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Research Article

Finite Series Representation of the Inverse Mittag-Leffler Function

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The inverse Mittag-Leffler function $E_{\alpha,\beta}^{-1}(z)$ is valuable in determining the value of the argument of a Mittag-Leffler function given the value of the function and it is not an easy problem. A finite series representation of the inverse Mittag-Leffler function has been found for a range of the parameters α and β; specifically, $0 < \alpha < 1/2$ for $\beta = 1$ and for $\beta = 2$. This finite series representation of the inverse Mittag-Leffler function greatly expedites its evaluation and has been illustrated with a number of examples. This represents a significant advancement in the understanding of Mittag-Leffler functions.

1. Introduction

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by the power series [1]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad z \in \mathbb{C}. \tag{1}$$

While the argument z and the parameters α and β can in general be complex provided $\operatorname{Re}\alpha > 0$, in this work z, α, and β will be restricted to those values most commonly found in physical problems; namely, the argument z will be restricted to real numbers and α and β will be restricted to positive real numbers. The Mittag-Leffler function is a generalization of the exponential function and arises frequently in the solutions of differential and/or integral equations of fractional (noninteger) order in much the same way as the exponential function appears in solutions of differential equations of integer order. Thus, Mittag-Leffler functions play a fundamental role in the theory of fractional differential equations. Consequently, books devoted to the subject of fractional differential equations (i.e., Podlubny [2], Kilbas et al. [3], and Diethelm [4]) all contain sections on the Mittag-Leffler functions. In addition to their inherent mathematical interest, Mittag-Leffler functions are also important in theoretical and applied physics and all the sciences (i.e., Hilfer [5], Mainardi [6], and Magin [7]). The works of Mainardi and Gorenflo [8], Magin [9], Berberan-Santos [10], Gupta and Debnath [11], and Haubold et al. [12] are a few of the numerous articles also worth noting.

The inverse Mittag-Leffler function $E_{\alpha,\beta}^{-1}(z)$ is defined as the solution of (2) [13]

$$E_{\alpha,\beta}^{-1}\left[E_{\alpha,\beta}(z)\right] = z. \tag{2}$$

Despite the inherent importance of Mittag-Leffler functions in fractional differential equations, with the wealth of analytical information about $E_{\alpha,\beta}(z)$, the inverse $E_{\alpha,\beta}^{-1}(z)$ has been largely unexplored. The one exception is the excellent work of Hilfer and Seybold [13] who have determined its principal branch numerically.

The power series representation of any Mittag-Leffler function can be inverted yielding an infinite series for the inverse. However, these infinite series are slow to converge and terminating the series always introduces error which is hard to evaluate. This present work identifies regions in the domain of α and β where the inverse of the Mittag-Leffler function can be written as a finite series. This represents the first time the inverse Mittag-Leffler function has been written as a finite series as opposed to an infinite series which greatly expedites its evaluation. Before deriving these expressions for the inverse Mittag-Leffler function, a brief review of the theory of power series and their inverses is in order.

2. Theory

Consider the convergent series which expresses the function $w = f(z)$ in terms of powers of $(z - z_o)$ with the corresponding coefficients a_k given by

$$w = f(z) = \sum_{k=0}^{\infty} a_k (z - z_o)^k = a_o + a_1 (z - z_o) + a_2 (z - z_o)^2 + \cdots . \tag{3}$$

The inversion of the function $f(z)$ requires only the sole assumption that $a_1 \neq 0$. That is, there exists one and only one function which represents the inverse of the $f(z)$, $z - z_o = f^{-1}(w)$, which is expressible by a convergent power series of the form [14]

$$z - z_o = f^{-1}(w) = \sum_{k=1}^{\infty} b_k (w - a_o)^k = b_1 (w - a_o) + b_2 (w - a_o)^2 + \cdots . \tag{4}$$

The process of finding the series expansion for $f^{-1}(w)$ is called reversion of the series. The coefficients b_k can be determined in terms of the coefficients a_k by substituting (3) into (4) and equating coefficients of like powers of $(z - z_o)^k$ on both sides of the equation yielding

$$b_1 = \frac{1}{a_1}, \qquad b_3 = \frac{1}{a_1^5}\left(2a_2^2 - a_1 a_3\right), \qquad b_2 = -\frac{a_2}{a_1^3}, \qquad b_4 = \frac{1}{a_1^7}\left(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3\right). \tag{5}$$

The coefficients $b_1, b_2, b_3, \ldots, b_7$ can be found in the literature [15–17]. An explicit expression for the coefficients b_k can be derived using the Lagrange inversion theorem. If $f(z)$ is analytic at $z = z_o$ and $f'(z_o) \neq 0$, then the inverse of $f(z)$ exists and is analytic about $f(z_o)$. Furthermore, if $f(z) = w$, the Lagrange inversion theorem gives the Taylor series expansion of the inverse function $f^{-1}(w)$ as [15]

$$f^{-1}(w) = z - z_o = \sum_{k=1}^{\infty} \frac{(w - a_o)^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left\{ \frac{(z - z_o)^k}{\left[f(z) - a_o\right]^k} \right\}_{z=z_o} . \tag{6}$$

The coefficients b_k are determined by comparing (6) and (4) yielding

$$b_k = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \left\{ \frac{(z - z_o)^k}{\left[f(z) - a_o\right]^k} \right\}_{z=z_o} . \tag{7}$$

Substituting $f(z) - a_o$ from (3) yields

$$b_k = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \left\{ \left[a_1 + a_2 (z - z_o) + a_3 (z - z_o)^2 + \cdots \right]^{-k} \right\}_{z=z_o} . \tag{8}$$

Factoring out a_1^k in (8) and defining $x = z - z_o$ yields

$$b_k = \frac{1}{a_1^k k!} \frac{d^{k-1}}{dz^{k-1}} \times \left\{ \left[1 + \left(\frac{a_2}{a_1}\right) x + \left(\frac{a_3}{a_1}\right) x^2 + \left(\frac{a_4}{a_1}\right) x^3 + \cdots \right]^{-k} \right\}_{x=0} . \tag{9}$$

Using the multinomial expansion and performing the required differentiation yields the desired result [18]

$$b_k = \frac{1}{k a_1^k} \times \sum_{s,t,u,\ldots} (-1)^{s+t+u+\cdots} \frac{(k)(k+1)\cdots(k-1+s+t+u\cdots)}{s!t!u!\cdots} \times \left(\frac{a_2}{a_1}\right)^s \left(\frac{a_3}{a_1}\right)^t \left(\frac{a_3}{a_1}\right)^u \cdots , \tag{10}$$

where $s + 2t + 3u + \cdots = k - 1$ and the numbers $s, t, u, \ldots$ are nonnegative integers and the summation extends over all partitions of $k - 1$. For example, b_5 contains 5 terms since the Diophantine equation $s + 2t + 3u + 4v = 4$ has 5 integer solutions or partitions. The number of partitions for $k = 11$ is 42; for $k = 51$ there are 204226 partitions and for $k = 101$ the number of partitions is 190569292. Consequently, the explicit tabulation of the full expression for the coefficients b_k rapidly becomes a rather tedious task. Nevertheless, the coefficients $b_1, b_2, b_3, \ldots, b_{14}$ are given in Table 1. An equivalent expression for the general term b_k in the reversion of series is given in a different form by McMahon [19].

By an appropriate change of variables it is always possible to write the power series in a form which results in simplified expressions for the coefficients in the reversed power series. Equation (3) can be rewritten as

$$\frac{w - a_o}{a_1} = (z - z_o) \left[1 + \frac{a_2}{a_1}(z - z_o) + \frac{a_3}{a_1}(z - z_o)^2 + \cdots \right] . \tag{11}$$

Defining the new variables $W = (w - a_o)/a_1$, $A_1 = -a_2/a_1$, $A_2 = -a_3/a_1$, and so forth, (11) becomes

$$W = (z - z_o) \left[1 - \sum_{k=1}^{\infty} A_k (z - z_o)^k \right] \tag{12}$$

and the reversed series is given by

$$(z - z_o) = W \left[1 - \sum_{k=1}^{\infty} B_k W^k \right] . \tag{13}$$

TABLE 1: Coefficients of the inverse function for a power series.

k	Coefficient b_k
1	$\dfrac{1}{a_1}$
2	$-\dfrac{a_2}{a_1^3}$
3	$\dfrac{1}{a_1^5}\left(2a_2^2 - a_1a_3\right)$
4	$\dfrac{1}{a_1^7}\left(-5a_2^3 + 5a_1a_2a_3 - a_1^2a_4\right)$
5	$\dfrac{1}{a_1^9}\left(14a_2^4 - 21a_1a_2^2a_3 + 3a_1^2a_3^2 + 6a_1^2a_2a_4 - a_1^3a_5\right)$
6	$\dfrac{1}{a_1^{11}}\left(-42a_2^5 + 84a_1a_2^3a_3 - 28a_1^2a_2a_3^2 - 28a_1^2a_2^2a_4 + 7a_1^3a_3a_4 + 7a_1^3a_2a_5 - a_1^4a_6\right)$
7	$\dfrac{1}{a_1^{13}}(132a_2^6 - 330a_1a_2^4a_3 + 180a_1^2a_2^2a_3^2 - 12a_1^3a_3^3 + 120a_1^2a_2^3a_4 - 72a_1^3a_2a_3a_4 + 4a_1^4a_4^2 - 36a_1^3a_2^2a_5 + 8a_1^4a_3a_5 + 8a_1^4a_2a_6 - a_1^5a_7)$
8	$\dfrac{1}{a_1^{15}}(-429a_2^7 + 1287a_1a_2^5a_3 - 990a_1^2a_2^3a_3^2 + 165a_1^3a_2a_3^3 - 495a_1^2a_2^4a_4 + 495a_1^3a_2^2a_3a_4 - 45a_1^4a_3^2a_4 - 45a_1^4a_2a_4^2 + 165a_1^3a_2^3a_5$ $-90a_1^4a_2a_3a_5 + 9a_1^5a_4a_5 - 45a_1^4a_2^2a_6 + 9a_1^5a_3a_6 + 9a_1^5a_2a_7 - a_1^6a_8)$
9	$\dfrac{1}{a_1^{17}}(1430a_2^8 - 5005a_1a_2^6a_3 + 5005a_1^2a_2^4a_3^2 - 1430a_1^3a_2^2a_3^3 + 55a_1^4a_3^4 + 2002a_1^2a_2^5a_4 - 2860a_1^3a_2^3a_3a_4 + 660a_1^4a_2a_3^2a_4 + 330a_1^4a_2^2a_4^2$ $-55a_1^5a_3a_4^2 - 715a_1^3a_2^4a_5 + 660a_1^4a_2^2a_3a_5 - 55a_1^5a_3^2a_5 - 110a_1^5a_2a_4a_5 + 5a_1^6a_5^2 + 220a_1^4a_2^3a_6 - 110a_1^5a_2a_3a_6 + 10a_1^6a_4a_6$ $-55a_1^5a_2^2a_7 + 10a_1^6a_3a_7 + 10a_1^6a_2a_8 - a_1^7a_9)$
10	$\dfrac{1}{a_1^{19}}(-4862a_2^9 + 19448a_1a_2^7a_3 - 24024a_1^2a_2^5a_3^2 + 10010a_1^3a_2^3a_3^3 - 1001a_1^4a_2a_3^4 - 8008a_1^2a_2^6a_4 + 15015a_1^3a_2^4a_3a_4 - 6006a_1^4a_2^2a_3^2a_4$ $+286a_1^5a_3^3a_4 - 2002a_1^4a_2^3a_4^2 + 858a_1^5a_2a_3a_4^2 - 22a_1^6a_4^3 + 3003a_1^3a_2^5a_5 - 4004a_1^4a_2^3a_3a_5 + 858a_1^5a_2a_3^2a_5 + 858a_1^5a_2^2a_4a_5$ $-132a_1^6a_3a_4a_5 - 66a_1^6a_2a_5^2 - 1001a_1^4a_2^4a_6 + 858a_1^5a_2^2a_3a_6 - 66a_1^6a_3^2a_6 - 132a_1^6a_2a_4a_6 + 11a_1^7a_5a_6 + 286a_1^5a_2^3a_7$ $-132a_1^6a_2a_3a_7 + 11a_1^7a_4a_7 - 66a_1^6a_2^2a_8 + 11a_1^7a_3a_8 + 11a_1^7a_2a_9 - a_1^8a_{10})$
11	$\dfrac{1}{a_1^{21}}(16796a_2^{10} - 75582a_1a_2^8a_3 + 111384a_1^2a_2^6a_3^2 - 61880a_1^3a_2^4a_3^3 + 10920a_1^4a_2^2a_3^4 - 273a_1^5a_3^5 + 31824a_1^2a_2^7a_4 - 74256a_1^3a_2^5a_3a_4$ $+43680a_1^4a_2^3a_3^2a_4 - 5460a_1^5a_2a_3^3a_4 + 10920a_1^4a_2^4a_4^2 + -8190a_1^5a_2^2a_3a_4^2 + 546a_1^6a_3^2a_4^2 + 364a_1^6a_2a_4^3 - 12376a_1^3a_2^6a_5$ $+21840a_1^4a_2^4a_3a_5 + -8190a_1^5a_2^2a_3^2a_5 + 364a_1^6a_3^3a_5 - 5460a_1^5a_2^3a_4a_5 + 2184a_1^6a_2a_3a_4a_5 - 78a_1^7a_4^2a_5 + 546a_1^6a_2^2a_5^2$ $-78a_1^7a_3a_5^2 + 4368a_1^4a_2^5a_6 - 5460a_1^5a_2^3a_3a_6 + 1092a_1^6a_2a_3^2a_6 + 1092a_1^6a_2^2a_4a_6 - 156a_1^7a_3a_4a_6 - 156a_1^7a_2a_5a_6 + 6a_1^8a_6^2$ $-1365a_1^5a_2^4a_7 + 1092a_1^6a_2^2a_3a_7 + -78a_1^7a_3^2a_7 - 156a_1^7a_2a_4a_7 + 12a_1^8a_5a_7 + 364a_1^6a_2^3a_8 - 156a_1^7a_2a_3a_8 + 12a_1^8a_4a_8$ $+ - 78a_1^7a_2^2a_9 + 12a_1^8a_3a_9 + 12a_1^8a_2a_{10} - a_1^9a_{11})$
12	$\dfrac{1}{a_1^{23}}(-58786a_2^{11} + 293930a_1a_2^9a_3 - 503880a_1^2a_2^7a_3^2 + 352716a_1^3a_2^5a_3^3 - 92820a_1^4a_2^3a_3^4 + 6188a_1^5a_2a_3^5 + -125970a_1^2a_2^8a_4$ $+352716a_1^3a_2^6a_3a_4 - 278460a_1^4a_2^4a_3^2a_4 + 61880a_1^5a_2^2a_3^3a_4 - 1820a_1^6a_3^4a_4 + -55692a_1^4a_2^5a_4^2 + 61880a_1^5a_2^3a_3a_4^2$ $-10920a_1^6a_2a_3^2a_4^2 - 3640a_1^6a_2^2a_4^3 + 455a_1^7a_3a_4^3 + 50388a_1^3a_2^7a_5 - 111384a_1^4a_2^5a_3a_5 + 61880a_1^5a_2^3a_3^2a_5 - 7280a_1^6a_2a_3^3a_5$ $+30940a_1^5a_2^4a_4a_5 + -21840a_1^6a_2^2a_3a_4a_5 + 1365a_1^7a_3^2a_4a_5 + 1365a_1^7a_2a_4^2a_5 - 3640a_1^6a_2^3a_5^2 + 1365a_1^7a_2a_3a_5^2 + -91a_1^8a_4a_5^2$ $-18564a_1^4a_2^6a_6 + 30940a_1^5a_2^4a_3a_6 - 10920a_1^6a_2^2a_3^2a_6 + 455a_1^7a_3^3a_6 + -7280a_1^6a_2^3a_4a_6 + 2730a_1^7a_2a_3a_4a_6 - 91a_1^8a_4^2a_6$ $+1365a_1^7a_2^2a_5a_6 - 182a_1^8a_3a_5a_6 + -91a_1^8a_2a_6^2 + 6188a_1^5a_2^5a_7 - 7280a_1^6a_2^3a_3a_7 + 1365a_1^7a_2a_3^2a_7 + 1365a_1^7a_2^2a_4a_7$ $+ - 182a_1^8a_3a_4a_7 - 182a_1^8a_2a_5a_7 + 13a_1^9a_6a_7 - 1820a_1^6a_2^4a_8 + 1365a_1^7a_2^2a_3a_8 - 91a_1^8a_3^2a_8 + -182a_1^8a_2a_4a_8 + 13a_1^9a_5a_8$ $+455a_1^7a_2^3a_9 - 182a_1^8a_2a_3a_9 + 13a_1^9a_4a_9 - 91a_1^8a_2^2a_{10} + 13a_1^9a_3a_{10} + 13a_1^9a_2a_{11} - a_1^{10}a_{12})$

Table 1: Continued.

k	Coefficient b_k
13	$\frac{1}{a_1^{25}}(208012a_2^{12} - 1144066a_1a_2^{10}a_3 + 2238390a_1^2a_2^8a_3^2 - 1899240a_1^3a_2^6a_3^3 + 678300a_1^4a_2^4a_3^4 + -81396a_1^5a_2^2a_3^5 + 1428a_1^6a_3^6 +497420a_1^2a_2^9a_4 - 1627920a_1^3a_2^7a_3a_4 + 1627920a_1^4a_2^5a_3^2a_4 + -542640a_1^5a_2^3a_3^3a_4 + 42840a_1^6a_2a_3^4a_4 + 271320a_1^4a_2^6a_4^2 -406980a_1^5a_2^4a_3a_4^2 + 128520a_1^6a_2^2a_3^2a_4^2 - 4760a_1^7a_3^3a_4^2 + 28560a_1^6a_2^3a_4^3 - 9520a_1^7a_2a_3a_4^3 + 140a_1^8a_4^4 + -203490a_1^3a_2^8a_5 +542640a_1^4a_2^6a_3a_5 - 406980a_1^5a_2^4a_3^2a_5 + 85680a_1^6a_2^2a_3^3a_5 - 2380a_1^7a_3^4a_5 + -162792a_1^5a_2^5a_4a_5 + 171360a_1^6a_2^3a_3a_4a_5 -28560a_1^7a_2a_3^2a_4a_5 - 14280a_1^7a_2^2a_4^2a_5 + 1680a_1^8a_3a_4^2a_5 + 21420a_1^6a_2^4a_5^2 - 14280a_1^7a_2^2a_3a_5^2 + 840a_1^8a_3^2a_5^2 + 1680a_1^8a_2a_4a_5^2 -35a_1^9a_5^3 + 77520a_1^4a_2^7a_6 - 162792a_1^5a_2^5a_3a_6 + 85680a_1^6a_2^3a_3^2a_6 - 9520a_1^7a_2a_3^3a_6 + 42840a_1^6a_2^4a_4a_6 + -28560a_1^7a_2^2a_3a_4a_6 +1680a_1^8a_3^2a_4a_6 + 1680a_1^8a_2a_4^2a_6 - 9520a_1^7a_2^3a_5a_6 + 3360a_1^8a_2a_3a_5a_6 + -210a_1^9a_4a_5a_6 + 840a_1^8a_2^2a_6^2 - 105a_1^9a_3a_6^2 -27132a_1^5a_2^6a_7 + 42840a_1^6a_2^4a_3a_7 - 14280a_1^7a_2^2a_3^2a_7 + 560a_1^8a_3^3a_7 - 9520a_1^7a_2^3a_4a_7 + 3360a_1^8a_2a_3a_4a_7 - 105a_1^9a_4^2a_7 +1680a_1^8a_2^2a_5a_7 - 210a_1^9a_3a_5a_7 + -210a_1^9a_2a_6a_7 + 7a_1^{10}a_7^2 + 8568a_1^6a_2^5a_8 - 9520a_1^7a_2^3a_3a_8 + 1680a_1^8a_2a_3^2a_8 + 1680a_1^8a_2^2a_4a_8 + - 210a_1^9a_3a_4a_8 - 210a_1^9a_2a_5a_8 + 14a_1^{10}a_6a_8 - 2380a_1^7a_2^4a_9 + 1680a_1^8a_2^2a_3a_9 - 105a_1^9a_3^2a_9 + -210a_1^9a_2a_4a_9 + 14a_1^{10}a_5a_9 +560a_1^8a_2^3a_{10} - 210a_1^9a_2a_3a_{10} + 14a_1^{10}a_4a_{10} - 105a_1^9a_2^2a_{11} + 14a_1^{10}a_3a_{11} + 14a_1^{10}a_2a_{12} - a_1^{11}a_{13})$
14	$\frac{1}{a_1^{27}}(-742900a_2^{13} + 4457400a_1a_2^{11}a_3 - 9806280a_1^2a_2^9a_3^2 + 9806280a_1^3a_2^7a_3^3 - 4476780a_1^4a_2^5a_3^4 + 813960a_1^5a_2^3a_3^5 - 38760a_1^6a_2a_3^6 -1961256a_1^2a_2^{10}a_4 + 7354710a_1^3a_2^8a_3a_4 - 8953560a_1^4a_2^6a_3^2a_4 + 4069800a_1^5a_2^4a_3^3a_4 - 581400a_1^6a_2^2a_3^4a_4 + 11628a_1^7a_3^5a_4 -1279080a_1^4a_2^7a_4^2 + 2441880a_1^5a_2^5a_3a_4^2 - 1162800a_1^6a_2^3a_3^2a_4^2 + 116280a_1^7a_2a_3^3a_4^2 - 193800a_1^6a_2^4a_4^3 + 116280a_1^7a_2^2a_3a_4^3 + - 6120a_1^8a_3^2a_4^3 - 3060a_1^8a_2a_4^4 + 817190a_1^3a_2^9a_5 - 2558160a_1^4a_2^7a_3a_5 + 2441880a_1^5a_2^5a_3^2a_5 + -775200a_1^6a_2^3a_3^3a_5 +58140a_1^7a_2a_3^4a_5 + 813960a_1^5a_2^6a_4a_5 - 1162800a_1^6a_2^4a_3a_4a_5 + 348840a_1^7a_2^2a_3^2a_4a_5 - 12240a_1^8a_3^3a_4a_5 + 116280a_1^7a_2^3a_4^2a_5 -36720a_1^8a_2a_3a_4^2a_5 + 680a_1^9a_4^3a_5 - 116280a_1^6a_2^5a_5^2 + 116280a_1^7a_2^3a_3a_5^2 - 18360a_1^8a_2a_3^2a_5^2 - 18360a_1^8a_2^2a_4a_5^2 + 2040a_1^9a_3a_4a_5^2 +680a_1^9a_2a_5^3 - 319770a_1^4a_2^8a_6 + 813960a_1^5a_2^6a_3a_6 - 581400a_1^6a_2^4a_3^2a_6 + 116280a_1^7a_2^2a_3^3a_6 - 3060a_1^8a_3^4a_6 - 232560a_1^6a_2^5a_4a_6 +232560a_1^7a_2^3a_3a_4a_6 + -36720a_1^8a_2a_3^2a_4a_6 - 18360a_1^8a_2^2a_4^2a_6 + 2040a_1^9a_3a_4^2a_6 + 58140a_1^7a_2^4a_5a_6 - 36720a_1^8a_2^2a_3a_5a_6 +2040a_1^9a_3^2a_5a_6 + 4080a_1^9a_2a_4a_5a_6 - 120a_1^{10}a_5^2a_6 - 6120a_1^8a_2^3a_6^2 + 2040a_1^9a_2a_3a_6^2 - 120a_1^{10}a_4a_6^2 + 116280a_1^5a_2^7a_7 -232560a_1^6a_2^5a_3a_7 + 116280a_1^7a_2^3a_3^2a_7 - 12240a_1^8a_2a_3^3a_7 + 58140a_1^7a_2^4a_4a_7 + -36720a_1^8a_2^2a_3a_4a_7 + 2040a_1^9a_3^2a_4a_7 +2040a_1^9a_2a_4^2a_7 - 12240a_1^8a_2^3a_5a_7 + 4080a_1^9a_2a_3a_5a_7 + -240a_1^{10}a_4a_5a_7 + 2040a_1^9a_2^2a_6a_7 - 240a_1^{10}a_3a_6a_7 - 120a_1^{10}a_2a_7^2 -38760a_1^6a_2^6a_8 + 58140a_1^7a_2^4a_3a_8 - 18360a_1^8a_2^2a_3^2a_8 + 680a_1^9a_3^3a_8 - 12240a_1^8a_2^3a_4a_8 + 4080a_1^9a_2a_3a_4a_8 + -120a_1^{10}a_4^2a_8 +2040a_1^9a_2^2a_5a_8 - 240a_1^{10}a_3a_5a_8 - 240a_1^{10}a_2a_6a_8 + 15a_1^{11}a_7a_8 + 11628a_1^7a_2^5a_9 + -12240a_1^8a_2^3a_3a_9 + 2040a_1^9a_2a_3^2a_9 +2040a_1^9a_2^2a_4a_9 - 240a_1^{10}a_3a_4a_9 - 240a_1^{10}a_2a_5a_9 + 15a_1^{11}a_6a_9 - 3060a_1^8a_2^4a_{10} + 2040a_1^9a_2^2a_3a_{10} - 120a_1^{10}a_3^2a_{10} -240a_1^{10}a_2a_4a_{10} + 15a_1^{11}a_5a_{10} + 680a_1^9a_2^3a_{11} - 240a_1^{10}a_2a_3a_{11} + 15a_1^{11}a_4a_{11} - 120a_1^{10}a_2^2a_{12} + 15a_1^{11}a_3a_{12} + 15a_1^{11}a_2a_{13} - a_1^{12}a_{14})$

The resulting coefficients B_k for $k = 1, 2, 3$, and 4 are given by

$$\begin{aligned} -B_1 &= A_1, \\ -B_3 &= A_3 + 5A_1A_2 + 5A_1^3, \\ -B_2 &= A_2 + 2A_1^2, \\ -B_4 &= A_4 + 6A_1A_3 + 3A_2^2 + 21A_1^2A_2 + 14A_1^4. \end{aligned} \tag{14}$$

These can be shown to be equivalent to (5) by setting $-B_k = b_{k+1}a_1^{k+1}$ and $A_k = -a_{k+1}/a_1$. The coefficients B_k for $k = 1, 2, \ldots, 7$ can be found tabulated in [20], for $k = 1, 2, \ldots, 9$ in [21] and for $k = 1, 2, \ldots, 12$ they are tabulated in [22] with a different choice of the sign of B_k. Müller [23] has reported an alternative expression for B_k and some symmetry relations for the coefficients.

3. Application to Mittag-Leffler Functions

Many Mittag-Leffler functions can be represented in terms of elementary functions. For example,

$$E_{1,1}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(k+1)} = \operatorname{Exp}(-z)$$

$$\begin{aligned} E_{1/2,3}(-z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(k/2+3)} \\ &= \frac{\operatorname{Exp}\left(z^2\right)\operatorname{erfc}(z) - 1}{z^4} - \frac{1}{z^2} + \frac{4}{3z\sqrt{\pi}} + \frac{2}{z^3\sqrt{\pi}}. \end{aligned} \quad (15)$$

Applying (10) from the above theory to these functions whose values can be determined as accurately as possible using their alternative representations yields

$$\begin{aligned} -z = E_{1,1}^{-1}(w) &= (w-1) - \frac{1}{2}(w-1)^2 + \frac{1}{3}(w-1)^3 \\ &\quad - \frac{1}{4}(w-1)^4 + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(w-1)^k}{k}, \end{aligned} \quad (16)$$

$$\begin{aligned} -z = E_{1/2,3}^{-1}(w) &= \frac{15\sqrt{\pi}}{8}\left(w - \frac{1}{2}\right) - \frac{1125\pi^{3/2}}{1024}\left(w - \frac{1}{2}\right)^2 \\ &\quad + \frac{3375\pi^{3/2}(175\pi - 256)}{458752}\left(w - \frac{1}{2}\right)^3 + \cdots. \end{aligned} \quad (17)$$

A few observations are in order. Equations (16) and (17) are typical of the inverse of most infinite series; that is, they are also infinite series and do not converge rapidly. This can be easily illustrated by the following examples. For $w = \operatorname{Exp}(-1)$, (16) should yield $-z = -1$ (equivalently $z = 1$). However, (16) requires 20 terms before the value of z is as large as 0.99999 (5 nines), 44 terms for 10 nines, 68 terms for 15 nines, and 92 terms for 20 nines. Whereas for $w = \operatorname{Exp}(-10)$, where (16) should yield $z = 10$, 156995 terms are required before the value of z is as large as 9.9999 (5 nines), 391895 terms for 10 nines, 635259 for 15 nines, and 881815 terms for 20 nines. Similarly, for $w = \operatorname{Exp}(-15)$ where (16) should yield $z = 15$, 16730862 terms are required before the value of z is as large as 14.999 (3 nines), 51041531 terms for 8 nines, 87009540 terms for 13 nines, and 123532970 terms for 18 nines. For $w = \operatorname{Exp}(-z)$, as z becomes large (or equivalently $w \to 0$), the number of terms in (16) required to yield a value accurate to a given number of significant digits becomes astronomically large.

A similar behavior is exhibited in (17). For $w = 0.30821552131\ldots$, (17) should yield $z = 1$. To obtain a value of z as large as 0.99999 (5 nines), 12 terms are required, 24 terms for 10 nines, 36 terms for 15 nines, and 48 terms for 20 nines. For $w = 0.0662592710\ldots$, (17) should yield $z = 10$, but

Table 2: Coefficients b_k for the inverse Mittag-Leffler function $-z = E_{1/7,1}^{-1}(w)$.

k	b_k
1	+0.93543756289254634824
2	+0.90975389394768139194
3	+0.90540301580659885103
4	+0.90454074680764978103
5	+0.90437439055401830557
6	+0.90434827833630659461
7	+0.90434699795298307168
8	+0.90434836056866111562
9	+0.90434917779666970118
10	+0.90434948952806441367
11	+0.90434957941529405394
12	+0.90434959664285150118
13	+0.90434959619870743701
14	+0.90434959373645610938
15	+0.90434959223258048967
16	+0.90434959159682667227
17	+0.90434959139076386922
18	+0.90434959134523887081
19	+0.90434959134495719874
20	+0.90434959135163221718
21	+0.90434959135628153678
22	+0.90434959135847703425
23	+0.90434959135928171696
24	+0.90434959135949807367
25	+0.90434959135952104631
26	+0.90434959135950171101
27	+0.90434959135948382131
28	+0.90434959135947403392
29	+0.90434959135946990785
30	+0.90434959135946855244
31	+0.90434959135946826616
32	+0.90434959135946829135
33	+0.90434959135946836127
34	+0.90434959135946840959
35	+0.90434959135946843362
36	+0.90434959135946844315
37	+0.90434959135946844604
38	+0.90434959135946844650
39	+0.90434959135946844632
40	+0.90434959135946844609
41	+0.90434959135946844595
42	+0.90434959135946844588
43	+0.90434959135946844585
44	+0.90434959135946844585
45	+0.90434959135946844584
46	+0.90434959135946844585

81 terms are required to obtain a value of z as large as 9.9999 (5 nines), 162 for 10 nines, 243 terms for 15 nines, and 324 terms for 20 nines. For $w = 0.007423646216\ldots$, (17) should

Table 3: Number of terms required in the finite representation of $E^{-1}_{\alpha,\beta}(z)$ for 20-significant-digit accuracy.

α	$\beta = 1$	$\beta = 2$
1/100	11	9
1/10	32	25
1/9	35	27
1/8	40	28
1/7	46	33
1/6	52	38
1/5	64	46
1/4	92	63
1/3	156	91
1/2	562	262
4/7	1051	429
3/5	1469	548

Table 4: Index of Mittag-Leffler inverse examples.

α	$\beta = 1$	$\beta = 2$
1/3	Equation (22) and Table 5	Equation (29) and Table 12
1/4	Equation (23) and Table 6	Equation (30) and Table 13
1/5	Equation (24) and Table 7	Equation (31) and Table 14
1/6	Equation (25) and Table 8	Equation (32) and Table 15
1/7	Equation (19) and Table 2	Equation (33) and Table 16
1/8	Equation (26) and Table 9	Equation (34) and Table 17
1/9	Equation (27) and Table 10	Equation (35) and Table 18
1/10	Equation (28) and Table 11	Equation (36) and Table 19

yield $z = 100$, but 770 terms are required to obtain a value of z as large as 99.999 (5 nines) and 1540 terms for 10 nines. There is, however, one big difference between (16) and (17). Equation (16) is one of the few inverses of a Mittag-Leffler function, where the coefficients b_k in the inverse given in (10) and itemized in Table 1 for $b_1 - b_{14}$ simplify to a tractable expression; in this case $b_k = (-1)^{k+1}/k$. The mathematical manipulations required to obtain the coefficients b_k in (17) using (10) become algebraically intensive as k becomes large. Whereas b_{14} given in Table 1 contains 101 terms, b_{1000} contains more than 2.4×10^{31} terms. Consequently, although the infinite series given in (17) correctly represents the inverse Mittag-Leffler function, it is impractical to use for anything other than small z where only a reasonable number of terms are needed for the required accuracy. This is the case for most of the inverse Mittag-Leffler functions.

Consider the inverse of the Mittag-Leffler function $E_{1/7,1}(-z)$. The coefficients b_k calculated from (10) are given in Table 2 (truncated to 20 significant digits).

It is obvious in looking at the coefficients b_k in Table 2 that they are approaching a constant as k becomes large. In this case, the constant is $1/\Gamma(6/7)$. Subsequently, the first 20 significant digits for all coefficients after b_{46} are identical

Table 5: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/3,1}(w)$.

k	b_k
1	+0.89297951156924921122
2	+0.78878610417460496420
3	+0.75763354875769329328
4	+0.74579778773344787841
5	+0.74098130749558031810
6	+0.73904720310555344414
7	+0.73834522959265981505
8	+0.73816142472165138055
9	+0.73817790739448622896
10	+0.73825511074871374070
11	+0.73833571434942144250
12	+0.73839997927465491398
13	+0.73844449496113714586
14	+0.73847210628873099793
15	+0.73848730878287129409
16	+0.73849432235279002530
17	+0.73849644605470171587
18	+0.73849598893480155549
19	+0.73849442620461442205
20	+0.73849261331276901456
21	+0.73849098486630158241
22	+0.73848971269933349619
23	+0.73848881978568953731
24	+0.73848825605202219473
25	+0.73848794494184281702
26	+0.73848780930951379358
27	+0.73848778373908570968
28	+0.73848781862086802257
29	+0.73848787970874775507
30	+0.73848794558318757255
31	+0.73848800448141685314
32	+0.73848805128840606965
33	+0.73848808504905157547
34	+0.73848810710188503719
35	+0.73848811979412736597
36	+0.73848812567490815232
37	+0.73848812704699497915
38	+0.73848812576600764535
39	+0.73848812319593135382
40	+0.73848812025242302623
41	+0.73848811748639447885
42	+0.73848811517762375779
43	+0.73848811342117360995
44	+0.73848811219849196547
45	+0.73848811143091641775
46	+0.73848811101668555873
47	+0.73848811085420482928
48	+0.73848811085483076012

Table 5: Continued.

k	b_k
49	+0.73848811094828991640
50	+0.73848811108336786221
51	+0.73848811122590646140
52	+0.73848811135556505150
53	+0.73848811146230003348
54	+0.73848811154312398420
55	+0.73848811159941978280
56	+0.73848811163489404655
57	+0.73848811165413834046
58	+0.73848811166170643685
59	+0.73848811166159371995
60	+0.73848811165700633342
61	+0.73848811165032209352
62	+0.73848811164316504107
63	+0.73848811163653598310
64	+0.73848811163095974872
65	+0.73848811162662488952
66	+0.73848811162350289072
67	+0.73848811162144189450
68	+0.73848811162023502242
69	+0.73848811161966626697
70	+0.73848811161953822181
71	+0.73848811161968617262
72	+0.73848811161998269778
73	+0.73848811162033624249
74	+0.73848811162068634439
75	+0.73848811162099743490
76	+0.73848811162125249050
77	+0.73848811162144729066
78	+0.73848811162158565606
79	+0.73848811162167577734
80	+0.73848811162172758004
81	+0.73848811162175098150
82	+0.73848811162175485904
83	+0.73848811162174654725
84	+0.73848811162173170072
85	+0.73848811162171438724
86	+0.73848811162169730770
87	+0.73848811162168206810
88	+0.73848811162166945421
89	+0.73848811162165967945
90	+0.73848811162165259141
91	+0.73848811162164783248
92	+0.73848811162164495684
93	+0.73848811162164350958
94	+0.73848811162164307502
95	+0.73848811162164330226
96	+0.73848811162164391339
97	+0.73848811162164470224

Table 5: Continued.

k	b_k
98	+0.73848811162164552587
99	+0.73848811162164629360
100	+0.73848811162164695521
101	+0.73848811162164748994
102	+0.73848811162164789692
103	+0.73848811162164818753
104	+0.73848811162164837957
105	+0.73848811162164849309
106	+0.73848811162164854766
107	+0.73848811162164856073
108	+0.73848811162164854687
109	+0.73848811162164851751
110	+0.73848811162164848112
111	+0.73848811162164844359
112	+0.73848811162164840865
113	+0.73848811162164837840
114	+0.73848811162164835374
115	+0.73848811162164833471
116	+0.73848811162164832086
117	+0.73848811162164831146
118	+0.73848811162164830565
119	+0.73848811162164830260
120	+0.73848811162164830154
121	+0.73848811162164830182
122	+0.73848811162164830293
123	+0.73848811162164830447
124	+0.73848811162164830614
125	+0.73848811162164830777
126	+0.73848811162164830923
127	+0.73848811162164831046
128	+0.73848811162164831145
129	+0.73848811162164831220
130	+0.73848811162164831273
131	+0.73848811162164831309
132	+0.73848811162164831331
133	+0.73848811162164831342
134	+0.73848811162164831345
135	+0.73848811162164831343
136	+0.73848811162164831338
137	+0.73848811162164831331
138	+0.73848811162164831323
139	+0.73848811162164831316
140	+0.73848811162164831310
141	+0.73848811162164831304
142	+0.73848811162164831300
143	+0.73848811162164831297
144	+0.73848811162164831294
145	+0.73848811162164831292
146	+0.73848811162164831291

Table 5: Continued.

k	b_k
147	+0.73848811162164831291
148	+0.73848811162164831291
149	+0.73848811162164831291
150	+0.73848811162164831291
151	+0.73848811162164831291
152	+0.73848811162164831292
153	+0.73848811162164831292
154	+0.73848811162164831292
155	+0.73848811162164831292
156	+0.73848811162164831293

differing only after the first 20 digits. Thus, applying (4) with $z_o = 0$, $a_o = 1$, the inverse for $E_{1/7,1}(-z)$ can be written as

$$-z = E_{1/7,1}^{-1}(w) = -\sum_{k=1}^{46} b_k (1-w)^k - \sum_{k=47}^{\infty} \frac{(1-w)^k}{\Gamma(6/7)}. \quad (18)$$

Equation (18) assumes that all coefficients b_k for $k > 46$ can be approximated by $1/\Gamma(6/7)$. The approximation is valid provided that an answer accurate to no more than 20 significant digits is sufficient. The last term in (18) is a geometric series which can be replaced by its corresponding sum yielding

$$-z = E_{1/7,1}^{-1}(w) = -\sum_{k=1}^{46} b_k (1-w)^k - \frac{1}{\Gamma(6/7)} \frac{(1-w)^{47}}{w}. \quad (19)$$

Equation (19) represents a finite series for the inverse Mittag-Leffler function for $w \le 1$ or equivalently $-z \le 0$ accurate to 20 significant digits. The series has been tested numerically and in all cases tested gives the correct answer to at least 20 significant digits $0 \ge -z < -\infty$ or equivalently $0 < w \le 1$. This finite series representation of the inverse Mittag-Leffler function has at least 3 advantages over the infinite series representation: (1) the finite series greatly expedites the evaluation of the inverse, (2) it is not limited to small $|-z|$, and (3) there is no ambiguity concerning the number of terms needed in the series to obtain a required accuracy in the final answer.

Note that if the required accuracy is only 10 significant digits, the first 10 digits of the coefficients b_k after b_{17} are identical differing only after the first 10 digits. In this case, the equation for the inverse can be written as

$$-z = E_{1/7,1}^{-1}(w) = -\sum_{k=1}^{17} b_k (1-w)^k - \frac{1}{\Gamma(6/7)} \frac{(1-w)^{18}}{w}. \quad (20)$$

The fact that the coefficients b_k approached a constant as k becomes large allowed the infinite series to be written as a finite series. For what other Mittag-Leffler functions do the coefficients in the inverse approach a constant?

Table 6: Coefficients b_k for the inverse Mittag-Leffler function $-z = E_{1/4,1}^{-1}(w)$.

k	b_k
1	0.90640247705547707798
2	0.84026894007589891391
3	0.82351018992990700207
4	0.81828957550795105766
5	0.81660707076917509278
6	0.81610221036029616113
7	0.81598545033147361884
8	0.81598379842062516042
9	0.81600622129819404882
10	0.81602620538545447950
11	0.81603889114688882765
12	0.81604558885224673170
13	0.81604856137343865455
14	0.81604957197906280172
15	0.81604970482268525625
16	0.81604953186530296411
17	0.81604931188945210690
18	0.81604913762546883873
19	0.81604902563306262589
20	0.81604896432490207114
21	0.81604893634591082941
22	0.81604892711529213868
23	0.81604892680747229621
24	0.81604892974411138067
25	0.81604893311589531898
26	0.81604893581695267902
27	0.81604893761527447364
28	0.81604893864655309011
29	0.81604893914547956993
30	0.81604893932600511441
31	0.81604893934331182207
32	0.81604893929407644732
33	0.81604893923044766688
34	0.81604893917568606393
35	0.81604893913673190214
36	0.81604893911270302848
37	0.81604893909990779088
38	0.81604893909439699044
39	0.81604893909300259361
40	0.81604893909356459299
41	0.81604893909479532456
42	0.81604893909603395768
43	0.81604893909701259830
44	0.81604893909767829618
45	0.81604893909807584283
46	0.81604893909827997471
47	0.81604893909836178143
48	0.81604893909837594019

Table 6: Continued.

k	b_k
49	0.81604893909835900689
50	0.81604893909833251654
51	0.81604893909830737526
52	0.81604893909828785893
53	0.81604893909827462118
54	0.81604893909826667548
55	0.81604893909826254932
56	0.81604893909826086189
57	0.81604893909826054632
58	0.81604893909826087970
59	0.81604893909826142714
60	0.81604893909826196124
61	0.81604893909826238717
62	0.81604893909826268505
63	0.81604893909826287085
64	0.81604893909826297283
65	0.81604893909826301912
66	0.81604893909826303243
67	0.81604893909826302882
68	0.81604893909826301846
69	0.81604893909826300704
70	0.81604893909826299725
71	0.81604893909826298997
72	0.81604893909826298513
73	0.81604893909826298224
74	0.81604893909826298074
75	0.81604893909826298012
76	0.81604893909826298002
77	0.81604893909826298015
78	0.81604893909826298038
79	0.81604893909826298060
80	0.81604893909826298078
81	0.81604893909826298092
82	0.81604893909826298100
83	0.81604893909826298105
84	0.81604893909826298108
85	0.81604893909826298109
86	0.81604893909826298109
87	0.81604893909826298109
88	0.81604893909826298109
89	0.81604893909826298108
90	0.81604893909826298108
91	0.81604893909826298108
92	0.81604893909826298107

4. Inverse Mittag-Leffler Functions for Which b_k Approach a Constant

Evaluation of great many inverse Mittag-Leffler functions reveals several important points. (1) It has been shown that the Mittag-Leffer function with these α and β parameters,

Table 7: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/5,1}(w)$.

k	b_k
1	+0.91816874239976061064
2	+0.87239815820597071525
3	+0.86241404655813210186
4	+0.85979515758722241525
5	+0.85910249498870304439
6	+0.85894043515591457893
7	+0.85891680448152655590
8	+0.85892230156211349359
9	+0.85892962600662844119
10	+0.85893406838573395241
11	+0.85893613453863527233
12	+0.85893690887578873062
13	+0.85893711768226131089
14	+0.85893712714123260150
15	+0.85893708951453251359
16	+0.85893705517782142799
17	+0.85893703407253609369
18	+0.85893702365426492211
19	+0.85893701950167094637
20	+0.85893701834496455469
21	+0.85893701834146563837
22	+0.85893701863535859269
23	+0.85893701890846377513
24	+0.85893701908567480745
25	+0.85893701917914447317
26	+0.85893701921959129502
27	+0.85893701923250482423
28	+0.85893701923367892385
29	+0.85893701923119867846
30	+0.85893701922846356042
31	+0.85893701922650554665
32	+0.85893701922537418952
33	+0.85893701922482812463
34	+0.85893701922461848499
35	+0.85893701922457079400
36	+0.85893701922458522240
37	+0.85893701922461381792
38	+0.85893701922463819190
39	+0.85893701922465413251
40	+0.85893701922466286636
41	+0.85893701922466687324
42	+0.85893701922466827905
43	+0.85893701922466848269
44	+0.85893701922466825732
45	+0.85893701922466796298
46	+0.85893701922466773028
47	+0.85893701922466758222
48	+0.85893701922466750209

Table 7: Continued.

k	b_k
49	+0.85893701922466746565
50	+0.85893701922466745311
51	+0.85893701922466745162
52	+0.85893701922466745405
53	+0.85893701922466745708
54	+0.85893701922466745948
55	+0.85893701922466746103
56	+0.85893701922466746188
57	+0.85893701922466746229
58	+0.85893701922466746243
59	+0.85893701922466746246
60	+0.85893701922466746244
61	+0.85893701922466746241
62	+0.85893701922466746238
63	+0.85893701922466746236
64	+0.85893701922466746235

namely, $0 < \alpha < 1$ and $\beta > \alpha$, is a completely monotonic decreasing function [24, 25], and thus the inverse is guaranteed to be single valued. (2) The coefficients b_k in the inverse approach a constant only when the parameter β is either 1 or 2. (3) The coefficients b_k approach a constant only when the parameter $\alpha < 1$. (4) The coefficients b_k approach a constant given by

$$\lim_{k\to\infty} b_k = \frac{1}{\Gamma(\beta-\alpha)}. \tag{21}$$

Consequently, as $\alpha \to 0$, the coefficient $b_k \to 1$ for both $\beta = 1$ and 2. However, for $\beta = 1$ the coefficient b_k is always less than 1 while for $\beta = 2$, b_k is always greater than 1 as $\alpha \to 0$. (5) The smaller the value of α, the fewer the numerical terms required in the inverse series to obtain a given significant digit accuracy. This is illustrated in Table 3 which gives the number of terms required in the finite representation of the inverse Mittag-Leffler function for 20-significant-digit accuracy for various values of α with $\beta = 1$ and $\beta = 2$.

Extending this logic to its natural conclusion implies that at $\alpha = 0$ no terms will be required in the series. To see that this is correct, note that using (1) both $w = E_{0,1}(-z)$ and $w = E_{0,2}(-z)$ reduce to $w = 1/(1+z)$ when $\alpha = 0$. Inverting and solving for $-z$ yield $-z = -(1-w)/w$. This is consistent with (19) which reduces to this same result when the upper limit on the summation is $k = 0$ (no terms in the summation) and the factor $1/\Gamma(6/7)$ is replaced by the more general equation (21) which gives unity for $\alpha = 0$ and $\beta = 1$ or $\beta = 2$.

Conversely, as α approaches 1, an increasingly larger number of numerical terms are required in the inverse series to obtain a given significant digit accuracy as Table 3 illustrates. (6) Consequently, as α increases above 1/2, the inverse Mittag-Leffler function described by a finite series requires more and more terms becoming less practical. For example, for $\alpha = 0.74$ and $\beta = 1$, for b_k to converge to just 5 significant digits requires 2215 terms while, for $\alpha = 0.825$ and $\beta = 2$, requiring 1828 terms for the same convergence.

Table 8: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/6,1}(w)$.

k	b_k
1	+0.92771933363003920070
2	+0.89414577241424278746
3	+0.88773763213642664587
4	+0.88628977973846244289
5	+0.88596635420228565868
6	+0.88590523526021224793
7	+0.88589982030554044972
8	+0.88590275825406298752
9	+0.88590511036557706322
10	+0.88590621213903655148
11	+0.88590660928393383455
12	+0.88590671636180511607
13	+0.88590672946995710075
14	+0.88590672153714995480
15	+0.88590671360108075777
16	+0.88590670912379822800
17	+0.88590670718459732010
18	+0.88590670653013358196
19	+0.88590670638937664049
20	+0.88590670640476657586
21	+0.88590670644420859302
22	+0.88590670647244801433
23	+0.88590670648712373933
24	+0.88590670649321013736
25	+0.88590670649512780068
26	+0.88590670649542543061
27	+0.88590670649526508181
28	+0.88590670649506835081
29	+0.88590670649493875951
30	+0.88590670649487287445
31	+0.88590670649484578753
32	+0.88590670649483743413
33	+0.88590670649483637544
34	+0.88590670649483735333
35	+0.88590670649483843570
36	+0.88590670649483914752
37	+0.88590670649483951597
38	+0.88590670649483967198
39	+0.88590670649483972248
40	+0.88590670649483973021
41	+0.88590670649483972509
42	+0.88590670649483971872
43	+0.88590670649483971430
44	+0.88590670649483971188
45	+0.88590670649483971080
46	+0.88590670649483971040
47	+0.88590670649483971032
48	+0.88590670649483971034

Table 8: Continued.

k	b_k
49	+0.88590670649483971037
50	+0.88590670649483971040
51	+0.88590670649483971042
52	+0.88590670649483971043

Table 9: Coefficients b_k for the inverse Mittag-Leffler function $-z = E_{1/8,1}^{-1}(w)$.

k	b_k
1	+0.94174269984970148808
2	+0.92145833616345434435
3	+0.91837205036733785079
4	+0.91782768940815950206
5	+0.91773538138444524775
6	+0.91772307499596115692
7	+0.91772281742094063440
8	+0.91772345536444575339
9	+0.91772376557618704885
10	+0.91772386504763452141
11	+0.91772388849585357984
12	+0.91772389156330350231
13	+0.91772389092194344445
14	+0.91772389025312412600
15	+0.91772388994408363001
16	+0.91772388984022900150
17	+0.91772388981468002747
18	+0.91772388981165650624
19	+0.91772388981289452457
20	+0.91772388981401655984
21	+0.91772388981455209678
22	+0.91772388981474237399
23	+0.91772388981479260777
24	+0.91772388981479946414
25	+0.91772388981479716385
26	+0.91772388981479476203
27	+0.91772388981479352017
28	+0.91772388981479304158
29	+0.91772388981479290058
30	+0.91772388981479287486
31	+0.91772388981479287784
32	+0.91772388981479288351
33	+0.91772388981479288690
34	+0.91772388981479288835
35	+0.91772388981479288884
36	+0.91772388981479288896
37	+0.91772388981479288897
38	+0.91772388981479288896
39	+0.91772388981479288895
40	+0.91772388981479288894

Table 10: Coefficients b_k for the inverse Mittag-Leffler function $-z = E_{1/9,1}^{-1}(w)$.

k	b_k
1	0.94696534880216399450
2	0.93053890407728875727
3	0.92827170167793346183
4	0.92791158668373485876
5	0.92785715845937955393
6	0.92785088338236675169
7	0.92785088003507065240
8	0.92785119095520099570
9	0.92785131874466606751
10	0.92785135396452045173
11	0.92785136086263488623
12	0.92785136142419633245
13	0.92785136114025016525
14	0.92785136095166820529
15	0.92785136088086115418
16	0.92785136086138201458
17	0.92785136085777479380
18	0.92785136085772523465
19	0.92785136085805437723
20	0.92785136085823967523
21	0.92785136085830846162
22	0.92785136085832754930
23	0.92785136085833100710
24	0.92785136085833091281
25	0.92785136085833047865
26	0.92785136085833023736
27	0.92785136085833014532
28	0.92785136085833011870
29	0.92785136085833011354
30	0.92785136085833011358
31	0.92785136085833011420
32	0.92785136085833011458
33	0.92785136085833011473
34	0.92785136085833011478
35	0.92785136085833011479

(7) For the same α, the number of terms in the inverse for a desired accuracy is less for $\beta = 2$ than for $\beta = 1$. (8) According to (21), when $\alpha = 1$ and $\beta = 1$, the coefficients b_k in the inverse for the Mittag-Leffler function $E_{1,1}(-z)$ approach the constant zero as $k \to \infty$ as seen in (16) while for $\alpha = 1$ and $\beta = 2$ the coefficients b_k in the inverse for the Mittag-Leffler function $E_{1,2}(-z)$ approach 1 as $k \to \infty$. (9) As noted above, according to (21), for $\beta = 2$ the coefficients b_k as $k \to \infty$ approach 1 as $\alpha \to 0$ and as $\alpha \to 1$ and b_k is greater than 1 for $0 < \alpha < 1$. This implies that there exists a relative maximum value of b_k as $k \to \infty$ in the range $0 < \alpha < 1$. This maximum occurs at $\alpha = 0.5383678550\ldots$ and corresponds to $b_k = 1.129173885\ldots$ as $k \to \infty$. Illustrating the above observations are numerous examples in the next section.

Table 11: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/10,1}(w)$.

k	b_k
1	+0.95135076986687318362
2	+0.93777687277778653379
3	+0.93606310059788083658
4	+0.93581557876046714164
5	+0.93578185282959558380
6	+0.93577844037062189666
7	+0.93577848991569929215
8	+0.93577864855859718780
9	+0.93577870513836571828
10	+0.93577871876872861069
11	+0.93577872101504047108
12	+0.93577872110471635717
13	+0.93577872099247192203
14	+0.93577872093543535964
15	+0.93577872091738487402
16	+0.93577872091324677459
17	+0.93577872091268732351
18	+0.93577872091275010310
19	+0.93577872091282731969
20	+0.93577872091286047983
21	+0.93577872091287052264
22	+0.93577872091287272009
23	+0.93577872091287295145
24	+0.93577872091287287167
25	+0.93577872091287280852
26	+0.93577872091287278263
27	+0.93577872091287277485
28	+0.93577872091287277316
29	+0.93577872091287277300
30	+0.93577872091287277308
31	+0.93577872091287277314
32	+0.93577872091287277317

Table 12: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/3,2}(w)$.

k	b_k
1	1.1906393487589989482
2	1.1218291259372159490
3	1.1091651079345480360
4	1.1070518842541741977
5	1.1071094825114241570
6	1.1074303857224016589
7	1.1076404430424683811
8	1.1077314700191564271
9	1.1077553955944287187
10	1.1077524578488089226
11	1.1077434947715842855
12	1.1077363499227504156
13	1.1077324896384468086
14	1.1077310965875528477
15	1.1077310087501244588
16	1.1077313864890338427
17	1.1077317908641062832
18	1.1077320654748424695
19	1.1077322009818539445
20	1.1077322408861236149
21	1.1077322323951713722
22	1.1077322084738571023
23	1.1077321862732534401
24	1.1077321716839450156
25	1.1077321645662504099
26	1.1077321625611127907
27	1.1077321631931161511
28	1.1077321646957392674
29	1.1077321661093029166
30	1.1077321670760136784
31	1.1077321675814921360
32	1.1077321677525601222
33	1.1077321677362930981
34	1.1077321676460497809
35	1.1077321675494545998
36	1.1077321674764692276
37	1.1077321674332138537
38	1.1077321674142290519
39	1.1077321674107002345
40	1.1077321674147729438
41	1.1077321674210887571
42	1.1077321674267605861
43	1.1077321674307081625
44	1.1077321674329036944
45	1.1077321674337785530
46	1.1077321674338524401
47	1.1077321674335523501
48	1.1077321674331583587

5. Results for Specific α and β

In this section, specific examples of various inverse Mittag-Leffler functions calculated using (10) will be given. Since the number of terms in the finite series for the inverse increases dramatically for $\alpha \geq 1/2$, then all examples will be for $\alpha < 1/2$. All equations for the inverses are written assuming a desired 20-significant-digit accuracy. This is far greater accuracy than most requirements might call for; however, the equations can then be easily modified to any degree of accuracy less than 20 as outlined in the discussion of (20). Each Mittag-Leffler inverse $-z = E^{-1}_{\alpha,\beta}(w)$ example includes the equation of the form given in (19) valid for $0 \geq -z < -\infty$ (equivalently $0 < w \leq 1$) representing the finite series representation of the inverse and a table with the corresponding coefficients b_k truncated to 20 significant digits. The specific values of α and β in each example are itemized in Table 4 which includes references to the

Table 12: Continued.

k	b_k
49	1.1077321674328171346
50	1.1077321674325809580
51	1.1077321674324479205
52	1.1077321674323926738
53	1.1077321674323855472
54	1.1077321674324019711
55	1.1077321674324254123
56	1.1077321674324467750
57	1.1077321674324623978
58	1.1077321674324719027
59	1.1077321674324764817
60	1.1077321674324777720
61	1.1077321674324772490
62	1.1077321674324759894
63	1.1077321674324746516
64	1.1077321674324735591
65	1.1077321674324728113
66	1.1077321674324723822
67	1.1077321674324721933
68	1.1077321674324721578
69	1.1077321674324722031
70	1.1077321674324722781
71	1.1077321674324723525
72	1.1077321674324724118
73	1.1077321674324724522
74	1.1077321674324724754
75	1.1077321674324724857
76	1.1077321674324724876
77	1.1077321674324724851
78	1.1077321674324724809
79	1.1077321674324724766
80	1.1077321674324724731
81	1.1077321674324724707
82	1.1077321674324724692
83	1.1077321674324724685
84	1.1077321674324724683
85	1.1077321674324724684
86	1.1077321674324724686
87	1.1077321674324724688
88	1.1077321674324724691
89	1.1077321674324724692
90	1.1077321674324724693
91	1.1077321674324724694

corresponding equations and table numbers for each example inverse.

For $\alpha = 1/3$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E_{1/3,1}^{-1}(w) = -\sum_{k=1}^{156} b_k (1-w)^k - \frac{1}{\Gamma(2/3)} \frac{(1-w)^{157}}{w}, \quad (22)$$

Table 13: Coefficients b_k for the inverse Mittag-Leffler function $-z = E_{1/4,2}^{-1}(w)$.

k	b_k
1	1.1330030963193463474
2	1.0941001823904933774
3	1.0884969259036715641
4	1.0878558386282093564
5	1.0879299260072536857
6	1.0880196901607404297
7	1.0880580603477134188
8	1.0880679256317365431
9	1.0880681582727259222
10	1.0880666922539625425
11	1.0880656824079111948
12	1.0880652661105853859
13	1.0880651718209129462
14	1.0880651893915501135
15	1.0880652225919339689
16	1.0880652438081294238
17	1.0880652525894990987
18	1.0880652545116835036
19	1.0880652539426656281
20	1.0880652530160160191
21	1.0880652524013356965
22	1.0880652521279001619
23	1.0880652520573054501
24	1.0880652520688854981
25	1.0880652520973313877
26	1.0880652521186658457
27	1.0880652521294678814
28	1.0880652521330822285
29	1.0880652521332880539
30	1.0880652521324663677
31	1.0880652521316757550
32	1.0880652521311980579
33	1.0880652521309915951
34	1.0880652521309409874
35	1.0880652521309543925
36	1.0880652521309814562
37	1.0880652521310025596
38	1.0880652521310142006
39	1.0880652521310188181
40	1.0880652521310196801
41	1.0880652521310191068
42	1.0880652521310183025
43	1.0880652521310177100
44	1.0880652521310173883
45	1.0880652521310172608
46	1.0880652521310172369
47	1.0880652521310172534
48	1.0880652521310172768

Table 13: Continued.

k	b_k
49	1.0880652521310172946
50	1.0880652521310173047
51	1.0880652521310173090
52	1.0880652521310173101
53	1.0880652521310173098
54	1.0880652521310173091
55	1.0880652521310173086
56	1.0880652521310173082
57	1.0880652521310173081
58	1.0880652521310173080
59	1.0880652521310173080
60	1.0880652521310173080
61	1.0880652521310173080
62	1.0880652521310173080
63	1.0880652521310173081

where $1/\Gamma(2/3) = 0.73848811162164831293\ldots$ and the coefficients b_k are given in Table 5.

For $\alpha = 1/4$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/4,1}(w) = -\sum_{k=1}^{92} b_k (1-w)^k - \frac{1}{\Gamma(3/4)} \frac{(1-w)^{93}}{w}, \tag{23}$$

where $1/\Gamma(3/4) = 0.81604893909826298107\ldots$ and the coefficients b_k are given in Table 6.

For $\alpha = 1/5$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/5,1}(w) = -\sum_{k=1}^{64} b_k (1-w)^k - \frac{1}{\Gamma(4/5)} \frac{(1-w)^{65}}{w}, \tag{24}$$

where $1/\Gamma(4/5) = 0.85893701922466746235\ldots$ and the coefficients b_k are given in Table 7.

For $\alpha = 1/6$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/6,1}(w) = -\sum_{k=1}^{52} b_k (1-w)^k - \frac{1}{\Gamma(5/6)} \frac{(1-w)^{53}}{w}, \tag{25}$$

where $1/\Gamma(5/6) = 0.88590670649483971043\ldots$ and the coefficients b_k are given in Table 8.

For $\alpha = 1/8$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/8,1}(w) = -\sum_{k=1}^{40} b_k (1-w)^k - \frac{1}{\Gamma(7/8)} \frac{(1-w)^{41}}{w}, \tag{26}$$

where $1/\Gamma(7/8) = 0.91772388981479288894\ldots$ and the coefficients b_k are given in Table 9.

Table 14: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/5,2}(w)$.

k	b_k
1	1.1018024908797127327
2	1.0767885838427981399
3	1.0738409976565079202
4	1.0735918587371880661
5	1.0736317220842957450
6	1.0736613247947013790
7	1.0736705276938335590
8	1.0736719763838959650
9	1.0736717194169915685
10	1.0736714224350730386
11	1.0736712961819915731
12	1.0736712656573585536
13	1.0736712658337183805
14	1.0736712703837243345
15	1.0736712730893571885
16	1.0736712740367576320
17	1.0736712741927198324
18	1.0736712741349301121
19	1.0736712740710214284
20	1.0736712740390593289
21	1.0736712740290815387
22	1.0736712740281108537
23	1.0736712740292769055
24	1.0736712740302628755
25	1.0736712740307308734
26	1.0736712740308721433
27	1.0736712740308820173
28	1.0736712740308614201
29	1.0736712740308445464
30	1.0736712740308363495
31	1.0736712740308337294
32	1.0736712740308334598
33	1.0736712740308337982
34	1.0736712740308341087
35	1.0736712740308342730
36	1.0736712740308343326
37	1.0736712740308343434
38	1.0736712740308343390
39	1.0736712740308343332
40	1.0736712740308343296
41	1.0736712740308343281
42	1.0736712740308343277
43	1.0736712740308343277
44	1.0736712740308343278
45	1.0736712740308343278
46	1.0736712740308343279

Table 15: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/6,2}(w)$.

k	b_k
1	1.0823392225683790674
2	1.0649027775627960975
3	1.0631672235921470234
4	1.0630530019556810014
5	1.0630738792005841325
6	1.0630852477868224812
7	1.0630879790229538911
8	1.0630882395356188174
9	1.0630881350924325915
10	1.0630880687069684272
11	1.0630880487821691319
12	1.0630880460752193502
13	1.0630880468406782122
14	1.0630880475089981383
15	1.0630880477592756779
16	1.0630880478102509927
17	1.0630880478069487489
18	1.0630880477988781918
19	1.0630880477949003960
20	1.0630880477937454972
21	1.0630880477936234680
22	1.0630880477937097855
23	1.0630880477937763019
24	1.0630880477938030729
25	1.0630880477938094515
26	1.0630880477938094111
27	1.0630880477938084584
28	1.0630880477938078846
29	1.0630880477938076743
30	1.0630880477938076305
31	1.0630880477938076352
32	1.0630880477938076449
33	1.0630880477938076504
34	1.0630880477938076523
35	1.0630880477938076527
36	1.0630880477938076527
37	1.0630880477938076526
38	1.0630880477938076525

For $\alpha = 1/9$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/9,1}(w) = -\sum_{k=1}^{35} b_k (1-w)^k - \frac{1}{\Gamma(8/9)} \frac{(1-w)^{36}}{w}, \tag{27}$$

where $1/\Gamma(8/9) = 0.92785136085833011479\ldots$ and the coefficients b_k are given in Table 10.

Table 16: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/7,2}(w)$.

k	b_k
1	1.0690715004486243979
2	1.0562222079392582146
3	1.0551161172778792415
4	1.0550572191430135037
5	1.0550686438367760540
6	1.0550735765579806638
7	1.0550745305844060375
8	1.0550745824286880798
9	1.0550745442484434695
10	1.0550745270114378657
11	1.0550745231398621545
12	1.0550745229039529066
13	1.0550745231150647888
14	1.0550745232236878646
15	1.0550745232523526379
16	1.0550745232552728533
17	1.0550745232539308640
18	1.0550745232530327777
19	1.0550745232527427584
20	1.0550745232526945061
21	1.0550745232527008669
22	1.0550745232527090401
23	1.0550745232527124342
24	1.0550745232527132611
25	1.0550745232527133099
26	1.0550745232527132427
27	1.0550745232527132021
28	1.0550745232527131884
29	1.0550745232527131859
30	1.0550745232527131861
31	1.0550745232527131865
32	1.0550745232527131867
33	1.0550745232527131868

For $\alpha = 1/10$ and $\beta = 1$, the equation for the inverse is given by

$$-z = E^{-1}_{1/10,1}(w) = -\sum_{k=1}^{32} b_k (1-w)^k - \frac{1}{\Gamma(9/10)} \frac{(1-w)^{33}}{w}, \tag{28}$$

where $1/\Gamma(9/10) = 0.93577872091287277317\ldots$ and the coefficients b_k are given in Table 11.

For $\alpha = 1/3$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/3,2}(w) = -\sum_{k=1}^{91} b_k (1-w)^k - \frac{1}{\Gamma(5/3)} \frac{(1-w)^{92}}{w}, \tag{29}$$

where $1/\Gamma(5/3) = 1.1077321674324724694\ldots$ and the coefficients b_k are given in Table 12.

Table 17: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/8,2}(w)$.

k	b_k
1	1.0594605373309141740
2	1.0495986360361847141
3	1.0488511439374145246
4	1.0488179924913977120
5	1.0488245771415334177
6	1.0488269350920946859
7	1.0488273133611761220
8	1.0488273231954621562
9	1.0488273085970012075
10	1.0488273034622550713
11	1.0488273025690472404
12	1.0488273025660372711
13	1.0488273026212509980
14	1.0488273026418176097
15	1.0488273026458036695
16	1.0488273026458799051
17	1.0488273026456156560
18	1.0488273026455029277
19	1.0488273026454771037
20	1.0488273026454752946
21	1.0488273026454766352
22	1.0488273026454773654
23	1.0488273026454775715
24	1.0488273026454775998
25	1.0488273026454775944
26	1.0488273026454775895
27	1.0488273026454775877
28	1.0488273026454775873

Table 18: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/9,2}(w)$.

k	b_k
1	1.0521837208912933272
2	1.0443758743852848005
3	1.0438473880668843474
4	1.0438274217379570018
5	1.0438314036520292794
6	1.0438326220325679883
7	1.0438327879063812056
8	1.0438327888679432382
9	1.0438327828921939645
10	1.0438327811719258151
11	1.0438327809347620508
12	1.0438327809451382984
13	1.0438327809605757500
14	1.0438327809650783542
15	1.0438327809657296823
16	1.0438327809656925712
17	1.0438327809656403057
18	1.0438327809656237760
19	1.0438327809656210337
20	1.0438327809656210870
21	1.0438327809656212900
22	1.0438327809656213646
23	1.0438327809656213797
24	1.0438327809656213804
25	1.0438327809656213796
26	1.0438327809656213792
27	1.0438327809656213791

For $\alpha = 1/4$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/4,2}(w) = -\sum_{k=1}^{63} b_k (1-w)^k - \frac{1}{\Gamma(7/4)} \frac{(1-w)^{64}}{w}, \quad (30)$$

where $1/\Gamma(7/4) = 1.0880652521310173081\ldots$ and the coefficients b_k are given in Table 13.

For $\alpha = 1/5$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/5,2}(w) = -\sum_{k=1}^{46} b_k (1-w)^k - \frac{1}{\Gamma(9/5)} \frac{(1-w)^{47}}{w}, \quad (31)$$

where $1/\Gamma(9/5) = 1.0736712740308343279\ldots$ and the coefficients b_k are given in Table 14.

For $\alpha = 1/6$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/6,2}(w) = -\sum_{k=1}^{38} b_k (1-w)^k - \frac{1}{\Gamma(11/6)} \frac{(1-w)^{39}}{w}, \quad (32)$$

where $1/\Gamma(11/6) = 1.0630880477938076525\ldots$ and the coefficients b_k are given in Table 15.

For $\alpha = 1/7$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/7,2}(w) = -\sum_{k=1}^{33} b_k (1-w)^k - \frac{1}{\Gamma(13/7)} \frac{(1-w)^{34}}{w}, \quad (33)$$

where $1/\Gamma(13/7) = 1.0550745232527131868\ldots$ and the coefficients b_k are given in Table 16.

For $\alpha = 1/8$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/8,2}(w) = -\sum_{k=1}^{28} b_k (1-w)^k - \frac{1}{\Gamma(15/8)} \frac{(1-w)^{29}}{w}, \quad (34)$$

where $1/\Gamma(15/8) = 1.0488273026454775873\ldots$ and the coefficients b_k are given in Table 17.

TABLE 19: Coefficients b_k for the inverse Mittag-Leffler function $-z = E^{-1}_{1/10,2}(w)$.

k	b_k
1	1.0464858468535605019
2	1.0401508480560282303
3	1.0397635665515442126
4	1.0397508768185132109
5	1.0397533885835924940
6	1.0397540594403559415
7	1.0397541383662199550
8	1.0397541376605313080
9	1.0397541350429188308
10	1.0397541344064541458
11	1.0397541343357544813
12	1.0397541343417232130
13	1.0397541343464255068
14	1.0397541343475460327
15	1.0397541343476671587
16	1.0397541343476507388
17	1.0397541343476394778
18	1.0397541343476366678
19	1.0397541343476363331
20	1.0397541343476363726
21	1.0397541343476364046
22	1.0397541343476364135
23	1.0397541343476364148
24	1.0397541343476364147
25	1.0397541343476364146

For $\alpha = 1/9$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/9,2}(w) = -\sum_{k=1}^{27} b_k (1-w)^k - \frac{1}{\Gamma(17/9)} \frac{(1-w)^{28}}{w}, \tag{35}$$

where $1/\Gamma(17/9) = 1.0438327809656213791\ldots$ and the coefficients b_k are given in Table 18.

For $\alpha = 1/10$ and $\beta = 2$, the equation for the inverse is given by

$$-z = E^{-1}_{1/10,2}(w) = -\sum_{k=1}^{25} b_k (1-w)^k - \frac{1}{\Gamma(19/10)} \frac{(1-w)^{26}}{w}, \tag{36}$$

where $1/\Gamma(19/10) = 1.0397541343476364146\ldots$ and the coefficients b_k are given in Table 19.

6. Summary

A finite series representation of the inverse Mittag-Leffler function has been found for a range of the parameters α and β; specifically $0 < \alpha < 1/2$ for $\beta = 1$ and for $\beta = 2$. Various properties of the coefficients b_k in the finite series have been examined. In addition, a formula for b_k as $k \to \infty$ is established and the limiting cases were investigated. These properties are illustrated in 16 examples of inverse Mittag-Leffler functions. Determining the value of the argument of a Mittag-Leffler function given the value of the function is not an easy problem and the finite series representation of the inverse Mittag-Leffler function greatly expedites their evaluation and represents a significant advancement.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] A. Erdélyi, Ed., *Higher Transcendental Functions*, vol. 3, McGraw-Hill, New York, NY, USA, 1955.

[2] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations*, vol. 2004 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 2010.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[6] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, UK, 2010.

[7] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, Redding, Conn, USA, 2006.

[8] F. Mainardi and R. Gorenflo, "On Mittag-Leffler-type functions in fractional evolution processes," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 283–299, 2000.

[9] R. L. Magin, "Fractional calculus in bioengineering," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 1, pp. 1–104, 2004.

[10] M. N. Berberan-Santos, "Properties of the Mittag-Leffler relaxation function," *Journal of Mathematical Chemistry*, vol. 38, no. 4, pp. 629–635, 2005.

[11] I. S. Gupta and L. Debnath, "Some properties of the Mittag-Leffler functions," *Integral Transforms and Special Functions*, vol. 18, no. 5-6, pp. 329–336, 2007.

[12] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[13] R. Hilfer and H. J. Seybold, "Computation of the generalized Mittag-Leffler function and its inverse in the complex plane," *Integral Transforms and Special Functions*, vol. 17, no. 9, pp. 637–652, 2006.

[14] K. Knopp, *Theory and Application of Infinite Series*, Dover, New York, NY, USA, 1990.

[15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, NY, USA, 1965.

[16] H. B. Dwight, *Tables of Integrals and Other mathematical Data*, The Macmillan Company, New York, NY, USA, 4th edition, 1961.

[17] W. H. Beyer, *CRC Standard Mathematical Tables*, CRC Press, Boca Raton, Fla, USA, 27th edition, 1984.

[18] P. Morse and H. Feshbach, *Methods of Theoretical Physics*, vol. 1, McGraw-Hill Book, New York, NY, USA, 1953.

[19] J. McMahon, "On the general term in the reversion of series," *Bulletin of the American Mathematical Society*, vol. 3, no. 7, pp. 170–172, 1894.

[20] E. P. Adams and R. L. Hippisley, *Smithsonian Mathematical Formulae and Tables of Elliptical Functions*, Smithsonian miscellaneous Collections 74(1), Publication 2672, Smithsonian Institution, Washington, DC, USA, 1939.

[21] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, NY, USA, 1968.

[22] C. E. Van Orstrand, "Reversion of power series," *Philosophical Magazine*, vol. 19, series 6, pp. 366–376, 1910.

[23] J. W. Müller, "Some observations on the reversion of series," Rapport BIPM-85/1, Bureau International des Poids et Mesures, Sèvres, France, 1985.

[24] K. S. Miller and S. G. Samko, "A note on the complete monotonicity of the generalized Mittag-Leffler function," *Real Analysis Exchange*, vol. 23, no. 2, pp. 753–755, 1997.

[25] K. S. Miller and S. G. Samko, "Completely monotonic functions," *Integral Transforms and Special Functions*, vol. 12, no. 4, pp. 389–402, 2001.