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Research Article

A Study on Some Fundamental Properties of Continuity and Differentiability of Functions of Soft Real Numbers

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We introduce a new type of functions from a soft set to a soft set and study their properties under soft real number setting. Firstly, we investigate some properties of soft real sets. Considering the partial order relation of soft real numbers, we introduce concept of soft intervals. Boundedness of soft real sets is defined, and the celebrated theorems like nested intervals theorem and Bolzano-Weierstrass theorem are extended in this setting. Next, we introduce the concepts of limit, continuity, and differentiability of functions of soft sets. It has been possible for us to study some fundamental results of continuity of functions of soft sets such as Bolzano's theorem, intermediate value property, and fixed point theorem. Because the soft real numbers are not linearly ordered, several twists in the arguments are required for proving those results. In the context of differentiability of functions, some basic theorems like Rolle's theorem and Lagrange's mean value theorem are also extended in soft setting.

1. Introduction

Following the seminal work of Zadeh [1] on fuzzy set theory, development of mathematical theory and their applications in handling the problems under uncertain environment have been gaining momentum day by day. Considering some difficulties in the parametrization process in fuzzy set theory, Molodtsov in 1999 [2] introduced an idea of soft set as a parametrized family of sets where parameter set takes values from an arbitrary set. He also showed the applications of soft sets in fields like smoothness of functions, probability theory, measure theory, and game theory.

After that Maji et al. [3, 4] defined some operations on soft sets based on which Shabir and Naz [5] introduced soft topologies, Aktas and Cagman [6] soft group, and Nazmul and Samanta [7] soft topological group. Recently Das et al. [8, 9] introduced the idea of soft metric and soft normed linear space. In [10] Das and Samanta also introduced the concept of soft real numbers. Using this concept they studied some basic properties of soft real numbers. However in their study on functions they have considered functions over crisp sets and have used extension principle for getting images of soft sets. In 2016 [11] Tantawy and Hassan studied some basic operations like supremum and infimum in soft setting.

However, no studies have been found for functions from soft numbers to soft numbers. To deal with this type of functions is something different from the previous one. In this paper we have considered such type of functions and have studied some fundamental properties of continuous functions, like Bolzano's property, intermediate value property, and fixed point property. The extension of Bolzano's theorem in soft setting is crucial and from which some other properties, like intermediate value property and fixed point property, follow. We have also introduced the concept of differentiation of such functions and have extended Rolle's theorem and Lagrange's theorem in soft settings.

2. Preliminaries

Unless otherwise stated all over this paper A is taken as the parameter set and any soft set taken in this paper is assumed to be parameterwise nonempty.

Definition 1 (see [2] (soft set)). Let X be a nonempty set and A be a set which is called index set. A pair (F, A), where $F : A \to 2^X$ is a mapping, is called a soft set on X.

Definition 2 (see [10] (soft element)). Let X be a nonempty set and A be a nonempty parameter set. Then a function $\epsilon : A \to X$ is said to be a soft element of X. A soft element ϵ of X is said to belong to a soft set F of X, denoted by $\epsilon \in F$, if $\epsilon(e) \in F(e)$, $\forall e \in A$. Thus a soft set F of X with respect to the index set A can be expressed as $F(e) = \{\epsilon(e),\ \epsilon \in A\}$.

Definition 3 (see [10]). Let $\mathbb{R}$ be the set of real numbers, $P(\mathbb{R})$ be the collection of all nonempty bounded subsets of $\mathbb{R}$, and A be a set of parameters. Then a mapping $F : A \to P(\mathbb{R})$ is called a soft real set. It is denoted by (F, A).

If in particular (F, A) is a singleton soft set, then, identifying (F, A) with the corresponding soft element, it will be called a soft real number.

We use the notation $\tilde{x}, \tilde{y}, \tilde{z}$ to denote soft real numbers whereas $\overline{x}, \overline{y}, \overline{z}$ will denote a particular type of soft real numbers such that $\overline{x}(\lambda) = x$, for all $\lambda \in A$ etc. Note that, in general, $\tilde{x}$ is not related to x.

Definition 4 (see [8]). For two soft real numbers $\tilde{r}, \tilde{s}$ we define the following:

(1) $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$.

(2) $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$.

(3) $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

(4) $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$.

Definition 5 (see [8]). If $\tilde{x}, \tilde{y}$ are soft numbers, then modulus, sum, difference, product, and division of soft real numbers are denoted by $|\tilde{x}|$, $\tilde{x} + \tilde{y}$, $\tilde{x} - \tilde{y}$, $\tilde{x} \cdot \tilde{y}$ and $\tilde{x}/\tilde{y}$, respectively, and defined by the following:

(i) $|\tilde{x}|(\lambda) = |\tilde{x}(\lambda)|$, for all $\lambda \in A$.

(ii) $(\tilde{x} + \tilde{y})(\lambda) = \tilde{x}(\lambda) + \tilde{y}(\lambda)$, for all $\lambda \in A$.

(iii) $(\tilde{x} - \tilde{y})(\lambda) = \tilde{x}(\lambda) - \tilde{y}(\lambda)$, for all $\lambda \in A$.

(iv) $(\tilde{x} \cdot \tilde{y})(\lambda) = \tilde{x}(\lambda) \cdot \tilde{y}(\lambda)$, for all $\lambda \in A$.

(v) $(\tilde{x}/\tilde{y})(\lambda) = \tilde{x}(\lambda)/\tilde{y}(\lambda)$, $\tilde{y}(\lambda) \neq 0$ for all $\lambda \in A$.

Theorem 6 (see [8]). *For any soft set* (F, A), $SS(SE((F, A))) = (F, A)$ *(where* $SE(F, A) = \{\tilde{x} : \tilde{x}(\lambda) \in F(\lambda),\ \forall \lambda \in A\}$ *and* $SS(B)(\lambda) = \{\tilde{x}(\lambda) : \tilde{x} \in B\}$, *where B is a set of soft real numbers).*

Definition 7 (see [3]). Let $(F, A), (G, A) \in R(A)$; then (F, A) is said to be subset of (G, A) and denoted by $(F, A) \tilde{\subseteq} (G, A)$, if $F(\lambda) \subseteq G(\lambda)$, $\forall \lambda \in A$.

Definition 8 (see [3] (equality of soft real sets)). Let $(F, A), (G, A) \in R(A)$; then (F, A) is said to be equal to (G, A) and denoted by $(F, A) = (G, A)$, if $F(\lambda) = G(\lambda)$, $\forall \lambda \in A$.

Definition 9 (see [3]). The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases} \quad (1)$$

We write $(F, A) \cup (G, B) = (H, C)$.

Definition 10 (see [3]). The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$ $H(e) = F(e) \cap G(e)$. We write $(F, A) \cap (G, B) = (H, C)$.

Definition 11 (see [11]). A real soft set (F, A) is said to be bounded from above if there exists a soft real number $\tilde{r}$ such that $\tilde{x} \tilde{\leq} \tilde{r}$, $\forall \tilde{x} \tilde{\in} (F, A)$.

Definition 12 (see [11]). A real soft set (F, A) is said to be bounded from below if there exists a soft real number $\tilde{r}$ such that $\tilde{x} \tilde{\geq} \tilde{r}$, $\forall \tilde{x} \tilde{\in} (F, A)$.

Definition 13 (see [12]). In a soft normed linear space a sequence $\{\tilde{x}_n\}$ of soft elements is said to be convergent and converges to a soft element $\tilde{x}$ if, for any soft real number $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists a soft natural number $\tilde{n}_0$ such that $\|\tilde{x}_n - \tilde{x}\|(\lambda) < \tilde{\epsilon}(\lambda)$, $\forall n \geq \tilde{n}_0(\lambda)$, $\forall \lambda \in A$ and is denoted by $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$ or $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$, where $\tilde{x}$ is called the soft limit of the sequence $\{\tilde{x}_n\}$.

Theorem 14. *Let* $\{\tilde{x}_n\}$ *be a sequence in soft real set over a finite parameter set converging to* $\tilde{x}$*; then* $\{\tilde{x}_n\}$ *converge to* $\tilde{x}$ *uniformly over parameterwise; that is, for any* $\tilde{\epsilon} \tilde{>} \overline{0}$ *there exists a* $n_0 \in \mathbb{N}$ *(set of natural numbers) such that* $|\tilde{x}_n - \tilde{x}| \tilde{<} \tilde{\epsilon}$, $\forall n \geq \tilde{n}_0$.

Theorem 15 (see [12]). *A sequence* $\tilde{x}_n \to \tilde{x}$ *iff* $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$ *for all* $\lambda \in A$.

3. Elementary Set Theory

From this section all the study has done on soft real sets.

Definition 16. If $\tilde{a}, \tilde{b}$ are two soft real numbers with $\tilde{a} \tilde{<} \tilde{b}$; then $[\tilde{a}, \tilde{b}] = \{\tilde{x} : \tilde{a} \tilde{\leq} \tilde{x} \tilde{\leq} \tilde{b}\}$ is said to be soft closed interval with boundary points $\tilde{a}$ and $\tilde{b}$. Clearly $[\tilde{a}, \tilde{b}](\lambda) = [\tilde{a}(\lambda), \tilde{b}(\lambda)]$. So any closed soft interval $[\tilde{a}, \tilde{b}]$ can be taken as a soft set, where $[\tilde{a}, \tilde{b}](\lambda) = [\tilde{a}(\lambda), \tilde{b}(\lambda)]$.

Definition 17. A length function L of a soft interval $[\tilde{a}, \tilde{b}]$ is defined by $L[\tilde{a}, \tilde{b}](\lambda)$ = length of the interval $[\tilde{a}(\lambda), \tilde{b}(\lambda)]$ and denoted by $L[\tilde{a}, \tilde{b}]$.

Definition 18. A soft real set S is said to be bounded if it is bounded from above and bounded from below.

Proposition 19. *A soft real set S is bounded iff $S(\lambda)$ are bounded sets in $\mathbb{R}$, $\forall \lambda \in A$.*

Proof. We have a soft real set S being bounded if there exists a soft real number $\widetilde{m}$ such that $|\widetilde{x}| \widetilde{\leq} \widetilde{m}$ for all $\widetilde{x} \in S$ iff for any $\lambda \in A$, $|\widetilde{x}|(\lambda) \widetilde{\leq} \widetilde{m}(\lambda)$ for all $\widetilde{x} \in S$ iff $S(\lambda)$ are bounded sets in $\mathbb{R}$, $\forall \lambda \in A$. □

Definition 20. A soft number $\widetilde{M}$ is said to be the least upper bound of a soft set S if $\widetilde{x} \widetilde{\leq} \widetilde{M}$, $\forall \widetilde{x} \in S$, and for any $\widetilde{\epsilon} \widetilde{>} \overline{0}$ there exists a $\widetilde{y} \in S$ such that $\widetilde{M} - \widetilde{\epsilon} \widetilde{<} \widetilde{y} \leq \widetilde{M}$.

Definition 21. A soft number $\widetilde{m}$ is said to be greatest lower bound of a soft set S if $\widetilde{x} \widetilde{\geq} \widetilde{m}$, $\forall \widetilde{x} \in S$, and for any $\widetilde{\epsilon} \widetilde{>} \overline{0}$ there exists a $\widetilde{y} \in S$ such that $\widetilde{m} + \widetilde{\epsilon} \widetilde{>} \widetilde{y} \geq \widetilde{m}$.

Proposition 22. *Every bounded soft real set (F, A) has a least upper bound (l.u.b.) and a greatest lower bound (g.l.b.).*

Proof. Let (F, A) be a bounded soft real set. Since $F(\lambda)$'s are bounded set in $\mathbb{R}$, l.u.b. of $F(\lambda)$ exists; say m_λ. Take $\widetilde{m}(\lambda) = m_\lambda$, $\forall \lambda \in A$. Now for $\widetilde{\epsilon} \widetilde{>} \overline{0}$, there exists y_λ such that $m_\lambda - \widetilde{\epsilon}(\lambda) < y_\lambda \leq m_\lambda$, $\forall \lambda \in A$; that is, $\widetilde{m}(\lambda) - \widetilde{\epsilon}(\lambda) < \widetilde{y}(\lambda) \leq \widetilde{m}(\lambda)$ where $\widetilde{y}(\lambda) = y_\lambda$. The proof for greatest lower bound is similar. □

Note. Uniqueness of l.u.b and g.l.b in a soft set follows from the uniqueness of l.u.b and g.l.b of crisp set.

Theorem 23 (nested interval theorem). *If $I_n (= [\widetilde{a}_n, \widetilde{b}_n])$ is a sequence of nested soft closed intervals satisfying the properties $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \cdots$ and $L(I_n) \to \overline{0}$ as $n \to \infty$; then $\bigcap_{n=1}^{\infty} I_n$ is a singleton set $\widetilde{a}$.*

Proof. Since for any $\lambda \in A$, $[\widetilde{a}_n, \widetilde{b}_n](\lambda)$ is a closed interval $[\widetilde{a}_n(\lambda), \widetilde{b}_n(\lambda)]$, by the definition of subset of soft sets $\{[\widetilde{a}_n(\lambda), \widetilde{b}_n(\lambda)] : n \in \mathbb{N}\}$ is a sequence of sets satisfying the properties of nested intervals theorem. Hence $\bigcap_{n=1}^{\infty} I_n(\lambda) = \{a_\lambda\}$, which implies $\bigcap_{n=1}^{\infty} I_n = \widetilde{a}$ (where $\widetilde{a}(\lambda) = a_\lambda, \forall \lambda \in A$). □

Definition 24. A soft set is said to be soft closed if for any $\lambda \in A$, $S(\lambda)$ is closed set in $\mathbb{R}$.

Definition 25. Let $\{\widetilde{x}_n\}$ be sequence of soft real numbers. Then $\{\widetilde{y}_n\}$ is said to be a soft subsequence of $\{\widetilde{x}_n\}$ if $\{\widetilde{y}_n(\lambda)\}$ is a subsequence of $\{\widetilde{x}_n(\lambda)\}$ for all $\lambda \in A$.

Proposition 26 (Bolzano-Weierstrass's form). *Let $\{\widetilde{x}_n\}$ be a sequence of soft real numbers in a bounded soft closed set S. Then we can construct a soft subsequence $\{\widetilde{y}_n\}$ of $\{\widetilde{x}_n\}$ which is convergent in S.*

Proof. Let $\{\widetilde{x}_n\}$ be a sequence of soft real numbers in a bounded soft closed set S. Since the sequence $\{\widetilde{x}_n(\lambda)\}$ is bounded and in closed set $S(\lambda)$, by Bolzano-Weierstrass's theorem there exists a subsequence $\{\widetilde{x}_{n_k^\lambda}(\lambda)\}$ converging to some point $x_\lambda \in S(\lambda)$. Now if we construct a sequence of soft numbers $\{\widetilde{y}_n\}$ such that $\widetilde{y}_k(\lambda)$ is the kth member of the subsequence $\{\widetilde{x}_{n_k^\lambda}(\lambda)\}$, since for every $\lambda \in A$ sequence $\{\widetilde{x}_{n_k^\lambda}(\lambda)\}$ is convergent to a point $x_\lambda \in S(\lambda)$, then for any $\widetilde{\epsilon} > \overline{0}$ there exists $\widetilde{n}_0(\lambda)$ such that $|\widetilde{x}_{n_k^\lambda}(\lambda) - x_\lambda| < \widetilde{\epsilon}(\lambda)$, $\forall n_k^\lambda > \widetilde{n}_0(\lambda)$, which holds for all $\lambda \in A$. Hence $|\widetilde{y}_k(\lambda) - \widetilde{x}(\lambda)| < \widetilde{\epsilon}(\lambda)$, $\forall k > \widetilde{n}_0(\lambda)$, where $\widetilde{x}(\lambda) = x_\lambda$. Hence the sequence $\{\widetilde{y}_n\}$ is convergent in S. □

Proposition 27 (Bolzano-Weierstrass's theorem in finite parameter set). *If $\{\widetilde{x}_n\}$ is a sequence in a bounded soft closed set S, then there exists a subsequence $\{\widetilde{x}_{n_k}\}$ of $\{\widetilde{x}_n\}$ converging to some $\widetilde{x} \in S$.*

Proof. Let $\{\widetilde{x}_n\}$ be a sequence in soft bounded closed set S and the parameter set $A = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Then $\{\widetilde{x}_n(\lambda_1)\}$ is a sequence in bounded closed set $S(\lambda_1)$, so by Bolzano-Weierstrass's theorem $\{\widetilde{x}_n(\lambda_1)\}$ has a subsequence $\{\widetilde{x}_{n_{k_1}}(\lambda_1)\}$ converging to some point in $S(\lambda_1)$ (say x_{λ_1}). Next consider the subsequence $\{\widetilde{x}_{n_{k_1}}\}$ in soft bounded closed set S, and by the similar arguments $\{\widetilde{x}_{n_{k_1}}(\lambda_2)\}$ has a subsequence $\{\widetilde{x}_{n_{k_2}}(\lambda_2)\}$ converging to some point in $S(\lambda_2)$ (Say x_{λ_2}). Continuing this process, we have a subsequence $\{\widetilde{x}_{n_{k_m}}\}$ of $\{\widetilde{x}_n\}$, where $\{\widetilde{x}_{n_{k_m}}(\lambda_m)\}$ converges to some point in $S(\lambda_m)$ (Say x_{λ_m}). By the construction it is clear that $\{\widetilde{x}_{n_{k_m}}(\lambda_i)\}$ is a subsequence of $\{\widetilde{x}_{n_{k_i}}(\lambda_i)\}$ converging to x_{λ_i} for all $i = 1, 2, \ldots, m$. Hence $\{\widetilde{x}_{n_{k_m}}(\lambda_i)\}$ converges to x_{λ_i} for all $i = 1, 2, \ldots, m$, which implies $\widetilde{x}_{n_{k_m}} \to \widetilde{x}$, where $\widetilde{x}(\lambda_i) = x_{\lambda_i}$ for all $i = 1, 2, 3$. □

4. Limit and Continuity

Definition 28 (function of soft sets). Let (F, A) and (G, B) be two soft real sets. Then f is said to be a function of soft sets from (F, A) to (G, B) if f sends a soft element of (F, A) to a unique soft element of (G, B).

Definition 29. A soft set D is said to be a soft domain if, for each $\lambda \in A$, $D(\lambda)$ is an open set in $\mathbb{R}$.

Definition 30. A soft set S is said to be a neighborhood of $\widetilde{x}_0$ if $S(\lambda)$ is a neighborhood of $\widetilde{x}_0(\lambda)$, $\forall \lambda \in A$.

Definition 31. A function of soft sets $f : (F, A) \to (G, B)$ is said to be constant in a domain D if the function of soft sets $f(\widetilde{x}) = \widetilde{k}$, $\forall \widetilde{x} \in (F, A)$, where $\widetilde{k}$ is a fixed soft element of (G, B).

Definition 32 (limit of a function of soft sets). Let $f : (F, A) \to (G, A)$ be a function of soft sets and $\widetilde{x}_0 \widetilde{\in} (F, A)$. Let a function of soft sets f be defined in some deleted neighborhood of $\widetilde{x}_0$. Then $\widetilde{l}$ is said to be the soft limit of the function of soft sets f as $\widetilde{x}$ tends to $\widetilde{x}_0$ and is denoted by $f(\widetilde{x}) \to \widetilde{l}$ as $\widetilde{x} \to \widetilde{x}_0$ or $\lim_{\widetilde{x} \to \widetilde{x}_0} f(\widetilde{x}) = \widetilde{l}$ if for any $\widetilde{\epsilon} \widetilde{>} \overline{0}$ there exists $\widetilde{\delta} \widetilde{>} \overline{0}$ such that $|f(\widetilde{x}) - \widetilde{l}| \widetilde{<} \widetilde{\epsilon}$ whenever $\overline{0} \neq |\widetilde{x} - \widetilde{x}_0| \widetilde{<} \widetilde{\delta}$. Further if $f(\widetilde{x}_0) = \widetilde{l}$, then we call the function of soft sets f continuous at $\widetilde{x}_0$.

Proposition 33. *Soft limit of a function of soft sets is unique.*

Proposition 34. *Let $f : (F, A) \to (G, A)$ be a function of soft sets and $\widetilde{x}_0 \widetilde{\in} (F, A)$. Let a function of soft sets f be defined* *in some deleted neighborhood of* $\tilde{x}_0$. *If* $\lim_{\tilde{x}\to\tilde{x}_0} f(\tilde{x}) = \tilde{l}$, *then there exists soft deleted neighborhood of* $\tilde{x}_0$ *such that* f *is bounded therein.*

Proof. Since $\lim_{\tilde{x}\to\tilde{x}_0} f(\tilde{x}) = \tilde{l}$, so for $\tilde{\epsilon} = \bar{1}$ there exists a $\tilde{\delta} > \bar{0}$ such that $|f(\tilde{x}) - \tilde{l}| \tilde{<} \bar{1}$ whenever $\bar{0} \neq |\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$. Thus, $|f(\tilde{x})| \tilde{\leq} |f(\tilde{x}) - \tilde{l}| + |\tilde{l}| \tilde{<} \bar{1} + |\tilde{l}|$ whenever $\bar{0} \neq |\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$, which shows the result. □

Result 1. If $\lim_{\tilde{x}\to\tilde{x}_0} f(\tilde{x}) = \tilde{l}$ ($\tilde{l}(\lambda) \neq 0\ \forall\lambda \in A$), then for $\tilde{\epsilon} = (1/2)|\tilde{l}|$ there exists $\tilde{\delta} \tilde{>} \bar{0}$ such that $|f(\tilde{x}) - \tilde{l}| \tilde{<} (1/2)|\tilde{l}|$ whenever $\bar{0} \neq |\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$. Therefore, $||f(\tilde{x})| - |\tilde{l}|| \leq |f(\tilde{x}) - \tilde{l}| \tilde{<} (1/2)|\tilde{l}|$. Thus, $(1/2)|\tilde{l}| \leq |f(\tilde{x})| \tilde{<} (3/2)|\tilde{l}|$ whenever $\bar{0} \neq |\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$.

Definition 35. A function of soft sets f is said to be continuous in a soft set (F, A) if the function of soft sets f is continuous at any soft element $\tilde{x} \tilde{\in} (F, A)$.

Proposition 36 (limit theorem of functions of soft sets). *Let* f *and* g *be two functions of soft sets defined in some soft domain D. Let* $\tilde{x}_0$ *be a soft point of D. Let* $\lim_{\tilde{x}\to\tilde{x}_0} f(\tilde{x}) = \tilde{l}$ *and* $\lim_{\tilde{x}\to\tilde{x}_0} g(\tilde{x}) = \tilde{m}$. *Then,*

(1) $\lim_{\tilde{x}\to\tilde{x}_0}\{f(\tilde{x}) + g(\tilde{x})\} = \tilde{l} + \tilde{m}$;

(2) $\lim_{\tilde{x}\to\tilde{x}_0}\{f(\tilde{x}) - g(\tilde{x})\} = \tilde{l} - \tilde{m}$;

(3) $\lim_{\tilde{x}\to\tilde{x}_0}\{\tilde{c} \cdot \tilde{g}(\tilde{x})\} = \tilde{c} \cdot \tilde{l}$;

(4) $\lim_{\tilde{x}\to\tilde{x}_0}\{f(\tilde{x}) \cdot g(\tilde{x})\} = \tilde{l} \cdot \tilde{m}$;

(5) $\lim_{\tilde{x}\to\tilde{x}_0}\{f(\tilde{x})/g(\tilde{x})\} = \tilde{l}/\tilde{m}$ *if* $g(\tilde{x})(\lambda) \neq 0$ *for all* $\lambda \in A$ *in some deleted neighborhood of* $\tilde{x}_0$.

Proofs are exactly the same as the proofs of limit of crisp case.

Proposition 37. *If* f *is a function of soft sets continuous at* $\tilde{x}_0 \tilde{\in} (F, A)$ *and* $f(\tilde{x}_0) \tilde{<} \bar{0}$ *(or* $\tilde{>} \bar{0}$*), then there exists* $\tilde{\delta} \tilde{>} \bar{0}$ *such that* $f(\tilde{x}) \tilde{<} \bar{0}$ *(or* $\tilde{>} \bar{0}$*) satisfying* $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$.

Proof. Let $f(\tilde{x}_0) = -\tilde{\epsilon}$. Then $\tilde{\epsilon} \tilde{>} \bar{0}$. By the continuity of f, there exists a $\tilde{\delta} \tilde{>} \bar{0}$ such that $|f(\tilde{x}) - f(\tilde{x}_0)| \tilde{<} \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$; that is, $f(\tilde{x}_0) - \tilde{\epsilon} \tilde{<} f(\tilde{x}) \tilde{<} f(\tilde{x}_0) + \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$; that is, $-2\tilde{\epsilon} \tilde{<} f(\tilde{x}) \tilde{<} \bar{0}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$. The argument for $f(\tilde{x}_0) \tilde{>} \bar{0}$ is similar. □

Corollary 38. *If* f *is a function of soft sets continuous at* $\tilde{x}_0 \tilde{\in} (F, A)$ *and* $f(\tilde{x}_0)(\lambda) < 0$ *(or* > 0*), then there exists* $\tilde{\delta} \tilde{>} \bar{0}$ *such that* $f(\tilde{x})(\lambda) < 0$ *(or* > 0*) satisfying* $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$.

Proof. The proof directly follows from Proposition 37 by taking $\tilde{\epsilon} = \bar{\epsilon}$ where $f(\tilde{x}_0)(\lambda) = -\epsilon$. □

Proposition 39. *Let* f *be a continuous function of soft sets on* $[\tilde{a}, \tilde{b}]$. *If* $\tilde{x}_n \to \tilde{x}$ *uniformly over parameter in* $[\tilde{a}, \tilde{b}]$, *then* $f(\tilde{x}_n) \to f(\tilde{x})$.

Proof. Let f be continuous at $\tilde{x}_0$. Now for any $\tilde{\epsilon} \tilde{>} \bar{0}$ there exists a $\tilde{\delta} \tilde{>} \bar{0}$ such that $|f(\tilde{x}) - f(\tilde{x}_0)| \tilde{<} \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| < \tilde{\delta}$. Since $\tilde{x}_n \to \tilde{x}$, there exists a soft natural number n_0 such that $|\tilde{x}_n - \tilde{x}_0| \tilde{<} \tilde{\delta}$, $\forall n > n_0$. Hence $|f(\tilde{x}_n) - f(\tilde{x}_0)| < \tilde{\epsilon}$, $\forall n > n_0$. Thus, $f(\tilde{x}_n) \to f(\tilde{x}_0)$. □

Lemma 40. *If* $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$ *and* $\tilde{a}_n \tilde{>} \bar{0}$ $(\tilde{<} \bar{0})$, $\forall n \in \mathbb{N}$, *then* $\tilde{a} \tilde{\geq} \bar{0}$ $(\tilde{\leq} \bar{0})$.

Proof. Let $\tilde{a}_n \tilde{>} \bar{0}$, $\forall n \in \mathbb{N}$. If for any $\lambda \in A$, $\tilde{a}(\lambda) < 0$, then for $\epsilon (= -\tilde{a}(\lambda))$ there exists a soft natural number $\tilde{n}_0$ such that $|\tilde{a}_n - \tilde{a}|(\lambda) < \bar{\epsilon}(\lambda) = \epsilon$, $\forall n > \tilde{n}_0(\lambda) \Rightarrow \tilde{a}(\lambda) - \epsilon < \tilde{a}_n(\lambda) < \tilde{a}(\lambda) + \epsilon$, $\forall n > \tilde{n}_0(\lambda)$, which is a contradiction. Hence $\tilde{a}(\lambda) \geq 0$, $\forall \lambda \in A$; that is, $\tilde{a} \geq \bar{0}$. □

We are now in a position to consider the extension of Bolzano's theorem in soft setting. This result plays a crucial role for several other theorems to follow immediately. The proof of this theorem is interesting.

Proposition 41 (Bolzano's form). *If a function of soft sets* f *is continuous on* $[\tilde{a}, \tilde{b}]$ *and* $f(\tilde{a}) \cdot f(\tilde{b}) < \bar{0}$, *then for any* $\mu \in A$ *there exists* $\tilde{z} \in [\tilde{a}, \tilde{b}]$ *such that* $f(\tilde{z})(\mu) = 0$.

Proof. Without loss of generality, let $f(\tilde{a})(\mu) < 0$. Let us construct a set $S = \{\tilde{x} : f(\tilde{y})(\mu) < 0\ \forall \tilde{y} \in [\tilde{a}, \tilde{x}]\}$. Since f is continuous at $\tilde{a}$, there exists a $\tilde{\delta}_1 \tilde{>} \bar{0}$ $(\tilde{a} + \tilde{\delta}_1/\bar{2} \tilde{<} \tilde{b})$ such that $f(\tilde{x}) < \bar{0}$ for all $\tilde{x} \in [\tilde{a}, \tilde{a} + \tilde{\delta}_1/\bar{2}]$. Therefore, the set S is nonempty. Clearly the soft real numbers form a poset under the relation $\tilde{\leq}$ on S. So by Hausdorff maximality principle there exists a maximal totally ordered subset $\bar{S} = \{\tilde{r}_\alpha : \alpha \in \Lambda\}$ of S. Let $\tilde{m}(\lambda) = \sup\{\tilde{r}_\alpha(\lambda) : \alpha \in \Lambda\}$ for all $\lambda \in A$. Since $\tilde{r}_\alpha \in S\ \forall \alpha \in \Lambda$, $\tilde{a} \tilde{\leq} \tilde{r}_\alpha \tilde{\leq} \tilde{b}$. Hence clearly $\tilde{m} \in [\tilde{a}, \tilde{b}]$. Now if $f(\tilde{m})(\mu) = 0$, then the theorem is proved. If not, then either (i) $f(\tilde{m})(\mu) > 0$ or (ii) $f(\tilde{m})(\lambda) < 0$.

Since $\tilde{r}_\alpha$s are the members of the totally ordered set $\bar{S}$, for any $\alpha, \beta \in \Lambda$, $[\tilde{a}, \tilde{r}_\alpha] \subseteq$ or $\supseteq$ $[\tilde{a}, \tilde{r}_\beta]$. Hence $f(\tilde{y})(\mu) < 0$, $\forall \tilde{y} \in \bigcup_{\alpha\in\Lambda} [\tilde{a}, \tilde{r}_\alpha]$.

Now we shall show that

$$\text{for } \tilde{s} \in [\tilde{a}, \tilde{b}] \quad \text{with } \tilde{s} \tilde{<} \tilde{m}, \quad f(\tilde{s})(\mu) < 0. \tag{$*$}$$

□

Proof. Since $\tilde{s} \tilde{<} \tilde{m}$ and $\tilde{m}(\lambda) = \sup\{\tilde{r}_\alpha(\lambda) : \alpha \in \Lambda\}$ for all $\lambda \in A$, by the property of supremum for any $\lambda \in A$, there exists a $\tilde{r}_{\alpha_\lambda} \in \bar{S}$ such that $\tilde{s}(\lambda) < \tilde{r}_{\alpha_\lambda}(\lambda) \leq \tilde{m}(\lambda)$; that is, $\tilde{s}(\lambda) \in [\tilde{a}, \tilde{r}_{\alpha_\lambda}](\lambda)$, which implies $\tilde{s}(\lambda) \in \bigcup_{\alpha\in\Lambda} [\tilde{a}, \tilde{r}_\alpha](\lambda)$. Since λ is arbitrary, $\tilde{s} \in \bigcup_{\alpha\in\Lambda} [\tilde{a}, \tilde{r}_\alpha]$. Hence $f(\tilde{s})(\mu) < 0$.

Now consider the possibility (i) If $f(\tilde{m})(\mu) > 0$. Then by the continuity of f at $\tilde{m}$ there exists a $\tilde{\delta} \tilde{>} \bar{0}$ such that $f(\tilde{x})(\mu) > 0$ for every $|\tilde{x} - \tilde{m}| \tilde{<} \tilde{\delta}$ in $[\tilde{a}, \tilde{b}]$. Therefore, we can find a $\tilde{m}_0 \tilde{<} \tilde{m}$ with $|\tilde{m}_0 - \tilde{m}| \tilde{<} \tilde{\delta}$ in $[\tilde{a}, \tilde{b}]$ such that $f(\tilde{m}_0)(\mu) > 0$, which contradicts the $(*)$. Hence possibility (i) is not possible; that is, $f(\tilde{m})(\mu)$ cannot be greater than 0.

Next consider the possibility (ii) $f(\tilde{m})(\mu) < 0$. We show that either there exists a point to serve the theorem or $\tilde{m} \in S$.

By $(*)$ for any $\tilde{s} \tilde{<} \tilde{m}$ in $[\tilde{a}, \tilde{b}]$, $f(\tilde{s})(\mu) < 0$. Choose any $\tilde{x} \in [\tilde{a}, \tilde{m}]$ with $\tilde{x}(\lambda) = \tilde{m}(\lambda)$ for some $\lambda \in A$ and $\tilde{x}(\lambda) \neq \tilde{m}(\lambda)$ for some $\lambda \in A$. If $f(\tilde{x})(\mu) = 0$ for any such soft number, then the theorem is proved. If $f(\tilde{x})(\mu) > 0$, then proceeding as in possibility (i) we have a contradiction. So if $f(\tilde{x})(\mu) \neq 0$ for such soft numbers, then $f(\tilde{x})(\mu) < 0$; that is, $\tilde{x} \in [\tilde{a}, \tilde{m}]$ implies $f(\tilde{x})(\mu) < 0$. Thus, either there exists a soft number to serve the theorem or $\tilde{m} \in S$.

Now if $\tilde{m} \in S$ and since $\tilde{m} \neq \tilde{b}$ (as $f(\tilde{m})(\mu) < 0$ and $f(\tilde{b})(\mu) > 0$, which follows from the condition $f(\tilde{a}) \cdot f(\tilde{b}) < \overline{0}$ and $f(\tilde{a})(\mu) < 0$), then there exists a $\nu \in A$ such that $\tilde{m}(\nu) < \tilde{b}(\nu)$. Choose a $\epsilon > 0$ such that $\tilde{m}(\nu) + \epsilon < \tilde{b}(\nu)$. Now if we define $\tilde{k}$ as $\tilde{k}(\lambda) = \tilde{m}(\lambda)$ for $\lambda (\in A) \neq \nu$ and $\tilde{k}(\nu) = \tilde{m}(\nu) + \epsilon$, then $\tilde{k} \notin S$; otherwise, it contradicts that $\tilde{m}$ is the supremum of the maximal chain $\overline{S}$. Therefore, there exists a $\tilde{p}_1 \in [\tilde{a}, \tilde{k}]$ such that $f(\tilde{p}_1)(\mu) \nless 0$. If $f(\tilde{p}_1)(\mu) = 0$ again, the theorem is served. If $f(\tilde{p}_1)(\mu) > 0$, then, since $\tilde{p}_1 \in [\tilde{a}, \tilde{k}]$ but $\tilde{p}_1 \notin [\tilde{a}, \tilde{m}]$, by construction of $\tilde{k}$ it follows that $\tilde{m}(\nu) < \tilde{p}_1(\nu)$ and $\tilde{p}_1(\lambda) \leq \tilde{m}(\lambda)$, $\forall \lambda (\in A) \neq \mu$. Now construct another soft number $\tilde{q}_1$ such that $\tilde{q}_1(\lambda) = \tilde{p}_1(\lambda)$ for $\lambda (\in A) \neq \nu$ and $\tilde{q}_1(\nu) = \tilde{m}(\nu)$. Clearly $\tilde{q}_1 \in [\tilde{a}, \tilde{m}]$, so $f(\tilde{q}_1)(\mu) < 0$ as we are considering the remaining possibility $\tilde{m} \in S$.

Now $\tilde{p}_1, \tilde{q}_1 \in [\tilde{a}, \tilde{b}]$ and $f(\tilde{p}_1)(\mu) > 0$ and $f(\tilde{q}_1)(\mu) < 0$. Take $\tilde{r}_1 = (\tilde{p}_1 + \tilde{q}_1)/\overline{2}$ $(\in [\tilde{a}, \tilde{b}])$. If $f(\tilde{r}_1)(\mu) = 0$, then the theorem is proved. If $f(\tilde{r}_1)(\mu) > 0$ choose $\tilde{p}_2 = \tilde{r}_1$ and $\tilde{q}_2 = \tilde{q}_1$; otherwise $\tilde{p}_2 = \tilde{p}_1$ and $\tilde{q}_2 = \tilde{r}_1$. In the similar argument choose $\tilde{r}_n = (\tilde{p}_n + \tilde{q}_n)/\overline{2}$, and if $f(\tilde{r}_n)(\mu) = 0$, then the theorem is proved. If $f(\tilde{r}_n)(\mu) > 0$ choose $\tilde{p}_{n+1} = \tilde{r}_n$ and $\tilde{q}_{n+1} = \tilde{q}_n$; otherwise $\tilde{p}_{n+1} = \tilde{p}_n$ and $\tilde{q}_{n+1} = \tilde{r}_n$. In this way if $f(\tilde{r}_i)(\mu) = 0$, then the theorem is proved. Otherwise we get sequences $\{\tilde{p}_n\}$ and $\{\tilde{q}_n\}$ such that $f(\tilde{p}_n)(\mu) > 0$ and $f(\tilde{q}_n)(\mu) < 0$. Clearly the sequence $\{I_n\}$, where $I_n = [\tilde{q}_n(\nu), \tilde{p}_n(\nu)]$ for all $n \in \mathbb{N}$, satisfies the nested intervals theorem. Therefore, there exists a unique $z \in \bigcap_{n \in \mathbb{N}} [\tilde{q}_n(\nu), \tilde{p}_n(\nu)]$ and $\tilde{q}_n(\nu) \to z$, $p_n(\nu) \to z$ as $n \to \infty$. By the construction of the sequence $\{\tilde{p}_n\}, \{\tilde{q}_n\}$, $\tilde{p}_n(\lambda) = \tilde{q}_n(\lambda) = \tilde{p}_1(\lambda)$ for all $\lambda (\in A) \neq \nu$. Hence $\tilde{q}_n \to \tilde{z}$ and $\tilde{p}_n \to \tilde{z}$ as $n \to \infty$, where $\tilde{z}(\lambda) = \tilde{p}_1(\lambda)$ for all $\lambda (\in A) = \nu$ and $\tilde{z}(\nu) = z$. Since the sequences $\{\tilde{p}_n\}, \{\tilde{q}_n\}$ only vary in the νth parameter and are constant in the other parameter, the sequences $\{\tilde{p}_n\}, \{\tilde{q}_n\}$ converge to $\tilde{z}$ parameterwise uniformly. Consequently, $f(\tilde{q}_n) \to f(\tilde{z})$ and $f(p_n) \to f(\tilde{z})$ as $n \to \infty$, but $f(\tilde{q}_n)(\mu) < 0$ and $f(\tilde{p}_n)(\mu) > 0$ for all $n \in \mathbb{N}$, which shows that $f(\tilde{z})(\mu) \leq 0$ and $f(z)(\mu) \geq 0$. Hence $f(\tilde{z})(\mu) = 0$. □

Corollary 42. *If a function of soft sets f is continuous on $[\tilde{a}, \tilde{b}]$ and if for $\lambda \in A$, $f(\tilde{a})(\lambda) \cdot f(\tilde{b})(\lambda) < 0$, then there exists $\tilde{z} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{z})(\lambda) = 0$.*

Remark 43. If a function of soft sets f is continuous on $[\tilde{a}, \tilde{b}]$ and $f(\tilde{a}) \cdot f(\tilde{b}) \tilde{<} \overline{0}$, then there may not exist any $\tilde{z} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{z}) = \overline{0}$, which can be shown by the following example.

Example 1. Let f be a function of soft sets with parameter set $A = \{\lambda, \mu\}$ defined on $[\overline{0.1}, \overline{0.9}]$ by $f(\tilde{x})(\lambda) = \tilde{x}(\lambda) + \tilde{x}(\mu) - 1$ and $f(\tilde{x})(\mu) = (\tilde{x}(\lambda))^2 + (\tilde{x}(\mu))^2 - 1$. Clearly f is continuous on $[\overline{0.1}, \overline{0.9}]$, but $f(\tilde{x})(\lambda) = f(\tilde{x})(\mu) = 0$ only when $\tilde{x}(\lambda) = 1$ or 0, which does not belong to the domain set, hence showing the result.

Proposition 44. *If f is a continuous function on a soft interval $[\tilde{a}, \tilde{b}]$ over a finite parameter set A, then f is bounded.*

Proof. If possible let f be not bounded above. Then there does not exist any $\tilde{m}$ such that $|f(\tilde{x})| \tilde{\leq} \tilde{m}$ $\forall \tilde{x} \in [\tilde{a}, \tilde{b}]$; that is, there exists a $\lambda \in A$, such that for every $n \in \mathbb{N}$ there exists a sequence $\{\tilde{x}_n\} \in [\tilde{a}, \tilde{b}]$ with $|f(\tilde{x}_n)(\lambda)| > n$ $\forall n \in \mathbb{N}$ (set of natural numbers). Now since $\{\tilde{x}_n\}$ is sequence in $[\tilde{a}, \tilde{b}]$, by Proposition 27, there exists a subsequence $\{\tilde{x}_{n_k}\}$ of $\{\tilde{x}_n\}$ converging to some soft number $\tilde{x} \in [\tilde{a}, \tilde{b}]$. Since the set parameter A is finite, so the sequence $\tilde{x}_{n_k} \to \tilde{x}$ uniformly over parameter, which implies $f(\tilde{x}_{n_k}) \to f(\tilde{x})$. However, $|f(\tilde{x}_{n_k})(\lambda)| > n_k$, $\forall k \in \mathbb{N}$, which implies $|f(\tilde{x})(\lambda)| \geq n_k$, $\forall k \in \mathbb{N}$, a contradiction. Hence f is bounded. □

Note. If the parameter set is not finite then a continuous function on a soft interval $[\tilde{a}, \tilde{b}]$ may not be bounded, which will considered by Example 2.

Lemma 45. *Let $[0, 1]$ be the closed interval in $\mathbb{R}$ endowed with the usual topology. If $X = \prod_{n=1}^{\infty} [0, 1]$, then there exists an unbounded continuous function $F : X \to \mathbb{R}$ with respect to box topology in X.*

Proof. Clearly X is a normal space and $A = \prod_{n=1}^{\infty} \{0, 1\}$ is a closed subset of X. Now define a function $f : A \to \mathbb{R}$ such that $f(x_1, x_2, x_3, \ldots) = 0$ if $x_i = 0$, $\forall i \in \mathbb{N}$, $f(x_1, x_2, x_3, \ldots) = 2n$ if x_{2n} is the first nonzero in the sequence $\{x_i\}$, and $f(x_1, x_2, x_3, \ldots) = -(2n + 1)$ if x_{2n+1} is the first nonzero in the sequence $\{x_i\}$. Since, with the box topology, the subspace topology on A is discrete topology, f is continuous and by construction f is unbounded in A. Now since X is a normal space, by Tietze's extension theorem there exists a continuous function F defined on X such that $f(a) = F(a)$, $\forall a \in A$, so F is unbounded, hence the lemma. □

Example 2. Let us consider the soft interval $[\overline{0}, \overline{1}]$ and $A = \{\lambda_i : i \in \mathbb{N}\}$. If we define a function of soft sets g such that $g(\tilde{x})(\lambda) = F(\tilde{x}(\lambda_1), \tilde{x}(\lambda_2), \tilde{x}(\lambda_3), \ldots)$ if $\lambda = \lambda_1$ and $g(\tilde{x})(\lambda) = 0$ if $\lambda \neq \lambda_1$, where F is as in Lemma 45, taking any $\tilde{x}_0 \in [\overline{0}, \overline{1}]$, then since F is continuous, for any $\tilde{\epsilon} \tilde{>} \overline{0}$ there exists $\delta_i > 0$, $i = 1, 2, 3, \ldots$ such that $|F(\tilde{x}(\lambda_1), \tilde{x}(\lambda_2), \tilde{x}(\lambda_3), \ldots) - F(\tilde{x}_0(\lambda_1), \tilde{x}_0(\lambda_2), \tilde{x}_0(\lambda_3), \ldots)| < \tilde{\epsilon}(\lambda_1)$ whenever $|\tilde{x}(\lambda_i) - \tilde{x}_0(\lambda_i)| < \delta_i$, $\forall i = 1, 2, 3, \ldots$, which shows that $|g(\tilde{x})(\lambda_1) - g(\tilde{x}_0)(\lambda_1)| < \tilde{\epsilon}(\lambda_1)$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$ (where $\tilde{\delta}(\lambda_i) = \delta_i$ for $i = 1, 2, 3, \ldots$) and $|g(\tilde{x})(\lambda_i) - g(\tilde{x}_0)(\lambda_i)| = 0$ $\forall i \neq 1$. Hence g is continuous but unbounded.

Proposition 46 (intermediate value property). *If a function of soft real sets is continuous on $[\tilde{a}, \tilde{b}]$ and if $f(\tilde{a})(\lambda) < k < f(\tilde{b})(\lambda)$ for some $\lambda \in A$, then there exists $\tilde{c}$ between $\tilde{a}$ and $\tilde{b}$ such that $f(\tilde{c})(\lambda) = k$.*

Proof. Consider the function of soft sets $g(\tilde{x}) = f(\tilde{x}) - \overline{k}$. Then clearly $g(\tilde{x})$ is continuous in $[\tilde{a}, \tilde{b}]$ and

$g(\tilde{a})(\lambda) \cdot g(\tilde{b})(\lambda) < 0$. Therefore, by Corollary 42 there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $g(\tilde{c})(\lambda) = 0$; that is, $g(\tilde{c})(\lambda) = f(\tilde{c})(\lambda) - k = 0$; that is, $f(\tilde{c})(\lambda) = k$. □

Proposition 47 (fixed point theorem). *If a function of soft real sets f is continuous on $[\tilde{a}, \tilde{b}]$ and the value is also in the soft interval $[\tilde{a}, \tilde{b}]$, then for any $\lambda \in A$ there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{c})(\lambda) = \tilde{c}(\lambda)$.*

Proof. Consider the function of soft sets $g(\tilde{x}) = f(\tilde{x}) - \tilde{x}$. If $f(\tilde{a})(\lambda) = \tilde{a}(\lambda)$ or $f(\tilde{b})(\lambda) = \tilde{b}(\lambda)$, then the proof is over. If not, then clearly $g(\tilde{x})$ is continuous in $[\tilde{a}, \tilde{b}]$ and $g(\tilde{a})(\lambda) \cdot g(\tilde{b})(\lambda) < 0$. Thus, by Corollary 42 there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $g(\tilde{c})(\lambda) = 0$; that is, $f(\tilde{c})(\lambda) - \tilde{c}(\lambda) = 0$; that is, $f(\tilde{c})(\lambda) = \tilde{c}(\lambda)$. □

Result 2. Let f be a continuous function of soft sets on $[\tilde{a}, \tilde{b}]$. Further if f is bounded, there may not exist any soft real number in $[\tilde{a}, \tilde{b}]$ attaining the bound parameterwise.

Consider the function F as in Lemma 45 and $h(x) = \tan^{-1} x$ in $\mathbb{R}$. Clearly the range set of F is $\mathbb{R}$, so $h \circ F : X \to \mathbb{R}$ is continuous. Now if we consider the function g as in Example 2, taking $h \circ F$ in place of F, then exactly by the similar argument g is a continuous function of soft sets in $[\tilde{a}, \tilde{b}]$ with range set in λ_1 parameter being $(-\pi/2, \pi/2)$ by Proposition 46 (intermediate value property), which shows the result.

Proposition 48. *Let a function of soft sets f be continuous on a soft set S and k be a real number. If*

$$g(\tilde{x})(\lambda) = \begin{cases} k - f(\tilde{x})(\lambda) & \text{for } \lambda = \mu \\ f(\tilde{x})(\lambda) & \lambda \neq \mu, \end{cases} \tag{2}$$

then the function of soft sets g is continuous.

Proof. Let $\tilde{x}_0 \in S$. Since f is continuous for any $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists $\tilde{\delta} \tilde{>} \overline{0}$ such that $|f(\tilde{x}) - f(\tilde{x}_0)| \tilde{<} \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$. By the definition of g we have $|f(\tilde{x}) - f(\tilde{x}_0)| = |g(\tilde{x}) - g(\tilde{x}_0)|$. Hence g is continuous. □

Proposition 49. *Let a function of soft sets f be continuous on a soft set S. If*

$$g(\tilde{x})(\lambda) = \begin{cases} \dfrac{1}{f(\tilde{x})(\lambda)} & \text{for } \lambda = \mu \\ f(\tilde{x})(\lambda) & \lambda \neq \mu \end{cases} \tag{3}$$

$$f(\tilde{x}(\mu)) \neq 0 \ \forall \tilde{x} \in S,$$

then the function of soft sets g is continuous.

Proof. We have

$$|g(\tilde{x}) - g(\tilde{x}_0)|(\lambda) = \begin{cases} |f(\tilde{x})(\lambda) - f(\tilde{x}_0)(\lambda)| & \text{if } \lambda \neq \mu \\ \dfrac{|f(\tilde{x})(\lambda) - f(\tilde{x}_0)(\lambda)|}{|f(\tilde{x})(\lambda)| \, |f(\tilde{x}_0)(\lambda)|} & \text{if } \lambda = \mu. \end{cases} \tag{4}$$

Since f is continuous at $\tilde{x}_0$, there exists a $\tilde{\delta} > \overline{0}$ such that $(1/2)|f(\tilde{x}_0)(\mu)| \leq |f(\tilde{x})(\mu)| \tilde{<} (3/2)|f(\tilde{x}_0)(\mu)|$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$ (by result 4.8), which implies $1/|f(\tilde{x})(\mu)| < 2/|f(\tilde{x}_0)(\mu)|$. Hence for any $\tilde{\epsilon} > \overline{0}$, $|g(\tilde{x}) - g(\tilde{x}_0)|(\mu) \leq (2/|f(\tilde{x}_0)|(\mu))(1/|f(\tilde{x}_0)(\mu)|)|f(\tilde{x})(\mu) - f(\tilde{x}_0)(\mu)|$, and for $\lambda \neq \mu$, $|f(\tilde{x}) - f(\tilde{x}_0)|(\lambda) = |g(\tilde{x}) - g(\tilde{x}_0)|(\lambda)$ can be made less than $\tilde{\epsilon}$ directly by the continuity of f. □

Remark 50. Let a function of soft sets f from (F, A) to (G, A) satisfy the condition that for any $\lambda \in A$, $\tilde{y}(\lambda) = \tilde{z}(\lambda)$ $(\tilde{z}, \tilde{y} \in (G, A))$ implies $f(\tilde{y})(\lambda) = f(\tilde{z})(\lambda)$. Then if we define $f_\lambda : F(\lambda) \to G(\lambda)$ for any $x \in F(\lambda)$ $f_\lambda(\tilde{x}(\lambda)) = f(\tilde{x})(\lambda)$ if $x = \tilde{x}(\lambda)$, then clearly f_λ is a function from $F(\lambda)$ to $G(\lambda)$.

Proposition 51. *A function of soft sets f satisfying the property of Remark 50 is continuous at $\tilde{x}_0$ iff the function f_λ is continuous at $\tilde{x}_0(\lambda)$, $\forall \lambda \in A$.*

Proof. Let the function of soft sets f be continuous at $\tilde{x}_0$. Take any $\epsilon > 0$; then for $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists a soft element $\tilde{\delta} \tilde{>} \overline{0}$ such that $|f(\tilde{x}) - f(\tilde{x}_0)| \tilde{<} \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$; that is, $|f(\tilde{x}) - f(\tilde{x}_0)|(\lambda) < \tilde{\epsilon}(\lambda) = \epsilon$ whenever $|\tilde{x} - \tilde{x}_0|(\lambda) \tilde{<} \tilde{\delta}(\lambda)$, $\forall \lambda \in A$; that is, $|f_\lambda(\tilde{x}(\lambda)) - f_\lambda(\tilde{x}_0(\lambda))| < \epsilon$ whenever $|\tilde{x}(\lambda) - \tilde{x}_0(\lambda)| < \tilde{\delta}(\lambda)$ (since $f_\lambda(\tilde{x}(\lambda)) = f(\tilde{x})(\lambda)$ is independent for the other parameter), which shows that f_λ is continuous at $\tilde{x}_0(\lambda)$, $\forall \lambda \in A$.

Conversely, let f_λ be continuous at $\tilde{x}_0(\lambda)$, $\forall \lambda \in A$. Take any $\tilde{\epsilon} \tilde{>} \overline{0}$. Since f_λ is continuous at $\tilde{x}_0(\lambda)$, $\forall \lambda \in A$, so for every $\tilde{\epsilon}(\lambda) > 0$, there exists δ_λ such that $|f_\lambda(\tilde{x}(\lambda)) - f_\lambda(\tilde{x}_0(\lambda))| < \tilde{\epsilon}(\lambda)$ whenever $|\tilde{x}(\lambda) - \tilde{x}_0(\lambda)| < \delta_\lambda$. Now if we take $\tilde{\delta}(\lambda) = \delta_\lambda$, $\forall \lambda \in A$, then we have $|f(\tilde{x}) - f(\tilde{x}_0)| \tilde{<} \tilde{\epsilon}$ whenever $|\tilde{x} - \tilde{x}_0| \tilde{<} \tilde{\delta}$. □

Proposition 52. *Let f be continuous function of soft sets on $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50 iff $\tilde{x}_n \to \tilde{x}$ in $[\tilde{a}, \tilde{b}]$ implies $f(\tilde{x}_n) \to f(\tilde{x})$.*

Proof. Let f be continuous at $\tilde{x}_0$. By Proposition 51, for any $\lambda \in A$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(\tilde{x})(\lambda) - f(\tilde{x}_0)(\lambda)| < \epsilon$ whenever $|\tilde{x}(\lambda) - \tilde{x}_0(\lambda)| < \delta$. Since $\tilde{x}_n \to \tilde{x}$, for any $\delta > 0$, there exists a soft natural number n_0 such that $|\tilde{x}_n(\lambda) - \tilde{x}_0(\lambda)| \tilde{<} \tilde{\delta}(\lambda) = \delta$, $\forall n > n_0(\lambda)$. Hence $|f(\tilde{x}_n)(\lambda) - f(\tilde{x}_0)(\lambda)| < \epsilon$, $\forall n > n_0$. Therefore, $f(\tilde{x}_n)(\lambda) \to f(\tilde{x}_0)(\lambda)$. Since $\lambda \in A$ is arbitrary, $f(\tilde{x}_n)(\lambda) \to f(\tilde{x}_0)(\lambda)$, $\forall \lambda \in A$; that is, $f(\tilde{x}_n) \to f(\tilde{x}_0)$.

Conversely, let $\tilde{x}_n \to \tilde{x}$ in $[\tilde{a}, \tilde{b}]$ implies $f(\tilde{x}_n) \to f(\tilde{x})$ hold. Since the function of soft sets f satisfies the property of Remark 50, $f(\tilde{x})(\lambda)$ is independent of other parameters for all $\lambda \in A$. Thus, $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$ in $[\tilde{a}(\lambda), \tilde{b}(\lambda)]$ implies $f(\tilde{x}_n)(\lambda) \to f(\tilde{x})(\lambda)$ for all $\lambda \in A$; that is, $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$ in $[\tilde{a}(\lambda), \tilde{b}(\lambda)]$ implies $f_\lambda(\tilde{x}_n(\lambda)) \to f_\lambda(\tilde{x}(\lambda))$ for all $\lambda \in A$; that is, f_λ is continuous for all $\lambda \in A$. Hence by Proposition 51 f is continuous on $[\tilde{a}, \tilde{b}]$. □

Proposition 53. *If a function of soft sets f is continuous on a closed soft interval $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50, then f is bounded therein.*

Proof. Without loss of generality, let f be not bounded from above, then there does not exist any $\widetilde{M}$ such that $f(\tilde{x}) \tilde{\leq} \widetilde{M}$, $\forall \tilde{x} \in [\tilde{a}, \tilde{b}]$. That is, there exists some $\lambda \in A$ such that there does not exist any real number M_λ such that $f(\tilde{x})(\lambda) \leq M_\lambda$, $\forall \tilde{x} \in [\tilde{a}, \tilde{b}]$. That is, for any $n \in \mathbb{N}$ there exists $\{\tilde{x}_n\} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{x}_n)(\lambda) > n$. However, the sequence of soft numbers $\{\tilde{x}_n\}$ is defined on bounded soft closed interval, so by Proposition 26 there exists soft subsequence of soft numbers $\{\tilde{y}_n\}$ converging to some soft real number $\tilde{x} (\in [\tilde{a}, \tilde{b}])$. Since f is continuous and $\tilde{y}_n \to \tilde{x}$, we have $f(\tilde{y}_n) \to f(\tilde{x})$. However, $f(\tilde{x}_n)(\lambda) > n$, $\forall n \in \mathbb{N} \Rightarrow \mathbb{f}(\tilde{y}_n)(\lambda) > n$, $\forall n \in \mathbb{N} \Rightarrow \mathbb{f}(\tilde{x})(\lambda) \geq n$ $n \in \mathbb{N}$, which is a contradiction. Hence f is bounded in $[\tilde{a}, \tilde{b}]$. □

Proposition 54. *Let f be a continuous function of soft sets on $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50 and $\tilde{m}$ be a soft supremum of the set $S = \{f(\tilde{x}) : \tilde{x} \in [\tilde{a}, \tilde{b}]\}$; that is, $(\tilde{m}(\lambda) = \sup\{f(\tilde{x})(\lambda) : \tilde{x} \in [\tilde{a}, \tilde{b}]\})$; then there exists a $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{c}) = \tilde{m}$.*

Proof. Let $\tilde{m}$ be supremum of S. Take $\lambda \in A$; then by the l.u.b property for any $n \in \mathbb{N}$, there exists $\tilde{x}_n^\lambda \in [\tilde{a}, \tilde{b}]$ such that $\tilde{m}(\lambda) - 1/n \tilde{<} f(\tilde{x}_n^\lambda)(\lambda)$ $(= f_\lambda(\tilde{x}_n^\lambda(\lambda))) \tilde{\leq} \tilde{m}(\lambda)$. If we take $\tilde{x}_n(\lambda) = \tilde{x}_n^\lambda(\lambda)$, then $\tilde{m}(\lambda) - 1/n \tilde{<} f(\tilde{x}_n)(\lambda)$ $(= f_\lambda(\tilde{x}_n^\lambda(\lambda))) \tilde{\leq} \tilde{m}(\lambda)$ for all $\lambda \in A$. That is, $\tilde{m} - \overline{1}/n \tilde{<} f(\tilde{x}_n) \tilde{\leq} \tilde{m}$. Since the sequence $\{\tilde{x}_n\}$ is in bounded soft set $[\tilde{a}, \tilde{b}]$, by Proposition 26, there exists a soft subsequence $\{\tilde{y}_n\}$ of $\{\tilde{x}_n\}$ such that $\{\tilde{y}_n\}$ converges to some soft number in $[\tilde{a}, \tilde{b}]$ (say $\tilde{c}$). That is, $\tilde{y}_n \to \tilde{c}$ in $[\tilde{a}, \tilde{b}]$. Since $\tilde{m} - \overline{1}/n \tilde{<} f(\tilde{y}_n) \tilde{\leq} \tilde{m}$ (by the construction of the sequence $\{\tilde{y}_n\}$), by Sandwich theorem clearly $f(\tilde{y}_n) \to \tilde{m}$. However, f is continuous on $[\tilde{a}, \tilde{b}]$ satisfying Remark 50 and $\tilde{y}_n \to \tilde{c}$ in $[\tilde{a}, \tilde{b}]$, so $f(\tilde{y}_n) \to f(\tilde{c})$. Hence by the uniqueness of the limit $f(\tilde{c}) = \tilde{m}$. □

Proposition 55. *If a function of soft sets is continuous on $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50 and $f(\tilde{a}) \cdot f(\tilde{b}) < \overline{0}$, then there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{c}) = \overline{0}$.*

Proof. By Proposition 41 for any $\lambda \in A$ there exists a $\tilde{c}_\lambda$ such that $f(\tilde{c}_\lambda)(\lambda) = 0$. Now if we consider a soft real number $\tilde{c}$ such that $\tilde{c}(\lambda) = \tilde{c}_\lambda(\lambda)$, then clearly $f_\lambda(\tilde{c}(\lambda)) = f(\tilde{c})(\lambda) = f(\tilde{c}_\lambda)(\lambda) = 0$. □

Similarly, from Propositions 46 and 47, we have the following.

Proposition 56 (intermediate value property). *Let f be a continuous function of soft sets on $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50. If $f(\tilde{a}) < \tilde{k} < f(\tilde{b})$, then there exists $\tilde{c}$ between $\tilde{a}$ and $\tilde{b}$ such that $f(\tilde{c}) = \tilde{k}$.*

Proposition 57 (fixed point theorem). *If a function of soft sets f is continuous on $[\tilde{a}, \tilde{b}]$ satisfying the property of Remark 50 and the value is also in the soft interval $[\tilde{a}, \tilde{b}]$, then there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{c}) = \tilde{c}$.*

5. Differentiation

Definition 58. A function of soft sets f defined on a soft neighborhood of $\tilde{x}$ is said to be differentiable at $\tilde{x}$ iff f can be written as $f(\tilde{x} + \tilde{h}) - f(\tilde{x}) = \tilde{h}(g(\tilde{x}) + \tilde{\epsilon})$ for some function of soft sets g, where $\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$. Furthermore the function of soft sets g is said to be the derivative of f at $\tilde{x}$ and denoted by $f'(\tilde{x}) = g(\tilde{x})$.

Example 3 ($f(\tilde{x}) = \tilde{x}$ is differentiable). $f(\tilde{x} + \tilde{h}) - f(\tilde{x}) = \tilde{h}(\overline{1} + \tilde{\epsilon})$ where $\tilde{\epsilon} = \overline{0}$ as $\tilde{h} \to \overline{0}$. Therefore, f is differentiable and $f'(\tilde{x}) = \overline{1}$.

Proposition 59. *If f and g are differentiable at $\tilde{c}$, then scalar product, sum, and differences are also differentiable at $\tilde{c}$ and*

(1) $(\tilde{k}f)'(\tilde{c}) = \tilde{k}f'(\tilde{c})$, *where $\tilde{k}$ is a soft number;*
(2) $(f(\tilde{c}) + g(\tilde{c}))' = f'(\tilde{c}) + g'(\tilde{c})$;
(3) $(f(\tilde{c}) - g(\tilde{c}))' = f'(\tilde{c}) - g'(\tilde{c})$.

Proof. (1) Since f is differentiable, $f(\tilde{c} + \tilde{h}) - f(\tilde{c}) = \tilde{h}(f'(\tilde{c}) + \tilde{\epsilon})$ where $\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$, which implies $\tilde{k}f(\tilde{c} + \tilde{h}) - \tilde{k}f(\tilde{c}) = \tilde{h}(\tilde{k}f'(\tilde{c}) + \tilde{k}\tilde{\epsilon})$ where $\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$, that is, implies $g(\tilde{c} + \tilde{h}) - g(\tilde{c}) = \tilde{h}(\tilde{k}f'(\tilde{c}) + \tilde{k}\tilde{\epsilon})$ where $\tilde{k}\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$. Hence f is differentiable and $g'(\tilde{c}) = \tilde{k}f'(\tilde{c})$.

Proof of (2) and (3) is trivial. □

Proposition 60. *If a function of soft sets f is constant in S, then f is differentiable in S and $f'(\tilde{x}) = \overline{0}$, $\forall \tilde{x} \in S$.*

Proof. Since f is constant in S, for all $\tilde{x} \in S$, $f(\tilde{x} + \tilde{h}) - f(\tilde{x}) = \overline{0}$ for all $\tilde{h}$ such that $\tilde{x} + \tilde{h} \in S$. Choosing $g(\tilde{x}) = \overline{0}$, $\forall \tilde{x} \in S$, and $\tilde{\epsilon} = \overline{0}$, we can write $f(\tilde{x} + \tilde{h}) - f(\tilde{x}) = \tilde{h}(g(\tilde{x}) + \tilde{\epsilon})$. Hence f is differentiable at $\tilde{x} \in S$ and $f'(\tilde{x}) = \overline{0}$, $\forall \tilde{x} \in S$. □

Proposition 61. *If f is differentiable at $\tilde{x}$, then f is continuous at $\tilde{x}$.*

Proof. Since f is differentiable so there exists a function of soft sets g such that $|f(\tilde{x} + \tilde{h}) - f(\tilde{x})| = |\tilde{h}||g(\tilde{x}) + \tilde{\epsilon}|$, where $\epsilon \to \overline{0}$ as $\tilde{h} \to \overline{0}$. Which shows that $|f(\tilde{x} + \tilde{h}) - f(\tilde{x})| \to \overline{0}$ as $\tilde{h} \to \overline{0}$. Hence f is continuous. □

Proposition 62. *If $f'(\tilde{c})(\lambda) > 0$, then there exists $\tilde{\delta} > \overline{0}$ such that for any $\tilde{x} \in (\tilde{c} - \tilde{\delta}, \tilde{c} + \tilde{\delta})$, $f(\tilde{x})(\lambda) \gtrless f(\tilde{c})(\lambda)$ according to $\tilde{x}(\lambda) \gtrless \tilde{c}(\lambda)$.*

Proof. Since f is differentiable so f can be written as $f(\tilde{c}+\tilde{h})-f(\tilde{c}) = \tilde{h}(f'(\tilde{c}) + \tilde{\epsilon})$ where $\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$ and $f'(\tilde{c})(\lambda) > 0$. Thus, there exists $\tilde{\delta} > \overline{0}$ such that $(f'(\tilde{c}) + \tilde{\epsilon})(\lambda) > 0$ whenever $\overline{0} < |\tilde{h}| \tilde{<} \tilde{\delta}$. Hence $f(\tilde{c} + \tilde{h})(\lambda) - f(\tilde{c})(\lambda) \gtrless 0$ according to $\tilde{h}(\lambda) \gtrless 0$. That is, $f(\tilde{x})(\lambda) \gtrless f(\tilde{c})(\lambda)$ according to $\tilde{x}(\lambda) \gtrless \tilde{c}(\lambda)$. □

Definition 63. A function of soft sets f is said to be differentiable on $[\tilde{a}, \tilde{b}]$ iff f is differentiable at $\tilde{x}$, $\forall \tilde{x} \in [\tilde{a}, \tilde{b}]$.

Proposition 64. *Let f be a bounded function of soft sets differentiable on $[\tilde{a}, \tilde{b}]$. If for some $\lambda \in A$, $f(\tilde{a})(\lambda) = f(\tilde{b})(\lambda)$ and f attains bound for parameter λ in an interior point of $[\tilde{a}, \tilde{b}]$, then there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ with $\tilde{c}(\lambda) \in (\tilde{a}(\lambda), \tilde{b}(\lambda))$ such that $f'(\tilde{c})(\lambda) = 0$.*

Proof. Choose $\lambda \in A$. Let $M = \sup\{f(\tilde{x})(\lambda) : \tilde{x} \in [\tilde{a}, \tilde{b}]\}$ and $m = \inf\{f(\tilde{x})(\lambda) : \tilde{x} \in [\tilde{a}, \tilde{b}]\}$. If $m = M$, that is, $f(\tilde{x})(\lambda) = m$, $\forall \tilde{x} \in [\tilde{a}, \tilde{b}]$, then for any point $\tilde{x}$, $\tilde{x} + \tilde{h} \in [\tilde{a}, \tilde{b}]$ $f(\tilde{x} + \tilde{h})(\lambda) - f(\tilde{x})(\lambda) = 0$; that is, $\tilde{h}(\lambda)(f'(\tilde{x})(\lambda) + \tilde{\epsilon}(\lambda)) = 0$ where $\epsilon \to \overline{0}$ as $\tilde{h} \to \overline{0}$. Since $\tilde{h}$ is arbitrary and $\tilde{\epsilon} \to \overline{0}$ as $\tilde{h} \to \overline{0}$, $f'(\tilde{x})(\lambda) = 0$ for all $\tilde{x} \in [\tilde{a}, \tilde{b}]$. If $M \neq m$, then either m or M is different from $f(\tilde{a})(\lambda) = f(\tilde{b})(\lambda)$. Without loss of generality, let $M \neq f(\tilde{a})(\lambda) = f(\tilde{b})(\lambda)$, so there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{c})(\lambda) = M$. Now if $f'(\tilde{c})(\lambda) > 0$, then there exists $\tilde{\delta}_1 > \overline{0}$ such that $f(\tilde{c} + \tilde{h})(\lambda) > f(\tilde{c})(\lambda)$ when $\overline{0} < \tilde{h} < \tilde{\delta}_1$, which contradicts the supremum of f in $[\tilde{a}, \tilde{b}]$. Similarly it can be shown that $f'(\tilde{c})(\lambda) \nless 0$. Hence $f'(\tilde{c})(\lambda) = 0$. □

Proposition 65. *Let f be bounded function of soft sets attaining the bound parameterwise in an interior point of $[\tilde{a}, \tilde{b}]$ and differentiable on $[\tilde{a}, \tilde{b}]$. Then for any $\lambda \in A$ there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $f(\tilde{b})(\lambda) - f(\tilde{a})(\lambda) = (\tilde{b}(\lambda) - \tilde{a}(\lambda)) f'(\tilde{c})(\lambda)$.*

Proof. If we take $g(\tilde{x}) = f(\tilde{x}) - ((f(\tilde{b}) - f(\tilde{a}))/(\tilde{b} - \tilde{a}))\tilde{x}$, then $g(\tilde{a}) = g(\tilde{b})$. Therefore, by Proposition 64, for any $\lambda \in A$ there exists $\tilde{c} \in [\tilde{a}, \tilde{b}]$ such that $g'(\tilde{c})(\lambda) = 0$. That is, $f'(\tilde{c})(\lambda) - (f(\tilde{b})(\lambda) - f(\tilde{a})(\lambda))/(\tilde{b}(\lambda) - \tilde{a}(\lambda)) = 0$. That is, $f(\tilde{b})(\lambda) - f(\tilde{a})(\lambda) = (\tilde{b}(\lambda) - \tilde{a}(\lambda)) f'(\tilde{c})(\lambda)$. □

6. Conclusion

In this paper we have dealt with continuity and differentiability of functions of soft real sets and extended some celebrated theorems, like Bolzano's theorem, fixed point theorem, intermediate value property, and Rolle's theorem, in soft settings. There is a huge scope for further study such as defining higher order derivatives together with extending Taylors theorem, integration theory, and theory of functions of several variables in soft setting.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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