumour invasion and metastasis model [13] solved by MDM and HPM. The results from both methods are then compared and reveal their capability, effectiveness and convenience. Both methods give successive approximations of high accuracy solution.

2. Problem Formulation

Let us consider a system describing the interactions of the tumour cells (denoted by n), extra cellular matrix (ECM, denoted by f), and matrix degrading enzymes (MDE, denoted by m) is given by [13]

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - \chi \nabla \cdot (n \nabla f),$$
$$\frac{\partial f}{\partial t} = -\delta m f, \tag{1}$$
$$\frac{\partial m}{\partial t} = D_m \nabla^2 m + \mu n - \lambda m,$$

where D_n is the tumour cell random motility coefficient, D_m is the MDE diffusion coefficient, χ is the haptotactic coefficient, and λ, μ, δ are the positive constants.

Non-dimensionalise of (1) by setting

$$\tilde{n} = \frac{n}{n_o}, \quad \tilde{f} = \frac{f}{f_o}, \quad \tilde{m} = \frac{m}{m_o}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{\tau}, \tag{2}$$

where n_o is the tumour cell density, f_o is the ECM density, m_o is the MDE concentration, L is the length scale, and τ is the time ($\tau = L^2/D$, where D is a reference chemical diffusion coefficient). By dropping the tildes for notational convenience, we obtain the scaled system of equations:

$$\frac{\partial n}{\partial t} = d_n \nabla^2 n - \gamma \nabla \cdot (n \nabla f), \tag{3}$$
$$\frac{\partial f}{\partial t} = -\eta m f, \tag{4}$$
$$\frac{\partial m}{\partial t} = d_m \nabla^2 m + \omega n - \beta m, \tag{5}$$

where $d_n = D_n/D$, $\gamma = \chi f_o/D$, $\eta = \tau m_o \delta$, $d_m = D_m/D$, $\omega = \tau \mu n_o/m_o$, and $\beta = \tau\lambda$. The initial conditions of each equation are

$$n(x,0) = \exp\left(-\frac{x^2}{\varepsilon}\right),$$
$$f(x,0) = 1 - 0.5\exp\left(-\frac{x^2}{\varepsilon}\right), \tag{6}$$
$$m(x,0) = 0.5\exp\left(-\frac{x^2}{\varepsilon}\right),$$

where ε is a positive constant.

The approximate solutions of (3)–(5) are obtained by integrating each equation once with respect to t and using the initial condition. Hence we obtained

$$n(x,t) = n(x) + d_n \int_0^t \frac{\partial^2 n}{\partial x^2} dz - \gamma \int_0^t \frac{\partial n}{\partial x} \cdot \frac{\partial f}{\partial x} dz - \gamma \int_0^t n \frac{\partial^2 f}{\partial x^2} dz, \tag{7}$$

$$f(x,t) = f(x) - \eta \int_0^t m f \, dz, \tag{8}$$

$$m(x,t) = m(x) + d_m \int_0^t \frac{\partial^2 m}{\partial x^2} dz + \omega \int_0^t n \, dz - \beta \int_0^t m \, dz. \tag{9}$$

In (7)–(9), we assume $n(x)$, $f(x)$, and $m(x)$ are bounded for all x in $J = [0,T]$, $(T \in \mathfrak{R})$, and $|t-\tau| \le m'$, for all $0 \le t, \tau \le T$. The terms $(\partial^2 n/\partial x^2)$, $(\partial n/\partial x) \cdot (\partial f/\partial x)$, $n(\partial^2 f/\partial x^2)$, $F_1(mf) = mf$, $\partial^2 m/\partial x^2$, $F_2(n) = n$, and $F_3(m) = m$ are Lipschitz continuous with

$$\left|\frac{\partial^2 n}{\partial x^2} - \frac{\partial^2 n^*}{\partial x^2}\right| \le L_1 |n - n^*|,$$
$$\left|\frac{\partial n}{\partial x} \cdot \frac{\partial f}{\partial x} - \frac{\partial n^*}{\partial x} \cdot \frac{\partial f^*}{\partial x}\right| \le L_2 |nf - n^* f^*|,$$
$$\left|n \frac{\partial^2 f}{\partial x^2} - n^* \frac{\partial^2 f^*}{\partial x^2}\right| \le L_3 |nf - n^* f^*|,$$
$$|F_1(m,f) - F_1(m^*, f^*)| \le L_4 |mf - mf^*|,$$
$$\left|\frac{\partial^2 m}{\partial x^2} - \frac{\partial^2 m^*}{\partial x^2}\right| \le L_5 |m - m^*|, \tag{10}$$
$$|F_2(n) - F_2(n^*)| \le L_6 |n - n^*|,$$
$$|F_3(m) - F_3(m^*)| \le L_7 |m - m^*|,$$
$$\alpha = T\left(m' L_1 + m' L_2 + m' L_3 + m' L_4 + m' L_5 + m' L_6 + m' L_7\right),$$
$$\zeta = 1 - T(1 - \alpha).$$

3. Mathematical Methods

3.1. Adomian Decomposition Method (ADM). The Adomian decomposition method is applied in (3)–(5):

$$L_t n = d_n \frac{\partial^2 n}{\partial x^2} - \gamma \left[\frac{\partial n}{\partial x}\frac{\partial f}{\partial x} + n \frac{\partial^2 f}{\partial x^2}\right], \tag{11}$$
$$L_t f = -\eta m f, \tag{12}$$
$$L_t m = d_m \frac{\partial^2 m}{\partial x^2} + \omega n - \beta m, \tag{13}$$

where $L_t = \partial/\partial t$ is integrable differential operator with $L_t^{-1} = \int_0^t (\cdot) dt$.

Operating on both sides of (11)–(13) with the integral operator L^{-1} leads to

$$n(x,t) = n(x,0) + d_n L_t^{-1}\left(\frac{\partial^2 n}{\partial x^2}\right) - \gamma \left[L_t^{-1}[N_1(n,f)] + L_t^{-1}[N_2(n,f)]\right],$$
$$f(x,t) = f(x,0) - \eta L_t^{-1}[N_3(m,f)],$$
$$m(x,t) = m(x,0) + L_t^{-1}\left(d_m \frac{\partial^2 m}{\partial x^2}\right) + L_t^{-1}[\omega n - \beta m], \tag{14}$$

where

$$N_1(n,f) = \frac{\partial n}{\partial x}\frac{\partial f}{\partial x},$$
$$N_2(n,f) = n\frac{\partial^2 f}{\partial x^2}, \tag{15}$$
$$N_3(m,f) = mf$$

are the nonlinear terms. The solutions $n(x,t)$, $f(x,t)$, and $m(x,t)$ can be decomposed by an infinite series as follows [7]:

$$n(x,t) = \sum_{i=0}^{\infty} n_i(x,t),$$
$$f(x,t) = \sum_{i=0}^{\infty} f_i(x,t), \tag{16}$$
$$m(x,t) = \sum_{i=0}^{\infty} m_i(x,t),$$

where $n_i(x,t)$, $f_i(x,t)$, and $m_i(x,t)$ are the components of $n(x,t)$, $f(x,t)$, and $m(x,t)$ that will elegantly be determined. The nonlinear term $N(x,t)$ is decomposed by the following infinite series:

$$N_k(x,t) = \sum_{l=0}^{\infty} A_{kl}, \quad k = 1,2,3, \tag{17}$$

where A_{kl} is called Adomian's polynomial and defined by

$$A_{kl} = \frac{1}{l!}\left[\frac{d^l}{d\psi^l} N_k\left(\sum_{i=0}^{\infty}\psi^i n_i, \sum_{i=0}^{\infty}\psi^i f_i, \sum_{i=0}^{\infty}\psi^i m_i\right)\right]_{\psi=0}, \quad i \geq 0. \tag{18}$$

From the above consideration, the decomposition method defines the components $n_i(x,t)$, $f_i(x,t)$, and $m_i(x,t)$ for $i \geq 0$ by the following recursive relationships.

Anderson et al. [12] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. The modified method (MDM) was established based on the assumption that if the zeroth component $u_o = g$ and the function g is possible to divide into two parts such as g_1 and g_2, one can formulate the recursive algorithm for u_o and general term u_{n+1} in a form of the modified recursive scheme as follows:
for $n_i(x,t)$,

$$n_0 = g_1,$$
$$n_1 = g_2 + \int_0^t \left[d_n \frac{\partial^2 n_0}{\partial x^2} - \gamma\left\{A_{1,0}(n,f) + A_{2,0}(n,f)\right\}\right] d\tau,$$
$$n_{l+1}(x,t) = \int_0^t \left[d_n \frac{\partial^2 n_l}{\partial x^2} - \gamma\left\{A_{1,l}(n,f) + A_{2,l}(n,f)\right\}\right] d\tau, \quad l \geq 1; \tag{19}$$

for $f_i(x,t)$,

$$f_0 = g_1',$$
$$f_1 = g_2' - \eta \int_0^t \left[A_{3,0}(m,f)\right] d\tau, \tag{20}$$
$$f_{l+1}(x,t) = -\eta \int_0^t \left[A_{3,l}(m,f)\right] d\tau, \quad l \geq 1;$$

for $m_i(x,t)$,

$$m_0 = g_1'',$$
$$m_1 = g_2'' + \int_0^t \left[d_m \frac{\partial^2 m_l}{\partial x^2} + \omega n_l - \beta m_l\right] d\tau, \tag{21}$$
$$m_{l+1}(x,t) = \int_0^t \left[d_m \frac{\partial^2 m_l}{\partial x^2} + \omega n_l - \beta m_l\right] d\tau, \quad l \geq 1.$$

This type of modification is giving more flexibility to the ADM in order to solve complicated nonlinear differential equations. MDM scheme avoids the unnecessary computation especially in calculation of the Adomian polynomials. The computation of these polynomials will be reduced very considerably by using the MDM.

3.2. Homotopy Perturbation Method (HPM). To solve (3)–(5) with the HPM method, we construct the following homotopy:

$$H_1(n,f,p) = (1-p)\left(\frac{\partial n}{\partial t} - \frac{\partial n_0}{\partial t}\right) + p\left(\frac{\partial n}{\partial t} - d_n\frac{\partial^2 n}{\partial x^2} + \gamma\frac{\partial n}{\partial x}\cdot\frac{\partial f}{\partial x} + \gamma n\frac{\partial^2 f}{\partial x^2}\right) = 0, \tag{22}$$

$$H_2(f,m,p) = (1-p)\left(\frac{\partial f}{\partial t} - \frac{\partial f_0}{\partial t}\right) + p\left(\frac{\partial f}{\partial t} + \eta m f\right) = 0, \tag{23}$$

$$H_3(m,n,p) = (1-p)\left(\frac{\partial m}{\partial t} - \frac{\partial m_0}{\partial t}\right) + p\left(\frac{\partial m}{\partial t} - d_m\frac{\partial^2 m}{\partial x^2} - \omega n + \beta m\right) = 0 \tag{24}$$

or

$$H_1(n,f,p) = \frac{\partial n}{\partial t} - \frac{\partial n_0}{\partial t} + p\left(-d_n\frac{\partial^2 n}{\partial x^2} + \gamma\frac{\partial n}{\partial x}\cdot\frac{\partial f}{\partial x} + \gamma n\frac{\partial^2 f}{\partial x^2} + \frac{\partial n_o}{\partial t}\right) = 0, \tag{25}$$

$$H_2(f,m,p) = \frac{\partial f}{\partial t} - \frac{\partial f_0}{\partial t} + p\left(\eta m f + \frac{\partial f_0}{\partial t}\right) = 0, \tag{26}$$

$$H_3(m,n,p) = \frac{\partial m}{\partial t} - \frac{\partial m_0}{\partial t} + p\left(-d_m\frac{\partial^2 m}{\partial x^2} - \omega n + \beta m + \frac{\partial m_0}{\partial t}\right) = 0. \tag{27}$$

In HPM, the solutions of (25)–(28) are expressed as power series in p:

$$\begin{aligned} n(x,t) &= n_o(x,t) + pn_1(x,t) + p^2 n_2(x,t) \\ &\quad + p^3 n_3(x,t) + \cdots, \\ f(x,t) &= f_o(x,t) + pf_1(x,t) + p^2 f_2(x,t) \\ &\quad + p^3 f_3(x,t) + \cdots, \\ m(x,t) &= m_o(x,t) + pm_1(x,t) + p^2 m_2(x,t) \\ &\quad + p^3 m_3(x,t) + \cdots, \end{aligned} \tag{28}$$

where $p \in [0,1]$ is an embedding parameter and n_o, f_o, and m_o are the arbitrary initial approximation satisfying the given initial condition. As p approaches to 1, we obtained

$$n(x,t) = \lim_{p\to 1} n = n_0 + n_1 + n_2 + n_3 + \cdots = \sum_{i=0}^{\infty} n_i, \tag{29}$$

$$f(x,t) = \lim_{p\to 1} f = f_0 + f_1 + f_2 + f_3 + \cdots = \sum_{i=0}^{\infty} f_i, \tag{30}$$

$$m(x,t) = \lim_{p\to 1} m = m_0 + m_1 + m_2 + m_3 + \cdots = \sum_{i=0}^{\infty} m_i. \tag{31}$$

Substituting (29)–(31) into (25),

$$\begin{aligned} &\frac{\partial}{\partial t}\left(n_0 + pn_1 + p^2 n_2 + p^3 n_3 + \cdots\right) - \frac{\partial n_0}{\partial t} \\ &\quad + p\left[-d_n \frac{\partial^2}{\partial x^2}\left(n_0 + pn_1 + p^2 n_2 + p^3 n_3 + \cdots\right)\right. \\ &\qquad + \gamma \frac{\partial}{\partial x}\left(n_0 + pn_1 + p^2 n_2 + p^3 n_3 + \cdots\right) \\ &\qquad \times \frac{\partial}{\partial x}\left(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \cdots\right) \\ &\qquad + \gamma\left(n_0 + pn_1 + p^2 n_2 + p^3 n_3 + \cdots\right) \\ &\qquad \times \frac{\partial^2}{\partial x^2}\left(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \cdots\right) \\ &\qquad \left. + \frac{\partial n_0}{\partial t}\right] = 0. \end{aligned} \tag{32}$$

Substituting (30)-(31) into (26),

$$\begin{aligned} &\frac{\partial}{\partial t}\left(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \cdots\right) - \frac{\partial f_0}{\partial t} \\ &\quad + p\left[\eta\left(m_0 + pm_1 + p^2 m_2 + p^3 m_3 + \cdots\right)\right. \\ &\qquad \left. \times \left(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \cdots\right) + \frac{\partial f_0}{\partial t}\right] = 0. \end{aligned} \tag{33}$$

Substituting (29)–(31) into (27),

$$\begin{aligned} &\frac{\partial}{\partial t}\left(m_0 + pm_1 + p^2 m_2 + p^3 m_3 + \cdots\right) - \frac{\partial m_0}{\partial t} \\ &\quad + p\left[-d_m \frac{\partial^2}{\partial x^2}\left(m_0 + pm_1 + p^2 m_2 + p^3 m_3 + \cdots\right)\right. \\ &\qquad - \alpha\left(n_0 + pn_1 + p^2 n_2 + p^3 n_3 + \cdots\right) \\ &\qquad + \beta\left(m_0 + pm_1 + p^2 m_2 + p^3 m_3 + \cdots\right) \\ &\qquad \left. + \frac{\partial m_0}{\partial t}\right] = 0. \end{aligned} \tag{34}$$

Equating the coefficients of the terms in (32)–(34) with the identical powers of p, we obtained the following.

From (32),

$$\begin{aligned} p^o&: \frac{\partial n_0}{\partial t} - \frac{\partial n_0}{\partial t} = 0, \\ p^1&: \frac{\partial n_1}{\partial t} - d_n \frac{\partial^2 n_0}{\partial x^2} + \gamma \frac{\partial n_0}{\partial x}\frac{\partial f_0}{\partial x} + \gamma n_0 \frac{\partial^2 f_0}{\partial x^2} + \frac{\partial n_0}{\partial t} = 0, \\ p^2&: \frac{\partial n_2}{\partial t} - d_n \frac{\partial^2 n_1}{\partial x^2} + \gamma \frac{\partial n_0}{\partial x}\frac{\partial f_1}{\partial x} + \gamma \frac{\partial n_1}{\partial x}\frac{\partial f_0}{\partial x} \\ &\quad + \gamma n_0 \frac{\partial^2 f_1}{\partial x^2} + \gamma n_1 \frac{\partial^2 f_0}{\partial x^2} = 0, \\ p^3&: \frac{\partial n_3}{\partial t} - d_n \frac{\partial^2 n_2}{\partial x^2} + \gamma \frac{\partial n_0}{\partial x}\frac{\partial f_2}{\partial x} + \gamma \frac{\partial n_1}{\partial x}\frac{\partial f_1}{\partial x} \\ &\quad + \gamma \frac{\partial n_2}{\partial x}\frac{\partial f_0}{\partial x} + \gamma n_0 \frac{\partial^2 f_2}{\partial x^2} \\ &\quad + \gamma n_1 \frac{\partial^2 f_1}{\partial x^2} + \gamma n_2 \frac{\partial^2 f_0}{\partial x^2} = 0, \\ p^4&: \frac{\partial n_4}{\partial t} - d_n \frac{\partial^2 n_3}{\partial x^2} + \gamma \frac{\partial n_0}{\partial x}\frac{\partial f_3}{\partial x} + \gamma \frac{\partial n_1}{\partial x}\frac{\partial f_2}{\partial x} + \gamma \frac{\partial n_2}{\partial x}\frac{\partial f_1}{\partial x} \\ &\quad + \gamma \frac{\partial n_3}{\partial x}\frac{\partial f_0}{\partial x} + \gamma n_0 \frac{\partial^2 f_3}{\partial x^2} + \gamma n_1 \frac{\partial^2 f_2}{\partial x^2} + \gamma n_2 \frac{\partial^2 f_1}{\partial x^2} = 0. \end{aligned} \tag{35}$$

From (33),

$$\begin{aligned} p^o&: \frac{\partial f_0}{\partial t} - \frac{\partial f_0}{\partial t} = 0, \\ p^1&: \frac{\partial f_1}{\partial t} + \eta m_0 f_0 + \frac{\partial f_0}{\partial t} = 0, \\ p^2&: \frac{\partial f_2}{\partial t} + \eta m_0 f_1 + \eta m_1 f_0 = 0, \\ p^3&: \frac{\partial f_3}{\partial t} + \eta m_0 f_2 + \eta m_1 f_1 + \eta m_2 f_0 = 0, \\ p^4&: \frac{\partial f_4}{\partial t} + \eta m_0 f_3 + \eta m_1 f_2 + \eta m_2 f_1 + \eta m_3 f_0 = 0. \end{aligned} \tag{36}$$

From (34),

$$\begin{aligned}
&p^o: \frac{\partial m_0}{\partial t} - \frac{\partial m_0}{\partial t} = 0,\\
&p^1: \frac{\partial m_1}{\partial t} - d_m \frac{\partial^2 m_0}{\partial x^2} - \omega n_0 + \beta m_0 + \frac{\partial m_0}{\partial t} = 0,\\
&p^2: \frac{\partial m_2}{\partial t} - d_m \frac{\partial^2 m_1}{\partial x^2} - \omega n_1 + \beta m_1 = 0,\\
&p^3: \frac{\partial m_3}{\partial t} - d_m \frac{\partial^2 m_2}{\partial x^2} - \omega n_2 + \beta m_2 = 0,\\
&p^4: \frac{\partial m_4}{\partial t} - d_m \frac{\partial^2 m_3}{\partial x^2} - \omega n_3 + \beta m_3 = 0.
\end{aligned} \tag{37}$$

4. Existence and Convergence of MDM and HPM

Theorem 1. *Let $0 < \alpha < 1$; then (3)–(5) have a unique solution.*

Proof. (I) Let n and n^* be two different solutions of (7) then

$$\begin{aligned}
|n - n^*| &= \left| d_n \int_0^t \left[\frac{\partial^2 n}{\partial x^2} - \frac{\partial^2 n^*}{\partial x^2} \right] dz \right.\\
&\quad - \gamma \int_0^t \left(\frac{\partial n}{\partial x} \cdot \frac{\partial f}{\partial x} - \frac{\partial n^*}{\partial x} \cdot \frac{\partial f^*}{\partial x} \right) dz\\
&\quad \left. -\gamma \int_0^t \left(n \frac{\partial^2 f}{\partial x^2} - n^* \frac{\partial^2 f^*}{\partial x^2} \right) dz \right|\\
&\leq |d_n| \int_0^t \left| \left[\frac{\partial^2 n}{\partial x^2} - \frac{\partial^2 n^*}{\partial x^2} \right] \right| dz\\
&\quad + |\gamma| \int_0^t \left| \frac{\partial n}{\partial x} \cdot \frac{\partial f}{\partial x} - \frac{\partial n^*}{\partial x} \cdot \frac{\partial f^*}{\partial x} \right| dz\\
&\quad + |\gamma| \int_0^t \left| n \frac{\partial^2 f}{\partial x^2} - n^* \frac{\partial^2 f^*}{\partial x^2} \right| dz\\
&\leq T \left(m' L_1 + m' L_2 + m' L_3 \right) |n - n^*| \, |f - f^*|\\
&= \alpha |n - n^*|,
\end{aligned} \tag{38}$$

from which we get $(1 - \alpha)|n - n^*| \leq 0$. Since $0 < \alpha < 1$, $|n - n^*| = 0$, implies $n = n^*$ and completes the proof.

(II) Let f and f^* be two different solutions of (8) then

$$\begin{aligned}
|f - f^*| &= \left| -\eta \int_0^t \left(F_1(mf) - F_1(m^* f^*) \right) dz \right|\\
&\leq |\eta| \left| \int_0^t \left(F_1(mf) - F_1(m^* f^*) \right) dz \right|\\
&\leq T \left(m' L_4 \right)\\
&= \alpha |f - f^*|,
\end{aligned} \tag{39}$$

from which we get $(1 - \alpha)|f - f^*| \leq 0$. Since $0 < \alpha < 1$, $|f - f^*| = 0$, implies $f = f^*$ and completes the proof.

(III) Let m and m^* be two different solutions of (9); then

$$\begin{aligned}
|m - m^*| &= \left| d_m \int_0^t \left[\frac{\partial^2 m}{\partial x^2} - \frac{\partial^2 m^*}{\partial x^2} \right] dz \right.\\
&\quad + \gamma \int_0^t \left(\left| F_2(n) - F_2(n^*) \right| \right) dz\\
&\quad \left. -\beta \int_0^t \left(\left| F_3(m) - F_3(m^*) \right| \right) dz \right|\\
&\leq |d_m| \int_0^t \left| \left[\frac{\partial^2 m}{\partial x^2} - \frac{\partial^2 m^*}{\partial x^2} \right] \right| dz\\
&\quad + |\alpha| \int_0^t \left| F_2(n) - F_2(n^*) \right| dz\\
&\quad + |\beta| \int_0^t \left| F_4(n) - F_4(n^*) \right| dz\\
&\leq T \left(m' L_5 + m' L_6 + m' L_7 \right)\\
&= \alpha |m - m^*|;
\end{aligned} \tag{40}$$

from which we get $(1 - \alpha)|m - m^*| \leq 0$. Since $0 < \alpha < 1$, $|m - m^*| = 0$, implies $m = m^*$ and completes the proof. □

Theorem 2. *The series solution $n(x,t) = \sum_{i=0}^{\infty} n_i(x,t)$, $f(x,t) = \sum_{i=0}^{\infty} f_i(x,t)$, and $m(x,t) = \sum_{i=0}^{\infty} m_i(x,t)$ of (3)–(5), respectively, using MDM converges if $0 < \alpha < 1$, $|n_1(x,t)| < \infty$, $|f_1(x,t)| < \infty$, and $|m_1(x,t)| < \infty$.*

Proof. Denote by $(C[J], \|\cdot\|)$ the Banach space of all continuous functions on J with the norm $\|f(t)\| = \max |f(t)| \forall_t \in J$. Define the sequence of partial series $\{S_b\}$; let S_b and S_a be arbitrary partial sums with $b \geq a$. We prove that S_b is a Cauchy sequence in this Banach space.

(I) For (11),

$$\begin{aligned}
\|S_b - S_a\| &= \max_{\forall t \in J} |S_b - S_a|\\
&= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} n_i(x,t) \right|\\
&= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} \left(\int_0^t d_n \frac{\partial^2 n_i}{\partial x^2} dz \right. \right.\\
&\qquad - \gamma \int_0^t \frac{\partial n_i}{\partial x} \cdot \frac{\partial f_i}{\partial x} dz\\
&\qquad \left. \left. -\gamma \int_0^t n \frac{\partial^2 f_i}{\partial x^2} dz \right) \right|
\end{aligned}$$

$$= \max_{\forall t \in J} \left| d_n \int_0^t \left(\sum_{i=k}^{b-1} \frac{\partial^2 n_i}{\partial x^2} \right) dz - \gamma \int_0^t \left(\sum_{i=k}^{b-1} \frac{\partial n_i}{\partial x} \cdot \frac{\partial f_i}{\partial x} \right) dz - \gamma \int_0^t \left(\sum_{i=k}^{b-1} n_i \frac{\partial^2 f_i}{\partial x^2} \right) dz \right|. \tag{41}$$

From [14], we have

$$\begin{aligned} \sum_{i=k}^{b-1} \frac{\partial^2 n_i}{\partial x^2} &= G_1^2 (S_{b-1}) - G_1^2 (S_{a-1}), \\ \sum_{i=k}^{b-1} \frac{\partial n_i}{\partial x} \cdot \frac{\partial f_i}{\partial x} &= G_2^2 (S_{b-1}) - G_2^2 (S_{a-1}), \\ \sum_{i=k}^{b-1} n_i \frac{\partial^2 f_i}{\partial x^2} &= G_3^2 (S_{b-1}) - G_3^2 (S_{a-1}). \end{aligned} \tag{42}$$

So,

$$\begin{aligned} \|S_b - S_a\| &= \max_{\forall t \in J} \left| d_n \int_0^t \left[G_1^2 (S_{b-1}) - G_1^2 (S_{a-1}) \right] dz - \gamma \int_0^t \left[G_2^2 (S_{b-1}) - G_2^2 (S_{a-1}) \right] dz - \gamma \int_0^t \left[G_3^2 (S_{b-1}) - G_3^2 (S_{a-1}) \right] dz \right| \\ &\leq |d_n| \int_0^t \left| G_1^2 (S_{b-1}) - G_1^2 (S_{a-1}) \right| dz + |\gamma| \int_0^t \left| G_2^2 (S_{b-1}) - G_2^2 (S_{a-1}) \right| dz + |\gamma| \int_0^t \left| G_3^2 (S_{b-1}) - G_3^2 (S_{a-1}) \right| dz \\ &\leq \alpha \|S_b - S_a\|. \end{aligned} \tag{43}$$

(II) For (12),

$$\begin{aligned} \|S_b - S_a\| &= \max_{\forall t \in J} |S_b - S_a| \\ &= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} f_i (x,t) \right| \\ &= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} \left(-\eta \int_0^t m_i f_i dz \right) \right| \\ &= \max_{\forall t \in J} \left| -\eta \int_0^t \left(\sum_{i=k}^{b-1} m_i f_i \right) dz \right|. \end{aligned} \tag{44}$$

From [14], we have

$$\sum_{i=k}^{b-1} m_i f_i = F_1 (S_{b-1}) - F_1 (S_{a-1}). \tag{45}$$

So,

$$\begin{aligned} \|S_b - S_a\| &= \max_{\forall t \in J} \left| -\eta \int_0^t \left[F_1 (S_{b-1}) - F_1 (S_{a-1}) \right] dz \right| \\ &\leq |\eta| \int_0^t \left| F_1 (S_{b-1}) - F_1 (S_{a-1}) \right| dz \\ &\leq \alpha \|S_b - S_a\|. \end{aligned} \tag{46}$$

(III) For (13),

$$\begin{aligned} \|S_b - S_a\| &= \max_{\forall t \in J} |S_b - S_a| \\ &= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} m_i (x,t) \right| \\ &= \max_{\forall t \in J} \left| \sum_{i=k+1}^{b} \left(\int_0^t d_m \frac{\partial^2 m_i}{\partial x^2} dz + \omega \int_0^t n_i dz - \beta \int_0^t m_i dz \right) \right| \\ &= \max_{\forall t \in J} \left| d_m \int_0^t \left(\sum_{i=k}^{b-1} \frac{\partial^2 m_i}{\partial x^2} \right) dz + \omega \int_0^t \left(\sum_{i=k}^{b-1} n_i \right) dz - \beta \int_0^t \left(\sum_{i=k}^{b-1} m_i \right) dz \right|. \end{aligned} \tag{47}$$

From [14], we have

$$\begin{aligned} \sum_{i=k}^{b-1} \frac{\partial^2 m_i}{\partial x^2} &= G_4^2 (S_{b-1}) - G_4^2 (S_{a-1}), \\ \sum_{i=k}^{b-1} n_i &= F_2 (S_{b-1}) - F_2 (S_{a-1}), \\ \sum_{i=k}^{b-1} m_i &= F_3 (S_{b-1}) - F_3 (S_{a-1}). \end{aligned} \tag{48}$$

So,

$$\begin{aligned}
&\|S_b - S_a\| \\
&\quad = \max_{\forall t \in J} \left| d_m \int_0^t \left[G_4^2 (S_{b-1}) - G_4^2 (S_{a-1}) \right] dz \right. \\
&\qquad\qquad + \omega \int_0^t \left[F_2 (S_{b-1}) - F_2 (S_{a-1}) \right] dz \\
&\qquad\qquad \left. -\beta \int_0^t \left[F_3 (S_{b-1}) - F_3 (S_{a-1}) \right] dz \right| \\
&\quad \le |d_m| \int_0^t \left| G_4^2 (S_{b-1}) - G_4^2 (S_{a-1}) \right| dz \\
&\qquad + |\omega| \int_0^t \left| F_2 (S_{b-1}) - F_2 (S_{a-1}) \right| dz \\
&\qquad + |\beta| \int_0^t \left| F_3 (S_{b-1}) - F_3 (S_{a-1}) \right| dz \\
&\quad \le \alpha \|S_b - S_a\| .
\end{aligned} \tag{49}$$

For (43), let $b = a + 1$; then

$$\begin{aligned}
\|S_{a+1} - S_a\| &\le \alpha \|S_a - S_{a-1}\| \\
&\le \alpha^2 \|S_{a-1} - S_{a-2}\| \\
&\quad \vdots \\
&\le \alpha^a \|S_1 - S_0\| .
\end{aligned} \tag{50}$$

From the triangle inequality, we have

$$\begin{aligned}
&\|S_b - S_a\| \\
&\quad \le \|S_{a+1} - S_a\| + \|S_{a+2} - S_{a+1}\| + \cdots + \|S_b - S_{b-1}\| \\
&\quad \le \left(\alpha^a + \alpha^{a+1} + \cdots + \alpha^{b-a-1}\right) \|S_1 - S_0\| \\
&\quad \le \alpha^a \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{b-a-1}\right) \|S_1 - S_0\| \\
&\quad \le \alpha^a \left(\frac{1 - \alpha^{b-a}}{1 - \alpha} \right) \|n_1 (x,t)\| ;
\end{aligned} \tag{51}$$

similar steps for (46)

$$\begin{aligned}
&\vdots \\
&\le \alpha^a \left(\frac{1 - \alpha^{b-a}}{1 - \alpha} \right) \|f_1 (x,t)\| ;
\end{aligned} \tag{52}$$

similar steps for (49)

$$\begin{aligned}
&\vdots \\
&\le \alpha^a \left(\frac{1 - \alpha^{b-a}}{1 - \alpha} \right) \|m_1 (x,t)\| .
\end{aligned} \tag{53}$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{b-a}) < 1$; then

$$\begin{aligned}
\|S_b - S_a\| &\le \frac{\alpha^a}{1 - \alpha} \max_{\forall t \in J} |n_1 (x,t)| , \\
\|S_b - S_a\| &\le \frac{\alpha^a}{1 - \alpha} \max_{\forall t \in J} |f_1 (x,t)| , \\
\|S_b - S_a\| &\le \frac{\alpha^a}{1 - \alpha} \max_{\forall t \in J} |m_1 (x,t)| .
\end{aligned} \tag{54}$$

But $|n_1(x,t), f_1(x,t), m_1(x,t)| < \infty$, so as $a \to \infty$ then $\|S_b - S_a\| \to 0$. We confide that $\{S_b\}$ is a Cauchy sequence in $C[J]$; therefore the series converges and the proof is completed. □

Theorem 3. *If $|n_a(x,t)| \le 1$, $|f_a(x,t)| \le 1$, $|m_a(x,t)| \le 1$, then the series solution $n(x,t) = \sum_{i=0}^{\infty} n_i(x,t)$, $f(x,t) = \sum_{i=0}^{\infty} f_i(x,t)$, and $m(x,t) = \sum_{i=0}^{\infty} m_i(x,t)$ of (3)–(5) converges to the exact solution by using HPM.*

Proof. (I) For (3), we set [14]

$$\begin{aligned}
\phi_b (x,t) &= \sum_{i=1}^{b} n_i (x,t) , \\
\phi_{b+1} (x,t) &= \sum_{i=1}^{b+1} n_i (x,t) .
\end{aligned} \tag{55}$$

So,

$$\begin{aligned}
&|\phi_{b+1} (x,t) - \phi_b (x,t)| \\
&\quad = |\phi_b + n_b - \phi_b| \\
&\quad = |n_b| \\
&\quad \le \sum_{k=0}^{a-1} \left(|d_n| \int_0^t \left| \frac{\partial^2 n_{a-k-1}}{\partial x^2} \right| dz \right. \\
&\qquad\qquad + |\gamma| \int_0^t \left| \frac{\partial n_{k-a-1}}{\partial x} \cdot \frac{\partial f_{k-a-1}}{\partial x} \right| dz \\
&\qquad\qquad \left. + |\gamma| \int_0^t \left| n_{k-a-1} \frac{\partial^2 f_{k-a-1}}{\partial x^2} \right| \right) dz.
\end{aligned} \tag{56}$$

Thus

$$\sum_{b=0}^{\infty} \|\phi_{b+1} (x,t) - \phi_b (x)\| \le (a-1)\, \alpha \,|f(x)| \sum_{b=0}^{\infty} \alpha^b . \tag{57}$$

Since $0 < \alpha < 1$, $\lim_{b \to \infty} n_b(x,t) = n(x,t)$.

(II) For (4), we set [14],

$$\begin{aligned}
\phi_b (x,t) &= \sum_{i=1}^{b} f_i (x,t) , \\
\phi_{b+1} (x,t) &= \sum_{i=1}^{b+1} f_i (x,t) .
\end{aligned} \tag{58}$$

So,

$$\begin{aligned}\left|\phi_{b+1}(x,t)-\phi_b(x,t)\right| &= \left|\phi_b+f_b-\phi_b\right| \\ &= \left|f_b\right| \\ &\le \sum_{k=0}^{a-1}|\eta|\int_0^t\left|m_{k-a-1}f_{k-a-1}\right|dz.\end{aligned} \tag{59}$$

Thus

$$\sum_{b=0}^{\infty}\left\|\phi_{b+1}(x,t)-\phi_b(x)\right\| \le (a-1)\,\alpha\left|f(x)\right|\sum_{b=0}^{\infty}\alpha^b. \tag{60}$$

Since $0<\alpha<1$, $\lim_{b\to\infty}f_b(x,t)=f(x,t)$.

(III) For (5), we set [14],

$$\begin{aligned}\phi_b(x,t) &= \sum_{i=1}^{b}m_i(x,t), \\ \phi_{b+1}(x,t) &= \sum_{i=1}^{b+1}m_i(x,t).\end{aligned} \tag{61}$$

So,

$$\begin{aligned}&\left|\phi_{b+1}(x,t)-\phi_b(x,t)\right| \\ &= \left|\phi_b+m_b-\phi_b\right| \\ &= \left|m_b\right| \\ &\le \sum_{k=0}^{a-1}\left(\left|d_m\right|\int_0^t\left|\frac{\partial^2 m_{a-k-1}}{\partial x^2}\right|dz \right.\\ &\qquad +|\omega|\int_0^t\left|n_{k-a-1}\right|dz \\ &\qquad \left.+|\beta|\int_0^t\left|m_{k-a-1}\right|\right)dz.\end{aligned} \tag{62}$$

Thus

$$\sum_{b=0}^{\infty}\left\|\phi_{b+1}(x,t)-\phi_b(x)\right\| \le (a-1)\,\alpha\left|f(x)\right|\sum_{b=0}^{\infty}\alpha^b. \tag{63}$$

Since $0<\alpha<1$, $\lim_{b\to\infty}m_b(x,t)=m(x,t)$. □

5. Numerical Experiment

In this section, we compute numerically (3)–(5) by the MDM and HPM methods.

5.1. MDM. From the ADM formula (18), we can obtain the first three terms of the Adomian polynomials:

$$\begin{aligned}A_{1,0} &= N_1(n_0,f_0) \\ &= -\frac{2x^2}{\varepsilon^2}e^{-2x^2/\varepsilon}, \\ A_{2,0} &= N_2(n_0,f_0) \\ &= \frac{e^{-2x^2/\varepsilon}}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right), \\ A_{3,0} &= N_3(m_0,f_0) \\ &= \frac{1}{2}e^{-x^2/\varepsilon}\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right).\end{aligned} \tag{64}$$

By the recursive formula in (19)–(21), we can obtain directly the components of n_i, f_i, and m_i.

From (22),

$$\begin{aligned}n_o &= 0, \\ n_1 &= n(x,0)+\int_0^t\left[d_n\frac{\partial^2 n_0}{\partial x^2}-\gamma\left(A_{10}+A_{20}\right)\right]d\tau \\ &= e^{-x^2/\varepsilon}-\left\{2d_n\left[1-\frac{2x^2}{\varepsilon}\right]\right. \\ &\qquad \left.+\gamma e^{-x^2/\varepsilon}\left[1-\frac{4x^2}{\varepsilon}\right]\right\}\frac{t}{\varepsilon}e^{-x^2/\varepsilon}, \\ n_2 &= -\left(\left\{\frac{-4d_n}{\varepsilon}\left\langle\frac{d_n}{\varepsilon}\left[3-\frac{6x^3}{\varepsilon}-\frac{2}{\varepsilon}\left(3x^2-\frac{2x^5}{\varepsilon}\right)\right]\right.\right.\right. \\ &\qquad \left.\left.+e^{-x^2/\varepsilon}\left[\gamma\left(3-\frac{8x^3}{\varepsilon}\right)+3x^2-\frac{4x^4}{\varepsilon}\right]\right\rangle\right\} \\ &\qquad +e^{-x^2/\varepsilon}\left\langle\frac{2}{\varepsilon^2}\gamma\mu x^2\left[e^{-x^2/\varepsilon}-1\right]\right. \\ &\qquad +\frac{4x}{\varepsilon}\left\{\frac{d_n}{\varepsilon}\left(3x-\frac{2x^3}{\varepsilon}\right)\right. \\ &\qquad \left.-\gamma e^{-x^2/\varepsilon}\left(3x-\frac{4x^3}{\varepsilon}\right)\right\} \\ &\qquad -\gamma\frac{\mu}{\varepsilon}\left[-1+\frac{2x^2}{\varepsilon}\right. \\ &\qquad \left.+e^{-x^2/\varepsilon}\left(1-\frac{4x^2}{\varepsilon}\right)\right] \\ &\qquad -\gamma\left(1-\frac{2x^2}{\varepsilon}\right) \\ &\qquad \times\left\{\frac{2d_n}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)\right. \\ &\qquad \left.\left.\left.+\gamma e^{-x^2/\varepsilon}\left(1-\frac{4x^2}{\varepsilon}\right)\right\}\right\rangle\right) \\ &\qquad \times\frac{t^2}{2}e^{-x^2/\varepsilon}.\end{aligned} \tag{65}$$

From (23),

$$\begin{aligned}
f_o &= 0,\\
f_1 &= f(x,0) - \eta \int_0^t A_{3,0}\,d\tau\\
&= 1 - \frac{1}{2}e^{-x^2/\varepsilon} - \left(1 - \frac{1}{2}e^{-x^2/\varepsilon}\right)\frac{\eta t}{2}e^{-x^2/\varepsilon},\\
f_2 &= \left(1 - \frac{1}{2}e^{-x^2/\varepsilon}\right)\\
&\quad \times \left[\frac{\eta}{4}e^{-x^2/\varepsilon} - \left(d_m\left(1-2x^2\right) - \omega + \frac{\beta}{2}\right)\right]\frac{\eta t^2}{2}e^{-x^2/\varepsilon}.
\end{aligned} \tag{66}$$

From (24),

$$\begin{aligned}
m_o &= 0,\\
m_1 &= \int_0^t \left[d_m \frac{\partial^2 m_0}{\partial x^2} + \omega n_0 - \beta m_0\right] d\tau\\
&= \frac{1}{2}e^{-x^2/\varepsilon} - \left[\frac{d_m}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right) - \omega + \frac{\beta}{2}\right] t e^{-x^2/\varepsilon},\\
m_2 &= -\left(\left\{d_m\left[-d_m\left\langle \frac{2}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right)\right.\right.\right.\right.\\
&\qquad \left.+4\left(1 - \frac{5x^2}{\varepsilon} + \frac{2x^4}{\varepsilon^2}\right)\right\rangle\\
&\qquad \left.\left.+\frac{1}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right)(2\omega - \beta)\right]\right\}\\
&\quad + \omega\left\{\frac{2d_n}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right)\right.\\
&\qquad \left.+\gamma e^{-x^2/\varepsilon}\left(1 - \frac{4x^2}{\varepsilon}\right)\right\}\\
&\quad \left.-\beta\left[d_m\left(1-2x^2\right) - \omega + \frac{\beta}{2}\right]\right)\frac{t^2}{2}e^{-x^2/\varepsilon}.
\end{aligned} \tag{67}$$

5.2. HPM Method. Following the HPM method, we can obtain the first three terms of the polynomials.

From (35)–(37),

$$\begin{aligned}
n_o &= e^{-x^2/\varepsilon},\\
n_1 &= \int_0^t \left(d_n \frac{2}{\varepsilon}e^{-x^2/\varepsilon}\left[1 - \frac{2x^2}{\varepsilon}\right] + \gamma\frac{2x^2}{\varepsilon^2}e^{-x^2/\varepsilon}e^{-x^2/\varepsilon}\right.\\
&\qquad \left.-\gamma e^{-x^2/\varepsilon}e^{-x^2/\varepsilon}\left[1 - \frac{2x^2}{\varepsilon}\right]\right)d\tau\\
&= -\left\{2d_n\left[1 - \frac{2x^2}{\varepsilon}\right] + \gamma e^{-x^2/\varepsilon}\left[1 - \frac{4x^2}{\varepsilon}\right]\right\}\frac{t}{\varepsilon}e^{-x^2/\varepsilon},
\end{aligned}$$

$$\begin{aligned}
n_2 = \int_0^t &\left(d_n \frac{4}{\varepsilon}te^{-x^2/\varepsilon}\right.\\
&\times\left\{\frac{d_n}{\varepsilon}\left[3 - \frac{6x^3}{\varepsilon} - \frac{2}{\varepsilon}\left(3x^2 - \frac{2x^5}{\varepsilon}\right)\right]\right.\\
&\qquad \left.-e^{-x^2/\varepsilon}\left[\gamma\left(3 - \frac{8x^3}{\varepsilon}\right) + 3x^2 - \frac{4x^4}{\varepsilon}\right]\right\}\\
&- \frac{2}{\varepsilon}\gamma x e^{-x^2/\varepsilon}\frac{\mu}{\varepsilon}txe^{-x^2/\varepsilon}\left[-1 + e^{-x^2/\varepsilon}\right]\\
&- \frac{4}{\varepsilon}te^{-x^2/\varepsilon}\left\{\frac{d_n}{\varepsilon}\left(3x - \frac{2x^3}{\varepsilon}\right)\right.\\
&\qquad \left.-\gamma e^{-x^2/\varepsilon}\left(3x - \frac{4x^3}{\varepsilon}\right)\right\}xe^{-x^2/\varepsilon}\\
&+ \gamma e^{-x^2/\varepsilon}\frac{\mu}{\varepsilon}te^{-x^2/\varepsilon}\\
&\times\left[-1 + \frac{2x^2}{\varepsilon} + e^{-x^2/\varepsilon} - \frac{4x^2}{\varepsilon}e^{-x^2/\varepsilon}\right]\\
&+ \gamma te^{-x^2/\varepsilon}\left\{\frac{2d_n}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right)\right.\\
&\qquad \left.+\gamma e^{-x^2/\varepsilon}\left(1 - \frac{4x^2}{\varepsilon}\right)\right\}e^{-x^2/\varepsilon}\\
&\left.\times\left(1 - \frac{2x^2}{\varepsilon}\right)\right)d\tau\\
= -\Bigg(&\left\{\frac{-4d_n}{\varepsilon}\left\langle\frac{d_n}{\varepsilon}\left[3 - \frac{6x^3}{\varepsilon}\right.\right.\right.\\
&\qquad \left.-\frac{2}{\varepsilon}\left(3x^2 - \frac{2x^5}{\varepsilon}\right)\right]\\
&\qquad \left.\left.+e^{-x^2/\varepsilon}\left[\gamma\left(3 - \frac{8x^3}{\varepsilon}\right) + 3x^2 - \frac{4x^4}{\varepsilon}\right]\right\rangle\right\}\\
&+ e^{-x^2/\varepsilon}\left\langle\frac{2}{\varepsilon^2}\gamma\mu x^2\left[e^{-x^2/\varepsilon} - 1\right]\right.\\
&\qquad + \frac{4x}{\varepsilon}\left\{\frac{d_n}{\varepsilon}\left(3x - \frac{2x^3}{\varepsilon}\right)\right.\\
&\qquad \left.-\gamma e^{-x^2/\varepsilon}\left(3x - \frac{4x^3}{\varepsilon}\right)\right\}\\
&\qquad - \gamma\frac{\mu}{\varepsilon}\left[-1 + \frac{2x^2}{\varepsilon} + e^{-x^2/\varepsilon}\left(1 - \frac{4x^2}{\varepsilon}\right)\right]\\
&\qquad - \gamma\left(1 - \frac{2x^2}{\varepsilon}\right)\\
&\qquad \times\left\{\frac{2d_n}{\varepsilon}\left(1 - \frac{2x^2}{\varepsilon}\right)\right.
\end{aligned}$$

$$+\gamma e^{-x^2/\varepsilon}\left(1-\frac{4x^2}{\varepsilon}\right)\Big\}\Big\rangle\Big)$$
$$\times\frac{t^2}{2}e^{-x^2/\varepsilon}. \tag{68}$$

From (43),

$$f_o = 1-\frac{1}{2}e^{-x^2/\varepsilon},$$
$$f_1 = -\int_0^t\left(\frac{\eta}{2}e^{-x^2/\varepsilon}\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right)\right)d\tau = -\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right)\frac{\eta t}{2}e^{-x^2/\varepsilon},$$
$$f_2 = \int_0^t\Big(\eta\Big[\frac{1}{2}e^{-x^2/\varepsilon}\frac{\eta}{2}e^{-x^2/\varepsilon}\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right)t + e^{-x^2/\varepsilon}\left(d_m\left(1-2x^2\right)-\omega+\frac{\beta}{2}\right)t \times\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right)\Big]\Big)d\tau$$
$$=\left(1-\frac{1}{2}e^{-x^2/\varepsilon}\right)\times\left[\frac{\eta}{4}e^{-x^2/\varepsilon}-\left(d_m\left(1-2x^2\right)-\omega+\frac{\beta}{2}\right)\right]\frac{\eta t^2}{2}e^{-x^2/\varepsilon}. \tag{69}$$

From (46),

$$m_o = \frac{1}{2}e^{-x^2/\varepsilon},$$
$$m_1 = -\int_0^t e^{-x^2/\varepsilon}\left[\frac{d_m}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)-\omega+\frac{\beta}{2}\right]d\tau = -\left[\frac{d_m}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)-\omega+\frac{\beta}{2}\right]te^{-x^2/\varepsilon},$$
$$m_2 = \int_0^t\Big(-d_m te^{-x^2/\varepsilon}\times\Big\{-d_m\Big\langle\frac{2}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)+4\left(1-\frac{5x^2}{\varepsilon}+\frac{2x^4}{\varepsilon^2}\right)\Big\rangle+\frac{1}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)(2\omega-\beta)\Big\}$$
$$-\omega te^{-x^2/\varepsilon}\times\left\{\frac{2d_n}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)+\gamma e^{-x^2/\varepsilon}\left(1-\frac{4x^2}{\varepsilon}\right)\right\}+\beta e^{-x^2/\varepsilon}t\left[d_m\left(1-2x^2\right)-\omega+\frac{\beta}{2}\right]\Big)d\tau,$$
$$m_2 = -\Big(\Big\{d_m\Big[-d_m\Big\langle\frac{2}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)+4\left(1-\frac{5x^2}{\varepsilon}+\frac{2x^4}{\varepsilon^2}\right)\Big\rangle+\frac{1}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)(2\omega-\beta)\Big]\Big\} \tag{70}$$
$$+\omega\left\{\frac{2d_n}{\varepsilon}\left(1-\frac{2x^2}{\varepsilon}\right)+\gamma e^{-x^2/\varepsilon}\left(1-\frac{4x^2}{\varepsilon}\right)\right\}-\beta\left[d_m\left(1-2x^2\right)-\omega+\frac{\beta}{2}\right]\Big)\times\frac{t^2}{2}e^{-x^2/\varepsilon}.$$

It is obvious that the first three terms' approximate solutions (65)–(67) obtained using MDM are the same as the first four terms' (68)–(70) of the HPM.

ADM and HPM provide analytical solution in terms of an infinite power series (see (16) for ADM and (29)–(31) for HPM). The series consists of both positive and negative terms, although not in a regular alternating fashion. The ratio test was applied to the absolute values of the series coefficient. This provides a sufficient condition for convergence of the series for a space interval ΔX in the form:

$$\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|<\frac{1}{\Delta X}. \tag{71}$$

However, the approach in this study was to replace (71) with

$$\lim_{m\to M}\left|\frac{a_{m+1}}{a_m}\right|<\frac{1}{\Delta X}, \tag{72}$$

where M is a large constant. Figures 1, 2, and 3 show the behavior of the function $f(m) = |a_{m+1}/a_m|$ for increasing values of m. It is clear from these figures that the ratio $f(m)$ decays as m increases, obviously indicating that the series is convergent.

Figures 4, 5, 6, and 7 show four snapshots in time of the tumour cell density, ECM density, and MDE concentration. The ECM profile shows clearly the degradation by the MDEs. As the MDEs degrade the ECM, the tumour cells invade via combination of diffusion and haptotaxis.

The tumour density distribution shows a small cluster of cells built up at the leading edge of the tumour due to haptotactic migration. As time evolves (Figures 5–7), this cluster of cells migrates further from the tumour main body and continues to invade the ECM at slower rate.

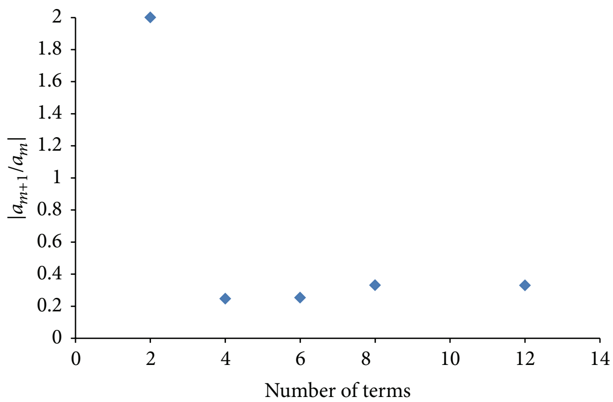

FIGURE 1: The ratio convergence test applied to the series coefficients (tumour) for MDM and HPM as a function of the number of terms in series.

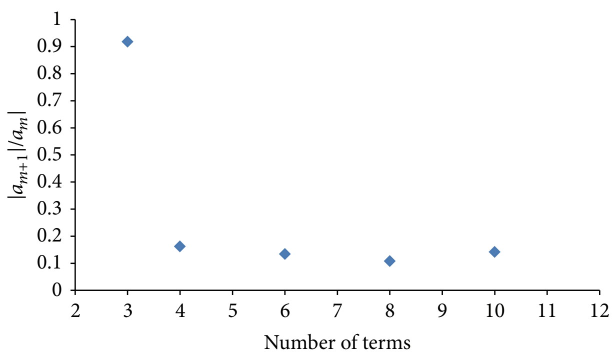

FIGURE 2: The ratio convergence test applied to the series coefficients (ECM) for MDM and HPM as a function of the number of terms in series.

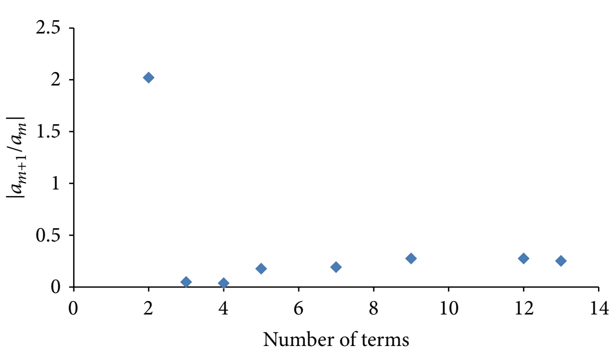

FIGURE 3: The ratio convergence test applied to the series coefficients (MDE) for MDM and HPM as a function of the number of terms in series.

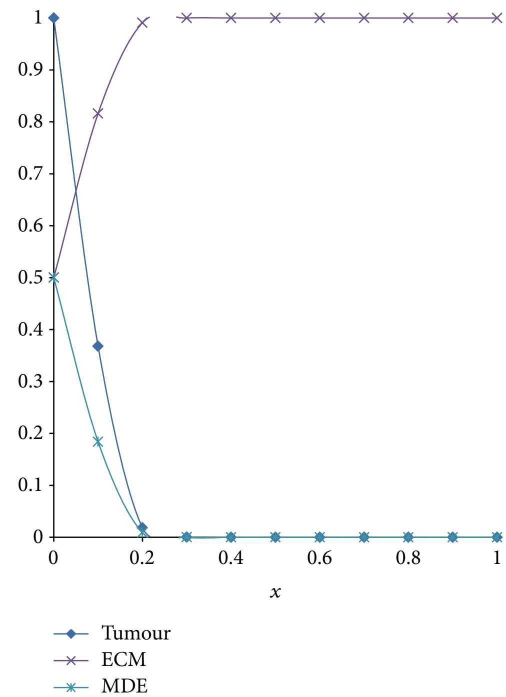

FIGURE 4: One-dimensional MDM and HPM solution of the system (3)–(5) with constant tumour cell diffusion showing the cell density, MDE concentration, and ECM density at $t = 0$.

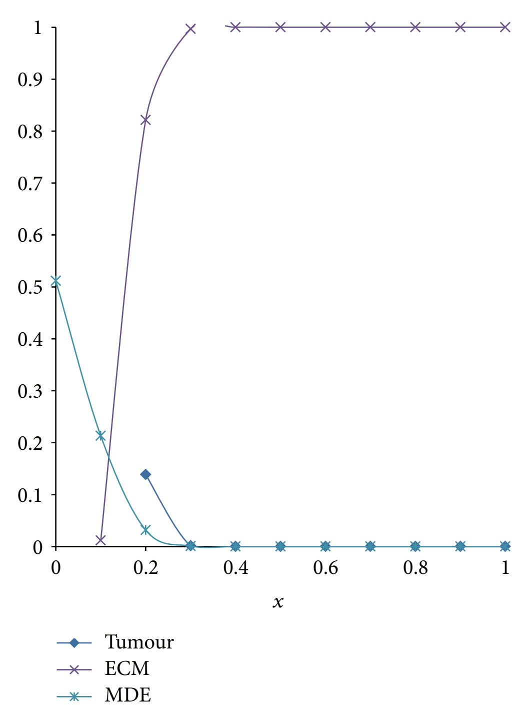

FIGURE 5: One-dimensional MDM and HPM solution of the system (3)–(5) with constant tumour cell diffusion showing the cell density, MDE concentration, and ECM density at $t = 1$.

6. Conclusion

In this paper, the modified decomposition method (MDM) and homotopy perturbation method (HPM) were used to obtain the solutions for the nonlinear model of tumour invasion and metastasis. Although the main difference between MDM and ADM is a slight variation in the definition of

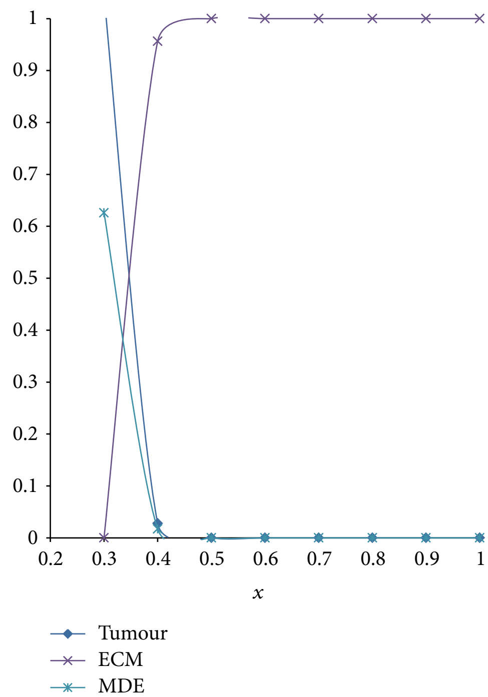

Figure 6: One-dimensional MDM and HPM solution of the system (3)–(5) with constant tumour cell diffusion showing the cell density, MDE concentration, and ECM density at $t = 10$.

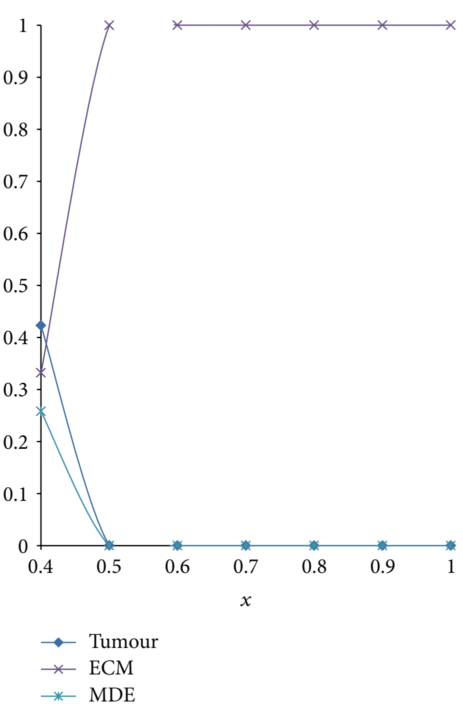

Figure 7: One-dimensional MDM and HPM solution of the system (3)–(5) with constant tumour cell diffusion showing the cell density, MDE concentration, and ECM density at $t = 20$.

the initial conditions, the modification demonstrates reliability and effectiveness in applying the present problem. This method thus eliminates the difficulties and massive computation work. Also it is shown that the obtained solution by MDM logically contains the solution obtained by HPM. The benefits of HPM with respect to MDM are HPM does not involve the Adomian polynomials which is a fundamental qualitative difference in analysis between HPM and MDM.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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