

Research Article

Modified Function Projective Synchronization for a Partially Linear and Fractional-Order Financial Chaotic System with Uncertain Parameters

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Received 1 June 2017; Revised 1 August 2017; Accepted 16 August 2017; Published 19 September 2017

Academic Editor: Jorge E. Macías-Díaz

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This paper investigates the modified function projective synchronization between fractional-order chaotic systems, which are partially linear financial systems with uncertain parameters. Based on the stability theory of fractional-order systems and the Lyapunov matrix equation, a controller is obtained for the synchronization between fractional-order financial chaotic systems. Using the controller, the error systems converged to zero as time tends to infinity, and the uncertain parameters were also estimated so that the phenomenon of parameter distortion was effectively avoided. Numerical simulations demonstrate the validity and feasibility of the proposed method.

1. Introduction

In recent years, study on chaos is one of the most interesting research topics in real and physical systems [1–4]. In 1985, chaotic behavior was discovered in financial systems [5]. In fact, chaotic phenomena often appear in financial systems [6], such as the supernormal shock of the international oil prices, stock market crash, and financial crisis. Owing to its randomness and unpredictability, these chaotic phenomena usually lead to financial crisis and instability in the financial system. Since financial risks come from uncertainties, it is important to introduce unknown parameters in financial systems [7, 8]. Therefore, chaotic behaviors and uncertainties in financial systems must be taken into account [9]. It is necessary to investigate chaos control strategies for financial systems to address financial crisis and other related problems. Hence, it is important to achieve a synchronized and healthy development of financial markets.

Since Pecora and Carroll [10] proposed synchronization between two chaotic systems in 1990, chaotic synchronization has been extensively and intensively studied in various fields [11, 12]. Different types of chaotic synchronization have

been reported, including generalized synchronization [13], lag synchronization [14], function projective synchronization [15], modified projective synchronization [16], and modified function projective synchronization (MFPS) [17, 18]. The modified function projective synchronization (MFPS) means that the drive and response systems could be synchronized up to a scaling function matrix, but not a constant matrix. Recent studies on complicated financial systems have shown satisfactory results using the nonlinear method [19]. Chaos synchronization in fractional-order financial systems has also been studied in recent years [20–22].

The synchronization of financial systems indicates that the financial systems in two different areas are to maintain synchronized development by the appropriate control conditions. That is, the drive system and the response system can be interpreted as a virtual economic target and a controlled objective, respectively. Hence, the goal of synchronization is to control a specific objective to a virtual economic target using the proposed controller. The dynamical behaviors of a financial system are more complex. Therefore, financial systems are not always completely synchronized, and some complex synchronization methods [20–22], such as function

projective synchronization and MFPS, should be considered. Meanwhile, because of the complexity of the financial environment, this study highlights its theoretical value and ensures the comprehensiveness of the theory.

Also, in practice, the parameter distortion phenomenon frequently appears in fractional-order chaotic systems. In other words, parameters may be uncertain or drift with time in the chaotic synchronization between the response and drive systems because of various kinds of interferences. Many projective synchronization methods of chaotic systems with unknown parameters have been proposed in recent years [23]. Hence, achieving synchronization and identifying parameters are very important in financial chaotic systems.

In this study, we investigated fractional-order financial systems proposed by Chen [24]. A new controller is proposed to achieve synchronization. The controller makes the drive and response systems remain asymptotically stable. Based on the stability theory of fractional-order systems and the Lyapunov matrix equation, the MFPS was realized for partially linear and fractional-order (PLFO) financial chaotic systems, and the unknown parameters were also estimated so that the phenomenon of parameter distortion was avoided. Numerical simulation results showed that the proposed method effectively eliminated chaos and stabilized two financial systems.

The rest of the paper is organized as follows. In Section 2, the definition of MFPS, the PLFO chaotic system, and a useful lemma are presented. In Section 3, the scheme of MFPS is presented for PLFO chaotic systems with uncertain parameters. Numerical simulations are presented in Section 4. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

The definitions of PLFO chaotic system and coupled PLFO chaotic system were given in [25]. We will give the definition of PLFO chaotic system with uncertain parameters and coupled PLFO chaotic system with uncertain parameters as follows.

Definition 1. The fractional-order chaotic systems with uncertain parameters are defined as follows:

$$\begin{aligned} \frac{d^q u}{dt^q} &= M(z, \theta) \cdot u, \\ \frac{d^q z}{dt^q} &= f(u, z). \end{aligned} \quad (1)$$

It is called an uncertain PLFO chaotic system, where the state vector is $u = (u_1, u_2, \dots, u_n)^T \in R^n$, $z \in R$ is a variable, and $f : R \times R^n \rightarrow R$ is a differentiable function. Its order is subject to $0 < q \leq 1$. $\theta \in R^n$ represents the vector of uncertain parameters. The coefficient matrix $M(z, \theta) \in R^{n \times n}$ is dependent on the variable z and the uncertain parameter θ .

System (1) can be written as

$$\begin{aligned} \frac{d^q u_1}{dt^q} &= m_{11}(z, \theta) u_1 + m_{12}(z, \theta) u_2 + \dots \\ &\quad + m_{1n}(z, \theta) u_n, \\ \frac{d^q u_2}{dt^q} &= m_{21}(z, \theta) u_1 + m_{22}(z, \theta) u_2 + \dots \\ &\quad + m_{2n}(z, \theta) u_n, \\ &\quad \vdots \\ \frac{d^q u_n}{dt^q} &= m_{n1}(z, \theta) u_1 + m_{n2}(z, \theta) u_2 + \dots \\ &\quad + m_{nn}(z, \theta) u_n, \\ \frac{d^q z}{dt^q} &= f(u, z), \end{aligned} \quad (2)$$

where $m_{ij}(z, \theta)$ is a coefficient of u_j in the i th differential equation, $i, j = 1, 2, \dots, n$. Hence, $M(z, \theta)$ can be described by

$$\begin{aligned} M(z, \theta) &= \begin{pmatrix} m_{11}(z, \theta) & m_{12}(z, \theta) & \dots & m_{1n}(z, \theta) \\ m_{21}(z, \theta) & m_{22}(z, \theta) & \dots & m_{2n}(z, \theta) \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}(z, \theta) & m_{n2}(z, \theta) & \dots & m_{nn}(z, \theta) \end{pmatrix}. \end{aligned} \quad (3)$$

Obviously, system (1) is an uncertain partially linear and integer-order chaotic system with $q = 1$.

Definition 2. Consider two uncertain PLFO chaotic systems (1), which can be described as

$$\begin{aligned} \frac{d^q u_r}{dt^q} &= M(z, \theta) \cdot u_r, \\ \frac{d^q z}{dt^q} &= f(u_r, z), \\ \frac{d^q u_s}{dt^q} &= \bar{M}(z, \bar{\theta}) \cdot u_s + \psi. \end{aligned} \quad (4)$$

The system is coupled by the variable z . The subscripts r and s in system (4) represent the drive and response systems, respectively. $\psi \in R^n$ is a controller. $\bar{\theta}$ is the estimated value of unknown parameters. The coefficient matrix $\bar{M}(z, \bar{\theta}) \in R^{n \times n}$ is dependent on z and $\bar{\theta}$. System (4) is the coupled PLFO chaotic system with uncertain parameters [26].

The synchronization error vector of the MFPS is defined as

$$e(t) = u_s - h(t) \cdot u_r, \quad (5)$$

where $h(t) = \text{diag}\{h_1(t), h_2(t), \dots, h_n(t)\}$, $h_i(t)$ are continuously differentiable functions, and $h(t)$ is a scaling function matrix. $e(t) = (e_1(t), e_2(t), \dots, e_n(t))^T \in R^n$ is an error state vector.

Definition 3. System (4) is said to be MFPS with respect to the scaling function matrix $h(t)$ if there exists a vector controller ψ such that

$$\lim_{t \rightarrow \infty} \|e(t)\| = \lim_{t \rightarrow \infty} \|u_s - h(t) \cdot u_r\| = 0, \quad (6)$$

where $\|\cdot\|$ is the Euclidean norm. It implies that the MFPS between the drive and response system (4) can be achieved or the error dynamic systems are globally asymptotically stable.

Remark 4. If $h_1(t) = h_2(t) = \dots = h_n(t)$, the MFPS converts to function projective synchronization. If $h_1(t), h_2(t), \dots, h_n(t)$ represent identical or different constants, the MFPS is simplified to modified projective synchronization (such as antisynchronization and complete synchronization). If $h_1(t) = h_2(t) = \dots = h_n(t) = 0$, the synchronization problem is transformed into a control problem.

Lemma 5 (see [27]). *Presume that the fractional-order autonomous system is*

$$\frac{d^q e}{dt^q} = A(e) \cdot e, \quad e(0) = e_0, \quad (7)$$

where $q_i \in (0, 1]$ ($i = 1, 2, \dots, n$) are the orders of the fractional derivative and $q = (q_1, q_2, \dots, q_n)^T$, $A(e) \in R^{n \times n}$ is a coefficient matrix which is a polynomial matrix and depends on the state vector e , and $e = (e_1, e_2, \dots, e_n)^T \in R^n$ is an n -dimension state vector. System (7) is asymptotically stable if and only if

$$|\arg(\lambda)| \geq \frac{\pi\alpha}{2}, \quad \alpha = \max(q_1, q_2, \dots, q_n), \quad (8)$$

where λ is an arbitrary eigenvalue of $A(e)$. In this case, each state vector decays toward 0, such as $t^{-\alpha}$. Furthermore, the system is stable if and only if $|\arg(\lambda)| \geq \pi\alpha/2$ and the critical eigenvalues that satisfy $|\arg(\lambda)| = \pi\alpha/2$ have geometric multiplicity one.

Then, the MFPS between the drive and response system (4) is transformed into the analysis of the asymptotical stability of zero solution of the error system (7).

Theorem 6 (see [28]). *Given the autonomous system (7), there exist a real symmetric positive definite matrix P and a positive definite matrix Q such that the MFPS between the drive and response system (4) can be achieved if $PA(e) + (A(e))^H P = -Q$, where H is the conjugate transpose of a matrix and $PA(e) + (A(e))^H P = -Q$ is called a continuous Lyapunov matrix equation.*

Proof. Assume that λ is one of the eigenvalues of the polynomial matrix $A(e)$ and the corresponding nonzero eigenvector is β ; that is,

$$A(e) \beta = \lambda \beta. \quad (9)$$

For (9), the Hermitian transpose is

$$\overline{(A(e) \beta)^T} = \bar{\lambda} \beta^H. \quad (10)$$

Multiplying the left side of (9) by $\beta^H P$, we can obtain

$$\beta^H PA(e) \beta = \lambda \beta^H P \beta. \quad (11)$$

Then, multiplying the right side of (10) by $P \beta$, we can also obtain

$$\beta^H (A(e))^H P \beta = \bar{\lambda} \beta^H P \beta. \quad (12)$$

From (11) and (12), we obtain

$$\beta^H (PA(e) + (A(e))^H P) \beta = (\lambda + \bar{\lambda}) \beta^H P \beta. \quad (13)$$

Since $(PA(e) + (A(e))^H P)^H = PA(e) + (A(e))^H P$, then $PA(e) + (A(e))^H P$ is a Hermitian matrix. At the same time, since $PA(e) + (A(e))^H P = -Q$, P is a real symmetric positive definite matrix, and Q is a positive definite matrix, then $\beta^H P \beta > 0$ and $\beta^H (-Q) \beta < 0$, so that

$$\lambda + \bar{\lambda} = \frac{\beta^H (-Q) \beta}{\beta^H P \beta} < 0. \quad (14)$$

From (14), we can obtain $|\arg(\lambda)| > \pi/2 \geq \pi\alpha/2$ ($\alpha \leq 1$). According to Lemma 5, the equilibrium point of system (7) is asymptotically stable. That is,

$$\lim_{t \rightarrow \infty} \|e(t)\| = \lim_{t \rightarrow \infty} \|u_s - h(t) \cdot u_r\| = 0, \quad (15)$$

which indicates that the MFPS between the drive and response system (4) can be achieved. The proof is completed. \square

Remark 7. Since $A(e)$ is a polynomial matrix, it is difficult to solve for the eigenvalues. However, based on Theorem 6, the equilibrium point in system (7) is asymptotically stable if and only if $PA(e) + (A(e))^H P = -Q$ is a continuous Lyapunov matrix equation.

3. Designing the Lyapunov Matrix Equation

Next, we introduce a scheme of MFPS for coupled PLFO chaotic systems with uncertain parameters (7).

Theorem 8. For a given scaling function matrix $h(t)$, we can select a controller (16) and a synchronization error system (17) as follows:

$$\begin{aligned} \psi = & -h(t) \widetilde{M}(z, \widetilde{\theta}) u_r - M(z, \theta) e + Ke \\ & + \frac{d^q(h(t) u_r)}{dt^q}, \end{aligned} \quad (16)$$

$$\frac{d^q e}{dt^q} = [\widetilde{M}(z, \widetilde{\theta}) - M(z, \theta) + K] e, \quad (17)$$

where $K = \text{diag}\{k_1, k_2, \dots, k_n\}$, $k_i < 0$, $0 < q \leq 1$. The error system (17) can be equivalently written as

$$\begin{aligned} \frac{d^q e_1}{dt^q} &= A_{11} e_{\theta_{11}} + A_{12} e_{\theta_{12}} + \dots + A_{1l_1} e_{\theta_{1l_1}} + k_1 e_1, \\ \frac{d^q e_2}{dt^q} &= A_{2(l_1+1)} e_{\theta_{2(l_1+1)}} + A_{2(l_1+2)} e_{\theta_{2(l_1+2)}} + \dots \\ &+ A_{2l_2} e_{\theta_{2l_2}} + k_2 e_2, \\ &\vdots \\ \frac{d^q e_n}{dt^q} &= A_{n(l_{n-1}+1)} e_{\theta_{n(l_{n-1}+1)}} + A_{n(l_{n-1}+2)} e_{\theta_{n(l_{n-1}+2)}} + \dots \\ &+ A_{nl_n} e_{\theta_{nl_n}} + k_n e_n, \end{aligned} \quad (18)$$

where $e_{\theta_{ij}} = \widetilde{\theta}_{ij} - \theta_{ij}$ ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, l_1, l_1 + 1, \dots, l_{n-1}, l_{n-1} + 1, \dots, l_n$), where l_1 stands for the number of unknown parameters in the first error equation, and $l_n - l_{n-1}$ stands for the number of unknown parameters in the n th error equation) and A_{ij} is a coefficient of $e_{\theta_{ij}}$ in the i th error equation. Therefore, we can obtain the systems with uncertain parameters as

$$\begin{aligned} \frac{d^q e_{\theta_{11}}}{dt^q} &= -A_{11} e_1 + k'_1 e_{\theta_{11}}, \\ \frac{d^q e_{\theta_{12}}}{dt^q} &= -A_{12} e_1 + k'_2 e_{\theta_{12}}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} \frac{d^q e_{\theta_{1l_1}}}{dt^q} &= -A_{1l_1} e_1 + k'_{l_1} e_{\theta_{1l_1}}, \\ \frac{d^q e_{\theta_{2(l_1+1)}}}{dt^q} &= -A_{2(l_1+1)} e_2 + k'_{(l_1+1)} e_{\theta_{2(l_1+1)}}, \\ &\vdots \\ \frac{d^q e_{\theta_{2l_2}}}{dt^q} &= -A_{2l_2} e_2 + k'_{l_2} e_{\theta_{2l_2}}, \\ &\vdots \\ \frac{d^q e_{\theta_{nl_n}}}{dt^q} &= -A_{nl_n} e_n + k'_{l_n} e_{\theta_{nl_n}}, \end{aligned} \quad (19)$$

where $k'_j < 0$ ($j = 1, 2, \dots, l_n$). We can achieve the MFPS of system (7) through the error system of (18) and the estimate of the system with unknown parameters (19). That is, the error system and the estimate of unknown parameters are asymptotically stable.

Proof. Assume that $e(t) = u_s - h(t) \cdot u_r$; we have

$$\begin{aligned} \frac{d^q e(t)}{dt^q} &= \frac{d^q u_s}{dt^q} - \frac{d^q(h(t) u_r)}{dt^q} \\ &= \widetilde{M}(z, \widetilde{\theta}) u_s + \psi - \frac{d^q(h(t) u_r)}{dt^q}. \end{aligned} \quad (20)$$

We select $\psi = -h(t) \widetilde{M}(z, \widetilde{\theta}) u_r - M(z, \theta) e + Ke + d^q(h(t) u_r)/dt^q$; then it has

$$\begin{aligned} \frac{d^q e(t)}{dt^q} &= \widetilde{M}(z, \widetilde{\theta}) u_s + \psi - \frac{d^q(h(t) u_r)}{dt^q} \\ &= \widetilde{M}(z, \widetilde{\theta}) u_s - h(t) \widetilde{M}(z, \widetilde{\theta}) u_r - M(z, \theta) e \\ &+ Ke + \frac{d^q(h(t) u_r)}{dt^q} - \frac{d^q(h(t) u_r)}{dt^q} \\ &= [\widetilde{M}(z, \widetilde{\theta}) - M(z, \theta) + K] e. \end{aligned} \quad (21)$$

Hence, we obtain a controller (16) and a synchronization error system (17).

From (18) and (19), the error system and the estimate of unknown parameters can be simply written as

$$\begin{pmatrix} \frac{d^q e}{dt^q} \\ \frac{d^q e_{\theta}}{dt^q} \end{pmatrix} = A \begin{pmatrix} e \\ e_{\theta} \end{pmatrix}, \quad (22)$$

where $e = (e_1, e_2, \dots, e_n)^T$, $e_{\theta} = (e_{\theta_{11}}, e_{\theta_{12}}, \dots, e_{\theta_{1l_1}}, e_{\theta_{2(l_1+1)}}, \dots, e_{\theta_{2l_2}}, \dots, e_{\theta_{nl_n}})^T$, and

$$A = \begin{pmatrix} k_1 & 0 & \cdots & 0 & A_{11} & \cdots & A_{1l_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 & 0 & \cdots & 0 & A_{2(l_1+1)} & \cdots & A_{2l_2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k_n & 0 & \cdots & 0 & 0 & \cdots & 0 & \vdots & A_{n(l_{n-1}+1)} & \cdots & A_{nl_n} \\ -A_{11} & 0 & \cdots & 0 & k'_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ -A_{1l_1} & 0 & \cdots & 0 & 0 & \cdots & k'_{l_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & -A_{2(l_1+1)} & \cdots & 0 & 0 & \cdots & 0 & k'_{l_1+1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -A_{2l_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & k'_{l_2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \vdots & -A_{n(l_{n-1}+1)} & 0 & \vdots & 0 & 0 & \cdots & 0 & \cdots & k'_{l_{n-1}+1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -A_{nl_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & k'_{l_n} \end{pmatrix}. \quad (23)$$

Hence, we can obtain $PA + A^H P = -Q$, where P is a real symmetric positive definite matrix and Q is a positive definite matrix. We usually choose $P = E$, where E is an identity matrix. Then,

$$Q = \text{diag} \{-2k_1, -2k_2, \dots, -2k_n, -2k'_1, \dots, -2k'_{l_n}\}. \quad (24)$$

According to Theorem 6, $PA + A^H P = -Q$ is a continuous Lyapunov matrix equation. Meanwhile, we can realize the MFPS of system (7) through controller (16). Therefore, $\lim_{t \rightarrow \infty} \|e(t)\| = 0$, which indicates that the MFPS within system (17) can be achieved. The proof is completed. \square

Remark 9. In the proof of Theorem 8, it is easy to see that the scaling function matrix $h(t)$ has no direct effect on $d^q e/dt^q$ and $d^q e_\theta/dt^q$. The results verify the validity and feasibility of the proposed control strategy.

Remark 10. Based on Theorem 8, we can choose the scaling function matrix to realize a variety of synchronizations by the method. For example, if $h_1(t) = h_2(t) = \dots = h_n(t)$ are continuously differentiable functions, the MFPS converts to function projective synchronization, and if $h(t) = \text{diag}\{h_1(t), h_2(t), \dots, h_n(t)\}$, $h_i(t)$ ($i = 1, 2, \dots, n$) being real constants, the MFPS converts to modified projective synchronization (such as complete synchronization and antisynchronization).

Remark 11. This method has universal applicability if and only if the chaotic system is a PLFO chaotic system (1).

4. Numerical Simulations

Huang and Li [29] proposed a dynamic model of finance comprising three first-order differential equations. Then, following the dynamic model, Chen [24] proposed a fractional-order financial system to describe the running of a financial system as follows:

$$\begin{aligned} \frac{d^{q_1} x_1}{dt^{q_1}} &= nx_3 + mx_1 x_2 - ax_1, \\ \frac{d^{q_2} x_2}{dt^{q_2}} &= 1 - bx_2 - x_1^2, \\ \frac{d^{q_3} x_3}{dt^{q_3}} &= px_1 - cx_3. \end{aligned} \quad (25)$$

It has three nonlinear equations of fractional-order chaotic systems. The state variables x_1, x_2, x_3 represent the interest rate, the investment demand, and the price index, respectively. Changes in x_1 are mostly influenced by two factors: contradictions from the investment market, which indicate a surplus between investment and savings, and structural adjustment from prices of goods. The changing rate of x_2 is proportional to the rate of investment but inversely proportional to the cost of investment and interest rates. Changes in x_3 are controlled by a contradiction between supply and demand in commercial markets and are affected by inflation rates. Particularly, the negative interest rate x_1 , negative investment demand x_2 , and negative price index x_3 mean that the banks' deposit rates are below the level

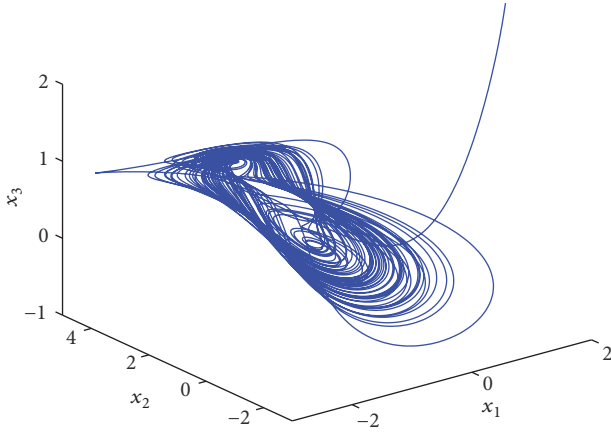


FIGURE 1: Three-dimensional phase diagram of fractional-order financial system (25).

of inflation, the investment market presents overinvestment, and the rate of deflation is increasing, respectively.

The constants a, b, c are representatives of the saving amount, the per-investment cost, and the elasticity of demand of the commercial markets. The fractional-order financial system (25) will exhibit chaotic behavior when $q_i = q \geq 0.85$, $i = 1, 2, 3$, and $n = 1, m = 1, p = -1, a = 3, b = 0.1$, and $c = 1$. When $q = 0.9$, the three-dimensional phase diagram of system (25) is illustrated in Figure 1 and the largest Lyapunov exponent is $L = 0.14$. The results highlight the chaotic attractor of fractional-order financial system (25).

It is obvious that system (25) is a PLFO chaotic system. Therefore, the drive and response systems can be constructed as follows:

$$\begin{aligned}
 \frac{d^{q_1} x_1}{dt^{q_1}} &= nx_3 + mx_1 x_2 - ax_1, \\
 \frac{d^{q_3} x_3}{dt^{q_3}} &= px_1 - cx_3, \\
 \frac{d^{q_2} x_2}{dt^{q_2}} &= 1 - bx_2 - x_1^2, \\
 \frac{d^{q_1} x_4}{dt^{q_1}} &= \tilde{n}x_5 + \tilde{m}x_4 x_2 - \tilde{a}x_4 + \psi_1, \\
 \frac{d^{q_3} x_5}{dt^{q_3}} &= \tilde{p}x_4 - \tilde{c}x_5 + \psi_2,
 \end{aligned} \tag{26}$$

where ψ_i ($i = 1, 2$) is the controller. In the coupled system (26), the drive system evolves independently, while the response system is governed by the drive system through the variable x_2 . That is, the drive system is

$$\begin{aligned}
 \frac{d^{q_1} x_1}{dt^{q_1}} &= nx_3 + mx_1 x_2 - ax_1, \\
 \frac{d^{q_3} x_3}{dt^{q_3}} &= px_1 - cx_3, \\
 \frac{d^{q_2} x_2}{dt^{q_2}} &= 1 - bx_2 - x_1^2,
 \end{aligned} \tag{27}$$

and the response system is

$$\begin{aligned}
 \frac{d^{q_1} x_4}{dt^{q_1}} &= \tilde{n}x_5 + \tilde{m}x_4 x_2 - \tilde{a}x_4 + \psi_1, \\
 \frac{d^{q_3} x_5}{dt^{q_3}} &= \tilde{p}x_4 - \tilde{c}x_5 + \psi_2.
 \end{aligned} \tag{28}$$

System (26) becomes a coupled PLFO chaotic system with uncertain parameters. Based on Theorem 8 and q_i ($i = 1, 2, 3$) = q , we have $u_r = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, $M(x_2, \theta) = \begin{pmatrix} mx_2 - a & n \\ p & -c \end{pmatrix}$, $\tilde{M}(x_2, \tilde{\theta}) = \begin{pmatrix} \tilde{m}x_2 - \tilde{a} & \tilde{n} \\ \tilde{p} & -\tilde{c} \end{pmatrix}$, $h(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix}$, $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} x_1 - h_1(t)x_4 \\ x_3 - h_2(t)x_5 \end{pmatrix}$, $e_m = \tilde{m} - m$, $e_a = \tilde{a} - a$, $e_n = \tilde{n} - n$, $e_p = \tilde{p} - p$, and $e_c = \tilde{c} - c$. According to (16), we can describe the controller as

$$\begin{aligned}
 \psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
 &= -h(t) \tilde{M}(x_2, \tilde{\theta}) u_r - M(x_2, \theta) e + Ke \\
 &\quad + \frac{d^q (h(t) u_r)}{dt^q}.
 \end{aligned} \tag{29}$$

From (17) to (19), we can obtain the error systems as

$$\begin{aligned}
 \frac{d^q e_1}{dt^q} &= x_2 e_1 e_m + (-e_1) e_a + e_2 e_n + k_1 e_1, \\
 \frac{d^q e_2}{dt^q} &= e_1 e_p + (-e_2) e_c + k_2 e_2.
 \end{aligned} \tag{30}$$

It shows that $A_{11} = x_2 e_1$, $A_{12} = -e_1$, $A_{13} = e_2$, $A_{24} = e_1$, and $A_{25} = -e_2$. Therefore, the unknown parameters systems are given by

$$\begin{aligned}
 \frac{d^q e_m}{dt^q} &= -x_2 e_1^2 + k'_1 e_m, \\
 \frac{d^q e_a}{dt^q} &= e_1^2 + k'_2 e_a, \\
 \frac{d^q e_n}{dt^q} &= -e_1 e_2 + k'_3 e_n, \\
 \frac{d^q e_p}{dt^q} &= -e_1 e_2 + k'_4 e_p, \\
 \frac{d^q e_c}{dt^q} &= e_2^2 + k'_5 e_c,
 \end{aligned} \tag{31}$$

where $K' = \text{diag}(k'_1, k'_2, k'_3, k'_4, k'_5)$ ($k'_i < 0$).

The initial conditions are $x_1(0) = -0.5$, $x_2(0) = -0.2$, $x_3(0) = 8$, $x_4(0) = 3$, $x_5(0) = -1$, and $q = 0.9$ and the estimated parameters have initial conditions $\tilde{n}(0) = -1$, $\tilde{m}(0) = 3$, $\tilde{a}(0) = -0.5$, $\tilde{p}(0) = 3$, $\tilde{c}(0) = 3.5$, and $k_i = -10$ ($i = 1, 2$) and $k'_i = -20$ ($i = 1, 2, 3, 4, 5$). Let the scaling function factors $h_1(t) = 0.1 \cos(2t)$ and $h_2(t) = \sqrt[3]{t}$; then the simulation results are shown in Figures 2 and 3. Obviously, in Figure 2, the errors converge to zero as time goes to infinity, which implies that the MFPS of system (26) is achieved with

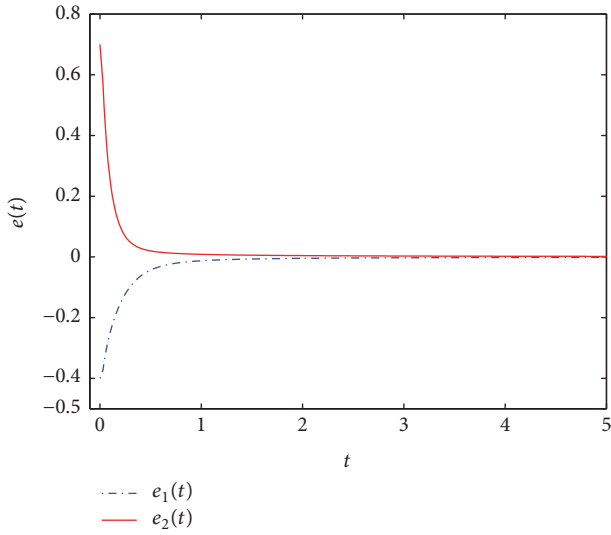


FIGURE 2: MFPS errors of system (30) with $h_1(t) = 0.1 \cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

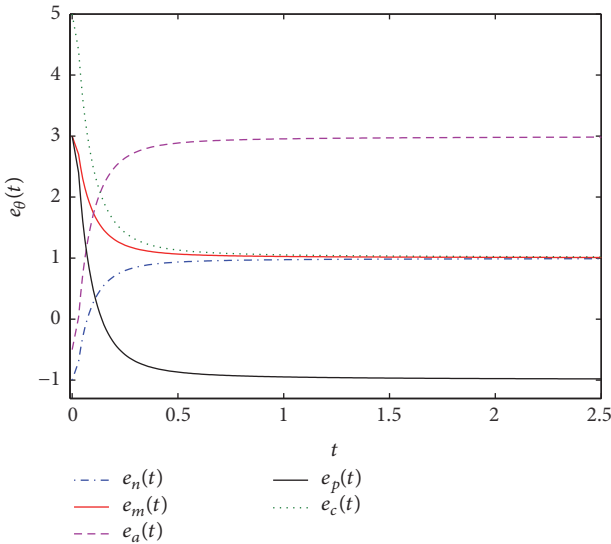


FIGURE 3: Unknown parameters of system (31) with $h_1(t) = 0.1 \cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

the scaling function factors. Figure 3 shows that the estimated values of unknown parameters \tilde{n} , \tilde{m} , \tilde{a} , \tilde{p} , and \tilde{c} converge to $n = 1$, $m = 1$, $a = 3$, $p = -1$, and $c = 1$, respectively, as $t \rightarrow \infty$. Therefore, the uncertain parameters n , m , a , p , and c are identified. Additionally, with the scaling function factors $h_1(t) = 0.1 \cos(2t)$ and $h_2(t) = \sqrt[3]{t}$, phase portrait of the drive and response systems is illustrated in Figure 4.

5. Conclusion

In this paper, the MFPS was investigated for the coupled uncertain fractional-order financial system. Based on the stability criterion of the fractional-order system and the continuous Lyapunov matrix equation, a synchronization

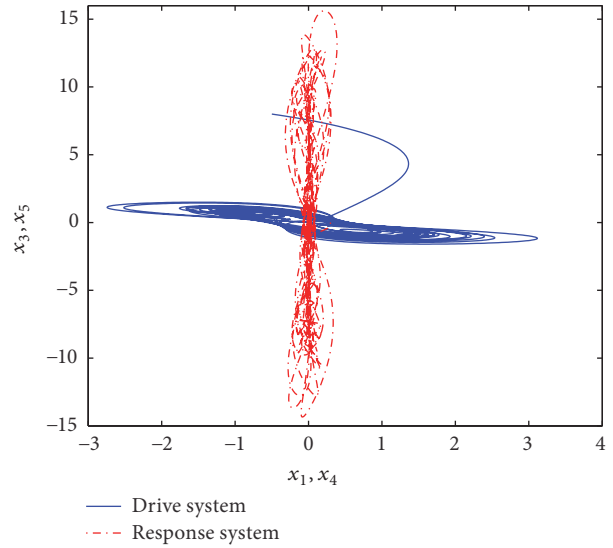


FIGURE 4: Phase portrait of the drive and response systems in $x_1 - x_3$ ($x_4 - x_5$) plane with $h_1(t) = 0.1 \cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

controller was proposed. Then, using an uncertain financial system, we verified the validity of the proposed controller. The unknown parameters were also estimated so that the phenomenon of parameter distortion was effectively avoided. The results showed that the controller was feasible and effective. In theory, the proposed approach can realize a synchronized and healthy development of financial markets. In the case of possessing homogeneous investment demand, we can keep profits with price index synchronous development so that interference factors can be avoided.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

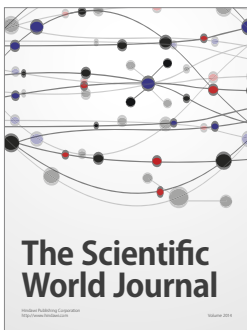
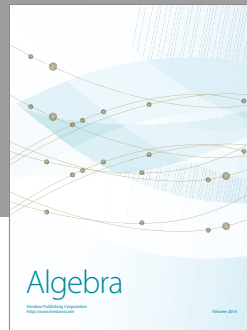
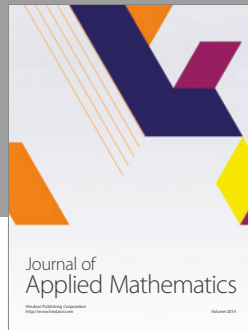
Acknowledgments

This research was supported by the State Key Program of the National Natural Science of China (no. 71232004) and Fundamental Research Funds for the Central Universities of China (no. CDJSK1001).

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