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Research Article

$\mathscr{H}_\infty$ Filter Design with Minimum Entropy for Continuous-Time Linear Systems

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We deal with the design problem of minimum entropy $\mathscr{H}_\infty$ filter in terms of linear matrix inequality (LMI) approach for linear continuous-time systems with a state-space model subject to parameter uncertainty that belongs to a given convex bounded polyhedral domain. Given a stable uncertain linear system, our attention is focused on the design of full-order and reduced-order robust minimum entropy $\mathscr{H}_\infty$ filters, which guarantee the filtering error system to be asymptotically stable and are required to minimize the filtering error system entropy (at $s_0 = \infty$) and to satisfy a prescribed $\mathscr{H}_\infty$ disturbance attenuation performance. Sufficient conditions for the existence of desired full-order and reduced-order filters are established in terms of LMIs, respectively, and the corresponding filter synthesis is cast into a convex optimization problem which can be efficiently handled by using standard numerical software. Finally, an illustrative example is provided to show the usefulness and effectiveness of the proposed design method.

1. Introduction

Estimation is the process of inferring the value of a quantity of interests from indirect, inaccurate, and uncertain observations [1]. State estimation of dynamic systems with both process and measurement noise inputs is a very important and challenging problem in engineering applications. In the past decades, quite a few attention has been devoted to estimation methods that are based on the minimization of the variance of the estimation error, that is, the well-known Kalman filtering approach [2, 3]. Unfortunately, it is recognized soon that the performance of Kalman filter can deteriorate significantly when the process parameters are subjected to relatively small modeling errors. In order to cope with this problem, over the past few years interest has been devoted to the design of robust estimators. There are essentially two approaches to the robust estimation problem. The first is robust $\mathscr{H}_2$ filtering, which minimizes the upper bound of the estimation error variance for all possible parametric uncertainties under the assumption that the noise processes have known power spectral densities [4, 5]. In many practical situations, however, we may not be able to have exactly known information on the spectral densities of the noise processes. In such cases, an alternative is to reformulate the estimation problem in an $\mathscr{H}_\infty$ filtering framework, which has been well recognized to be most appropriate for systems with noise input whose stochastic information is not precisely known. It minimizes the worst-case energy gain from the noise inputs to the estimation error [6–8]. There are many results reported on the problem of $\mathscr{H}_\infty$ filtering; for example, it has been addressed for linear systems [9], linear systems with uncertain parameters [10, 11], delay systems [6, 12], and stochastic systems [8]. Although the $\mathscr{H}_\infty$ filter is known to be less sensitive to modeling errors than $\mathscr{H}_2$ filter, it is generally so conservative as to lead to a large intolerable estimation error variance when the system is driven by white noise signals.

Similar to $\mathscr{H}_\infty$ control problem, in the $\mathscr{H}_\infty$ filtering problem [13–17], the family of filters that satisfy a filtering error system with an $\mathscr{H}_\infty$-norm bound is characterized by a linear fractional transformation of a "ball in $\mathscr{H}_\infty$," and then a natural question is which element of this ball to choose. One choice that has been considered in a closely related problem in mathematics is to choose that which minimizes an entropy integral; that is, the filter is well selected such that not only the filtering error system is asymptotically stable and the $\mathscr{H}_\infty$-norm of its transfer function is below a prescribed level, but also the entropy of the filtering error system at infinity is minimized. This kind of optimal filters is referred to as minimum entropy $\mathscr{H}_\infty$ filters in the present paper. In fact, minimum entropy $\mathscr{H}_\infty$ filtering provides a means of trading off some of the features of other filtering problems, namely, $\mathscr{H}_2$ filtering and $\mathscr{H}_\infty$ filtering. As for the minimum entropy $\mathscr{H}_\infty$ control problem, there are many important results that have been reported in the literature; for example, controllers which minimize the entropy of the closed-loop transfer function have been studied extensively for linear time-invariant (LTI) systems, both in the continuous and discrete-time cases [18–20]. Minimum entropy control for time-varying systems has been investigated in [21]. To the best of our knowledge, however, there is not any result reported on minimum entropy $\mathscr{H}_\infty$ filtering problem in the literature; research in this area should be important and challenging; this motivates us to carry out the present study.

In this paper, we make an attempt to investigate the design of minimum entropy $\mathscr{H}_\infty$ filters by using linear matrix inequality (LMI) approach for linear continuous-time systems with a state-space model subject to parameter uncertainty that belongs to a given convex bounded polyhedral domain. Given a stable uncertain linear system, our attention is focused on the design of full-order and reduced-order robust minimum entropy $\mathscr{H}_\infty$ filters, which guarantee the filtering error system to be asymptotically stable and are required to minimize the filtering error system entropy (at $s_0 = \infty$) as well as to satisfy a prescribed $\mathscr{H}_\infty$ disturbance attenuation performance. Sufficient conditions for the existence of desired full-order and reduced-order filters are established in terms of LMIs, respectively, and the corresponding filter synthesis is cast into a convex optimization problem which can be efficiently handled by using the well-known interior-point algorithms [22]. A numerical simulation example is provided to show the usefulness and effectiveness of the proposed design method.

The rest of this paper is organized as follows. In Section 2, the minimum entropy $\mathscr{H}_\infty$ filtering problem is formulated. Section 3 presents our main results of the full-order and reduced-order minimum entropy $\mathscr{H}_\infty$ filters design. Section 4 provides an illustrative example. Finally, conclusions are drawn in Section 5.

Notations. The notation used here is fairly standard except where otherwise stated. A^T represents the transpose of A; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. the notation $P>0$ means that P is a real symmetric and positive definite matrix; trace(P) represents the trace of P; diag$\{F_1, F_2, \ldots\}$ stands for a block-diagonal matrix whose diagonal blocks are given by $F_1, F_2, \ldots$. I and 0 represent identity matrix and zero matrix; $|\cdot|$ refers to the Euclidean vector norm and $\|\cdot\|_2$ denotes the $\mathscr{L}_2$ norm of a differential signal; The signals that are square integrable over $[0,\infty)$ are denoted by $\mathscr{L}_2[0,\infty)$ with the norm $\|\cdot\|_2$; the symbol $*$ in a matrix means that the corresponding term of the matrix can be obtained by symmetric property.

2. Problem Formulation

Consider the following linear time-invariant (LTI) system (Σ):

$$(\Sigma): \begin{cases} \dot{x}(t) = Ax(t) + B\omega(t), \\ y(t) = Cx(t) + D\omega(t), \\ z(t) = Lx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^m$ is the measured output; $z(t) \in \mathbb{R}^q$ is a linear combination of the state variables to be estimated; $\omega(t) \in \mathbb{R}^l$ is the disturbance input which belongs to $\mathscr{L}_2[0,\infty)$; A, B, C, D, and L are constant real matrices of appropriate dimensions, where L is a known matrix and A, B, C, D are unknown matrices such that the system matrix

$$\mathscr{G} \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2}$$

belongs to a given polytope $\mathscr{D}$ described by

$$\mathscr{D} \triangleq \left\{ \mathscr{G} \mid \mathscr{G} = \sum_{i=1}^{n_s} \lambda_j \mathscr{G}_j; \sum_{i=1}^{n_s} \lambda_j = 1, \lambda_j \geq 0 \right\}; \tag{3}$$

that is, any admissible system matrix $\mathscr{G}$ can be written as an unknown convex combination of n_s vertices $\mathscr{G}_j$, $j = 1, 2, \ldots, n_s$, given by

$$\mathscr{G}_j \triangleq \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \tag{4}$$

where A_j, B_j, C_j, and D_j, $j = 1, 2, \ldots, n_s$, are given matrices. Clearly, $n_s = 1$ corresponds to the case where the system (Σ) is perfectly known.

Before formulating the problem of this paper, we first give some definitions and existing results of the minimum entropy problem. Consider the following LTI system (Π):

$$(\Pi): \begin{cases} \dot{x}(t) = Ax(t) + B\omega(t), \\ y(t) = Lx(t), \end{cases} \tag{5}$$

where $y(t)$ is the output; the other notations are defined as in (1), and let $\mathbb{G}$ denote its transfer function.

Definition 1 (entropy at $s_0 \in (0,\infty)$). Let $\mathbb{G} \in \mathscr{RL}_\infty$ and $\gamma > 0$ be a real scalar such that $\|\mathbb{G}\|_\infty < \gamma$. Then the entropy of $\mathbb{G}$ at s_0 is defined by

$$\mathscr{I}(\mathbb{G}; \gamma; s_0) \triangleq -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| \det \left(I - \gamma^{-2} \mathbb{G}^*(j\varpi)\, \mathbb{G}(j\varpi) \right) \right| \times \left[\frac{s_0^2}{s_0^2 + \varpi^2} \right] d\varpi. \tag{6}$$

When $s_0 \rightarrow \infty$, we obtain the entropy at infinity of the system, which has the following definition.

Definition 2 (entropy at infinity). Let $\mathbb{G} \in \mathscr{RL}_\infty$ and $\gamma > 0$ be a real scalar such that $\|\mathbb{G}\|_\infty < \gamma$. Then the entropy of $\mathbb{G}$ at infinity is defined by

$$\mathcal{I}(\mathbb{G};\gamma;\infty) \triangleq -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln\left|\det\left(I-\gamma^{-2}\mathbb{G}^*(j\varpi)\mathbb{G}(j\varpi)\right)\right| d\varpi. \quad (7)$$

It is well known that for LTI system (Π) in (5), we have the following results for the $\mathscr{H}_2$ optimization problem:

$$\|\mathbb{G}\|_2^2 = \operatorname{trace}\left(B^T Q B\right), \quad (8)$$

where $Q = Q^T \geq 0$ is the solution to the Lyapunov equation

$$A^T Q + QA + L^T L = 0. \quad (9)$$

In fact, Q is just the controllability Gramian.

Now, considering the $\mathscr{H}_\infty$ optimization problem of LTI system, we knew that $\|\mathbb{G}\|_\infty < \gamma$ if there exists a stabilizing solution $P = P^T \geq 0$ to the algebraic Riccati equation

$$A^T P + PA + \gamma^{-2} PBB^T P + L^T L = 0, \quad (10)$$

and it is easy to prove that any P solving (10) overbounds the controllability Gramian Q; that is, $P \geq Q$. To evaluate the entropy cost, we define the following auxiliary cost.

Definition 3 (auxiliary cost). Let $\mathbb{G} \in \mathscr{RL}_\infty$ and $\gamma > 0$ be a real scalar such that $\|\mathbb{G}\|_\infty < \gamma$. Then the auxiliary cost with $\mathbb{G}$ is defined by

$$\mathcal{J}(\mathbb{G};\gamma) \triangleq \operatorname{trace}\left(B^T P B\right), \quad (11)$$

where P is a positive symmetric matrix with the smallest possible maximum singular value among all solutions of the algebraic Riccati equality (10).

The following lemma gives the equivalence between the auxiliary cost and the entropy defined in (7) and (11), respectively, which plays a key role in deriving our main results subsequently.

Lemma 4 (see [20]). *Let $\mathbb{G} \in \mathscr{RL}_\infty$ and $\gamma > 0$ be a real scalar such that $\|\mathbb{G}\|_\infty < \gamma$. Then the entropy equals the auxiliary cost; that is,*

$$\mathcal{I}(\mathbb{G};\gamma;\infty) = \mathcal{J}(\mathbb{G};\gamma). \quad (12)$$

Moreover, according to the result in [22], the minimum entropy $\mathscr{H}_\infty$ optimization problem for the LTI system (Π) can be formulated as follows:

$$\min_{P>0,\, R>0} \operatorname{trace}(R), \quad (13)$$

subject to

$$\begin{bmatrix} A^T P + PA & PB & L^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} -R & B^T P \\ \star & -P \end{bmatrix} \leq 0. \quad (14)$$

The objective of this paper is to design a full-order (or reduced-order) minimum entropy $\mathscr{H}_\infty$ filter the system (Σ) in (1) of the following form:

$$(\widehat{\Sigma}): \begin{cases} \dot{\widehat{x}}(t) = A_f \widehat{x}(t) + B_f y(t), & \widehat{x}(0) = 0, \\ \widehat{z}(t) = L_f \widehat{x}(t), \end{cases} \quad (15)$$

where $\widehat{x}(t) \in \mathbb{R}^k$ is the filter state vector, $A_f \in \mathbb{R}^{k\times k}$, $B_f \in \mathbb{R}^{k\times m}$, and $C_f \in \mathbb{R}^{q\times k}$ are filter parameters to be determined later. In the case where $k = n$, the filter will be referred to as a full-order filter and as a reduced-order filter when $k < n$.

Augmenting the model of (1) to include the state of the filter $(\widehat{\Sigma})$, we obtain the filtering error system as

$$(\widetilde{\Sigma}): \begin{cases} \dot{\xi}(t) = \widetilde{A}\xi(t) + \widetilde{B}\omega(t), \\ e(t) = \widetilde{L}\xi(t), \end{cases} \quad (16)$$

where $\xi(t) \triangleq [x^T(t)\widehat{x}^T(t)]^T$, $e(t) \triangleq z(t) - \widehat{z}(t)$ and

$$\widetilde{A} \triangleq \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \qquad \widetilde{B} \triangleq \begin{bmatrix} B \\ B_f D \end{bmatrix}, \qquad \widetilde{L} \triangleq \begin{bmatrix} L & -L_f \end{bmatrix}. \quad (17)$$

Our aim in this paper is to determine the matrices A_f, B_f, and L_f of the full-order (or reduced-order) minimum entropy $\mathscr{H}_\infty$ filter $(\widehat{\Sigma})$ in (15) such that the filtering error system $(\widetilde{\Sigma})$ in (16) is asymptotically stable with a prescribed $\mathscr{H}_\infty$ disturbance attenuation level $\gamma > 0$ and a guaranteed minimum entropy at infinity; that is, $\mathcal{I}(\mathbb{E};\gamma;\infty)$ is minimized for a given scalar $\gamma > 0$ (where $\mathbb{E}$ is defined as the transfer function of the filtering error system $(\widetilde{\Sigma})$ in (16)).

3. Main Results

3.1. Full-Order Robust Filter Design. In this section, we will first study the design of a full-order filter. In order to pave the way for deriving the robust filter, initially we consider the case where the system matrix $\mathscr{G}$ is perfectly known; that is, $n_s = 1$. We first give the following result which will play a key role in deriving our subsequent results. Since the result can be easily obtained according to the analysis in Section 2, we omit the proof.

Theorem 5. *The filtering error system* $(\widetilde{\Sigma})$ *in* (16) *is asymptotically stable with an* $\mathscr{H}_\infty$ *disturbance attenuation level* $\gamma > 0$ *and a guaranteed minimum entropy, if there exist matrices* $P > 0$ *and* $R > 0$ *such that the following optimization problem has feasible solution:*

$$\min_{P>0,\,R>0,\,A_f,\,B_f,\,L_f} trace\,(R), \tag{18}$$

subject to

$$\begin{bmatrix} \widetilde{A}^T P + P\widetilde{A} & P\widetilde{B} & \widetilde{L}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} -R & \widetilde{B}^T P \\ \star & -P \end{bmatrix} \leq 0. \tag{20}$$

Notice that the inequalities constraints of (19)-(20) are not convex on the decision variables P, A_f, B_f, and L_f. In what follows, we will present the design method of the minimum entropy $\mathscr{H}_\infty$ filter, and give the following result.

Theorem 6. *Consider the system* (Σ) *with perfectly known system matrix* $\mathscr{G}$. *There exists a full-order minimum entropy* $\mathscr{H}_\infty$ *filter of the form of* (15) *such that the filtering error system* $(\widetilde{\Sigma})$ *in* (16) *is asymptotically stable with an* $\mathscr{H}_\infty$ *disturbance attenuation level* $\gamma > 0$ *and a guaranteed minimum entropy, if there exist matrices* $\mathscr{U} > 0$ *and* $\mathscr{V} > 0$ *and matrices* $\mathscr{A}_f$, $\mathscr{B}_f$, *and* $\mathscr{L}_f$ *such that the following optimization problem has feasible solution:*

$$\min_{\mathscr{U}>0,\,\mathscr{V}>0,\,\mathscr{A}_f,\,\mathscr{B}_f,\,\mathscr{L}_f} trace\,(R), \tag{21}$$

subject to

$$\begin{bmatrix} \mathscr{U}A + A^T\mathscr{U} + \mathscr{B}_f C + C^T\mathscr{B}_f^T & \mathscr{A}_f + A^T\mathscr{V} + C^T\mathscr{B}_f^T & \mathscr{U}B + \mathscr{B}_f D & L^T \\ \star & \mathscr{A}_f + \mathscr{A}_f^T & \mathscr{V}B + \mathscr{B}_f D & -\mathscr{L}_f^T \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} -R & B^T\mathscr{U} + D^T\mathscr{B}_f^T & B^T\mathscr{V} + D^T\mathscr{B}_f^T \\ \star & -\mathscr{U} & -\mathscr{V} \\ \star & \star & -\mathscr{V} \end{bmatrix} \leq 0. \tag{23}$$

Moreover, a desired full-order minimum entropy $\mathscr{H}_\infty$ *filter can be computed from*

$$\begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} \triangleq \begin{bmatrix} \mathscr{V}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_f \\ \mathscr{L}_f & 0 \end{bmatrix}. \tag{24}$$

Proof. According to Theorem 5, P is nonsingular if the optimization problem of (18)–(20) has feasible solutions since $P > 0$. Now, partition P as

$$P \triangleq \begin{bmatrix} P_1 & P_3 \\ \star & P_2 \end{bmatrix}, \tag{25}$$

where P_1 and P_2 are $n \times n$ symmetric positive definite matrices. Without loss of generality, we assume that P_3 is nonsingular. To see this, let the matrix $Q \triangleq P + \alpha W$, where α is a positive scalar and

$$W \triangleq \begin{bmatrix} 0 & I \\ \star & 0 \end{bmatrix}, \qquad Q \triangleq \begin{bmatrix} Q_1 & Q_3 \\ \star & Q_2 \end{bmatrix}. \tag{26}$$

Observe that, since $P > 0$, we have $Q > 0$ for $\alpha > 0$ in a neighborhood of the origin. Thus, it can be easily verified that there exists an arbitrarily small $\alpha > 0$ such that Q_3 is nonsingular and inequalities (19)-(20) are feasible with P replaced by Q and such that the objective function of (18) will be increased only by an arbitrarily small quantity. Since Q_3 is nonsingular, we thus conclude that there is no loss of generality to assume the matrix P_3 to be nonsingular.

Define the following matrices which are all nonsingular:

$$\Gamma \triangleq \begin{bmatrix} I & 0 \\ 0 & P_2^{-1}P_3^T \end{bmatrix}, \qquad \mathscr{U} \triangleq P_1, \qquad \mathscr{V} \triangleq P_3 P_2^{-1} P_3^T, \tag{27}$$

$$\begin{bmatrix} \mathscr{A}_f & \mathscr{B}_f \\ \mathscr{L}_f & 0 \end{bmatrix} \triangleq \begin{bmatrix} P_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} \begin{bmatrix} P_2^{-1}P_3^T & 0 \\ 0 & I \end{bmatrix}. \tag{28}$$

Performing congruence transformations to (19)-(20) by $\mathrm{diag}(\Gamma, I, I)$ and $\mathrm{diag}(I, \Gamma)$, respectively, we have

$$\begin{bmatrix} \Gamma^T\widetilde{A}^T P\Gamma + \Gamma^T P\widetilde{A}\Gamma & \Gamma^T P\widetilde{B} & \Gamma^T\widetilde{L}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} -R & \widetilde{B}^T P\Gamma \\ \star & -\Gamma^T P\Gamma \end{bmatrix} \leq 0. \tag{30}$$

Considering (17) and (25)–(28), we have

$$\Gamma^T P\widetilde{A}\Gamma \triangleq \begin{bmatrix} \mathscr{U}A + \mathscr{B}_f C & \mathscr{A}_f \\ \mathscr{V}A + \mathscr{B}_f C & \mathscr{A}_f \end{bmatrix}, \qquad \Gamma^T P\widetilde{B} \triangleq \begin{bmatrix} \mathscr{U}B + \mathscr{B}_f D \\ \mathscr{V}B + \mathscr{B}_f D \end{bmatrix},$$

$$\Gamma^T P\Gamma \triangleq \begin{bmatrix} \mathscr{U} & \mathscr{V} \\ \star & \mathscr{V} \end{bmatrix}, \qquad \widetilde{L}\Gamma \triangleq \begin{bmatrix} L & -\mathscr{L}_f \end{bmatrix}. \tag{31}$$

Substituting (31) into (29)-(30) yields (22)-(23), respectively.

On the other hand, (28) is equivalent to

$$\begin{aligned} \begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} &= \begin{bmatrix} P_3^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_f \\ \mathscr{L}_f & 0 \end{bmatrix} \begin{bmatrix} P_3^{-T}P_2 & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \left(P_3^{-T}P_2\right)^{-1}\mathscr{V}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_f \\ \mathscr{L}_f & 0 \end{bmatrix} \begin{bmatrix} P_3^{-T}P_2 & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{32}$$

Note that the filter matrices of (15) can be written as (32), which implies that $P_3^{-T}P_2$ can be viewed as a similarity transformation on the state-space realization of the filter and, as such, has no effect on the filter mapping from y to $\widehat{z}$. Without loss of generality, we can set $P_3^{-T}P_2 = I$ and thus obtain (24). This completes the proof. □

Now, we will consider the design of a full-order robust filter. To this end, consider the system (Σ) in (1) and the uncertainty domain $\mathscr{D}$ in (3). According to the previous arguments, we have the following result.

Theorem 7. *Consider the system (Σ) with system matrix $\mathscr{G} \in \mathscr{D}$. There exists a full-order minimum entropy $\mathscr{H}_\infty$ filter of the form of (15) such that the filtering error system $(\widetilde{\Sigma})$ in (16) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attenuation level $\gamma > 0$ and a guaranteed minimum entropy, if there exist matrices $\mathscr{U} > 0$ and $\mathscr{V} > 0$ and matrices $\mathscr{A}_f$, $\mathscr{B}_f$, and $\mathscr{L}_f$ such that, for $j = 1, 2, \ldots, n_s$, the following optimization problem has a feasible solution:*

$$\min_{\mathscr{U}>0,\mathscr{V}>0,\mathscr{A}_f,\mathscr{B}_f,\mathscr{L}_f} trace\,(R)\,, \tag{33}$$

subject to

$$\begin{bmatrix} \mathscr{U}A_j + A_j^T\mathscr{U} + \mathscr{B}_f C_j + C_j^T\mathscr{B}_f^T & \mathscr{A}_f + A_j^T\mathscr{V} + C_j^T\mathscr{B}_f^T & \mathscr{U}B_j + \mathscr{B}_f D_j & L^T \\ \star & \mathscr{A}_f + \mathscr{A}_f^T & \mathscr{V}B_j + \mathscr{B}_f D_j & -\mathscr{L}_f^T \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -R & B_j^T\mathscr{U} + D_j^T\mathscr{B}_f^T & B_j^T\mathscr{V} + D_j^T\mathscr{B}_f^T \\ \star & -\mathscr{U} & -\mathscr{V} \\ \star & \star & -\mathscr{V} \end{bmatrix} \leq 0. \tag{34}$$

Moreover, a desired full-order minimum entropy $\mathscr{H}_\infty$ filter can be computed from (24).

Proof. Employing the same arguments as in the proof of Theorem 6, it follows that the optimization problem of (33)-(34) is equivalent to

$$\min_{P>0,\,R>0,\,A_f,\,B_f,\,L_f} \text{trace}\,(R)\,, \tag{35}$$

subject to $j = 1, 2, \ldots, n_s$,

$$\begin{bmatrix} \widetilde{A}_j^T P + P\widetilde{A}_j & P\widetilde{B}_j & \widetilde{L}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -R & \widetilde{B}_j^T P \\ \star & -P \end{bmatrix} \leq 0, \tag{36}$$

where $\widetilde{A}_j$ and $\widetilde{B}_j$ are as in (17) with A, B, C, and D replaced by A_j, B_j, C_j, and D_j, respectively.

Now, in view of the convexity of the uncertainty domain and considering that the inequalities (36) are affine in the matrices A_j, B_j, C_j, and D_j, we have the result that the optimization problem of (35)–(36) is equivalent to that of (18)–(20) with system matrix $\mathscr{G} \in \mathscr{D}$. This completes the proof. □

3.2. Reduced-Order Robust Filter Design. In this section, we will consider the design of reduced-order robust filter, that is, the case where the order of the filter k is smaller than the order n of the original system model. As the former, we first consider the case where the system matrix $\mathscr{G}$ is perfectly known, that is, $n_s = 1$, which have the following result.

Theorem 8. *Consider system (Σ) with perfectly known system matrix $\mathscr{G}$. There exists a reduced-order minimum entropy $\mathscr{H}_\infty$ filter of the form of (15) such that the filtering error system $(\widetilde{\Sigma})$ in (16) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attenuation level $\gamma > 0$ and a guaranteed minimum entropy, if there exist matrices $\mathscr{M} > 0$ and $\mathscr{N} > 0$ and matrices $\mathscr{A}_f$, $\mathscr{B}_f$, and $\mathscr{L}_f$ such that the following optimization problem has feasible solution:*

$$\min_{\mathscr{M}>0,\mathscr{N}>0,\mathscr{A}_f,\mathscr{B}_f,\mathscr{L}_f} trace\,(R)\,, \tag{37}$$

subject to

$$\begin{bmatrix} \Pi_{11} & \mathscr{K}\mathscr{A}_f + A^T\mathscr{K}\mathscr{N} + C^T\mathscr{B}_f^T & \mathscr{M}B + \mathscr{K}\mathscr{B}_f D & L^T \\ \star & \mathscr{A}_f + \mathscr{A}_f^T & \mathscr{N}^T\mathscr{K}^T B + \mathscr{B}_f D & -\mathscr{L}_f^T \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -I \end{bmatrix} \tag{38}$$
$$< 0,$$

$$\begin{bmatrix} -R & B^T\mathscr{M} + D^T\mathscr{B}_f^T & B^T\mathscr{K}\mathscr{N} + D^T\mathscr{B}_f^T \\ \star & -\mathscr{M} & -\mathscr{K}\mathscr{N} \\ \star & \star & -\mathscr{N} \end{bmatrix} \leq 0, \tag{39}$$

where $\mathscr{K} \triangleq \begin{bmatrix} I_{k\times k} \\ 0_{(n-k)\times k} \end{bmatrix}$ and $\Pi_{11} \triangleq \mathscr{M}A + A^T\mathscr{M} + \mathscr{K}\mathscr{B}_f C + C^T\mathscr{B}_f^T\mathscr{K}^T$. Moreover, a desired reduced-order minimum entropy $\mathscr{H}_\infty$ filter can be computed from

$$\begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} \triangleq \begin{bmatrix} \mathscr{N}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_f \\ \mathscr{L}_f & 0 \end{bmatrix}. \tag{40}$$

Proof. The proof is along the same lines as in the proof of Theorem 6. According to Theorem 5, P is nonsingular if the optimization problem of (18)–(20) has feasible solutions since $P > 0$. Now, partition P as

$$P \triangleq \begin{bmatrix} P_1 & P_3 \\ \star & P_2 \end{bmatrix}, \qquad P_3 \triangleq \begin{bmatrix} P_4 \\ 0_{(n-k)\times k} \end{bmatrix}, \tag{41}$$

where $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{k\times k}$ are symmetric positive definite matrices and $P_4 \in \mathbb{R}^{k\times k}$. Without loss of generality, we assume that P_4 is nonsingular; to see this, let the matrix $\mathcal{Q} \triangleq P + \alpha\mathcal{W}$, where α is a positive scalar and

$$\mathcal{W} \triangleq \left[\begin{array}{c|c} 0_{n\times n} & \mathcal{K} \\ \hline \star & 0_{k\times k} \end{array}\right], \qquad \mathcal{Q} \triangleq \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_3 \\ \star & \mathcal{Q}_2 \end{bmatrix}, \qquad \mathcal{Q}_3 \triangleq \begin{bmatrix} \mathcal{Q}_4 \\ 0_{(n-k)\times k} \end{bmatrix}. \tag{42}$$

With the same principle as in the proof of Theorem 6, it can be seen that there exists an arbitrarily small $\alpha > 0$ such that $\mathcal{Q}_4$ is nonsingular and (18)–(20) have feasible solutions with P replaced by $\mathcal{Q}$; thus, we can conclude that there is no loss of generality to assume the matrix P_4 to be nonsingular.

Define the following matrices:

$$\widetilde{\Gamma} \triangleq \begin{bmatrix} I & 0 \\ 0 & P_2^{-1}P_4^T \end{bmatrix}, \qquad \mathcal{M} \triangleq P_1, \qquad \mathcal{N} \triangleq P_4P_2^{-1}P_4^T, \tag{43}$$

$$\begin{bmatrix} \mathcal{A}_f & \mathcal{B}_f \\ \mathcal{L}_f & 0 \end{bmatrix} \triangleq \begin{bmatrix} P_4 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} \begin{bmatrix} P_2^{-1}P_4^T & 0 \\ 0 & I \end{bmatrix}. \tag{44}$$

Performing congruence transformations to (19) and (20) by $\operatorname{diag}(\widetilde{\Gamma}, I, I)$ and $\operatorname{diag}(I, \widetilde{\Gamma})$, respectively, we have

$$\begin{bmatrix} \widetilde{\Gamma}^T\widetilde{A}^TP\widetilde{\Gamma} + \widetilde{\Gamma}^TP\widetilde{A}\widetilde{\Gamma} & \widetilde{\Gamma}^TP\widetilde{B} & \widetilde{\Gamma}^T\widetilde{L}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0, \tag{45}$$

$$\begin{bmatrix} -R & \widetilde{B}^TP\widetilde{\Gamma} \\ \star & -\widetilde{\Gamma}^TP\widetilde{\Gamma} \end{bmatrix} \le 0. \tag{46}$$

Considering (17) and (41)–(44), we have

$$\widetilde{\Gamma}^TP\widetilde{A}\widetilde{\Gamma} \triangleq \begin{bmatrix} \mathcal{M}A + \mathcal{K}\mathcal{B}_fC & \mathcal{K}\mathcal{A}_f \\ \mathcal{N}^T\mathcal{K}^TA + \mathcal{B}_fC & \mathcal{A}_f \end{bmatrix},$$

$$\widetilde{\Gamma}^TP\widetilde{B} \triangleq \begin{bmatrix} \mathcal{M}B + \mathcal{K}\mathcal{B}_fD \\ \mathcal{N}^T\mathcal{K}^TB + \mathcal{B}_fD \end{bmatrix}, \tag{47}$$

$$\widetilde{\Gamma}^TP\widetilde{\Gamma} \triangleq \begin{bmatrix} \mathcal{M} & \mathcal{K}\mathcal{N} \\ \star & \mathcal{N} \end{bmatrix}, \qquad \widetilde{L}\widetilde{\Gamma} \triangleq \begin{bmatrix} L & -\mathcal{L}_f \end{bmatrix}.$$

Substituting (47) into (45)-(46), we obtain (38)-(39), respectively.

The remainder of the proof follows along the same lines as in the proof of Theorem 6. □

Now, considering the design of reduced-order robust filter, we give the following result without proof, which can be obtained by employing the same techniques used as those in Theorems 7 and 8.

Theorem 9. *Consider the system (Σ) with system matrix $\mathcal{G} \in \mathcal{D}$. There exists a reduced-order minimum entropy $\mathcal{H}_\infty$ filter of the form of* (15) *such that the filtering error system $(\widetilde{\Sigma})$ in* (16) *is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level $\gamma > 0$ and a guaranteed minimum entropy, if there exist matrices $\mathcal{M} > 0$ and $\mathcal{N} > 0$ and matrices $\mathcal{A}_f$, $\mathcal{B}_f$, and $\mathcal{L}_f$ such that, for $j = 1, 2, \ldots, n_s$, the following optimization problem has feasible solution:*

$$\min_{\mathcal{M}>0,\, \mathcal{N}>0,\, \mathcal{A}_f,\, \mathcal{B}_f,\, \mathcal{L}_f} \operatorname{trace}(R), \tag{48}$$

subject to

$$\begin{bmatrix} \Pi_{11j} & \mathcal{K}\mathcal{A}_f + A_j^T\mathcal{K}\mathcal{N} + C_j^T\mathcal{B}_f^T & \mathcal{M}B_j + \mathcal{K}\mathcal{B}_fD_j & L^T \\ \star & \mathcal{A}_f + \mathcal{A}_f^T & \mathcal{N}^T\mathcal{K}^TB_j + \mathcal{B}_fD_j & -\mathcal{L}_f^T \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -I \end{bmatrix} \le 0,$$

$$\begin{bmatrix} -R & B_j^T\mathcal{M} + D_j^T\mathcal{B}_f^T\mathcal{K}^T & B_j^T\mathcal{K}\mathcal{N} + D_j^T\mathcal{B}_f^T \\ \star & -\mathcal{M} & -\mathcal{K}\mathcal{N} \\ \star & \star & -\mathcal{N} \end{bmatrix} < 0, \tag{49}$$

where $\Pi_{11j} \triangleq \mathcal{M}A_j + A_j^T\mathcal{M} + \mathcal{K}\mathcal{B}_fC_j + C_j^T\mathcal{B}_f^T\mathcal{K}^T$ and $\mathcal{K}$ is defined in (38). *Moreover, a desired reduced-order minimum entropy $\mathcal{H}_\infty$ filter can be computed from* (40).

4. Numerical Example

In this section, we present an illustrative example to demonstrate the effectiveness of the proposed algorithm. Consider the linear continuous-time system (Σ) in (1) with parameter matrix $\mathcal{G}$ belonging to polyhedral domain $\mathcal{D}$, and assume $n_s = 3$, then the system data $\mathcal{G}_j$, $(j = 1, 2, 3)$ are given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.3 & 0.2 & -0.3 \\ -0.4 & -0.6 & 0.0 \\ 0.0 & 0.5 & -1.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.6 \\ 0.3 \\ -0.9 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1.0 & 0.3 & 1.0 \end{bmatrix}, \quad D_1 = 0.2, \\ A_2 &= \begin{bmatrix} -2.5 & 0.0 & 0.6 \\ 0.0 & -1.3 & -0.2 \\ -0.2 & 0.5 & -1.6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.4 \\ 0.6 \\ -0.3 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.4 & 1.2 & -0.7 \end{bmatrix}, \quad D_2 = 0.8, \\ A_3 &= \begin{bmatrix} -1.6 & 0.5 & -0.2 \\ 0.3 & -1.6 & 0.2 \\ 0.2 & 0.0 & -0.6 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.1 \\ -0.3 \\ 0.6 \end{bmatrix}, \\ C_3 &= \begin{bmatrix} 0.3 & -1.5 & 0.8 \end{bmatrix}, \quad D_3 = -0.5, \\ L &= \begin{bmatrix} 0.3 & 0.5 & 0.8 \end{bmatrix}. \end{aligned} \tag{50}$$

First, we consider the full-order filter design; solving the LMIs condition in Theorem 7 by applying the well-developed LMI Toolbox in the MATLAB environment directly, we obtain that the minimum γ is $\gamma^* = 0.4666$, the minimum entropy of

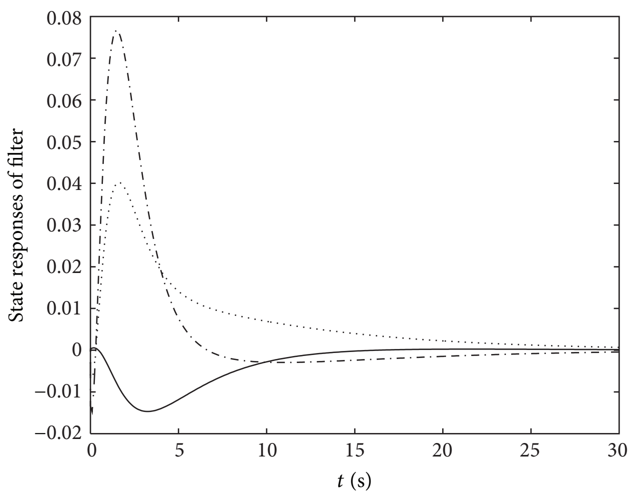

Figure 1: States of the designed full-order filter.

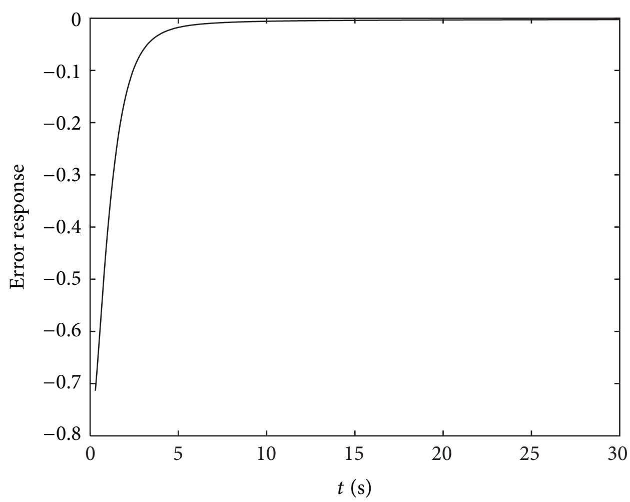

Figure 2: Filtering error of the full-order filtering.

the filtering error system is $\mathcal{I}(\mathbb{E};\gamma^*;\infty) = 0.3618$ (where $\mathbb{E}$ denotes the transfer function of filtering error system), and

$$A_f = \begin{bmatrix} -0.3015 & -0.0948 & -0.0196 \\ -0.3450 & -1.0629 & -0.6874 \\ -0.2279 & -0.4217 & -0.4166 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.0125 \\ -0.5696 \\ -0.2754 \end{bmatrix}, \quad L_f = \begin{bmatrix} -0.3007 & -0.5002 & -0.8002 \end{bmatrix}. \tag{51}$$

With $\lambda_1 = 0.7$, $\lambda_2 = 0.1$, and $\lambda_3 = 0.2$ in (3), the states of the full-order designed filter are given in Figure 1, where the initial condition is $[1.0\ \ -0.5\ \ -1.0]^T$, and the exogenous disturbance input $\omega(t)$ is given as $\omega(t) = 1/(0.5 + 1.8t)$, $t \geq 0$. Figure 2 shows the error response of $e(t)$.

Now, we consider the reduced-order filter design, and two cases of such filters are considered.

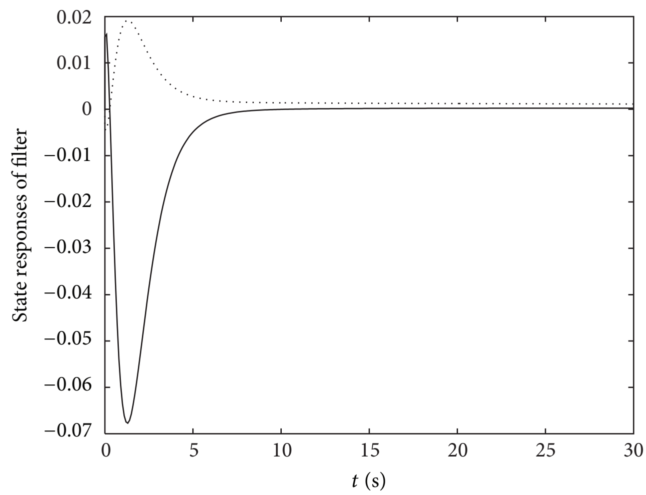

Figure 3: States of the designed reduced-order filter (with the order of $k = 2$).

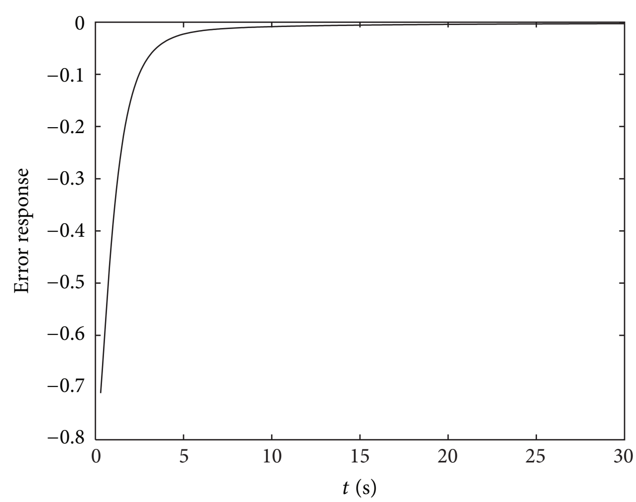

Figure 4: Filtering error of the reduced-order filtering (with the order of $k = 2$).

Case 1. Set $k = 2$; that is, the order of the reduced filter is $k = 2$; solving the LMIs condition in Theorem 9, we obtain that the minimum γ is $\gamma^* = 0.5069$, the minimum entropy of the filtering error system is $\mathcal{I}(\mathbb{E};\gamma^*;\infty) = 0.9401$, and

$$A_f = \begin{bmatrix} -1.9483 & 0.5354 \\ 0.5302 & -0.1546 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.6662 \\ -0.1850 \end{bmatrix}, \quad L_f = \begin{bmatrix} 1.1058 & -0.3003 \end{bmatrix}. \tag{52}$$

Under the same conditions as in the full-order filter design, the states of the designed reduced-order filter are given in Figure 3. Figure 4 shows the error response of $e(t)$.

Case 2. Set $k = 1$; that is, the order of the reduced filter is $k = 1$; by solving the LMIs condition in Theorem 9, we obtain

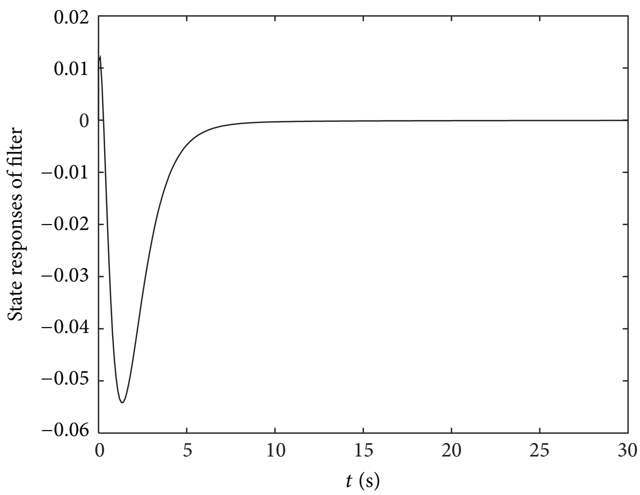

Figure 5: States of the designed reduced-order filter (with the order of $k = 1$).

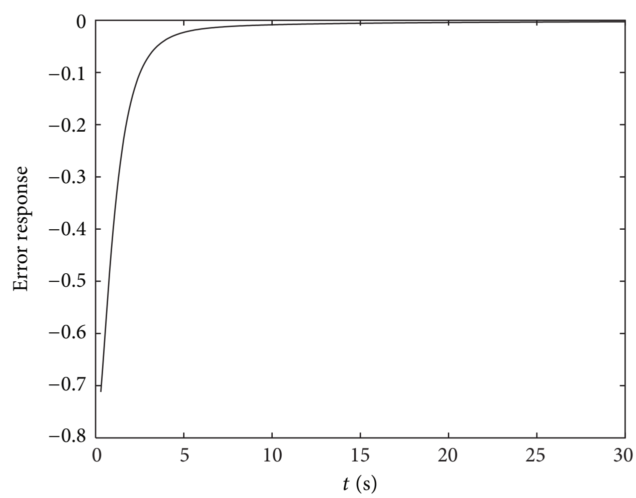

Figure 6: Filtering error of the reduced-order filtering (with the order of $k = 1$).

that the minimum γ is $\gamma^* = 0.5148$, the minimum entropy of the filtering error system is $\mathcal{I}(\mathbb{E}; \gamma^*; \infty) = 0.8677$, and

$$A_f = -1.8368, \qquad B_f = 0.4853, \qquad L_f = 1.2617. \quad (53)$$

The states and the filtering error of the designed reduced-order filter are given in Figures 5 and 6, respectively.

5. Conclusion

In this paper, the robust minimum entropy $\mathcal{H}_\infty$ filter has been designed for linear continuous-time systems with polytopic parameter uncertainty. Sufficient conditions have been established for the existence of general full- and reduced-order minimum entropy $\mathcal{H}_\infty$ filters in terms of LMIs, which guarantee the filtering error system to be robustly asymptotically stable and to have a prescribed $\mathcal{H}_\infty$ performance as well as a guaranteed minimum entropy. The filler design could be cast into a convex optimization problem and a numerical example has been provided to demonstrate the effectiveness of the proposed design method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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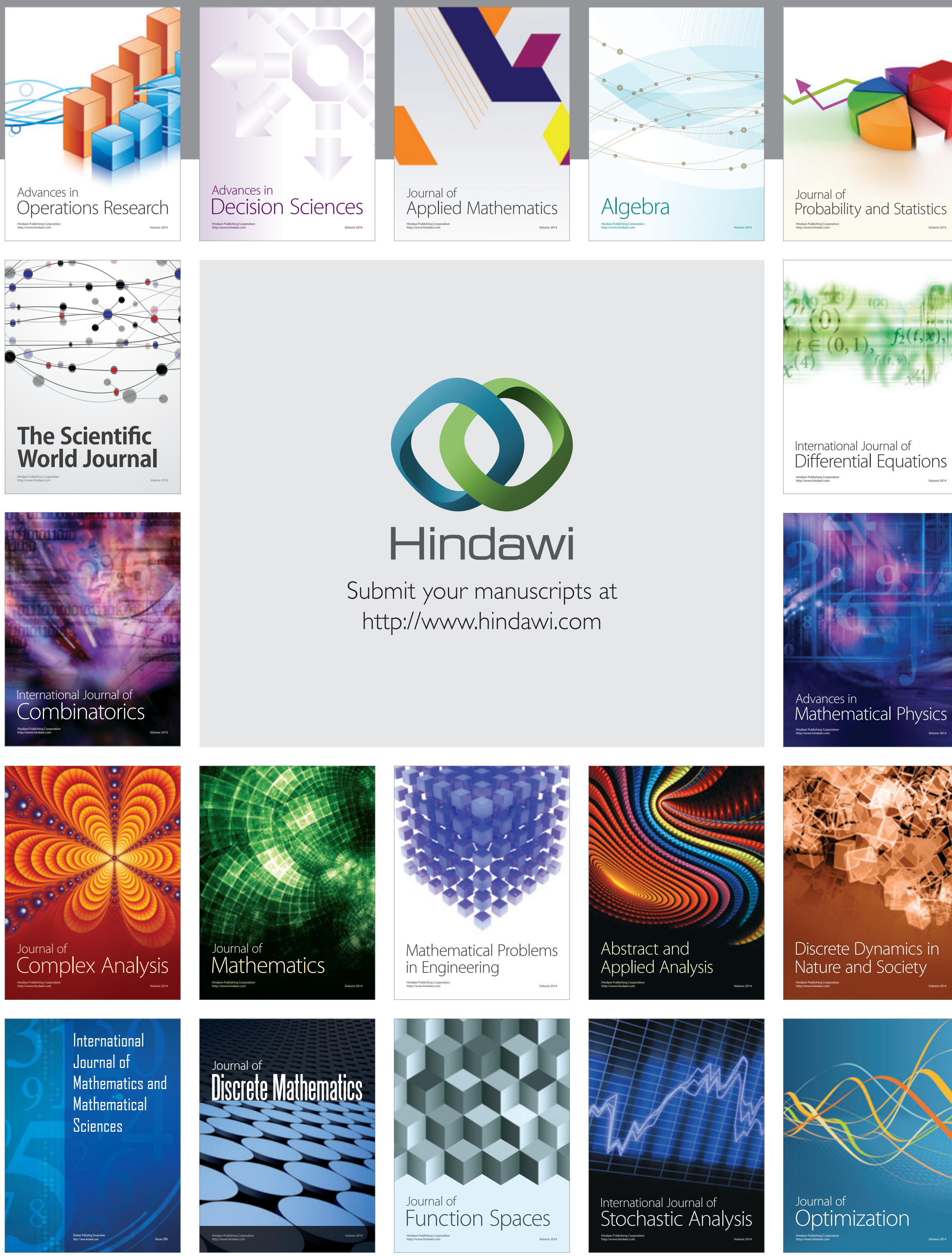
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