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Research Article

Existence of Solution via Integral Inequality of Volterra-Fredholm Neutral Functional Integrodifferential Equations with Infinite Delay

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In this work we study existence results for mixed Volterra-Fredholm neutral functional integrodifferential equations with infinite delay in Banach spaces. To obtain a priori bounds of solutions required in Krasnoselski-Schaefer type fixed point theorem, we have used an integral inequality established by B. G. Pachpatte. The variants for obtained results are given. An example is considered to illustrate the obtained results.

1. Introduction

In this paper we establish existence results for the mixed Volterra-Fredholm neutral functional integrodifferential equations with infinite delay of the form

$$\frac{d}{dt}\left[x(t)-g\left(t,x_t,\int_0^t e(t,s,x_s)\,ds\right)\right] = Ax(t)+f\left(t,x_t,\int_0^t k(t,s,x_s)\,ds,\int_0^b w(t,s,x_s)\,ds\right), \quad t\in J=[0,b],$$
$$x_0=\phi\in\mathscr{B}_h, \tag{1}$$

where A is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$, $t \geq 0$ in a Banach space X, $g : J\times\mathscr{B}_h\times X \to X$, $e, k, h : \Delta\times\mathscr{B}_h \to X$, and $f : J\times\mathscr{B}_h\times X\times X \to X$ are given functions, $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$, and $\mathscr{B}_h$ is a phase space defined later. The histories $x_t : (-\infty, 0] \to X$, $x_t(s) = x(t + s)$, $s \leq 0$, belong to the abstract phase space $\mathscr{B}_h$.

Due to the importance of neutral functional differential and integrodifferential equations with infinite delay in diverse fields of applied mathematics, these equations have generated considerable interest among researchers. Excellent account on the work with infinite delay can be found in [1–5]. The work in partial neutral functional differential equations with unbounded delay was initiated by Hernández and Henríquez [6, 7] and they have investigated the results pertaining to existence of mild, strong, and periodic solutions to the neutral functional differential equations. Recently, several works reported on existence results and controllability problem for various special forms of (1) and their variants with impulse or inclusion. Hernández [8] proved existence results for special form of (1) with $e = 0$, $h = 0$, by using the Leray-Schauder alternative. Li et al. [9] investigated the controllability problem when $e = 0$ and $f = \int_0^t k(s,x_s)ds$ by applying Sadovskii fixed point theorem. Henríquez [10, 11] has studied approximation and regularity of solutions of functional differential equations with unbounded delay. Chang et al. [12] established existence results for neutral functional integrodifferential equations with infinite delay using the resolvent operators and Krasnoselski-Schaefer type fixed point theorem. The work related to existence and controllability results with the impulse effect and infinite delay can be found in [13–15] and some of the references cited therein. The recent investigations on this theme can also be found in the work of Henriquez and dos Santos [16].

The authors [17–20] have studied existence, uniqueness, continuous dependence, and other properties of the solution of special forms of (1) with finite delay.

In this paper we investigate the existence results for (1) by using Krasnoselski-Schaefer type fixed point theorem via integral inequality by Pachpatte. We further prove existence results for the same equation without using integral inequality with different assumptions on the functions involved in the equation. To study (1), we use an abstract phase space $\mathscr{B}_h$ given by Yan [21] instead of seminormed space, introduced by Hale and Kato in [3].

The paper is organized as follows. In Section 2, we present the preliminaries. Section 3 is concerned with main results and proof. In Section 4, we present an example to illustrate the application of our results.

2. Preliminaries

We give some preliminaries from [21, 22] that will be used in our subsequent discussion. Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < +\infty$. For any $a > 0$, we define

$$\mathscr{B}_a = \{\psi : [-a, 0] \longrightarrow X \text{ such that } \psi(t) \text{ is bounded and measurable}\} \tag{2}$$

and equip the space $\mathscr{B}_a$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|, \quad \forall \psi \in \mathscr{B}_a. \tag{3}$$

Let us define

$$\mathscr{B}_h = \left\{\psi : (-\infty, 0] \longrightarrow X \text{ such that for any } c > 0, \; \psi|_{[-c,0]} \in \mathscr{B}_c, \int_{-\infty}^{0} h(s) \|\psi\|_{[s,0]} ds < +\infty\right\}. \tag{4}$$

If $\mathscr{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \|\psi\|_{[s,0]} ds, \quad \forall \psi \in \mathscr{B}_h, \tag{5}$$

then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a Banach space.

Now we consider the space

$$\mathscr{B}'_h = \{x : (-\infty, b] \longrightarrow X \text{ such that } x|_J \in C(J, X), \; x_0 \in \mathscr{B}_h\}. \tag{6}$$

Set $\|\cdot\|_b$ to be a seminorm in $\mathscr{B}'_h$ defined by

$$\|x\|_b = \|x_0\|_{\mathscr{B}_h} + \sup\{\|x(s)\| : s \in [0, b]\}, \quad x \in \mathscr{B}'_h. \tag{7}$$

Let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$, $t \geq 0$ on a Banach space X with the norm $\|\cdot\|$, and let $0 \in \rho(A)$; then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear invertible operator with domain $D(-A)^\alpha$ dense in X. The closedness of $D(-A)^\alpha$ implies that $D(-A)^\alpha$ endowed with the graph norm $\|x\|_X = \|x\| + \|(-A)^\alpha x\|$ is a Banach space. Since $(-A)^\alpha$ is invertible, its graph norm $\|x\|_X$ is equivalent to the norm $|x| = \|(-A)^\alpha x\|$. Thus $D(-A)^\alpha$ equipped with the norm $|\cdot|$ is a Banach space which we denote by X_α.

The following lemmas play an important role in our further discussions.

Lemma 1 (see [22]). *The following properties hold.*

(i) *If $0 < \beta < \alpha \leq 1$, then $X_\alpha \subset X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.*

(ii) *For every $0 < \alpha \leq 1$, there exists $C_\alpha > 0$ such that*

$$\left\|(-A)^\alpha T(t)\right\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq b. \tag{8}$$

Lemma 2 (see [23]). *Let X be a Banach space, and let Φ_1, Φ_2 be two operators on X such that*

(a) *Φ_1 is contraction, and*

(b) *Φ_2 is completely continuous.*

Then either

(i) *the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution or*

(ii) *the set $G = \{x \in X : \lambda\Phi_1(x/\lambda) + \lambda\Phi_2 x = x, \lambda \in (0, 1)\}$ is unbounded.*

Lemma 3 (see [8]). *Let $u(\cdot)$, $v(\cdot)$: $[0, b] \to [0, \infty)$ be continuous functions. If $v(\cdot)$ is nondecreasing and there are constants $\theta > 0$, $0 < \alpha < 1$ such that*

$$u(t) \leq v(t) + \theta \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad t \in J, \tag{9}$$

then

$$u(t) \leq \exp\left[\frac{\theta^n (\Gamma(\alpha))^n t^{n\alpha}}{\Gamma(n\alpha)}\right] \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j v(t), \tag{10}$$

for every $t \in [0, b]$ and every $n \in N$ such that $n\alpha > 1$, and $\Gamma(\cdot)$ is the Gamma function.

Lemma 4 (see [24]). *Assume $x \in \mathscr{B}'_h$; then for $t \in J$, $x_t \in \mathscr{B}_h$. Moreover,*

$$l\|x(t)\| \leq \|x_t\|_{\mathscr{B}_h} \leq \|x_0\|_{\mathscr{B}_h} + l \sup_{s\in[0,t]} \|x(s)\|, \tag{11}$$

where $l = \int_{-\infty}^{0} h(t)dt < +\infty$.

Lemma 5 (see [25], p-47). *Let $z(t)$, $u(t)$, $v(t)$, $w(t)$ $\in$ $C([\alpha,\beta],R_+)$, and $k \geq 0$ be a real constant and*

$$z(t) \leq k + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma) z(\sigma)\, d\sigma + \int_\alpha^\beta w(\sigma) z(\sigma)\, d\sigma\right] ds, \quad \text{for } t \in [\alpha,\beta]. \tag{12}$$

If

$$r^* = \int_\alpha^\beta w(\sigma) \exp\left(\int_\alpha^\sigma [u(\tau) + v(\tau)]\, d\tau\right) d\sigma < 1, \tag{13}$$

then

$$z(t) \leq \frac{k}{1-r^*} \exp\left(\int_\alpha^t [u(s) + v(s)]\, ds\right), \quad \text{for } t \in [\alpha,\beta]. \tag{14}$$

Definition 6. A function $x : (-\infty, b] \to X$ is called a mild solution of the problem (1) if $x_0 = \phi \in \mathscr{B}_h$ on $(-\infty, 0]$, the restriction of $x(\cdot)$ to the interval J is continuous, and for each $s \in [0,t)$ the function $AT(t-s)g(s, x_s, \int_0^s e(s,\tau,x_\tau)d\tau)$ is integrable and the integral equation

$$\begin{aligned} x(t) = {} & T(t)[\phi(0) - g(0,\phi,0)] + g\left(t, x_t, \int_0^t e(t,s,x_s)\, ds\right) \\ & + \int_0^t AT(t-s)\, g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\, d\tau\right) ds \\ & + \int_0^t T(t-s)\, f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, d\tau, \int_0^b w(s,\tau,x_\tau)\, d\tau\right) ds, \quad t \in J, \end{aligned} \tag{15}$$

is satisfied.

Definition 7. A map $f: J \times \mathscr{B}_h \times X \times X \to X$ is said to be an L^1 -Caratheodory if

(i) for each $t \in J$, the function $f(t,\cdot,\cdot,\cdot) : \mathscr{B}_h \times X \times X \to X$ is continuous;

(ii) for each $(\psi, x, y) \in \mathscr{B}_h \times X \times X$; the function $f(\cdot,\psi,x,y) : J \to X$ is strongly measurable;

(iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1(J, R_+)$ such thatand for almost all $t \in J$.

$$\|f(t,\psi,x,y)\| \leq \alpha_m(t), \quad \forall \|\psi\|_{\mathscr{B}_h} \leq m, \quad \|x\| \leq m, \quad \|y\| \leq m, \tag{16}$$

3. Existence Results

In this section we state and prove our main results. We list the following hypotheses for our convenience.

(H1) A is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$, $t > 0$ in X and $0 \in \rho(A)$ such that

$$\|T(t)\| \leq M, \quad \forall t \geq 0, \qquad \left\|(-A)^{1-\alpha} T(t-s)\right\| \leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}, \quad 0 < t \leq b, \tag{17}$$

where $0 \leq \alpha < 1$.

(H2) There exist constants $0 < \beta < 1$, $L_1, L_2 > 0$ such that g is X_β-valued, $(-A)^\beta g$ is continuous, and

(i) $\|e(t,s,\psi) - e(t,s,\chi)\| \leq L_1 \|\psi - \chi\|_{\mathscr{B}_h}$, $t,s \in J$, ψ, $\chi \in \mathscr{B}_h$,

(ii) $\|(-A)^\beta g(t,\psi,x) - (-A)^\beta g(t,\chi,y)\| \leq L_2[\|\psi - \chi\|_{\mathscr{B}_h} + \|x - y\|]$, $t \in J$, $\psi, \chi \in \mathscr{B}_h$ with

$$C_0 = lL_2(1 + bL_1)\left[\left\|(-A)^{-\beta}\right\| + \frac{\left(C_{1-\beta} b^\beta\right)}{\beta}\right] < 1. \tag{18}$$

(H3) There exist integrable functions $p, q, r: J \to [0,\infty)$ such that

(i) $\|k(t,s,\psi)\| \leq q(s)\|\psi\|_{\mathscr{B}_h}$, $(t,s) \in \Delta$, $\psi \in \mathscr{B}_h$,

(ii) $\|w(t,s,\psi)\| \leq r(s)\|\psi\|_{\mathscr{B}_h}$, $(t,s) \in \Delta$, $\psi \in \mathscr{B}_h$,

(iii) $\|f(t,\psi,x,y)\| \leq p(t)\,(\|\psi\|_{\mathscr{B}_h} + \|x\| + \|y\|)$, for each $(t,\psi,x,y) \in J \times \mathscr{B}_h \times X \times X$.

(H4) For each $(t,s) \in \Delta$, the functions $k(t,s,\cdot)$, $w(t,s,\cdot)$: $\mathscr{B}_h \to X$ are continuous and for each $\psi \in \mathscr{B}_h$, the functions $k(\cdot,\cdot,\psi)$, $w(\cdot,\cdot,\psi)$: $\Delta \to X$ are strongly measurable.

(H5) f is an L^1-Caratheodory.

(H6) The condition $\int_0^b r(\sigma)\exp(\int_0^\sigma [B_0 K_1 p(\tau) + q(\tau)]d\tau)d\sigma := R < 1$ holds, where

$$\begin{aligned} B_0 &= \exp\left[\frac{K_2^n (\Gamma(\beta))^n b^{n\beta}}{\Gamma(n\beta)}\right] \sum_{j=0}^{n-1} \left(\frac{K_2 b^\beta}{\beta}\right)^j, \\ & \qquad n \in N \text{ such that } n\beta > 1, \\ K_1 &= \frac{lM}{1 - lL_2(1 + bL_1)\left\|(-A)^{-\beta}\right\|}, \\ K_2 &= \frac{lL_2(1 + bL_1)C_{1-\beta}}{1 - lL_2(1 + bL_1)\left\|(-A)^{-\beta}\right\|}. \end{aligned} \tag{19}$$

We set $c_1 = b\,\sup_{(t,s)\in\Delta}\|e(t,s,0)\|$ and $c_2 = \|(-A)^\beta\|\sup_{t\in J}\|g(t,0,0)\|$.

Using the hypotheses (H1), (H2) and Lemma 1, we have the following inequality:

$$\begin{aligned}
&\left\| AT(t-s)\, g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\, d\tau\right)\right\| \\
&\leq \left\|(-A)^{1-\beta} T(t-s)\right\| \left\{ \left\| (-A)^{\beta} g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\, d\tau\right) \right.\right. \\
&\qquad \left.\left. -(-A)^{\beta} g(s,0,0) \right\| + c_2 \right\} \\
&\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\{ L_2 \left[\|x_s\|_{\mathscr{B}_h} + \left\| \int_0^s e(s,\tau,x_\tau)\, d\tau \right\| \right] + c_2 \right\} \\
&\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\{ L_2 \left[\|x_s\|_{\mathscr{B}_h} + \int_0^s \left\| e(s,\tau,x_\tau) - e(s,\tau,0)\right\| d\tau \right.\right. \\
&\qquad \left.\left. + c_1 \right] + c_2 \right\} \\
&\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\{ L_2 \left[(1+bL_1) \|x_s\|_{\mathscr{B}_h} + c_1 \right] + c_2 \right\}.
\end{aligned} \tag{20}$$

Thus from Bochner theorem, it follows that $AT(t-s)g(s,x_s,\int_0^s e(s,\tau,x_\tau)d\tau)$ is integrable on $[0,t)$.

Throughout this paper, for brevity we set

$$\begin{aligned}
F &= M\|\phi(0)\| + M\left\|(-A)^{-\beta}\right\| \left[L_2 \|\phi\|_{B_h} + c_2 \right] \\
&\quad + \left\|(-A)^{-\beta}\right\| (L_2 c_1 + c_2) + \frac{C_{1-\beta}(L_2 c_1 + c_2)\, b^{\beta}}{\beta}.
\end{aligned} \tag{21}$$

In the following theorem we establish a priori bound for the mild solution of the following system by using Pachpatte inequality:

$$\begin{aligned}
&\frac{d}{dt}\left[x(t) - \lambda g\left(t, x_t, \int_0^t e(t,s,x_s)\, ds\right)\right] \\
&= Ax(t) + \lambda f\left(t, x_t, \int_0^t k(t,s,x_s)\, ds, \int_0^b w(t,s,x_s)\, ds\right), \\
&\qquad t \in J = [0,b], \\
&\qquad x_0 = \phi \in \mathscr{B}_h,
\end{aligned} \tag{22}$$

where $\lambda \in (0,1)$. By Definition 6, the mild solution of the system (22) is given by

$$\begin{aligned}
x(t) &= T(t)\left[\phi(0) - \lambda g(0,\phi,0)\right] \\
&\quad + \lambda g\left(t, x_t, \int_0^t e(t,s,x_s)\, ds\right) \\
&\quad + \lambda \int_0^t AT(t-s)\, g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\, d\tau\right) ds \\
&\quad + \lambda \int_0^t T(t-s)\, f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, d\tau, \right. \\
&\qquad \left. \int_0^b w(s,\tau,x_\tau)\, d\tau\right) ds, \\
&\qquad t \in J, \\
x_0 &= \phi \in \mathscr{B}_h.
\end{aligned} \tag{23}$$

Theorem 8. *If hypotheses (H1)–(H6) are satisfied and letting $x(t)$ be a mild solution of the system (22), then $\|x_t\|_{\mathscr{B}_h} \leq B_0 K_3/(1-R)\exp(\int_0^b [B_0 K_1 p(s) + q(s)] ds) := K$, $t \in J$, where*

$$K_3 = \frac{\|\phi\|_{\mathscr{B}_h} + lF}{1 - lL_2(1+bL_1)\left\|(-A)^{-\beta}\right\|}. \tag{24}$$

Proof. Using the hypotheses (H1)–(H3) in (23), we get

$$\begin{aligned}
&\|x(t)\| \\
&\leq M\|\phi(0)\| + M\left\|(-A)^{-\beta}\right\| \\
&\quad \times \left[\left\| (-A)^{\beta} g(0,\phi,0) - (-A)^{\beta} g(t,0,0)\right\| + c_2 \right] \\
&\quad + \left\|(-A)^{-\beta}\right\| \left\{ \left\| (-A)^{\beta} g\left(t, x_t, \int_0^t e(t,s,x_s)\, ds\right) \right.\right. \\
&\qquad \left.\left. - (-A)^{\beta} g(t,0,0) \right\| + c_2 \right\} \\
&\quad + \int_0^t \left\|(-A)^{1-\beta} T(t-s)\right\| \\
&\quad \times \left\{ \left\| (-A)^{\beta} g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\, d\tau\right) \right.\right. \\
&\qquad \left.\left. -(-A)^{\beta} g(t,0,0) \right\| + c_2 \right\} ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|T(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\, d\tau, \right.\right. \\
&\qquad\qquad \left.\left. \int_0^b w(s,\tau,x_\tau)\, d\tau\right)\right\| ds \\
&\leq M\|\phi(0)\| + M\left\|(-A)^{-\beta}\right\|\left[L_2\|\phi\|_{\mathscr{B}_h} + c_2\right] \\
&+\left\|(-A)^{-\beta}\right\|\left\{L_2\left[\|x_t\|_{\mathscr{B}_h} \right.\right. \\
&\qquad\qquad \left.\left. + \left\|\int_0^t e(t,s,x_s)\, ds\right\|\right] + c_2\right\} \\
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \\
&\times\left\{L_2\left[\|x_s\|_{\mathscr{B}_h} + \left\|\int_0^s e(s,\tau,x_\tau)\, d\tau\right\|\right] + c_2\right\} ds \\
&+\int_0^t Mp(s)\left[\|x_s\|_{\mathscr{B}_h} + \int_0^s q(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau\right] ds \\
&\leq M\|\phi(0)\| + M\left\|(-A)^{-\beta}\right\|\left[L_2\|\phi\|_{\mathscr{B}_h} + c_2\right] \\
&+\left\|(-A)^{-\beta}\right\|\left\{L_2\left[(1+bL_1)\|x_t\|_{\mathscr{B}_h} + c_1\right] + c_2\right\} \\
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \\
&\times\left\{L_2\left[(1+bL_1)\|x_s\|_{\mathscr{B}_h} + c_1\right] + c_2\right\} ds \\
&+\int_0^t Mp(s)\left[\|x_s\|_{\mathscr{B}_h} + \int_0^s q(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau\right] ds \\
&\leq M\|\phi(0)\| + M\left\|(-A)^{-\beta}\right\|\left[L_2\|\phi\|_{\mathscr{B}_h} + c_2\right] \\
&+\left\|(-A)^{-\beta}\right\|(L_2c_1 + c_2) + \frac{C_{1-\beta}(L_2c_1+c_2)b^\beta}{\beta} \\
&+\left\|(-A)^{-\beta}\right\|L_2(1+bL_1)\|x_t\|_{\mathscr{B}_h} \\
&+C_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{B_h}}{(t-s)^{1-\beta}} ds \\
&+\int_0^t Mp(s)\left[\|x_s\|_{\mathscr{B}_h} + \int_0^s q(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau\right] ds \\
&= F + \left\|(-A)^{-\beta}\right\|L_2(1+bL_1)\|x_t\|_{\mathscr{B}_h} \\
&+C_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{B_h}}{(t-s)^{1-\beta}} ds \\
&+\int_0^t Mp(s)\left[\|x_s\|_{\mathscr{B}_h} + \int_0^s q(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau\right] ds.
\end{aligned}
\tag{25}
$$

From inequality (25) and Lemma 4, we have

$$
\begin{aligned}
\|x_t\|_{\mathscr{B}_h} &\leq \|\phi\|_{\mathscr{B}_h} + l\sup_{s\in[0,t]}|x(s)| \\
&\leq \|\phi\|_{\mathscr{B}_h} + lF \\
&\quad + l\left\|(-A)^{-\beta}\right\|L_2(1+bL_1)\sup_{s\in[0,t]}\|x_s\|_{\mathscr{B}_h} \\
&\quad + lC_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{\mathscr{B}_h}}{(t-s)^{1-\beta}} ds \\
&\quad + \int_0^t lMp(s)\left[\|x_s\|_{\mathscr{B}_h} + \int_0^s q(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\|x_\tau\|_{\mathscr{B}_h} d\tau\right] ds, \\
&\qquad\qquad t \in J.
\end{aligned}
\tag{26}
$$

Define the function $\mu(t) = \sup\{\|x_s\|_{\mathscr{B}_h} : 0 \leq s \leq t\}$, $t \in J$; then $\mu(t)$ is nondecreasing on J, and we get

$$
\begin{aligned}
\mu(t) &\leq \|\phi\|_{\mathscr{B}_h} + lF + l\left\|(-A)^{-\beta}\right\|L_2(1+bL_1)\mu(t) \\
&\quad + lC_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} ds \\
&\quad + \int_0^t lMp(s)\left[\mu(s) + \int_0^s q(\tau)\mu(\tau)\, d\tau \right. \\
&\qquad\qquad \left. + \int_0^b r(\tau)\mu(\tau)\, d\tau\right] ds.
\end{aligned}
\tag{27}
$$

Therefore,

$$\begin{aligned}\mu(t) \leq{}& \frac{\|\phi\|_{\mathscr{B}_h} + lF}{1 - lL_2(1+bL_1)\left\|(-A)^{-\beta}\right\|} \\ &+ \frac{lL_2(1+bL_1)C_{1-\beta}}{1 - lL_2(1+bL_1)\left\|(-A)^{-\beta}\right\|}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds \\ &+ \frac{lM}{1 - lL_2(1+bL_1)\left\|(-A)^{-\beta}\right\|} \\ &\times \int_0^t p(s)\left[\mu(s) + \int_0^s q(\tau)\mu(\tau)\,d\tau \right. \\ &\qquad\left. + \int_0^b r(\tau)\mu(\tau)\,d\tau\right] ds \\ ={}& K_3 + K_1\int_0^t p(s)\left[\mu(s) + \int_0^s q(\tau)\mu(\tau)\,d\tau \right. \\ &\qquad\left. + \int_0^b r(\tau)\mu(\tau)\,d\tau\right] ds \\ &+ K_2\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds. \end{aligned} \tag{28}$$

Using Lemma 3, we have

$$\begin{aligned}\mu(t) \leq{}& B_0\left(K_3 + K_1\int_0^t p(s)\right. \\ &\times\left[\mu(s) + \int_0^s q(\tau)\mu(\tau)\,d\tau\right. \\ &\left.\left. + \int_0^b r(\tau)\mu(\tau)\,d\tau\right] ds\right) \end{aligned} \tag{29}$$

$$\begin{aligned} ={}& B_0K_3 + \int_0^t B_0K_1 p(s) \\ &\times\left[\mu(s) + \int_0^b r(\tau)\mu(\tau)\,d\tau\right. \\ &\left. + \int_0^s q(\tau)\mu(\tau)\,d\tau\right] ds, \end{aligned}$$

where

$$B_0 = \exp\left[\frac{K_2^n(\Gamma(\beta))^n b^{n\beta}}{\Gamma(n\beta)}\right]\sum_{j=0}^{n-1}\left(\frac{K_2 b^\beta}{\beta}\right)^j. \tag{30}$$

Thanks to Pachpatte's inequality given in Lemma 5 and applying it with $z(t) = \mu(t)$ and using hypothesis (H6), we obtain

$$\mu(t) \leq \frac{B_0K_3}{1-R}\exp\left(\int_0^t \left[B_0K_1 p(s) + q(s)\right] ds\right), \quad t \in J. \tag{31}$$

This implies that $\|x_t\|_{\mathscr{B}_h} \leq K$, $t \in J$. □

Now we define the operator $\Psi : B_h' \to B_h'$ by

$$\Psi x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ T(t)\left[\phi(0) - g(0,\phi,0)\right] & \\ \quad + g\left(t, x_t, \int_0^t e(t,s,x_s)\,ds\right) & \\ \quad + \int_0^t AT(t-s) & \\ \quad \times g\left(s, x_s, \int_0^s e(s,\tau,x_\tau)\,d\tau\right) ds & \\ \quad + \int_0^t T(t-s) & \\ \quad \times f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)\,d\tau,\right. & \\ \qquad\qquad \left.\int_0^b w(s,\tau,x_\tau)\,d\tau\right) ds, & t \in J. \end{cases} \tag{32}$$

For $\phi \in \mathscr{B}_h$, define $\widetilde{\phi}$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T(t)\phi(0), & t \in J = [0, b]; \end{cases} \tag{33}$$

then $\widetilde{\phi} \in \mathscr{B}_h'$. Let $x(t) = y(t) + \widetilde{\phi}(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies (15) if and only if y satisfies $y_0 = 0$ and

$$\begin{aligned} y(t) ={}& -T(t)g(0,\phi,0) \\ &+ g\left(t, y_t + \widetilde{\phi}_t, \int_0^t e\left(t,s,y_s + \widetilde{\phi}_s\right) ds\right) \\ &+ \int_0^t AT(t-s)\, g\left(s, y_s + \widetilde{\phi}_s, \int_0^s e\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds \\ &+ \int_0^t T(t-s)\, f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau,\right. \\ &\qquad \left.\int_0^b w\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds. \end{aligned} \tag{34}$$

Let $\mathscr{B}_h'' = \{y \in \mathscr{B}_h' : y_0 = 0 \in \mathscr{B}_h\}$; then for any $y \in \mathscr{B}_h''$ we have

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathscr{B}_h} + \sup\{\|y(s)\| : 0 \leq s \leq b\} \\ &= \sup\{\|y(s)\| : 0 \leq s \leq b\}; \end{aligned} \tag{35}$$

thus $(\mathscr{B}_h'', \|\cdot\|_b)$ is a Banach space. Define $B_m = \{y \in \mathscr{B}_h'' : \|y\|_b \le m\}$ for some $m > 0$; then $B_m \subseteq \mathscr{B}_h''$ is uniformly bounded, and for $y \in B_m$, from Lemma 4, we have

$$\begin{aligned}\left\|y_t + \tilde{\phi}_t\right\|_{\mathscr{B}_h} &\le \|y_t\|_{\mathscr{B}_h} + \left\|\tilde{\phi}_t\right\|_{\mathscr{B}_h} \\ &\le \|y_0\|_{\mathscr{B}_h} + l \sup_{s\in[0,t]} \|y(s)\| \\ &\quad + \left\|\tilde{\phi}_0\right\|_{\mathscr{B}_h} + l \sup_{s\in[0,t]} \left\|\tilde{\phi}(s)\right\| \\ &\le lm + \left\|\tilde{\phi}_0\right\|_{\mathscr{B}_h} + l \sup_{s\in[0,b]} \|T(s)\phi(0)\| \\ &\le l(m + M\|\phi(0)\|) + \|\phi\|_{\mathscr{B}_h} = m'.\end{aligned} \tag{36}$$

Define the operator $\overline{\Psi} : \mathscr{B}_h'' \to \mathscr{B}_h''$ by

$$\overline{\Psi} y(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ -T(t)\, g(0, \phi, 0) \\ \quad + g\left(t, y_t + \tilde{\phi}_t, \int_0^t e\left(t, s, y_s + \tilde{\phi}_s\right) ds\right) \\ \quad + \int_0^t AT(t-s) \\ \quad\quad \times g\left(s, y_s + \tilde{\phi}_s, \int_0^s e\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau\right) ds \\ \quad + \int_0^t T(t-s) \\ \quad\quad \times f\left(s, y_s + \tilde{\phi}_s, \int_0^s k\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau\right) ds, & t \in J. \end{cases} \tag{37}$$

In the view of Krasnoselski-Schaefer type fixed point theorem, we decompose $\overline{\Psi}$ as $\overline{\Psi}_1 + \overline{\Psi}_2$, where $\overline{\Psi}_1$ and $\overline{\Psi}_2$ are defined on $\mathscr{B}_h''$, respectively, by

$$\begin{aligned}\overline{\Psi}_1 y(t) &= -T(t)\, g(0, \phi, 0) \\ &\quad + g\left(t, y_t + \tilde{\phi}_t, \int_0^t e\left(t, s, y_s + \tilde{\phi}_s\right) ds\right) \\ &\quad + \int_0^t AT(t-s)\, g\left(s, y_s + \tilde{\phi}_s, \int_0^s e\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau\right) ds, \\ &\qquad t \in J,\end{aligned} \tag{38}$$

$$\begin{aligned}\overline{\Psi}_2 y(t) &= \int_0^t T(t-s) \\ &\quad \times f\left(s, y_s + \tilde{\phi}_s, \int_0^s k\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s, \tau, y_\tau + \tilde{\phi}_\tau\right) d\tau\right) ds, \quad t \in J.\end{aligned} \tag{39}$$

Observe that the operator Ψ having a fixed point is equivalent to $\overline{\Psi}$ having one. Next, our aim is to prove that the operator $\overline{\Psi}_1$ is a contraction, while $\overline{\Psi}_2$ is a completely continuous operator.

Theorem 9. *If the hypotheses (H1) and (H2) are satisfied, then $\overline{\Psi}_1$ is a contraction on $\mathscr{B}_h''$.*

Proof. Let any $u, v \in \mathscr{B}_h''$; then by using the hypotheses (H1), (H2) and Lemma 4, from (38) for each $t \in J$, we have

$$\begin{aligned}&\left\|\overline{\Psi}_1 u(t) - \overline{\Psi}_1 v(t)\right\| \\ &\le \left\|(-A)^{\beta}\left[(-A)^{\beta} g\left(t, u_t + \tilde{\phi}_t, \int_0^t e\left(t, s, u_s + \tilde{\phi}_s\right) ds\right) - (-A)^{\beta} g\left(t, v_t + \tilde{\phi}_t, \int_0^t e\left(t, s, v_s + \tilde{\phi}_s\right) ds\right)\right]\right\| \\ &\quad + \int_0^t \left\|(-A)^{1-\beta} T(t-s)\right\| \\ &\quad \times \left\|(-A)^{\beta} g\left(s, u_s + \tilde{\phi}_s, \int_0^s e\left(s, \tau, u_\tau + \tilde{\phi}_\tau\right) d\tau\right) - (-A)^{\beta} g\left(s, v_s + \tilde{\phi}_s, \int_0^s e\left(s, \tau, v_\tau + \tilde{\phi}_\tau\right) d\tau\right)\right\| ds \\ &\le \left\|(-A)^{-\beta}\right\| L_2 \left[\|u_t - v_t\|_{\mathscr{B}_h} + \int_0^t \left\|e\left(t, s, u_s + \tilde{\phi}_s\right) - e\left(t, s, v_s + \tilde{\phi}_s\right)\right\| ds\right] \\ &\quad + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_2 \left[\|u_s - v_s\|_{\mathscr{B}_h}\right.\end{aligned}$$

$$+\int_0^s \left\| e\left(s,\tau,u_\tau+\widetilde{\phi}_\tau\right) -e\left(s,\tau,v_\tau+\widetilde{\phi}_\tau\right)\right\| d\tau \Bigg] ds$$

$$\le \left\|(-A)^{-\beta}\right\| L_2\left(1+bL_1\right)\|u_t-v_t\|_{\mathscr{B}_h}$$
$$+C_{1-\beta}\, L_2\left(1+bL_1\right)\|u_t-v_t\|_{\mathscr{B}_h}\int_0^t \frac{ds}{(t-s)^{1-\beta}}$$
$$\le L_2\left(1+bL_1\right)\left[\left\|(-A)^{-\beta}\right\|+\frac{C_{1-\beta}b^\beta}{\beta}\right]\|u_t-v_t\|_{\mathscr{B}_h}$$
$$\le L_2\left(1+bL_1\right)\left[\left\|(-A)^{-\beta}\right\|+\frac{C_{1-\beta}b^\beta}{\beta}\right]$$
$$\times\left[\|u_0\|_{\mathscr{B}_h}+\|v_0\|_{\mathscr{B}_h}+l\sup_{s\in[0,t]}\|u(s)-v(s)\|\right]$$
$$\le lL_2\left(1+bL_1\right)\left[\left\|(-A)^{-\beta}\right\|+\frac{C_{1-\beta}b^\beta}{\beta}\right]\sup_{s\in[0,b]}\|u(s)-v(s)\|, \tag{40}$$

since $\|u_0\|_{\mathscr{B}_h}=0$, $\|v_0\|_{\mathscr{B}_h}=0$.

This implies that

$$\left\|\overline{\Psi}_1u-\overline{\Psi}_1v\right\|_b\le C_0\|u-v\|_b. \tag{41}$$

Since $C_0<1$, $\overline{\Psi}_1$ is contraction on $\mathscr{B}_h''$. □

Theorem 10. *If the hypotheses (H1), (H3)–(H5) are satisfied, then $\overline{\Psi}_2:\mathscr{B}_h''\to\mathscr{B}_h''$ is completely continuous operator.*

Proof. We give the proof in the following steps.

Step 1. $\overline{\Psi}_2$ maps bounded sets into bounded sets in $\mathscr{B}_h''$.

Let any $y\in B_m=\{y\in\mathscr{B}_h'':\|y\|_b\le m\}$. Then it is enough to prove that $\|\overline{\Psi}_2y\|_b\le\Lambda$ for some constant Λ. By using the hypotheses (H1), (H3) and condition (36) from (39), we have

$$\left\|\overline{\Psi}_2y(t)\right\|$$
$$\le\int_0^t Mp(s)\left[\left\|y_s+\widetilde{\phi}_s\right\|_{\mathscr{B}_h}+\int_0^s q(\tau)\left\|y_\tau+\widetilde{\phi}_\tau\right\|_{\mathscr{B}_h}d\tau +\int_0^b r(\tau)\left\|y_\tau+\widetilde{\phi}_\tau\right\|_{\mathscr{B}_h}d\tau\right]ds \tag{42}$$
$$\le\int_0^t Mp(s)\left[m'+\int_0^s m'q(\tau)\,d\tau+\int_0^b m'r(\tau)\,d\tau\right]ds$$
$$\le\int_0^b Mm'p(s)\left[1+\int_0^b\{q(\tau)+r(\tau)\}\,d\tau\right]ds:=\Lambda.$$

Thus for each $y\in B_m$, we have $\|\overline{\Psi}_2y\|_b\le\Lambda$.

Step 2. $\overline{\Psi}_2$ maps bounded sets into equicontinuous sets of $\mathscr{B}_h''$.

Let $y\in B_m$ and $t_1,t_2\in(-\infty,b]$. Then from (39), using the hypotheses (H1) and (H3) and condition (36), we have the following three cases.

Case 1. Let $0<t_1<t_2\le b$. Then, we have

$$\left\|\overline{\Psi}_2y(t_1)-\overline{\Psi}_2y(t_2)\right\|$$
$$\le\int_0^{t_1-\epsilon}\left\|T(t_1-s)-T(t_2-s)\right\| \times\left\|f\left(s,y_s+\widetilde{\phi}_s,\int_0^s k\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau, \int_0^b w\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau\right)\right\|ds$$
$$+\int_{t_1-\epsilon}^{t_1}\left\|T(t_1-s)-T(t_2-s)\right\| \times\left\|f\left(s,y_s+\widetilde{\phi}_s,\int_0^s k\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau, \int_0^b w\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau\right)\right\|ds$$
$$+\int_{t_1}^{t_2}\left\|T(t_2-s)\right\|\left\|f\left(s,y_s+\widetilde{\phi}_s,\int_0^s k\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau, \int_0^b w\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau\right)\right\|ds$$
$$\le\int_0^{t_1-\epsilon}\left\|T(t_1-s)-T(t_2-s)\right\|m'p(s) \times\left[1+\int_0^b\{q(\tau)+r(\tau)\}\,d\tau\right]ds$$
$$+\int_{t_1-\epsilon}^{t_1}\left\|T(t_1-s)-T(t_2-s)\right\|m'p(s) \times\left[1+\int_0^b\{q(\tau)+r(\tau)\}\,d\tau\right]ds$$
$$+\int_{t_1}^{t_2}\left\|T(t_2-s)\right\|m'p(s) \times\left[1+\int_0^b\{q(\tau)+r(\tau)\}\,d\tau\right]ds. \tag{43}$$

Case 2. Let $t_1\le 0\le t_2\le b$. Then, we have

$$\left\|\overline{\Psi}_2y(t_1)-\overline{\Psi}_2y(t_2)\right\|$$
$$\le\int_0^{t_2}\left\|T(t_2-s)\right\|\left\|f\left(s,y_s+\widetilde{\phi}_s,\int_0^s k\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau, \int_0^b w\left(s,\tau,y_\tau+\widetilde{\phi}_\tau\right)d\tau\right)\right\|ds$$
$$\le\int_0^{t_2} Mm'p(s)\left[1+\int_0^b\{q(\tau)+r(\tau)\}\,d\tau\right]ds. \tag{44}$$

Case 3. If $t_1 \le t_2 \le 0$, then $\|\overline{\Psi}_2 y(t_1) - \overline{\Psi}_2 y(t_2)\| = 0$.

From Cases 1–3, we deduce that the right-hand side of the above inequality tends to zero as $t_2 - t_1$ tends to 0 for ϵ sufficiently small, since the compactness of $T(t)$, $t > 0$, implies the continuity in the uniform operator topology. Thus the set $\{\overline{\Psi}_2 y : y \in B_m\}$ is equicontinuous.

Step 3. $\overline{\Psi}_2$ maps B_m into a precompact set in $\mathscr{B}''_h$.

Together with Arzela-Ascoli theorem and Steps 1–2 to prove $\overline{\Psi}_2 B_m$ is precompact in $\mathscr{B}''_h$, it is sufficient to show that the set $\{(\overline{\Psi}_2 y)(t) : y \in B_m\}$ is precompact in X. Let $0 < t \le b$ be fixed, and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_m$, we define the operators

$$\begin{aligned}(\overline{\Psi}_2^{\epsilon} y)(t) &= \int_0^{t-\epsilon} T(t-s) f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds \\ &= T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon) \times f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds.\end{aligned} \tag{45}$$

Since $T(\epsilon)$ is compact operator, the set $\overline{V}_\epsilon(t) = \{(\overline{\Psi}_2^{\epsilon} y)(t) : y \in B_m\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover, for each $y \in B_m$, we have

$$\begin{aligned}&\left\|(\overline{\Psi}_2 y)(t) - (\overline{\Psi}_2^{\epsilon} y)(t)\right\| \\ &\quad\le \int_{t-\epsilon}^t m' M p(s)\left[1 + \int_0^b \{q(\tau) + r(\tau)\} d\tau\right] ds \longrightarrow 0, \quad \text{as } \epsilon \to 0^+.\end{aligned} \tag{46}$$

Therefore there are precompact sets arbitrarily close to the set $\{(\overline{\Psi}_2 y)(t) : y \in B_m\}$. Thus the set $\{(\overline{\Psi}_2 y)(t) : y \in B_m\}$ is precompact in X.

Step 4. $\overline{\Psi}_2 : \mathscr{B}''_h \to \mathscr{B}''_h$ is continuous.

Let $\{y^{(n)}\}_{n=0}^{+\infty} \subseteq \mathscr{B}''_h$, with $y^{(n)} \to y$ in $\mathscr{B}''_h$. Then there is a number $m > 0$ such that $\|y^{(n)}(t)\| \le m$ for all n and a. e. $t \in J$, so $y^{(n)} \in B_m$ and $y \in B_m$.

By using the hypotheses (H4), (H5) and condition (36) we have

$$\begin{aligned}&f\left(t, y_t^{(n)} + \widetilde{\phi}_t, \int_0^t k\left(t,s,y_s^{(n)} + \widetilde{\phi}_s\right) ds, \int_0^b w\left(t,s,y_s^{(n)} + \widetilde{\phi}_s\right) ds\right) \\ &\quad\longrightarrow f\left(t, y_t + \widetilde{\phi}_t, \int_0^t k\left(t,s,y_s + \widetilde{\phi}_s\right) ds, \int_0^b w\left(t,s,y_s + \widetilde{\phi}_s\right) ds\right),\end{aligned} \tag{47}$$

for each $t \in J$, and since

$$\begin{aligned}&\left\| f\left(t, y_t^{(n)} + \widetilde{\phi}_t, \int_0^t k\left(t,s,y_s^{(n)} + \widetilde{\phi}_s\right) ds, \int_0^b w\left(t,s,y_s^{(n)} + \widetilde{\phi}_s\right) ds\right) \right. \\ &\quad \left. - f\left(t, y_t + \widetilde{\phi}_t, \int_0^t k\left(t,s,y_s + \widetilde{\phi}_s\right) ds, \int_0^b w\left(t,s,y_s + \widetilde{\phi}_s\right) ds\right)\right\| \\ &\quad\le 2m' p(s)\left[1 + \int_0^b \{q(\tau) + r(\tau)\} d\tau\right],\end{aligned} \tag{48}$$

we have by the dominated convergence theorem that

$$\begin{aligned}&\left\|\overline{\Psi}_2 y^{(n)} - \overline{\Psi}_2 y\right\|_b \\ &= \int_0^t \|T(t-s)\| \left\| \int_0^s f\left(s, y_s^{(n)} + \widetilde{\phi}_s, \int_0^s k\left(s,\tau,y_\tau^{(n)} + \widetilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s,\tau,y_\tau^{(n)} + \widetilde{\phi}_\tau\right) d\tau\right) \right. \\ &\quad \left. - f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau, \int_0^b w\left(s,\tau,y_\tau + \widetilde{\phi}_\tau\right) d\tau\right)\right\| ds \\ &\quad\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{49}$$

Therefore,

$$\left\|\overline{\Psi}_2 y^{(n)} - \overline{\Psi}_2 y\right\|_b = \sup_{t\in J} \left\|\overline{\Psi}_2 y^{(n)}(t) - \overline{\Psi}_2 y(t)\right\| \longrightarrow 0. \tag{50}$$

This implies that $\overline{\Psi}_2$ is continuous.

From Steps 1–4, we can conclude that the operator $\overline{\Psi}_2$ is completely continuous and thus satisfies condition (b) in Lemma 2. □

Theorem 11. *Assume that the hypotheses (H1)–(H6) hold. Then the problem* (1) *has at least one mild solution on* $(-\infty, b]$.

Proof. Let $G = \{y \in \mathscr{B}_h'' : y = \lambda\overline{\Psi}_1(y/\lambda) + \lambda\overline{\Psi}_2 y$ for some $\lambda \in (0,1)\}$. Then for any $y \in G$, the function $x = y + \widetilde{\phi}$ is a mild solution of the system (22) for which we have proved in Theorem 8 that $\|x_t\|_{\mathscr{B}_h} \le K, t \in J$, and hence from Lemma 4, we have

$$\begin{aligned}\|y\|_b &= \|y_0\|_{\mathscr{B}_h} + \sup\{\|y(t)\| : t \in [0,b]\}\\ &= \sup\{\|y(t)\| : t \in [0,b]\}\\ &\le \sup\{\|x(t)\| : t \in [0,b]\}\\ &\quad + \sup\{\|\widetilde{\phi}(t)\| : t \in [0,b]\}\\ &\le \sup\{l^{-1}\|x_t\|_{\mathscr{B}_h} : t \in [0,b]\}\\ &\quad + \sup\{\|T(t)\phi(0)\| : t \in [0,b]\}\\ &\le l^{-1}K + M\|\phi(0)\|,\end{aligned} \tag{51}$$

which yields that the set G is bounded.

Consequently, by virtue of Lemma 2, Theorem 9, and Theorem 10, the equation $\overline{\Psi}_1 y + \overline{\Psi}_2 y = y$ has a solution $y^* \in \mathscr{B}_h''$. Let $x(t) = y^*(t) + \widetilde{\phi}(t)$, $t \in (-\infty, b]$; then x is a fixed point of the operator Ψ which is a mild solution of the problem (1). □

Theorem 12. *Assume that (H1), (H2), (H4), (H5), and the following hypotheses are satisfied.*

$(H3)'$ *There exist integrable functions $p, q, r : J \to [0,\infty)$ such that*

(i) $\|\int_0^t k(t,s,\psi)\| \le q(t)\|\psi\|_{\mathscr{B}_h}$, $(t,s) \in \Delta, \psi \in \mathscr{B}_h$,

(ii) $\|\int_0^b w(t,s,\psi)\| \le r(t)\|\psi\|_{\mathscr{B}_h}$, $(t,s) \in \Delta, \psi \in \mathscr{B}_h$,

(iii) $\|f(t,\psi,x,y)\| \le p(t)\Omega(\|\psi\|_{\mathscr{B}_h} + \|x\| + \|y\|)$, *for each $(t,\psi,x,y) \in J \times \mathscr{B}_h \times X \times X$, where $\Omega : [0,\infty) \to [0,\infty)$ is continuous nondecreasing function such that*

$$\Omega(\gamma(t) r) \le \gamma(t)\Omega(r), \qquad \gamma(t) = 1 + p(t) + q(t), \tag{52}$$

for each $t \in J$ and $r \ge 0$.

$(H6)'$ *The condition $B_0 K_1 \int_o^b p(s)\gamma(s)ds < \int_{B_0K_3}^\infty ds/\Omega(s)$ holds, where B_0, K_1, and K_3 are as in (H6) and* (24).

Then the problem (1) *has at least one mild solution on* $(-\infty, b]$.

Proof. Let $x(t)$ be the solution of (22). By using the hypotheses (H1), (H2), and $(H3)'$ and (23), we obtain

$$\begin{aligned}&\|x(t)\|\\ &\le F + \left\|(-A)^{-\beta}\right\| L_2(1+bL_1)\|x_t\|_{\mathscr{B}_h}\\ &\quad + C_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{\mathscr{B}_h}}{(t-s)^{1-\beta}}ds\\ &\quad + \int_0^t Mp(s)\Omega\left(\|x_s\|_{\mathscr{B}_h} + q(s)\|x_s\|_{\mathscr{B}_h} + r(s)\|x_s\|_{\mathscr{B}_h}\right)ds\\ &\le F + \left\|(-A)^{-\beta}\right\| L_2(1+bL_1)\|x_t\|_{\mathscr{B}_h}\\ &\quad + C_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{\mathscr{B}_h}}{(t-s)^{1-\beta}}ds\\ &\quad + \int_0^t Mp(s)\gamma(s)\Omega\left(\|x_s\|_{\mathscr{B}_h}\right)ds.\end{aligned} \tag{53}$$

By an application of Lemma 4, we get

$$\begin{aligned}&\|x_t\|_{\mathscr{B}_h}\\ &\le \|\phi\|_{\mathscr{B}_h} + lF + l\left\|(-A)^{-\beta}\right\| L_2(1+bL_1)\|x_t\|_{\mathscr{B}_h}\\ &\quad + lC_{1-\beta}L_2(1+bL_1)\int_0^t \frac{\|x_s\|_{\mathscr{B}_h}}{(t-s)^{1-\beta}}ds\\ &\quad + \int_0^t lMp(s)\gamma(s)\Omega\left(\|x_s\|_{\mathscr{B}_h}\right)ds.\end{aligned} \tag{54}$$

Define the function μ as in the proof of Theorem 8 and, proceeding on the same line, we obtain

$$\begin{aligned}\mu(t) &\le K_3 + K_1\int_0^t p(s)\gamma(s)\Omega(\mu(s))ds\\ &\quad + K_2\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds, \quad t \in J.\end{aligned} \tag{55}$$

Applying Lemma 3 to the above inequality, we obtain

$$\mu(t) \le B_0\left(K_3 + K_1\int_0^t p(s)\gamma(s)\Omega(\mu(s))ds\right), \quad t \in J. \tag{56}$$

Let

$$w(t) = B_0\left(K_3 + K_1\int_0^t p(s)\gamma(s)\Omega(\mu(s))ds\right), \quad t \in J. \tag{57}$$

Then $w(0) = B_0K_3$, $\mu(t) \le w(t)$, $t \in J$, and

$$w'(t) = B_0K_1 p(t)\gamma(t)\Omega(\mu(t)), \quad t \in J. \tag{58}$$

Since Ω is nondecreasing function, we have

$$w'(t) \le B_0K_1 p(t)\gamma(t)\Omega(w(t)), \quad t \in J. \tag{59}$$

Therefore,

$$\frac{w'(t)}{\Omega(w(t))} \leq B_0 K_1 p(t)\gamma(t), \quad t \in J. \tag{60}$$

Integrating from 0 to t and using the change of variables $t \to s = w(t)$ and the hypothesis (H6)$'$, we obtain

$$\begin{aligned}\int_{B_0K_3}^{w(t)} \frac{ds}{\Omega(s)} ds &\leq B_0 K_1 \int_0^b p(s)\gamma(s)\,ds \\ &< \int_{B_0K_3}^{\infty} \frac{ds}{\Omega(s)} ds, \quad t \in J.\end{aligned} \tag{61}$$

This implies that $w(t) < \infty$. So there is constant K^* such that $w(t) \leq K^*$, $t \in J$, and hence $\|x_t\|_{\mathscr{B}_h} \leq \mu(t) \leq w(t) \leq K^*$, $t \in J$, where K^* depends on the functions p, γ, and Ω.

Define the operators $\overline{\Psi}$, $\overline{\Psi}_1$, and $\overline{\Psi}_2$ as discussed above. Note that the set $G = \{y \in \mathscr{B}_h'' : y = \lambda\overline{\Psi}_1(y/\lambda) + \lambda\overline{\Psi}_2 y$ for some $\lambda \in (0,1)\}$ is bounded by $l^{-1}K^* + M\|\phi(0)\|$. Theorem 9 satisfies condition (a) in Lemma 2. The proof of $\overline{\Psi}_2$ is completely continuous operator which can be completed using the hypotheses (H_1), $(H_3)'$, (H_4), and (H_5) and closely looking at the proof of Theorem 10. Finally by applying Lemma 2, the problem (1) has mild solution on $(-\infty, b]$. □

Theorem 13. *Assume that the hypotheses (H1), (H2), (H4), and (H5) are satisfied. In addition suppose the following.*

$(H3)''$ *There exist integrable functions $p, q, r : J \to [0,\infty)$ such that*

(i) $\| \int_0^t k(t,s,\psi)\| \leq q(t)\ \Omega(\|\psi\|_{\mathscr{B}_h}), (t,s) \in \Delta, \psi \in \mathscr{B}_h$,

(ii) $\| \int_0^b w(t,s,\psi)\| \leq r(t)\ \Omega(\|\psi\|_{\mathscr{B}_h}), (t,s) \in \Delta, \psi \in \mathscr{B}_h$,

(iii) $\|f(t,\psi,x,y)\| \leq p(t)\ \Omega(\|\psi\|_{\mathscr{B}_h}) + \|x\| + \|y\|$, *for each* $(t,\psi,x,y) \in J \times \mathscr{B}_h \times X \times X$, *where* Ω: $[0,\infty) \to [0,\infty)$ *is continuous nondecreasing function.*

$(H6)''$ *The condition* $B_0K_1 \int_o^b [p(s) + q(s) + r(s)]ds < \int_{B_0K_3}^{\infty}(ds/\Omega(s))$ *holds, where* B_0, K_1, *and* K_3 *are as in (H6) and* (24).

Then the problem (1) *has at least one mild solution on* $(-\infty, b]$.

Proof. Proceeding as in the proof of Theorem 12 with suitable modification, we can complete the proof. Hence we omit the details. □

4. Application

Consider the following partial neutral mixed integrodifferential equation of the form

$$\begin{aligned}&\frac{\partial}{\partial t}\Bigg[v(t,x) + G\Bigg(t, \int_{-\infty}^{0} P_1(s-t)\,v(s,x)\,ds\Bigg), \\ &\qquad \int_0^t \int_{-\infty}^{s} P_2(s,x,\tau-s)\,Q_1(v(\tau,x))\,d\tau ds\Bigg] \\ &= \frac{\partial^2}{\partial x^2} v(t,x) \\ &\quad + H\Bigg(t, \int_{-\infty}^{0} P_3(s-t)\,v(s,x)\,ds, \\ &\qquad \int_0^t \int_{-\infty}^{s} P_4(s,x,\tau-s)\,Q_2(v(\tau,x))\,d\tau ds, \\ &\qquad \int_0^b \int_{-\infty}^{s} P_5(s,x,\tau-s)\,Q_3(v(\tau,x))\,d\tau ds\Bigg), \\ &\qquad (x,t) \in [0,\pi]\times[0,b], \\ &\qquad v(t,0) = v(t,\pi) = 0, \quad t \geq 0 \\ &\quad v(t,x) = \phi(t,x), \quad t \in (-\infty,0], \quad x \in [0,\pi],\end{aligned} \tag{62}$$

where $\phi \in \mathscr{B}_h$. Consider the space $X = L^2[0,\pi]$ with the norm $|\cdot|_{L^2}$. Define the operator $A : X \to X$ by $Aw = w''$ with the domain $D(A) = \{w \in X : w, w'$ are absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$.

Then $Aw = \sum_{n=1}^{\infty} -n^2 < w, w_n > w_n$, and $w \in D(A)$, where $w_n(x) = \sqrt{2/\pi}\sin(nx)$, $n = 1, 2, \ldots$, is the orthogonal set of eigen vectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$, and is given by $T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, w_n > w_n$, and $w \in X$. Further for every $w \in X$, $(-A)^{-1/2}w = \sum_{n=1}^{\infty}(1/n) < w, w_n > w_n$, and $\|(-A)^{-1/2}\| = 1$. The operator $(-A)^{1/2}$ is given by

$$(-A)^{1/2}w = \sum_{n=1}^{\infty} n < w, \quad w_n > w_n, \tag{63}$$

with the domain $D((-A)^{1/2}) = \{w \in X : \sum_{n=1}^{\infty} n < w, w_n > w_n \in X\}$. It follows that $\|T(t)\| \leq 1$. Let $h(s) = e^{2s}$, $s < 0$; then $l = \int_{-\infty}^{0} h(s)ds = 1/2$, and define

$$\|\phi\|_h = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} |\phi(\theta)|_{L^2} ds. \tag{64}$$

Hence for $(t,\phi) \in [0,b] \times \mathscr{B}_h$, where $\phi(\theta)(x) = \phi(\theta,x)$, $(\theta,x) \in (-\infty,0] \times [0,\pi]$. Set

$$\begin{aligned}
v(t)(y) &= v(t,y)\\
e(t,s,\psi)(y) &= \int_{-\infty}^{0} P_2(t,y,\theta)\,Q_1(\psi(\theta)(y))\,d\theta,\\
k(t,s,\psi)(y) &= \int_{-\infty}^{0} P_4(t,y,\theta)\,Q_2(\psi(\theta)(y))\,d\theta,\\
w(t,s,\psi)(y) &= \int_{-\infty}^{0} P_5(t,y,\theta)\,Q_3(\psi(\theta)(y))\,d\theta,\\
&g\left(t,\psi,\int_0^t e(t,s,\psi)\,ds\right)(y)\\
&= G\left(t,\int_{-\infty}^{0} P_1(\theta)\,\psi(\theta)(y)\,d\theta,\right.\\
&\qquad \left.\int_0^t e(t,s,\psi)(y)\,ds\right)\\
&f\left(t,\psi,\int_0^t k(t,s,\psi)\,ds,\int_0^b w(t,s,\psi)\,ds\right)(y)\\
&= H\left(t,\int_{-\infty}^{0} P_3(\theta)\,\phi(\theta)(y)\,d\theta,\right.\\
&\qquad \int_0^t k(t,s,\psi)(y)\,ds,\\
&\qquad \left.\int_0^b w(t,s,\psi)(y)\,ds\right).
\end{aligned} \tag{65}$$

With these choices of functions, the system (1) is the abstract formulation of the system (62). By imposing suitable conditions on the above defined functions to verify the assumptions of Theorem 10, we conclude that system (62) has at least one mild solution on $(-\infty,b]$.

5. Conclusion

The integral inequality established by B. G. Pachpatte is used to obtain the bound and different existence results are established for more general equations which include the study of Volterra and Fredholm functional integrodifferential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] C. Corduneanu and V. Lakshmikantham, "Equations with unbounded delay: a survey," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 831–877, 1980.

[2] J. K. Hale, "Retarded equations with infinite delays," in *Functional Differential Equations and Approximation of Fixed Points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978)*, H.-O. Peitgen and H.-O. Walther, Eds., vol. 730 of *Lecture Notes in Mathematics*, pp. 157–193, Springer, Berlin, Germany, 1979.

[3] J. K. Hale and J. Kato, "Phase space for retarded equations with infinite delay," *Funkcialaj Ekvacioj*, vol. 21, no. 1, pp. 11–41, 1978.

[4] Y. Hino, S. Murakami, and T. Naito, *Functional-Differential Equations with Infinite Delay*, vol. 1473 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1991.

[5] V. Lakshmikantham, L. Z. Wen, and B. G. Zhang, *Theory of Differential Equations with Unbounded Delay*, vol. 298 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[6] E. Hernández and H. R. Henríquez, "Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay," *Journal of Mathematical Analysis and Applications*, vol. 221, no. 2, pp. 499–522, 1998.

[7] E. Hernández and H. R. Henríquez, "Existence results for partial neutral functional-differential equations with unbounded delay," *Journal of Mathematical Analysis and Applications*, vol. 221, no. 2, pp. 452–475, 1998.

[8] E. Hernández, "Existence results for partial neutral functional integrodifferential equations with unbounded delay," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 1, pp. 194–210, 2004.

[9] M. Li, Y. Duan, X. Fu, and M. Wang, "Controllability of neutral functional integrodifferential systems in abstract space," *Journal of Applied Mathematics & Computing*, vol. 23, no. 1-2, pp. 101–112, 2007.

[10] H. R. Henríquez, "Approximation of abstract functional-differential equations with unbounded delay," *Indian Journal of Pure and Applied Mathematics*, vol. 27, no. 4, pp. 357–386, 1996.

[11] H. R. Henríquez, "Regularity of solutions of abstract retarded functional-differential equations with unbounded delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 3, pp. 513–531, 1997.

[12] Y.-K. Chang, M. M. Arjunan, and V. Kavitha, "Existence results for neutral functional integrodifferential equations with infinite delay via fractional operators," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1-2, pp. 201–218, 2011.

[13] Y.-K. Chang, A. Anguraj, and M. M. Arjunan, "Existence results for impulsive neutral functional differential equations with infinite delay," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 1, pp. 209–218, 2008.

[14] J. Y. Park, K. Balachandran, and G. Arthi, "Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 3, pp. 184–194, 2009.

[15] J. Y. Park, K. Balachandran, and N. Annapoorani, "Existence results for impulsive neutral functional integrodifferential equations with infinite delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3152–3162, 2009.

[16] H. R. Henriquez and J. P. C. dos Santos, "Differentiability of solutions of abstract neutral integro-differential equations," *Journal of Integral Equations and Applications*, vol. 25, no. 1, pp. 47–77, 2013.

[17] M. B. Dhakne and K. D. Kucche, “Global existence for abstract nonlinear Volterra-Fredholm functional integrodifferential equation,” *Demonstratio Mathematica*, vol. 45, no. 1, pp. 117–127, 2012.

[18] M. B. Dhakne and K. D. Kucche, “Global existence for abstract nonlinear Volterra-Fredholm functional integrodifferential equation,” *Malaya Journal of Mathematic*, vol. 1, no. 1, pp. 1–8, 2012.

[19] K. D. Kucche and M. B. Dhakne, “On existence results and qualitative properties of mild solution of semilinear mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces,” *Applied Mathematics and Computation*, vol. 219, no. 22, pp. 10806–10816, 2013.

[20] K. D. Kucche and M. B. Dhakne, “Controllability of non-densely defined abstract mixed Volterra-Fredholm neutral functional integrodifferential equations,” *Asian-European Journal of Mathematics*, vol. 6, no. 2, Article ID 1350025, 17 pages, 2013.

[21] B. Yan, “Boundary value problems on the half-line with impulses and infinite delay,” *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 94–114, 2001.

[22] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[23] T. A. Burton and C. Kirk, “A fixed point theorem of Krasnoselskii-Schaefer type,” *Mathematische Nachrichten*, vol. 189, pp. 23–31, 1998.

[24] Y.-K. Chang, “Controllability of impulsive functional differential systems with infinite delay in Banach spaces,” *Chaos, Solitons & Fractals*, vol. 33, no. 5, pp. 1601–1609, 2007.

[25] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, vol. 205 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.