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RESEARCH NOTES

THE SPACES O_M AND O_C ARE ULTRABORNOLOGICAL A NEW PROOF

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ABSTRACT. In [1] Laurent Schwartz introduced the spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ of multiplication and convolution operators on temperate distributions. Then in [2] Alexandre Grothendieck used tensor products to prove that both $\mathcal{O}_M$ and $\mathcal{O}'_C$ are bornological. Our proof of this property is more constructive and based on duality.

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We use C, N, R, and Z, resp., for the set of all complex, nonnegative integer, real, and integer numbers. For each $q \in N$, the space

$$L_q = \{f: R^n \to C;\ \|f\|_q^2 = \sum_{|\alpha+\beta| \le q} \int_{R^n} x^{2\alpha} |D^\beta f(x)|^2\, dx < +\infty\} \text{ is Hilbert.}$$

Here $D^\beta f$ stands for the Sobolev generalized derivative. We denote by L_{-q} the strong dual of L_q and by $\|\cdot\|_{-q}$ the standard norm on L_{-q}. Then the space $\mathcal{S}$ of rapidly decreasing functions, resp. its strong dual $\mathcal{S}'$, is the $\operatorname{proj\,lim}_{q\to\infty} L_q$, resp. $\operatorname{ind\,lim}_{q\to\infty} L_{-q}$.

It is convenient to introduce the weight-function $W(x) = (1 + |x|^2)^{1/2}$, $x \in R^n$. The mapping $T_k : f \mapsto W^k f: \mathcal{S}' \to \mathcal{S}'$, $k \in Z$, is injective. We denote by $W^k L_m$, $k, m \in Z$, the image of L_m under T_k and provide it with the topology which makes $T_k : L_m \to W^k L_m$ a topological isomorphism. Further, $\mathcal{O}_q$, $q \in N$, stands for the $\operatorname{ind\,lim}_{p\to\infty} W^p L_q$, and $\mathcal{O}_{-q}$ for its strong dual. It is proved in [7] that for each $q \in N$, the space $\mathcal{O}_q$ is reflexive and $\mathcal{O}_{-q} = \operatorname{proj\,lim}_{p\to\infty} W^{-p} L_{-q}$. Finally, the space $\mathcal{O}_M$ of multiplication operators on $\mathcal{S}'$ equals $\operatorname{proj\,lim}_{q\to\infty} \mathcal{O}_q$, see [6].

PROPOSITION 1. The strong dual $\mathcal{O}'_M$ of $\mathcal{O}_M$ equals $\operatorname{ind\,lim}_{q\to\infty} \mathcal{O}_{-q}$.

PROOF. The space $\mathcal{S}$ is dense in each L_q, $q \in N$. Hence $\mathcal{S} = W^p \mathcal{S}$ is dense in $W^p L_q$ for each $p \in N$. Then $\mathcal{S}$, and à fortiori its superset $\mathcal{O}_M$, are dense in each

$\mathcal{O}_q = \operatorname{ind\,lim}_{p\to\infty} W^p L_q$, $q \in N$. By [3, ch. IV, 4.4], the dual of $\mathcal{O}_M$, equipped with the Mackey topology, equals $\operatorname{ind\,lim}_{q\to\infty} \mathcal{O}_{-q}$. The Mackey and strong topologies on $\mathcal{O}'_M$ coincide since $\mathcal{O}_M$, as a projective limit of reflexive spaces $\mathcal{O}_q$, is semireflexive, see [3, ch. IV, 5.5].

PROPOSITION 2. $\mathcal{O}_M$ is the strong dual of $\operatorname{ind\,lim}_{q\to\infty} \mathcal{O}_{-q}$.

PROOF. By [3, ch. IV, 4.5], the topology τ of $\mathcal{O}_M = \operatorname{proj\,lim}_{q\to\infty} \mathcal{O}_q$ is consistent with the duality $< \mathcal{O}'_M, \mathcal{O}_M >$. Hence τ is coarser than the strong topology $\beta(\mathcal{O}_M, \mathcal{O}'_M)$. On the other hand, it is proved in [5, Prop. 4] that τ is finer than $\beta(\mathcal{O}_M, \mathcal{O}'_M)$.

THEOREM 1. The space $\mathcal{O}_M$ is reflexive and $\mathcal{O}'_M$ is the strong dual.

LEMMA 1. Let $r = 1 + [\frac{1}{2} n]$, $q \in N$. Then $W^{-r}L_q \subset L_q$ and every set bounded in $W^{-r}L_q$ is relatively compact in L_q.

PROOF. Let B be an absolutely convex, bounded, and closed, set in $W^{-r}L_q$. Then B is weakly compact as a polar of a neighborhood in $W^r L_{-q}$. By [3, Ch. IV, 11.1, Cor 2], B is weakly sequentially compact and every sequence in B contains a subsequence $\{f_k\}$ which converges weakly to some $g \in B$. We may assume $g = 0$.

Since the set $\{W^{r+q}f;\ f \in B\}$ is bounded in $L^2(R^n)$, the set $\{W^q f;\ f \in B\}$ is bounded in $L^1(R^n)$ and for any $\alpha \in N^n$, $|\alpha| \le q$, the set $\{D^\alpha \mathcal{F} f;\ f \in B\}$, where $\mathcal{F}f$ is the Fourier transform of f, is uniformly bounded and locally equicontinuous on R^n. Hence $\{f_k\}$ contains a subsequence, let it be again $\{f_k\}$, such that $\{D^\alpha \mathcal{F} f_k(x)\}$ converges uniformly on R^n for all $\alpha \in N^n$, $|\alpha| \le q$.

Take a non-negative function $h \in \mathcal{S}$, $\int_{R^n} h(x)dx = 1$, and put $h_i(x) = i^n h(ix)$, $i \in N$. Then $f * h^i \to f$ as $i \to \infty$ in the topology of L_q uniformly on B. Given $\varepsilon > 0$, there is $i \in N$ such that $\|f - f * h_i\|_q < \varepsilon$ for any $f \in B$. We fix this i. For every $\alpha, \beta \in N^n$, $|\alpha + \beta| \le q$, the sequence $\{W^\alpha D^\beta(\mathcal{F}f_k \cdot \mathcal{F}h_i)\}$ converges uniformly to 0 on R^n as $k \to \infty$ and has an integrable majorant from $\mathcal{S}$. Hence $\mathcal{F}(f_k * h_i) \to 0$, and à fortiori $f_k * h_i \to 0$, both in the topology of L_q. If we choose $k_0 \in N$ so that $\|f_k * h_i\|_q < \varepsilon$ for $k > k_0$, then $\|f_k\|_q < 2\varepsilon$ for $k > k_0$.

LEMMA 2. Let $r = 1 + [\frac{1}{2} n]$, $q \in N$. Then $W^{-r}L_{-q} \subset L_{-q}$ and every set bounded in $W^{-r}L_{-q}$ is relatively compact in L_{-q}.

PROOF. Let B be an absolutely convex, bounded, and closed, set in $W^{-r}L_{-q}$. By the same argument as in Lemma 1, every sequence in B has a subsequence $\{f_k\}$ which converges weakly to some $g \in B$. We again assume $g = 0$.

Denote by $\|\cdot\|_{-r,-q}$, resp. $\|\cdot\|_{r,q}$, the norm in $W^{-r}L_{-q}$, resp. $W^r L_q$. Let A be the closed unit ball in L_q, B_0 the open unit ball in $W^r L_q$, and $a = \sup\{\|f\|_{-r,-q};\ f \in B\}$. Choose $\varepsilon > 0$. By Lemma 1, A is compact in the topology of $W^r L_q$. Since L_q is dense in $W^r L_q$, there exists a finite set $\{\varphi_i;\ i \in F\} \subset L_q$ such that $A \subset \cup\{\varphi_i + \varepsilon B_0;\ i \in F\}$. For any $\varphi \in A$, there exists φ_i such that $\|\varphi - \varphi_i\|r,q < \varepsilon$ and for any $k \in N$ we have $|< \varphi, f_k >| \le |< \varphi - \varphi_i, f_k >| + |< \varphi_i, f_k >| \le \|\varphi - \varphi_i\|_{r,q} \cdot \|f_k\|_{-r,-q} + |< \varphi_i, f_k <| \le$

$\le \varepsilon a + |< \varphi_i, f_k >|$. If we choose $k_0 \in N$ so that $|< \varphi_i, f_k >| < \varepsilon$ for all $i \in F$ and $k > k_0$ and the sequence $\{f_k\}$ converges in L_{-q}.

PROPOSITION 3. For each $q \in N$, $\mathcal{O}_{-q}$ is a Schwartz space.

PROOF. By Lemma 2, for every $p \in N$ the closed unit ball is $W^{-r-p}L_{-q}$, where $r = 1 + [\frac{1}{2}n]$, is compact in $w^{-p}L_{-q}$. By [4, Ch. 3.15, Prop. 9], the space $\mathcal{O}_{-q} = \operatorname{proj\,lim}_{p \to \infty} W^{-p}L_{-q}$ is Schwartz.

PROPOSITION 4. Let $E_1 \subset E_2 \subset \ldots$ be locally convex spaces with identity maps: $E_k \to E_{k+1}$, $k \in N$, continuous and $E = \operatorname{ind\,lim}_{k \to \infty} E_k$ Hausdorff. Assume:

(1) every set bounded in E is bounded in some E_k,

(2) every E_k is a Schwartz space.

Then E is a Schwartz space.

Proposition 4 is slightly more general than Prop. 8 in [4, Ch. 3.15] and its proof requires only minor changes of the proof presented in [4].

THEOREM 2. $\mathcal{O}'_M$ is a Schwartz space.

PROOF. We have $\mathcal{O}'_M = \operatorname{ind\,lim}_{q \to \infty} \mathcal{O}_{-q}$. Each space $\mathcal{O}_{-q}$ is Schwartz and Fríchet. Further, $\mathcal{O}'_M$ is reflexive, hence quasi-complete, which in turn implies fast completeness. By [8, Th. 1], the assumption (1) of Prop. 4 is satisfied and $\mathcal{O}'_M$ is a Schwartz space.

THEOREM 3. $\mathcal{O}'_M$ is complete.

PROOF. The space $\mathcal{B}$ of C^∞ - functions, whose derivatives vanish at ∞ was introduced in [1]. We denote the space $W^m\dot{\mathcal{B}}$ by $\dot{\mathcal{B}}_m$ and provide it with the topology for which $f \mapsto W^m f : \dot{\mathcal{B}} \to \dot{\mathcal{B}}_m$ is a topological isomorphism. Then the strong dual $\mathcal{O}_C$ of $\mathcal{O}'_C$ equals $\operatorname{ind\,lim} \dot{\mathcal{B}}_m$, see [2, Ch. 2, 4.4]. Also, $\mathcal{O}_C$ is isomorphic to $\mathcal{O}'_M$ via Fourier transformation. Hence it suffices to prove that $\operatorname{ind\,lim}_{m \to \infty} \dot{\mathcal{B}}_m$ is complete.

Let F be a Cauchy filter on $\mathcal{O}_C$, G a filter of all 0-neighborhoods in $\mathcal{O}_C$, and H the filter with base $\{A+B;\ A \in F,\ B \in G\}$. By [4, Ch. 2.12, Lemma 3], there exists $m \in N$ such that H induces a filter H_m on $\dot{\mathcal{B}}_m$ which is Cauchy in the topology inherited from $\mathcal{O}_C$. On each ball $\{x \in R^n,\ |x| \le n\}$, $r > 0$, the filter H_m converges uniformly pointwise to a function $f \in \dot{\mathcal{B}}_m$. Then f adheres to H_m on the subset $\dot{\mathcal{B}}_m$ of $\mathcal{O}_C$ and by [4, Ch. 2.9, Prop. 1] the filter F converges to f.

THEOREM 4. The spaces $\mathcal{O}_M$ and $\mathcal{O}'_M$ are ultrabornological.

PROOF. By Exercise 9 in [4, Ch. 3.15], the strong dual of a complete Schwartz space is ultrabornological. Hence $\mathcal{O}_M$ is ultrabornological by Theorems 1, 2, and 3.

The space $\mathcal{O}'_M$ is ultrabornological as an inductive limit of Fréchet spaces $\mathcal{O}_{-q}$, $q \in N$.

THEOREM 5. The spaces $\mathcal{O}_C$ and its strong dual $\mathcal{O}'_C$ are both complete, reflexive, and ultrabornological spaces.

PROOF. The space $\mathcal{O}_M$ is complete as a strong dual of a bornological space. Since the Fourier transformations $\mathcal{F}: \mathcal{O}_M \to \mathcal{O}'_C$ and $\mathcal{F}: \mathcal{O}'_M \to \mathcal{O}_C$ are topological isomorphisms, Theorem 5 follows from Theorems 1, 3, and 4.

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