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Research Article

Milloux Inequality of *E*-Valued Meromorphic Function

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The main purpose of this paper is to establish the Milloux inequality of E-valued meromorphic function from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space E with a Schauder basis. As an application, we study the Borel exceptional values of an E-valued meromorphic function and those of its derivatives; results are obtained to extend some related results for meromorphic scalar-valued function of Singh, Gopalakrishna, and Bhoosnurmath.

1. Introduction

In the 1970s, the Nevanlinna theory of meromorphic function is extended to the vector-valued meromorphic function from the complex plane $\mathbb{C}$ to a finite dimensional space $\mathbb{C}^n$ (see Ziegler [1]). After that, some works related to vector-valued meromorphic function in finite dimensional spaces were done by [2–5]. In 2006, C. G. Hu and Q. J. Hu [6] established Nevanlinna's first and second fundamental theorems for an E-valued meromorphic function from the disk $\mathbb{C}_r = \{|z| < r\}$, $0 < r \leq +\infty$, to infinite-dimensional Banach spaces E with a Schauder basis. Xuan and Wu [7] established Nevanlinna's first and second fundamental theorems for an E-valued meromorphic function from a generic domain $D \subseteq \mathbb{C}$ to E and generalized Chuang's inequality. Motivated by [6, 7], Bhoosnurmath and Pujari [8] studied the E-valued Borel exceptional values of meromorphic functions, Wu and Xuan [9, 10] studied the characteristic functions, exceptional values, and deficiency of E-valued meromorphic function, and Hu [11] surveyed the advancements of the Nevanlinna theory of E-valued meromorphic functions and studied its related Paley problems. In this paper, we will generalize Milloux's inequality (see [12] or [13]) to E-valued meromorphic function.

2. The Nevanlinna Theory in Banach Spaces

In this section, we introduce some fundamental definitions and notations of E-valued meromorphic function which was introduced by C. G. Hu and Q. J. Hu [6]. See also [7–10].

Let $(E, \|\bullet\|)$ be an infinite dimension complex Banach space with Schauder basis $\{e_j\}$ and the norm $\|\bullet\|$. Thus an E-valued meromorphic function $f(z)$ defined in $\mathbb{C}_r$, $0 < r \leq +\infty$, can be written as

$$f(z) = \left(f_1(z), f_2(z), \ldots, f_k(z), \ldots\right) \in E, \quad (1)$$

where $f_1(z), f_2(z), \ldots, f_k(z), \ldots$ are the component functions of $f(z)$. Let E_n be an n-dimensional projective space of E with a basis $\{e_j\}_1^n$. The projective operator $P_n : E \to E_n$ is a realization of E_n associated with the basis.

The elements of E are called vectors and are usually denoted by letters from the alphabet: $a, b, c, \ldots$. The symbol 0 denotes the zero vector of E. We denote vector infinity, complex number infinity, and the norm infinity by $\widehat{\infty}$, ∞, and $+\infty$, respectively. A vector-valued mapping is called holomorphic (meromorphic) if all component functions of $f(z)$ are holomorphic (some of component functions of $f(z)$ are meromorphic). The jth derivative of $f(z)$ is defined by

$$f^{(j)}(z) = \left(f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_k^{(j)}(z), \ldots\right), \quad (2)$$

where $j = 1, 2, \ldots$. A point $z_0 \in \mathbb{C}_r$ is called a pole (or $\widehat{\infty}$ point) of $f(z)$ if z_0 is a pole (or ∞ point) of at least one of the component functions of $f(z)$. A point $z_0 \in \mathbb{C}_r$ is called a zero of $f(z)$ if z_0 is a common zero of all the component functions of $f(z)$. A point $z_0 \in \mathbb{C}_r$ is called a pole or an $\widehat{\infty}$-point of $f(z)$ of multiplicity $q \in \mathbb{N}^+$ which means that in such a point z_0 at least one of the meromorphic component functions of $f(z)$ has a pole of this multiplicity in the ordinary sense of function theory. A point $z_0 \in \mathbb{C}_r$ is called a zero of $f(z)$ of multiplicity $q \in \mathbb{N}^+$ which means that in such a point z_0 all component functions of $f(z)$ vanish, each with at least this multiplicity.

An E-valued meromorphic function $f(z)$ in $\mathbb{C}$ is said to be of compact projection, if for any given $\varepsilon > 0$, $\|P_n(f(z)) - f(z)\| < \varepsilon$ as sufficiently large n in any fixed compact subset $D \subset \mathbb{C}$.

Let $n(r, f)$ or $n(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \le r$ and $n(r, a, f)$ denote the number of a-points of $f(z)$ in $|z| \le r$, counting with multiplicities. Define the volume function associated with E-valued meromorphic function $f(z)$ by

$$\begin{aligned} &V(r, \widehat{\infty}, f) \\ &\quad = V(r, f) = \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log \|f(\xi)\|\, dx \wedge dy, \\ &\qquad\qquad \xi = x + iy; \qquad (3) \\ &V(r, a, f) = \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log \|f(\xi) - a\|\, dx \wedge dy, \\ &\qquad\qquad \xi = x + iy, \end{aligned}$$

and the counting function of finite or infinite a-points by

$$\begin{aligned} N(r, f) &= n(0, f)\log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt, \\ N(r, \widehat{\infty}) &= n(0, \widehat{\infty})\log r + \int_0^r \frac{n(t, \widehat{\infty}) - n(0, \widehat{\infty})}{t} dt, \\ N(r, a, f) &= n(0, a, f)\log r + \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt, \end{aligned} \qquad (4)$$

respectively. Next, we define

$$\begin{aligned} m(r, f) = m(r, \widehat{\infty}, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left\|f\left(re^{i\theta}\right)\right\| d\theta; \\ m(r, a, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta}) - a\|} d\theta; \\ T(r, f) &= m(r, f) + N(r, f). \end{aligned} \qquad (5)$$

Let $\overline{n}(r, f)$ or $\overline{n}(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \le r$ and $\overline{n}(r, a, f)$ denote the number of a-points of $f(z)$ in $|z| \le r$, ignoring multiplicities. Similarly, we can define the counting functions $\overline{N}(r, f)$, $\overline{N}(r, \widehat{\infty})$, and $\overline{N}(r, a, f)$ of $\overline{n}(r, f)$, $\overline{n}(r, \widehat{\infty})$, and $\overline{n}(r, a, f)$.

Let $f(z)$ $(z \in \mathbb{C}_r)$ be an E-valued meromorphic function and $a \in E$; if k is a positive integer, let $\overline{n}_k(r, f)$ or $\overline{n}_k(r, \widehat{\infty})$ denote the number of distinct poles of $f(z)$ of order $\le k$ in $|z| \le r$ and $\overline{n}_k(r, a, f)$ denote the number of distinct a-points of $f(z)$ of order $\le k$ in $|z| \le r$. Similarly, we can define the counting functions $\overline{N}_k(r, f)$, $\overline{N}_k(r, \widehat{\infty})$, and $\overline{N}_k(r, a, f)$ of $\overline{n}_k(r, f)$, $\overline{n}_k(r, \widehat{\infty})$, and $\overline{n}_k(r, a, f)$.

If $f(z)$ is an E-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

$$\begin{aligned} \lambda(f) &= \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}; \\ \mu(f) &= \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}. \end{aligned} \qquad (6)$$

We call the E-valued meromorphic function f admissible if

$$\limsup_{r \to +\infty} \frac{T(r, f)}{\log r} = +\infty. \qquad (7)$$

Definition 1. Let $f(z)$ be an admissible E-valued meromorphic function in $\mathbb{C}$. One denotes by $S(r, f)$ any quantity such that

$$S(r, f) = O(\log T(r, f) + \log r), \quad r \longrightarrow +\infty, \qquad (8)$$

without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure.

In 2006, C. G. Hu and Q. J. Hu [6] proved the following theorems.

Theorem A (the E-valued Nevanlinna's first fundamental theorem). *Let $f(z)$ be a nonconstant E-valued meromorphic function in $\mathbb{C}_R = \{|z| < R\}$, $0 < R \le +\infty$. Then, for $0 < r < R$, $a \in E$, and $f(z) \not\equiv a$,*

$$\begin{aligned} T(r, f) &= V(r, a) + N(r, a) \\ &\quad + m(r, a) + \log^+ \left\|c_q(a)\right\| + \varepsilon(r, a). \end{aligned} \qquad (9)$$

Here $\varepsilon(r, a)$ is a function such that

$$|\varepsilon(r, a)| \le \log^+ \|a\| + \log 2, \qquad \varepsilon(r, 0) \equiv 0, \qquad (10)$$

and $c_q(a) \in E$ is the coefficient of the first term in the Laurent series at the point a.

Theorem B (the E-valued Nevanlinna's second fundamental theorem). *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}_R = \{|z| < R\}$, $0 < R \le +\infty$, and $a^{[k]} \in E (k = 1, 2, \ldots, q)$ be $q \ge 3$ distinct points. Then, for $0 < r < R$,*

$$\sum_{k=1}^{q} m\left(r, a^{[k]}, f\right) \le 2T(r, f) - N_1(r) + S(r, f), \qquad (11)$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, 0, f')$.

3. Milloux Inequality of E-Valued Meromorphic Function

In this section, we will establish the Milloux inequality of E-valued meromorphic function and prove the following theorems.

Theorem 2 (Milloux inequality). *Suppose that $f(z)$ is an admissible E-valued meromorphic function of compact projection in $\mathbb{C}_R = \{|z| < R\}$, $0 < R \le +\infty$. Let $a, b \in E$ be distinct points and $b \neq 0$. Then, for $0 < r < R$,*

$$T(r,f) \le \overline{N}(r,f) + (k+1)\left\{\overline{N}(r,a,f) + V(r,a,f)\right\} + \left\{\overline{N}\left(r,b,f^{(k)}\right) + V\left(r,b,f^{(k)}\right)\right\} + S(r,f). \tag{12}$$

In order to prove Theorem 2, we will prove the following general form of Milloux inequality of E-valued meromorphic function when the multiple values are considered.

Theorem 3 (general form of Milloux inequality). *Suppose that $f(z)$ is an admissible E-valued meromorphic function of compact projection in $\mathbb{C}_R = \{|z| < R\}$, $0 < R \le +\infty$. Let $a^{[i]}, b^{[j]} \in E\,(i = 1,2,\dots,p;\ j = 1,2,\dots,q)$ be distinct points such that $b^{[j]} \neq 0\,(j = 1,2,\dots,q)$ and let m_i, n_j $(i = 1,2,\dots,p;\ j = 1,2,\dots,q)$, and l be any positive integers. Then*

$$\left\{pq - \left(\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right)\right\}T(r,f)$$
$$\le \frac{l}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\overline{N}_l(r,f) + (kq+1)\sum_{i=1}^{p}\left\{\overline{N}_{m_i}\left(r,a^{[i]},f\right) + V\left(r,a^{[i]},f\right)\right\}$$
$$+ \sum_{j=1}^{q}\left\{\overline{N}_{n_j}\left(r,b^{[j]},f^{(k)}\right) + V\left(r,b^{[j]},f^{(k)}\right)\right\} + S(r,f). \tag{13}$$

By letting $p = q = 1$ and l, m_i, n_j tend to infinity in (13), we can get Theorem 2. In order to prove Theorem 3, we need the following lemma.

Lemma 4 (see [10]). *Let $f(z)$ be of compact projection in $\mathbb{C}$; then, for a positive integer k, one has*

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{\left\|f^{(k)}\left(re^{i\theta}\right)\right\|}{\left\|f\left(re^{i\theta}\right)\right\|}d\theta = S(r,f). \tag{14}$$

We are now in the position to prove Theorem 3.

Proof. We set

$$F(z) = \sum_{i=1}^{p}\frac{1}{\left\|f(z) - a^{[i]}\right\|}; \tag{15}$$

then

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+F\left(re^{i\theta}\right)d\theta \le m\left(r,0,f^{(k)}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log^+\left\{F\left(re^{i\theta}\right)\left\|f^{(k)}\left(re^{i\theta}\right)\right\|\right\}d\theta. \tag{16}$$

By [6], we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+F\left(re^{i\theta}\right)d\theta \ge \sum_{i=1}^{p}m\left(r,a^{[i]}\right) - \log^+\frac{2q}{\delta}. \tag{17}$$

From (16) and (17), we can get

$$\sum_{i=1}^{p}m\left(r,a^{[i]},f\right) \le m\left(r,0,f^{(k)}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log^+\left\{F\left(re^{i\theta}\right)\left\|f^{(k)}\left(re^{i\theta}\right)\right\|\right\}d\theta + \log^+\frac{2q}{\delta}. \tag{18}$$

Hence, we can get from the above inequality and Lemma 4 that

$$\sum_{i=1}^{p}m\left(r,a^{[i]},f\right) \le m\left(r,0,f^{(k)}\right) + S(r,f). \tag{19}$$

It follows from Theorem A that

$$T\left(r,f^{(k)}\right) = m\left(r,0,f^{(k)}\right) + N\left(r,0,f^{(k)}\right) + V\left(r,0,f^{(k)}\right) + O(1). \tag{20}$$

Thus from (19) and (20) we deduce

$$\sum_{i=1}^{p}m\left(r,a^{[i]},f\right) \le T\left(r,f^{(k)}\right) - N\left(r,0,f^{(k)}\right) - V\left(r,0,f^{(k)}\right) + S(r,f). \tag{21}$$

By Theorem A, we have

$$pT(r,f) \le T\left(r,f^{(k)}\right) + \sum_{i=1}^{p}\left[N\left(r,a^{[i]},f\right) + V\left(r,a^{[i]},f\right)\right] - N\left(r,0,f^{(k)}\right) - V\left(r,0,f^{(k)}\right) + S(r,f). \tag{22}$$

Now it follows from Theorems A and B and Lemma 4 that

$$\begin{aligned} qT\left(r,f^{(k)}\right) &\le \sum_{j=1}^{q}\left\{N\left(r,b^{[j]},f^{(k)}\right)+V\left(r,b^{[j]},f^{(k)}\right)\right\} \\ &\quad + N\left(r,0,f^{(k)}\right)+V\left(r,0,f^{(k)}\right)+N\left(r,f^{(k)}\right) \\ &\quad - \left(N\left(r,0,f^{(k+1)}\right)+2N\left(r,f^{(k)}\right)-N\left(r,f^{(k+1)}\right)\right) \\ &\quad + S\left(r,f^{(k)}\right) \\ &= \sum_{j=1}^{q}\left\{N\left(r,b^{[j]},f^{(k)}\right)+V\left(r,b^{[j]},f^{(k)}\right)\right\} \\ &\quad + N\left(r,0,f^{(k)}\right)+V\left(r,0,f^{(k)}\right)+N\left(r,f^{(k+1)}\right) \\ &\quad - N\left(r,f^{(k)}\right)+N\left(r,0,f^{(k+1)}\right)+S\left(r,f\right) \\ &\le \sum_{j=1}^{q}\left\{N\left(r,b^{[j]},f^{(k)}\right)+V\left(r,b^{[j]},f^{(k)}\right)\right\} \\ &\quad + N\left(r,0,f^{(k)}\right)+V\left(r,0,f^{(k)}\right)+\overline{N}\left(r,f\right) \\ &\quad - N\left(r,0,f^{(k+1)}\right)+S\left(r,f\right). \end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$\begin{aligned} pqT\left(r,f\right) &\le \overline{N}\left(r,f\right)+\left(q-1\right)\left\{\sum_{i=1}^{p}N\left(r,a^{[i]},f\right)-N\left(r,0,f^{(k)}\right)\right\} \\ &\quad + \left\{\sum_{i=1}^{p}N\left(r,a^{[i]},f\right) + \sum_{j=1}^{q}N\left(r,b^{[j]},f^{(k)}\right)-N\left(r,0,f^{(k+1)}\right)\right\} \\ &\quad + q\sum_{i=1}^{p}V\left(r,a^{[i]},f\right)+\sum_{j=1}^{q}V\left(r,b^{[j]},f^{(k)}\right)+S\left(r,f\right). \end{aligned} \tag{24}$$

A zero of $f-a$ of order $j>k$ is a zero of $f^{(k+1)}$ of order $j-(k+1)$ and a zero of $f^{(k)}-b$ of order m is a zero of $f^{(k+1)}$ of order $m-1$. Moreover, zeros of $f-a$ of order $>k$ are zeros of $f^{(k)}$ and so are not zeros of $f^{(k)}-b$ since $b\neq 0$. Hence

$$\begin{aligned} &\sum_{i=1}^{p}N\left(r,a^{[i]},f\right)+\sum_{j=1}^{q}N\left(r,b^{[j]},f^{(k)}\right)-N\left(r,0,f^{(k+1)}\right) \\ &\quad \le \sum_{i=1}^{p}N_{k+1}\left(r,a^{[i]},f\right)+\sum_{j=1}^{q}\overline{N}\left(r,b^{[j]},f^{(k)}\right), \\ &\sum_{i=1}^{p}N\left(r,a^{[i]},f\right)-N\left(r,0,f^{(k)}\right)\le \sum_{i=1}^{p}N_{k}\left(r,a^{[i]},f\right). \end{aligned} \tag{25}$$

Substituting (25) to (24), we obtain

$$\begin{aligned} pqT\left(r,f\right) &\le \overline{N}\left(r,f\right)+\left(q-1\right)\sum_{i=1}^{p}N_{k}\left(r,a^{[i]},f\right) \\ &\quad + \sum_{i=1}^{p}N_{k+1}\left(r,a^{[i]},f\right)+\sum_{j=1}^{q}\overline{N}\left(r,b^{[j]},f^{(k)}\right) \\ &\quad + q\sum_{i=1}^{p}V\left(r,a^{[i]},f\right) \\ &\quad + \sum_{j=1}^{q}V\left(r,b^{[j]},f^{(k)}\right)+S\left(r,f\right), \end{aligned} \tag{26}$$

since

$$\begin{aligned} &N_{k}\left(r,a^{[i]},f\right) \\ &\quad \le k\overline{N}\left(r,a^{[i]},f\right) \\ &\quad \le \frac{k}{m_i+1}\left\{m_i\overline{N}_{m_i}\left(r,a^{[i]},f\right)+N\left(r,a^{[i]},f\right)\right\} \\ &\quad \le \frac{k}{m_i+1}\left\{m_i\overline{N}_{m_i}\left(r,a^{[i]},f\right)+T\left(r,f\right)\right\}+O\left(1\right), \end{aligned} \tag{27}$$

$$\begin{aligned} &N_{k+1}\left(r,a^{[i]},f\right) \\ &\quad \le \left(k+1\right)\overline{N}\left(r,a^{[i]},f\right) \\ &\quad \le \frac{k+1}{m_i+1}\left\{m_i\overline{N}_{m_i}\left(r,a^{[i]},f\right)+N\left(r,a^{[i]},f\right)\right\} \\ &\quad \le \frac{k+1}{m_i+1}\left\{m_i\overline{N}_{m_i}\left(r,a^{[i]},f\right)+T\left(r,f\right)\right\}+O\left(1\right). \end{aligned} \tag{28}$$

Similarly, we can get

$$\begin{aligned} &\overline{N}\left(r,b^{[j]},f^{(k)}\right) \\ &\quad \le \frac{1}{n_j+1}\left\{n_j\overline{N}_{n_j}\left(r,b^{[j]},f^{(k)}\right)+T\left(r,f^{(k)}\right)\right\}+O\left(1\right), \\ &\overline{N}\left(r,f\right)\le \frac{1}{l+1}\left\{l\overline{N}_{l}\left(r,f\right)+T\left(r,f\right)\right\}. \end{aligned} \tag{29}$$

By Lemma 4, we can get

$$\begin{aligned} T\left(r, f^{(k)}\right) &= m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right) \\ &\leq m(r, f) + N\left(r, f^{(k)}\right) \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\left\| f^{(k)}\left(re^{i\theta}\right)\right\|}{\left\| f\left(re^{i\theta}\right)\right\|} d\theta \\ &\leq m(r, f) + N(r, f) + k\overline{N}(r, f) + S(r, f) \\ &\leq T(r, f) + k\overline{N}(r, f) + S(r, f). \end{aligned} \tag{30}$$

Substituting (27)–(30) into (26), we obtain

$$\begin{aligned} & pqT(r, f) \\ &\quad \leq \overline{N}(r, f) + (q-1) \\ &\qquad \times \sum_{i=1}^{p} \frac{k}{m_i+1}\left\{m_i \overline{N}_{m_i}\left(r, a^{[i]}, f\right) + T(r, f)\right\} \\ &\qquad + \sum_{i=1}^{p} \frac{k+1}{m_i+1}\left\{m_i \overline{N}_{m_i}\left(r, a^{[i]}, f\right) + T(r, f)\right\} \\ &\qquad + \sum_{j=1}^{q} \frac{1}{n_j+1}\left\{n_j \overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) + T\left(r, f^{(k)}\right)\right\} \\ &\qquad + q\sum_{i=1}^{p} V\left(r, a^{[i]}, f\right) + \sum_{j=1}^{q} V\left(r, b^{[j]}, f^{(k)}\right) + S(r, f) \\ &\quad \leq \left(1 + \sum_{j=1}^{q}\frac{k}{n_j+1}\right)\overline{N}(r, f) + (q-1) \\ &\qquad \times \sum_{i=1}^{p} \frac{km_i}{m_i+1}\overline{N}_{m_i}\left(r, a^{[i]}, f\right) \\ &\qquad + \sum_{i=1}^{p} \frac{k+1}{m_i+1} m_i \overline{N}_{m_i}\left(r, a^{[i]}, f\right) \\ &\qquad + \sum_{j=1}^{q} \frac{n_j}{n_j+1}\overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) \\ &\qquad + (q-1)\sum_{i=1}^{p}\frac{k}{m_i+1}T(r, f) \\ &\qquad + \sum_{i=1}^{p}\frac{k+1}{m_i+1}T(r, f) + \sum_{j=1}^{q}\frac{1}{n_j+1}n_j T(r, f) \\ &\qquad + q\sum_{i=1}^{p} V\left(r, a^{[i]}, f\right) + \sum_{j=1}^{q} V\left(r, b^{[j]}, f^{(k)}\right) + S(r, f) \\ &\quad \leq \left(1 + \sum_{j=1}^{q}\frac{k}{n_j+1}\right)\frac{l}{l+1}\overline{N}_l(r, f) + (kq+1) \\ &\qquad \times \sum_{i=1}^{p} \frac{m_i}{m_i+1}\overline{N}_{m_i}\left(r, a^{[i]}, f\right) \\ &\qquad + \sum_{j=1}^{q} \frac{n_j}{n_j+1}\overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) \\ &\qquad + q\sum_{i=1}^{p} V\left(r, a^{[i]}, f\right) + \sum_{j=1}^{q} V\left(r, b^{[j]}, f^{(k)}\right) \\ &\qquad + \left(\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right) \\ &\qquad \times T(r, f) + S(r, f). \end{aligned} \tag{31}$$

Since m_i, n_j, k, and q are positive integers, it follows from (31) that

$$\begin{aligned} & pqT(r, f) \\ &\quad \leq \left(\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right) \\ &\qquad \times T(r, f) \\ &\qquad + (kq+1)\sum_{i=1}^{p}\left[\overline{N}_{m_i}\left(r, a^{[i]}, f\right) + V\left(r, a^{[i]}, f\right)\right] \\ &\qquad + \sum_{j=1}^{q}\left[\overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) + V\left(r, b^{[j]}, f^{(k)}\right)\right] \\ &\qquad + \left(1 + \sum_{j=1}^{q}\frac{k}{n_j+1}\right)\frac{l}{l+1}\overline{N}_l(r, f) + S(r, f). \end{aligned} \tag{32}$$

Hence, (13) follows from (32). □

4. E-Valued Borel Exceptional Values of Meromorphic Function and Its Derivatives

Most recently, Bhoosnurmath and Pujari [8] studied the E-valued Borel exceptional values of meromorphic functions and gave the following definition.

Definition 5. Let $f(z)$ $(z \in \mathbb{C})$ be an E-valued meromorphic function and $a \in E \cup \{\widehat{\infty}\}$ k is a positive integer. One defines

$$\begin{aligned} \overline{\rho}_k(a, f) &= \limsup_{r\to\infty}\frac{\log^+\left[V(a, f) + \overline{N}_k(r, a)\right]}{\log r}; \\ \overline{\rho}(a, f) &= \limsup_{r\to\infty}\frac{\log^+\left[V(a, f) + \overline{N}(r, a)\right]}{\log r}; \\ \rho(a, f) &= \limsup_{r\to\infty}\frac{\log^+\left[V(a, f) + N(r, a)\right]}{\log r}. \end{aligned} \tag{33}$$

We say that a is an

(i) E-valued evB (exceptional value in the sense of Borel) for f for distinct zeros of order $\leq k$ if $\overline{\rho}_k(a, f) < \lambda(f)$;

(ii) E-valued evB for f for distinct zeros if $\overline{\rho}(a, f) < \lambda(f)$;

(iii) E-valued evB for f (for the whole aggregate of zeros) if $\rho(a, f) < \lambda(f)$.

Suppose that $f(z)$ is an E-valued meromorphic function with finite order $\rho > 0$ in $\mathbb{C}$. Xuan and Wu [7] proved that the order of f' is ρ. Hence for any positive integer l the order of $f^{(l)}$ is ρ. Therefore, we call a a vector-valued evB for $f^{(l)}$ for distinct zeros of order $\leq k$, if

$$\overline{\rho}_k\left(a, f^{(l)}\right) = \limsup_{r\to\infty} \frac{\log\left[V\left(r, a, f^{(l)}\right) + \overline{N}_k\left(r, a, f^{(l)}\right)\right]}{\log r} < \rho. \tag{34}$$

In this section, we will prove the following theorem.

Theorem 6. *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and the order of $f(z)$ is $\rho\,(0 < \rho < +\infty)$. Suppose that $\widehat{\infty}$ is an E-valued evB for f for distinct zeros of order $\leq l$, $a^{[i]} \in E\,(i = 1, 2, \ldots, p)$ are E-valued evB for f for distinct zeros of order $\leq m_i$, and $b^{[j]}(\neq 0) \in E\,(j = 1, 2, \ldots, q)$ are E-valued evB for $f^{(k)}$ for distinct zeros of order $\leq n_j$, where k, p, q, l and all of m_i, n_j are positive integers. Then*

$$\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right) \geq pq. \tag{35}$$

Proof. By Theorem 3, we obtain

$$\begin{aligned}
&\left\{pq - \left(\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right)\right\} T(r, f) \\
&\leq \frac{l}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\overline{N}_l(r, f) \\
&\quad + (kq+1)\sum_{i=1}^{p}\left\{\overline{N}_{m_i}\left(r, a^{[i]}, f\right) + V\left(r, a^{[i]}, f\right)\right\} \\
&\quad + \sum_{j=1}^{q}\left\{\overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) + V\left(r, b^{[j]}, f^{(k)}\right)\right\} + S(r, f).
\end{aligned} \tag{36}$$

Since $\widehat{\infty}$ is an E-valued evB for f for distinct zeros of order $\leq l$, $a^{[i]} \in E$ $(i = 1, 2, \ldots, p)$ is an E-valued evB for f for distinct zeros of order $\leq m_i$ and $b^{[j]}(\neq 0) \in E$ $(j = 1, 2, \ldots, q)$ is an E-valued evB for $f^{(k)}$ for distinct zeros of order $\leq n_j$. Thus there is a $0 < \mu < \rho$ such that for any i, j $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ we have

$$\begin{aligned}
\overline{N}_l(r, f) &\leq R^{\mu}, \\
\overline{N}_{m_i}\left(r, a^{[i]}, f\right) + V\left(r, a^{[i]}, f\right) &\leq R^{\mu}, \\
\overline{N}_{n_j}\left(r, b^{[j]}, f^{(k)}\right) + V\left(r, b^{[j]}, f^{(k)}\right) &\leq R^{\mu}.
\end{aligned} \tag{37}$$

It follows from $0 < \mu < \rho$ and (36) and (37) that

$$\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q}\frac{1}{n_j+1}\right) \geq pq. \tag{38}$$

□

Letting $p = q = 1$ in Theorem 6, we can get the following corollary.

Corollary 7. *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and the order of $f(z)$ is $\rho\,(0 < \rho < +\infty)$. Suppose that $\widehat{\infty}$ is an E-valued evB for f for distinct zeros of order $\leq l$, where l is an integer ≥ 1. If there exist $a, b \in E$, $b \neq 0$, such that a is an E-valued evB for f for distinct zeros of order $\leq p$ and b is a an E-valued evB for $f^{(k)}$ for distinct zeros of order $\leq q$, where p, q are positive integers, then*

$$\frac{q+1+k}{(q+1)(l+1)} + \frac{k+1}{p+1} + \frac{1}{q+1} \geq 1. \tag{39}$$

If $\widehat{\infty}$, a are E-valued evB for f for distinct zeros, that is, letting l, p tend to infinity in (39), we can get $1/(q+1) \geq 1$. This means that, for each integer $k, q \geq 1$, $\overline{\rho}_q(b, f^{(k)}) \geq \rho$, for all $b \neq 0, \neq \widehat{\infty}$. Hence, we can get the following corollary.

Corollary 8. *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and the order of $f(z)$ is $\rho\,(0 < \rho < +\infty)$. Suppose that $\widehat{\infty}$, $a \in E$ are E-valued evB for f for distinct zeros. Then, for all positive integers k and q, $\overline{\rho}_q(b, f^{(k)}) = \rho$ for all $b \neq 0, \neq \widehat{\infty}$.*

The corresponding results of Corollaries 7 and 8 for the meromorphic scalar value function were obtained by Gopalakrishna and Bhoosnurmath [14] and Singh and Gopalakrishna [15]. The corresponding results of Corollaries 7 and 8 for the meromorphic scalar value function on annuli were obtained by Chen and Wu [16].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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