

Oscillation of Certain Partial Difference Equations

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We first obtain sufficiency conditions for the oscillation of all solutions of linear partial difference equation

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + P_{m,n}A_{m-k,n-l} = 0.$$

Next, we establish a linearized oscillation result for the nonlinear partial difference equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + P_{m,n}f(A_{m-k,n-l}) = 0.$$

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1 INTRODUCTION

Partial difference equations have been posed from various practical problems [3,8] and in the approximation of solutions of partial difference equations by finite difference methods [1,2,8]. Recently, the qualitative analysis of partial difference equations has received much attention, see [4–7,9,10,12].

In this paper, we first consider the linear partial difference equation

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + P_{m,n}A_{m-k,n-l} = 0, \quad (1.1)$$

where $P_{m,n} > 0$ on N_0^2 , $k, l \in N_0$, $N_i = \{i, i+1, \dots\}$ and i is an integer. Equation (1.1) can be regarded

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as a discrete analogue of the delay partial difference equation

$$a \frac{\partial^2 u}{\partial y \partial x} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + P(x, y)u(x - \sigma, y - \tau) = 0. \tag{1.2}$$

Next, we consider the nonlinear partial difference equation

$$A_{m+1, n+1} + A_{m+1, n} + A_{m, n+1} - A_{m, n} + P_{m, n}f(A_{m-k, n-l}) = 0, \tag{1.3}$$

where $f \in C(\mathbb{R}, \mathbb{R})$.

The general theory of partial functional differential equations can be seen from Wu [11].

A double sequence $\{A_{m, n}\}$ is said to be a *solution* of (1.1) if it satisfies (1.1) for $m \geq m_0, n \geq n_0$. A solution $\{A_{i, j}\}$ of (1.1) is said to be *eventually positive* if $A_{i, j} > 0$ for all large i and j , and *eventually negative* if $A_{i, j} < 0$ for all large i and j . It is said to be *oscillatory* if it is neither eventually positive nor eventually negative.

In Section 2, we shall obtain sufficiency conditions for all solutions of (1.1) to be oscillatory. In Section 3, we shall show a linearized oscillation theorem for (1.3), i.e., we shall show that under some assumptions, (1.3) has the same oscillatory character as an associated linear equation.

2 EQUATION (1.1)

We assume that a, b, c, d and $P_{m, n}$ are positive in Eq. (1.1). Define a set E by

$$E = \{\lambda > 0 \mid d - \lambda P_{m, n} > 0, \text{ eventually}\}. \tag{2.1}$$

THEOREM 2.1 *Assume that*

- (i) $\lim_{m, n \rightarrow \infty} \sup P_{m, n} > 0$;
- (ii) for $k \geq l \geq 1$, there exist $M, N \in \mathbb{N}_1$ such that

$$\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \prod_{i=1}^l (d - \lambda P_{m-i, n-i}) \times \prod_{j=1}^{k-l} (d - \lambda P_{m-l-j, n-l-j}) < \left(a + \frac{2bc}{d}\right)^l b^{k-l} \tag{2.2}$$

and for $l \geq k$

$$\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \prod_{i=1}^k (d - \lambda P_{m-i, n-i}) \times \prod_{j=1}^{l-k} (d - \lambda P_{m-k, n-k-j}) < \left(a + \frac{2bc}{d}\right)^k c^{l-k}. \tag{2.3}$$

Then every solution of (1.1) oscillates.

Proof Suppose, to the contrary, we let $\{A_{m, n}\}$ be an eventually positive solution. We define a subset S of the positive numbers as follows:

$$S(A) = \{\lambda > 0 \mid aA_{m+1, n+1} + bA_{m+1, n} + cA_{m, n+1}(d - \lambda P_{m, n})A_{m, n} \leq 0, \text{ eventually}\}.$$

From (1.1), we have

$$aA_{m+1, n+1} + bA_{m+1, n} + cA_{m, n+1} < dA_{m, n}. \tag{2.4}$$

If $k \geq l$, then

$$A_{m-k, n-l} > \left(\frac{a}{d}\right)^l A_{m-k+l, n} > \left(\frac{a}{d}\right)^l \left(\frac{b}{d}\right)^{k-l} A_{m, n}. \tag{2.5}$$

If $l \geq k$, then

$$A_{m-k, n-l} > \left(\frac{a}{d}\right)^k A_{m, n-l+k} > \left(\frac{a}{d}\right)^k \left(\frac{c}{d}\right)^{l-k} A_{m, n}. \tag{2.6}$$

Substituting (2.5) and (2.6) into (1.1) we obtain

$$aA_{m+1, n+1} + bA_{m+1, n} + cA_{m, n+1} - dA_{m, n} + \left(\frac{a}{d}\right)^l \left(\frac{b}{d}\right)^{k-l} P_{m, n}A_{m, n} < 0 \tag{2.7}$$

and

$$aA_{m+1, n+1} + bA_{m+1, n} + cA_{m, n+1} - dA_{m, n} + \left(\frac{a}{d}\right)^k \left(\frac{c}{d}\right)^{l-k} P_{m, n}A_{m, n} < 0, \tag{2.8}$$

respectively. Inequalities (2.7) and (2.8) show that $S(A)$ is nonempty. For $\lambda \in S$, we have eventually

$$d - \lambda P_{m,n} > 0,$$

which implies that $S \subset E$. Due to the condition (i), the set E is bounded, and hence $S(A)$ is bounded. Let $\mu \in S$. Then, from (2.4), we have

$$A_{m+1,n+1} \leq \frac{d}{b} A_{m,n+1} \quad \text{and} \quad A_{m+1,n+1} \leq \frac{d}{c} A_{m+1,n}.$$

Hence we obtain

$$\begin{aligned} & \left(a + \frac{2bc}{d} \right) A_{m+1,n+1} \\ & \leq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ & \leq (d - \mu P_{m,n}) A_{m,n}. \end{aligned}$$

If $k \geq l$, then

$$A_{m,n} \leq \left(a + \frac{2bc}{d} \right)^{-l} \prod_{i=1}^l (d - \mu P_{m-i,n-i}) A_{m-l,n-l}$$

and

$$\begin{aligned} A_{m-l,n-l} & \leq \frac{1}{b} (d - \mu P_{m-l-1,n-l}) A_{m-l-1,n-l} \\ & \leq \dots \leq \left(\frac{1}{b} \right)^{k-l} \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) A_{m-k,n-l}. \end{aligned}$$

Hence

$$\begin{aligned} A_{m,n} & \leq \left(a + \frac{2bc}{d} \right)^{-l} b^{l-k} \prod_{i=1}^l (d - \mu P_{m-i,n-i}) \\ & \quad \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) A_{m-k,n-l}. \end{aligned} \quad (2.9)$$

Similarly, if $l \geq k$, then

$$\begin{aligned} A_{m,n} & \leq \left(a + \frac{2bc}{d} \right)^{-k} c^{k-l} \prod_{i=1}^k (d - \mu P_{m-i,n-i}) \\ & \quad \times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j}) A_{m-k,n-l}. \end{aligned} \quad (2.10)$$

Substituting (2.9) and (2.10) into (1.3) we find respectively

$$\begin{aligned} & aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} \\ & + P_{m,n} \left(a + \frac{2bc}{d} \right)^k c^{l-k} \left(\prod_{i=1}^k (d - \mu P_{m-i,n-i}) \right. \\ & \quad \left. \times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j}) \right)^{-1} A_{m,n} \leq 0, \quad \text{for } l \geq k \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} & aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} \\ & + P_{m,n} \left(a + \frac{2bc}{d} \right)^l b^{k-l} \left(\prod_{i=1}^l (d - \mu P_{m-i,n-i}) \right. \\ & \quad \left. \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) \right)^{-1} A_{m,n} \leq 0, \quad \text{for } k \geq l. \end{aligned} \quad (2.12)$$

Hence

$$\begin{aligned} & aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ & - \left(d - P_{m,n} \left(a + \frac{2bc}{d} \right)^k c^{l-k} \right. \\ & \quad \left. \times \sup_{m \geq M, n \geq N} \left[\left(\prod_{i=1}^k (d - \mu P_{m-i,n-i}) \right. \right. \right. \\ & \quad \left. \left. \left. \times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j}) \right)^{-1} \right] \right) A_{m,n} \leq 0, \quad l \geq k \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} & aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ & - \left(d - P_{m,n} \left(a + \frac{2bc}{d} \right)^l b^{k-l} \right. \\ & \quad \left. \times \sup_{m \geq M, n \geq N} \left[\left(\prod_{i=1}^l (d - \mu P_{m-i,n-i}) \right. \right. \right. \\ & \quad \left. \left. \left. \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) \right)^{-1} \right] \right) A_{m,n} \leq 0, \quad k \geq l. \end{aligned} \quad (2.14)$$

From (2.13) and (2.14) we obtain

$$\left(a + \frac{2bc}{d} \right)^k c^{l-k} \left(\sup_{m \geq M, n \geq N} \left[\left(\prod_{i=1}^k (d - \mu P_{m-i, n-i}) \right. \right. \right. \\ \left. \left. \left. \times \prod_{j=1}^{l-k} (d - \mu P_{m-k, n-k-j}) \right)^{-1} \right] \right) \in S, \quad l > k \tag{2.15}$$

and

$$\left(a + \frac{2bc}{d} \right)^l b^{k-l} \left(\sup_{m \geq M, n \geq N} \left[\left(\prod_{i=1}^l (d - \mu P_{m-i, n-i}) \right. \right. \right. \\ \left. \left. \left. \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j, n-l}) \right)^{-1} \right] \right) \in S, \quad k > l. \tag{2.16}$$

On the other hand, (2.2) implies that there exists $\beta \in (0, 1)$ such that

$$\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \prod_{i=1}^l (d - \lambda P_{m-i, n-i}) \prod_{j=1}^{k-l} (d - \lambda P_{m-l-j, n-l}) \\ \leq \beta \left(a + \frac{2bc}{d} \right)^l b^{k-l}, \quad k \geq l$$

and (2.3) implies that there exists $\beta \in (0, 1)$ such that

$$\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \prod_{i=1}^k (d - \lambda P_{m-i, n-i}) \prod_{j=1}^{l-k} (d - \lambda P_{m-k, n-k-j}) \\ \leq \beta \left(a + \frac{2bc}{d} \right)^k c^{l-k}, \quad l \geq k.$$

Hence, for $k \geq l$,

$$\sup_{m \geq M, n \geq N} \left(\prod_{i=1}^l (d - \mu P_{m-i, n-i}) \prod_{j=1}^{k-l} (d - \mu P_{m-l-j, n-l}) \right) \\ \leq \frac{\beta}{\mu} \left(a + \frac{2bc}{d} \right)^l b^{k-l} \tag{2.17}$$

and for $l \geq k$

$$\sup_{m \geq M, n \geq N} \prod_{i=1}^k (d - \mu P_{m-i, n-i}) \prod_{j=1}^{l-k} (d - \mu P_{m-k, n-k-j}) \\ \leq \frac{\beta}{\mu} \left(a + \frac{2bc}{d} \right)^k c^{l-k}. \tag{2.18}$$

From (2.15) and (2.18) for $l \geq k$, (2.16) and (2.17) for $k \geq l$, we have that $\mu/\beta \in S$. Repeating the above procedure, we conclude that $\mu(1/\beta)^r \in S$, $r = 1, 2, \dots$, which contradicts the boundedness of S . The proof is complete.

From Theorem 2.1, we can derive an explicit oscillation condition.

COROLLARY 2.1 *In addition to (i) of Theorem 2.1, assume that, for $k \geq l$,*

$$\liminf_{m, n \rightarrow \infty} P_{m, n} = P > d^{k+1} \left(\left(a + \frac{2bc}{d} \right)^l b^{k-l} \right)^{-1} \\ \times \frac{k^k}{(1+k)^{1+k}} \tag{2.19}$$

and for $l \geq k$,

$$\liminf_{m, n \rightarrow \infty} P_{m, n} = P > d^{l+1} \left(\left(a + \frac{2bc}{d} \right)^k b^{l-k} \right)^{-1} \\ \times \frac{l^l}{(1+l)^{1+l}}. \tag{2.20}$$

Then the conclusion of Theorem 2.1 holds.

Proof We see that

$$\max_{d/P > \lambda > 0} \lambda (d - \lambda P)^k = \frac{d^{k+1} k^k}{P(1+k)^{1+k}}.$$

Hence (2.19) and (2.20) imply that (2.2) and (2.3) hold. By Theorem 2.1, every solution of (1.1) oscillates. The proof is complete.

Remark Obviously, Theorem 2.1 is true for $a = 0$. Hence Theorem 2.1 includes Theorem 2.3 in

[9] as a special case. From (2.4) we have

$$\begin{aligned} A_{m,n} &< \frac{d}{b} A_{m-1,n} < \dots < \left(\frac{d}{b}\right)^k A_{m-k,n} \\ &< \left(\frac{d}{b}\right)^k \left(\frac{d}{c}\right) A_{m-k,n-1} \\ &< \dots < \left(\frac{d}{b}\right)^k \left(\frac{d}{c}\right)^l A_{m-k,n-l}. \end{aligned}$$

Let $\mu \in S$. Then

$$\begin{aligned} A_{m,n} &\leq \frac{1}{b} (d - \mu P_{m-1,n}) A_{m-1,n} \\ &\leq \left(\frac{1}{b}\right)^k \prod_{i=1}^k (d - \mu P_{m-i,n}) A_{m-k,n} \\ &\leq \left(\frac{1}{b}\right)^k \left(\frac{1}{c}\right) \prod_{i=1}^k (d - \mu P_{m-i,n}) \\ &\quad \times (d - \mu P_{m-k,n-1}) A_{m-k,n-1} \\ &\leq \left(\frac{1}{b}\right)^k \left(\frac{1}{c}\right)^l \prod_{i=1}^k (d - \mu P_{m-i,n}) \\ &\quad \times \prod_{j=1}^l (d - \mu P_{m-k,n-j}) A_{m-k,n-l}. \end{aligned}$$

Substituting the above inequality into (1.1) we obtain

$$\begin{aligned} &aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} \\ &+ P_{m,n} b^k c^l \left[\prod_{i=1}^k (d - \mu P_{m-i,n}) \right. \\ &\quad \left. \times \prod_{j=1}^l (d - \mu P_{m-k,n-j}) \right]^{-1} A_{m,n} \leq 0. \end{aligned}$$

Hence

$$\begin{aligned} &aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ &- \left(d - P_{m,n} b^k c^l \left[\sup_{m \geq M, n \geq N} \prod_{i=1}^k (d - \mu P_{m-i,n}) \right. \right. \\ &\quad \left. \left. \times \prod_{j=1}^l (d - \mu P_{m-k,n-j}) \right]^{-1} \right) A_{m,n} \leq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &b^k c^l \left[\sup_{m \geq M, n \geq N} \prod_{i=1}^k (d - \mu P_{m-i,n}) \right. \\ &\quad \left. \times \prod_{j=1}^l (d - \mu P_{m-k,n-j}) \right]^{-1} \in S. \end{aligned}$$

We are ready to state the following proposition.

THEOREM 2.2 *In addition to (i) of Theorem 2.1, further assume that (ii)^l there exist $M, N \in N_1$ such that*

$$\begin{aligned} &\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \prod_{i=1}^k (d - \lambda P_{m-i,n}) \\ &\quad \times \prod_{j=1}^l (d - \lambda P_{m-k,n-j}) < b^k c^l. \quad (2.21) \end{aligned}$$

Then every solution of (1.1) oscillates.

Since

$$\max_{d/q > \lambda > 0} \lambda (d - \lambda P)^{k+l} = \frac{d^{k+l+1} (k+l)^{k+l}}{p(1+k+l)^{1+k+l}}$$

and (2.21), we have the following result.

COROLLARY 2.2 *In addition to (i) of Theorem 2.1, assume that*

$$\liminf_{m,n \rightarrow \infty} P_{m,n} = P > \frac{d^{l+k+l} (k+l)^{k+l}}{b^k c^l (1+k+l)^{1+k+l}}. \quad (2.22)$$

Then every solution of (1.1) is oscillatory.

Example 2.1 Consider the partial difference equation

$$\begin{aligned} &A_{m+1,n+1} + eA_{m+1,n} + A_{m,n+1} \\ &- A_{m,n} + \frac{1+e}{e^4} A_{m-2,n-2} = 0. \quad (2.23) \end{aligned}$$

It is easy to see that (2.23) satisfies the conditions of Corollary 2.1, so every solution of this equation is oscillatory. In fact, $A_{m,n} = (-e)^{m+n}$ is such a solution.

3 EQUATION (1.3)

We consider (1.3) together with the associate linear equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + pA_{m-k,n-l} = 0, \tag{3.1}$$

where $p > 0$, k and l are positive integers.

For (3.1), the following result is known [12].

LEMMA 3.1 *Every solution of (3.1) oscillates if and only if its characteristic equation*

$$\lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l} = 0 \tag{3.2}$$

has no positive roots.

LEMMA 3.2 *Assume that every solution (3.1) oscillates. Then there exists $\varepsilon_0 \in (0, p)$ such that for each $\varepsilon \in [0, \varepsilon_0]$ every solution of the equation*

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p - \varepsilon)A_{m-k,n-l} = 0 \tag{3.3}$$

also oscillates.

Proof By Lemma 3.1, it is sufficient to prove that

$$\lambda\mu + \lambda + \mu - 1 + (p - \varepsilon)\lambda^{-k}\mu^{-l} = 0 \tag{3.4}$$

has no positive roots. Obviously, (3.4) has no positive roots in the region $\lambda\mu + \lambda + \mu - 1 \geq 0$.

Let

$$F(\lambda, \mu) = \lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l}.$$

Since (3.2) has no positive roots, so $F(\lambda, \mu) > 0$ for $(\lambda, \mu) \in (0, \infty) \times (0, \infty)$. Thus

$$\min_{\lambda>0, \mu>0} F(\lambda, \mu) = m > 0.$$

Let

$$G(s, t) = st + s + t - 1 + \frac{1}{2}ps^{-k}t^{-l},$$

$\varepsilon_0 \in (0, p/2)$ and $\varepsilon_0\alpha^{-(k+l)} \leq m/2$, where

$$0 < \alpha < \min\left\{\left(\frac{p}{2}\right)^{1/(k+l)}, \left(\frac{p}{2}\right)^{1/k}, \left(\frac{p}{2}\right)^{1/l}\right\}.$$

For $0 < \lambda \leq \alpha$, $0 < \mu \leq \alpha$, we have

$$\begin{aligned} &\lambda\mu + \lambda + \mu - 1 + (p - \varepsilon)\lambda^{-k}\mu^{-l} \\ &> \lambda\mu + \lambda + \mu - 1 + \frac{p}{2}\lambda^{-k}\mu^{-l} \\ &> -1 + \frac{p}{2}\alpha^{-(k+l)} > 0. \end{aligned}$$

For $\lambda > \alpha$, $\mu > \alpha$, we have

$$\begin{aligned} &\lambda\mu + \lambda + \mu - 1 + (p - \varepsilon)\lambda^{-k}\mu^{-l} \\ &\geq F(\lambda, \mu) - \varepsilon_0\lambda^{-k}\mu^{-l} \\ &\geq m - \varepsilon_0\alpha^{-(k+l)} \geq \frac{m}{2} > 0. \end{aligned}$$

For $0 < \lambda \geq \alpha$, $\mu > \alpha$, $\lambda\mu + \lambda + \mu < 1$, we have

$$\begin{aligned} &\lambda\mu + \lambda + \mu - 1 + (p - \varepsilon)\lambda^{-k}\mu^{-l} \\ &> \lambda\mu + \lambda + \mu - 1 + \frac{p}{2}\lambda^{-k}\mu^{-l} \\ &> 1 - \frac{p}{2}\alpha^{-k} > 0. \end{aligned}$$

Similarly, for $\lambda > \alpha$, $0 < \mu \leq \alpha$, $\lambda\mu + \lambda + \mu < 1$, we have

$$\lambda\mu + \lambda + \mu - 1 + (p - \varepsilon)\lambda^{-k}\mu^{-l} > 0.$$

We have proved that (3.4) has no positive roots.

We are ready to state the following result.

THEOREM 3.1 *Assume that*

- (i) $\lim_{m,n \rightarrow \infty} \inf P_{m,n} = P > 0$,
- (ii) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ as $x \neq 0$, $\lim_{x \rightarrow 0} f(x)/x = 1$.

Then every solution of (3.1) oscillates implies that every solution of (1.3) oscillates.

Proof If not, let $A_{m,n} > 0$, $m \geq m_0$, $n \geq n_0$ be a solution (1.3). Then $A_{m,n}$ is decreasing in m and n , and hence $\lim_{m,n \rightarrow \infty} A_{m,n} = 0$, $\lim_{m \rightarrow \infty} A_{m,n} = 0$ and $\lim_{n \rightarrow \infty} A_{m,n} = 0$. Let

$$\bar{P}_{m,n} = P_{m,n} \frac{f(A_{m-k,n-l})}{A_{m-k,n-l}}.$$

Then $\lim_{m,n \rightarrow \infty} \inf \bar{P}_{m,n} = P$. For each $\varepsilon \in (0, \varepsilon_0]$, there exist M and N such that $\bar{P}_{m,n} > p - \varepsilon$, for $m \geq M, n \geq N$. Therefore

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p - \varepsilon)A_{m-k,n-l} \leq 0, \quad m \geq M, \quad n \geq N.$$

Summing it in n from $n (\geq N)$ to ∞ , we have

$$\sum_{i=n}^{\infty} A_{m+1,i+1} + \sum_{i=n}^{\infty} A_{m+1,i} - A_{m,n} + (p - \varepsilon) \sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0.$$

We rewrite the last inequality in the form

$$\sum_{i=n}^{\infty} A_{m+1,i+1} + \sum_{i=n+1}^{\infty} A_{m+1,i} + A_{m+1,n} - A_{m,n} + (p - \varepsilon) \sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0.$$

Summing it in m from $m (\geq M)$ to ∞ , we obtain

$$\sum_{j,i=m,n}^{\infty} A_{j+1,i+1} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} A_{j+1,i} - A_{m,n} + (p - \varepsilon) \sum_{j,i=m,n}^{\infty} A_{j-k,i-l} \leq 0.$$

Hence

$$A_{m,n} \geq \sum_{j,i=m,n}^{\infty} A_{j+1,i+1} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} A_{j+1,i} + (p - \varepsilon) \sum_{j,i=m,n}^{\infty} A_{j-k,i-l}. \quad (3.5)$$

Define a set of real double sequences

$$X = \{ \{ B_{m,n} \} \mid 0 \leq B_{m,n} \leq 1, m \geq M - k, n \geq N - l \}$$

and an operator T on X by

$$(TB)_{m,n} = \begin{cases} \frac{1}{A_{m,n}} \left[2 \sum_{j,i=m+1,n+1}^{\infty} A_{j,i} B_{j,i} + (p - \varepsilon) \sum_{j,i=m,n}^{\infty} A_{j-k,i-l} B_{j-k,i-l} \right], \\ 1, \quad \text{otherwise.} \end{cases} \quad (3.6)$$

In view of (3.5), it is easy to see that $TX \subset X$. Define a set of sequences $\{B_{m,n}^{(r)}\}, r = 0, 1, 2, \dots$ as follows:

$$B_{m,n}^{(0)} \equiv 1, \quad B_{m,n}^{(r)} = (TB)_{m,n}^{(r-1)}, \quad r = 1, 2, \dots$$

In view of (3.5), we have

$$B_{m,n}^{(0)} \geq B_{m,n}^{(1)} \geq \dots \geq B_{m,n}^{(r)} \geq \dots, \quad \text{for } m \geq M - k, \quad n \geq N - l.$$

Hence $\lim_{r \rightarrow \infty} B_{m,n}^{(r)} = B_{m,n}$ exists, for $m \geq M - k, n \geq N - l$. From (3.6),

$$B_{m,n} = \begin{cases} \frac{1}{A_{m,n}} \left[2 \sum_{j,i=m+1,n+1}^{\infty} A_{j,i} B_{j,i} + (p - \varepsilon) \sum_{j,i=m,n}^{\infty} A_{j-k,i-l} B_{j-k,i-l} \right], \\ 1, \quad \text{otherwise.} \end{cases}$$

Clearly, $B_{m,n} > 0$ for $m \geq M - k, n \geq N - l$. Let $x_{m,n} = A_{m,n} B_{m,n}$. Then $x_{m,n} > 0$ for $m \geq M - k, n \geq N - l$ and

$$x_{m,n} = 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} x_{j,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} x_{j-k,i-l}.$$

Hence

$$x_{m+1,n} - x_{m,n} = -2 \sum_{i=n+1}^{\infty} x_{m+1,i} - (p - \varepsilon) \sum_{i=n}^{\infty} x_{m-k,i-l}$$

or

$$x_{m,n} = \sum_{i=n+1}^{\infty} x_{m+1,i} + \sum_{i=n}^{\infty} x_{m+1,i} + (p - \varepsilon) \sum_{i=n}^{\infty} x_{m-k,i-l}.$$

Then we have

$$\begin{aligned} x_{m,n+1} - x_{m,n} &= -x_{m+1,n+1} - x_{m+1,n} \\ &\quad - (p - \varepsilon)x_{m-k,n-l} = 0, \end{aligned}$$

which implies that (3.3) has a positive solution $\{x_{m,n}\}$. By Lemma 3.2, (3.1) has a positive solution, which is a contradiction. The proof is complete.

THEOREM 3.2 *Assume that*

- (i) $0 \leq P_{m,n} \leq P$,
- (ii) *there exists $h > 0$ such that $f(x)$ is nondecreasing and $0 \leq f(x)/x \leq 1$, for $0 < |x| < h$.*

If (3.1) has a positive solution, then (1.3) also has a positive solution.

Proof If (3.1) has a positive solution, by Lemma 3.1,

$$\lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l} = 0$$

has a positive root (λ, μ) with $0 < \lambda < 1$, $0 < \mu < 1$ and that $\{\lambda^m \mu^n\}$ is a positive solution of (3.1). Choose $a > 0$ such that

$$A_{m,n} = a\lambda^m \mu^n < h, \quad \text{for } m \geq -k, n - l.$$

$\{A_{m,n}\}$ is a positive solution of (3.1) and satisfies $f(A_{m,n}) \leq A_{m,n}$ by condition (ii). Similar to Theorem 3.1, summing (3.1) we get

$$\begin{aligned} A_{m,n} &= 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + p \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l}, \\ m \geq 0, \quad n \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} A_{m,n} &\geq 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} \\ &\quad + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} P_{j,i} f(A_{j-k,i-l}). \end{aligned} \quad (3.7)$$

Define

$$X = \{ \{B_{m,n}\} \mid 0 \leq B_{m,n} \leq 1, m \geq -k, n \geq -l \}$$

and an operator T on X by

$$(TB_{m,n}) = \begin{cases} \frac{1}{A_{m,n}} \left[2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} B_{j,i} \right. \\ \left. + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} P_{j,i} f(A_{j-k,i-l} B_{j-k,i-l}) \right], \\ m \geq 0, \quad n \geq 0, \\ 1, \quad \text{otherwise.} \end{cases}$$

In view of (3.7), $TX \subset X$. Similar to Theorem 3.1, we can prove that there exists $\{B_{m,n}\} \in X$ such that $(TB)_{m,n} = B_{m,n}$ for $m \geq 0, n \geq 0$. Let $x_{m,n} = A_{m,n} B_{m,n} > 0, m \geq 0, n \geq 0$. Then

$$x_{m,n} = 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} x_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} P_{j,i} f(x_{j-k,i-l}). \quad (3.8)$$

Similar to the proof of Theorem 3.1, (3.8) implies that $\{x_{m,n}\}$ is a positive solution of (1.3). The proof is complete.

From Theorems 3.1 and 3.2 we obtain

COROLLARY 3.1 *Assume that $P_{m,n} \equiv P > 0$, (ii) of Theorem 3.1 and (ii) of Theorem 3.2 hold. Then every solution of (1.3) oscillates if and only if every solution of (3.1) oscillates.*

Consider (1.3) together with the equation

$$\begin{aligned} A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} \\ - A_{m,n} + q_{m,n} f(A_{m-k,n-l}) = 0, \end{aligned} \quad (3.9)$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ as $x \neq 0$.

We have the comparison theorem as follows:

THEOREM 3.3 *Assume that $p_{m,n} \geq q_{m,n} > 0$ for all large m, n and that every solution of (3.9) oscillates. Then every solution of (1.3) oscillates.*

Proof Suppose to the contrary, $\{A_{m,n}\}$ is a positive solution, of (1.3). Summing (1.3), we

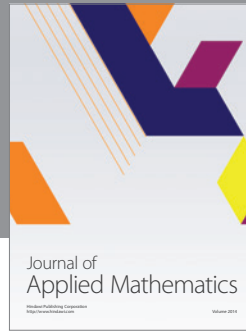
obtain

$$\begin{aligned} A_{m,n} &= 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} f(A_{j-k,i-l}) \\ &\geq 2 \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} q_{j,i} f(A_{j-k,i-l}), \end{aligned}$$

from which and using the method in the proof of Theorem 3.2, we obtain a positive solution of (3.9). This contradiction proves the theorem.

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