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Research Article

Bifurcation Analysis and Different Kinds of Exact Travelling Wave Solutions of a Generalized Two-Component Hunter-Saxton System

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This paper focuses on a generalized two-component Hunter-Saxton system. From a dynamic point of view, the existence of different kinds of periodic wave, solitary wave, and blow-up wave is proved and the sufficient conditions to guarantee the existence of the above solutions in different regions of the parametric space are given. Also, some exact parametric representations of the travelling waves are presented.

1. Introduction

The Hunter-Saxton equation,

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \quad (1)$$

was first introduced by Hunter and Saxton [1] as a model of the dynamics of a nematic liquid crystal. Geometrically, the HS equation (1) describes geodesic flow associated with the right-invariant metrics on a homogeneous space [2]. It is a particular case of the Euler-Poincaré equation on the diffeomorphisms in one spatial dimension [3]. Its integrability was proved by Hunter and Zheng [4].

The two-component Hunter-Saxton system is as follows:

$$\begin{gathered} u_{txx} + 2u_x u_{xx} + u u_{xxx} - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{gathered} \quad (2)$$

which is a generalization of the Hunter-Saxton equation, and was proposed by Wunsch [5] in a periodic setting. It is a particular case of the Gurevich-Zybin system [6] pertaining to nonlinear one-dimensional dynamics of dark matter as well as nonlinear ion-acoustic waves [7]. The two-component system in a periodic setting has received some attention lately [8, 9].

Very recently, the generalized two-component Hunter-Saxton system

$$\begin{gathered} u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + A u_x = 0, \\ \rho_t + (\rho u)_x = 0, \quad \sigma \in R, \ A \geq 0, \end{gathered} \quad (3)$$

was studied [10–12]. Moon and Liu [10] studied the wave-breaking phenomena and global existence for (3). Moon [11] determined the existence of solitary wave solutions for $\sigma = 0$ and classified the solitary waves for $\sigma \neq 0$. The existence of peaked solitary waves for $\sigma > 1$ was shown. The wave-breaking criterion and local well-posedness were studied in [12].

In this paper, we investigated the following generalized two-component Hunter-Saxton system:

$$\begin{gathered} u_{txx} + 2u_x u_{xx} + u u_{xxx} - \rho\rho_x + A u_x = 0, \\ \rho_t + (\rho u)_x = 0, \quad A \in R. \end{gathered} \quad (4)$$

Obviously, system (4) becomes system (2) when $A = 0$. We will prove the existence of different kinds of travelling wave of (4) and give some new exact travelling solutions using the approach of dynamical system [13–17].

Using the following independent variable transformation:

$$u(x,t)=u(\xi),\quad u(x,t)=\rho(\xi),\quad \xi=x-ct, \tag{5}$$

where $c\ (c\neq 0)$ is the wave speed, and substituting (5) into (4), we obtain

$$\begin{aligned}&-cu'''+2u'u''+uu'''-\rho\rho'+Au'=0,\\&-c\rho'+(\rho u)'=0,\end{aligned} \tag{6}$$

where "$'$" is the derivative with respect to ξ.

Integrating equations of (6) once with respect to ξ, respectively, and setting the integral constants that are both $-2c^2$, we have

$$\begin{aligned}&-cu''+\frac{1}{2}(u')^2+uu''-\frac{1}{2}\rho^2+Au=-2c^2,\\&-c\rho+(\rho u)=-2c^2.\end{aligned} \tag{7}$$

The second equation of (7) becomes

$$\rho=-\frac{2c^2}{c-u}. \tag{8}$$

Substituting (8) into the first equation of (7) yields

$$u''=\frac{u\left(Au^2-2c(A-c)u+c^2(A-4c)\right)+(1/2)(c-u)^2(u')^2}{(c-u)^3}. \tag{9}$$

Letting $y=du/d\xi$, we get the following planar dynamical system:

$$\begin{aligned}\frac{du}{d\xi}&=y,\\\frac{dy}{d\xi}&=\frac{u\left(Au^2-2c(A-c)u+c^2(A-4c)\right)+(1/2)(c-u)^2y^2}{(c-u)^3}.\end{aligned} \tag{10}$$

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcation sets and phase portraits of system (10), where explicit parametric conditions will be derived. In Section 3, we give all possible exact periodic wave, solitary wave, and blow-up wave solutions of system (4). A short conclusion will be given in Section 4.

2. Bifurcation Sets and Phase Portraits of System (10)

Using the transformation $d\xi=(c-u)^3d\tau$, it carries (10) into the Hamiltonian system:

$$\begin{aligned}\frac{du}{d\tau}&=(c-u)^3y,\\\frac{dy}{d\tau}&=u\left(Au^2-2c(A-c)u+c^2(A-4c)\right)+\frac{1}{2}(c-u)^2y^2,\end{aligned} \tag{11}$$

with the following first integral:

$$H(u,y)=\frac{(c-u)^2y^2+Au^3-c(A-4c)u^2-4c^3u+4c^4}{c-u}=h. \tag{12}$$

For a fixed h, the level curve $H(\phi,y)=h$ defined by (12) determines a set of invariant curves of system (11) which contains different branches of curves. As h is varied, it defines different families of orbits of (11) with different dynamical behaviors.

Write $\Delta=c(2A+c)$. Clearly, when $A\neq 0$ and $\Delta>0$, system (11) has three equilibrium points at $(0,0)$, $(u_1,0)$, and $(u_2,0)$ in u-axis, where $u_{1,2}=c((A-c)\pm\sqrt{\Delta})/A$. When $A\neq 0$ and $\Delta=0$, system (11) has two equilibrium points at $(0,0)$ and $(c(A-c)/A,0)$ in u-axis. When $A=0$, system (11) has only one equilibrium point at $(0,0)$ in u-axis. There is no any equilibrium point of system (11) in line $u=c$.

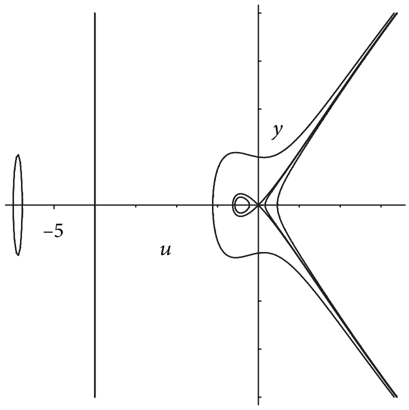
Let $M(\phi_e,y_e)$ be the coefficient matrix of the linearized system of (11) at equilibrium point (ϕ_e,y_e), then we have Trace $M(\phi_e,0)=0$ and

$$\begin{aligned}J(0,0)&=\det M(0,0)=-c^5(A-4c),\\J(u_1,0)&=\det M(u_1,0)\\&=\frac{2c^5(-c+\sqrt{\Delta})^3\left(c(2A+c)+(A-c)\sqrt{\Delta}\right)}{A^4},\\J(u_2,0)&=\det M(u_2,0)\\&=\frac{2c^5(c+\sqrt{\Delta})^3\left(-c(2A+c)+(A-c)\sqrt{\Delta}\right)}{A^4}.\end{aligned} \tag{13}$$

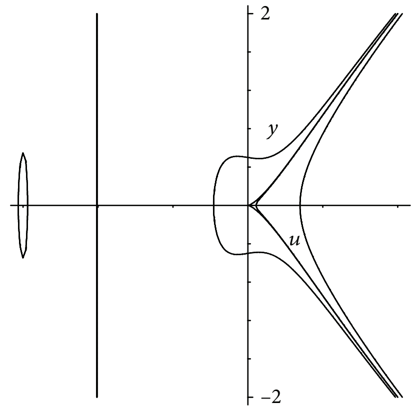
For an equilibrium point (ϕ_e,y_e) of a planar integrable system, we know that (ϕ_e,y_e) is a saddle point if $J(\phi_e,y_e)<0$, a center point if $J(\phi_e,y_e)>0$ and Trace $M(\phi_e,y_e)=0$, and a cusp if $J(\phi_e,y_e)=0$ and the Poincaré index of (ϕ_e,y_e) is zero.

Since both systems (10) and (11) have the same first integral (12), then the two systems above have the same topological phase portraits. Therefore we can obtain the bifurcation sets and phase portraits of system (10) from that of system (11).

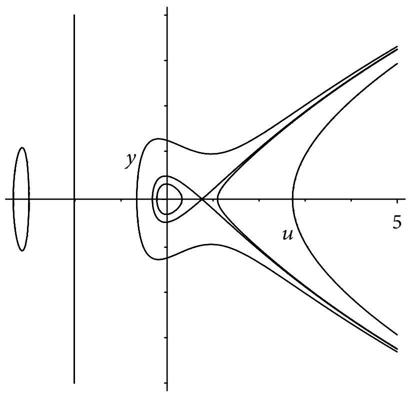
By using the properties of equilibrium points and bifurcation method of dynamical systems, we can show that bifurcation sets and phase portraits of system (10) are drawn in Figure 1.



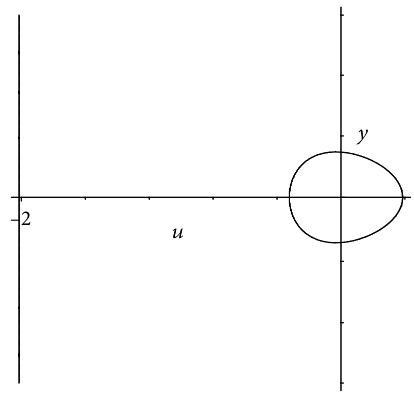
(a) $c < 0,\ A < 4c$



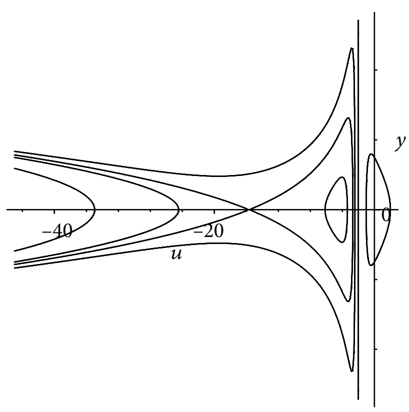
(b) $c < 0,\ A = 4c$



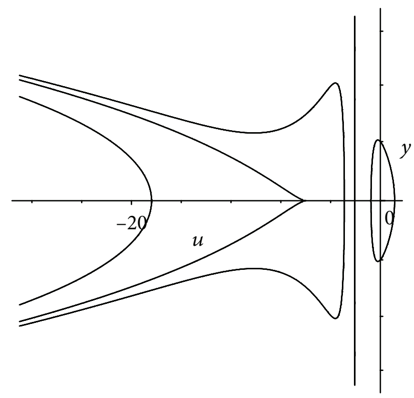
(c) $c < 0,\ 4c < A < 0$



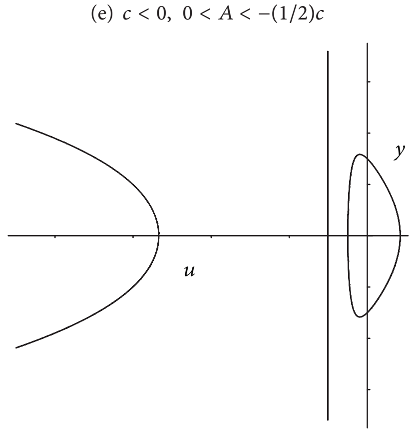
(d) $c < 0,\ A = 0$



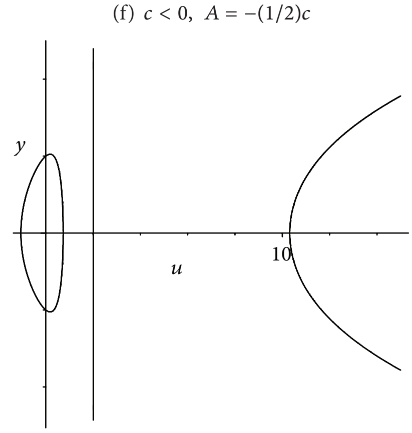
(e) $c < 0,\ 0 < A < -(1/2)c$



(f) $c < 0,\ A = -(1/2)c$

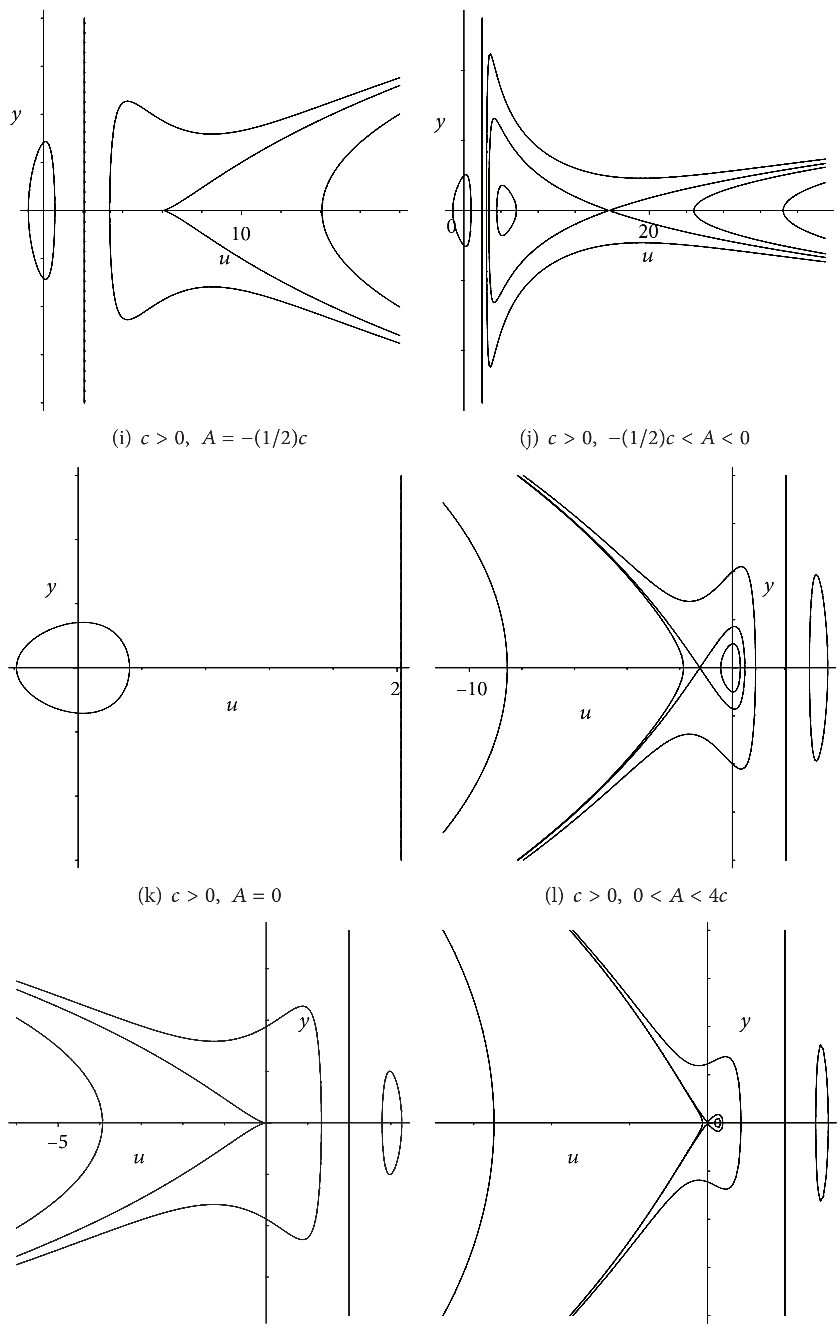


(g) $c < 0,\ A > -(1/2)c$



(h) $c > 0,\ A < -(1/2)c$

Figure 1: Continued.



(i) $c>0,\ A=-(1/2)c$ (j) $c>0,\ -(1/2)c<A<0$

(k) $c>0,\ A=0$ (l) $c>0,\ 0<A<4c$

(m) $c>0,\ A=4c$ (n) $c>0,\ A>4c$

FIGURE 1: Bifurcation sets and phase portraits of system (10).

3. Exact Travelling Wave Solutions of System (4)

Denote that

$$h_0 = 4c^3,$$

$$h_1 = \frac{c^2\left(c\left(7A^2+2Ac+4c^2\right)+\left(A^2+2Ac-4c^2\right)\sqrt{\Delta}\right)}{A\left(c-\sqrt{\Delta}\right)},$$

$$h_2 = \frac{c^2\left(-c\left(7A^2+2Ac+4c^2\right)+\left(A^2+2Ac-4c^2\right)\sqrt{\Delta}\right)}{A\left(-c-\sqrt{\Delta}\right)}. \tag{14}$$

From Figure 1, we give some exact travelling wave solutions of system (4) as follows.

3.1. Different Kinds of Periodic Wave Solutions. From Figures 1(d) and 1(k), we see that there is one periodic orbit of system (10) defined by $H(u,y)=h$ if and only if one of the following conditions holds:

(a_1) $c<0,\ A=0,\ h<h_0$;

(a_2) $c>0,\ A=0,\ h>h_0$.

The periodic orbit passes points $(u_m,0)$ and $(u_M,0)$, where $u_{M,m}=(-h+4c^3\pm\sqrt{h^2+8c^3h-48c^6})/8c^2$. Its expression is

$$y=\pm\frac{2c\sqrt{(u_M-u)(u-u_m)}}{c-u},\quad u_m\le u\le u_M. \tag{15}$$

If introducing a new parametric variable χ and letting

$$d\xi=(c-u)\,d\chi, \tag{16}$$

then we have

$$\frac{du}{d\chi}=(c-u)\,y. \tag{17}$$

Substituting (15) into (17) and integrating it along the periodic orbit yield the following:

$$\int_u^{u_M}\frac{ds}{\sqrt{(u_M-s)(s-u_m)}}=2|c\chi|. \tag{18}$$

Completing (18) and using (8) and (16), we can get a periodic wave solution of system (4) as follows:

$$\begin{aligned}u(x,t)&=u_m\sin^2(c\chi)+u_M\cos^2(c\chi),\\ \rho(x,t)&=\frac{2c^2}{u_m\sin^2(c\chi)+u_M\cos^2(c\chi)-c},\\ t&=\frac{1}{c}\left(x-\left(\left(c-\frac{1}{2}(u_m+u_M)\right)\chi+\frac{u_m-u_M}{4c}\sin(2c\chi)\right)\right).\end{aligned} \tag{19}$$

From Figures 1(a), 1(b), 1(c), 1(h), 1(i), and 1(j), we see that there are one periodic orbit and an open curve of system (10) defined by $H(u,y)=h$ if and only if one of the following conditions holds:

(b_1) $c<0,\ A<4c,\ h<h_2$;

(b_2) $c<0,\ A<4c,\ h_0<h<h_1$;

(b_3) $c<0,\ A=4c,\ h<h_2$;

(b_4) $c<0,\ 4c<A<0,\ h>h_2$;

(b_5) $c<0,\ 4c<A<0,\ h_1<h<h_0$;

(b_6) $c>0,\ A<-(1/2)c,\ h>h_0$;

(b_7) $c>0,\ A=-(1/2)c,\ h>h_0$;

(b_8) $c>0,\ -(1/2)c<A<0,\ h>h_0$;

(b_9) $c>0,\ -(1/2)c<A<0,\ h_2<h<h_1$.

The periodic orbit passing points $(\gamma_3,0)$ and $(\gamma_2,0)$ and the open curve passing point $(\gamma_1,0)$, where $\gamma_1,\gamma_2,\gamma_3$ $(\gamma_3<\gamma_2<\gamma_1)$, are three real roots of $Az^3+c(4c-A)z^2+(h-4c^3)z+c(4c^3-h)=0$. Their expressions are, respectively,

$$y=\pm\frac{\sqrt{-A(\gamma_1-u)(\gamma_2-u)(u-\gamma_3)}}{c-u},\quad \gamma_3\le u\le\gamma_2, \tag{20}$$

$$y=\pm\frac{\sqrt{-A(u-\gamma_1)(u-\gamma_2)(u-\gamma_3)}}{c-u},\quad u\ge\gamma_1. \tag{21}$$

Substituting (20) into (17) and integrating it along the periodic orbit yield equation

$$\int_{\gamma_3}^{u}\frac{ds}{\sqrt{(\gamma_1-s)(\gamma_2-s)(s-\gamma_3)}}=\sqrt{-A}\,|\chi|. \tag{22}$$

Completing (22) and using (8) and (16), we can get a periodic wave solution of system (4) as follows:

$$\begin{aligned}u(x,t)&=\gamma_3+(\gamma_2-\gamma_3)\,\mathrm{sn}^2(\omega\chi,k),\\ \rho(x,t)&=\frac{2c^2}{(\gamma_3-c)+(\gamma_2-\gamma_3)\,\mathrm{sn}^2(\omega\chi,k)},\\ t&=\frac{1}{c}\left(x-\left((c-\gamma_3)\chi+\frac{\gamma_3-\gamma_2}{\omega k^2}\right.\right.\\ &\qquad\left.\left.\times\left(\omega\chi-\mathrm{E}(\mathrm{am}(\omega\chi,k),k)\right)\right)\right),\end{aligned} \tag{23}$$

where $\omega=(1/2)\sqrt{-A(\gamma_1-\gamma_3)}$, $k=\sqrt{(\gamma_2-\gamma_3)/(\gamma_1-\gamma_3)}$, $\mathrm{sn}(\cdot,k)$ is the Jacobian elliptic function with the modulus k, $\mathrm{E}(\mathrm{am}(u_1,k),k)$ is the normal elliptic integral of the second kind, and $\mathrm{am}(u_1,k)$ reads amplitude u_1 (see [18]).

Substituting (21) into (17) and integrating it along the open curve yield the following:

$$\int_u^{+\infty}\frac{ds}{\sqrt{(s-\gamma_1)(s-\gamma_2)(s-\gamma_3)}}=\sqrt{-A}\,|\chi|. \tag{24}$$

Completing (24) and using (8) and (16), we can get a periodic blow-up wave solution of system (4) as follows:

$$\begin{aligned}u(x,t)&=\gamma_3+(\gamma_1-\gamma_3)\,\mathrm{ns}^2(\omega\chi,k),\\ \rho(x,t)&=\frac{2c^2}{(\gamma_3-c)+(\gamma_1-\gamma_3)\,\mathrm{ns}^2(\omega\chi,k)},\\ t&=\frac{1}{c}\left(x-\left((c-\gamma_1)\chi+\frac{\gamma_1-\gamma_3}{\omega}\right.\right.\\ &\qquad\left.\left.\times\left(\mathrm{E}(\mathrm{am}(\omega\chi,k),k)+\mathrm{dn}(\omega\chi,k)\,\mathrm{cs}(\omega\chi,k)\right)\right)\right),\end{aligned} \tag{25}$$

where $\omega=(1/2)\sqrt{-A(\gamma_1-\gamma_3)}$, $k=\sqrt{(\gamma_2-\gamma_3)/(\gamma_1-\gamma_3)}$, $\mathrm{ns}(\cdot,k)$, $\mathrm{dn}(\cdot,k)$, and $\mathrm{cs}(\cdot,k)$ are the Jacobian elliptic functions (see [18]).

From Figures 1(e), 1(f), 1(g), 1(l), 1(m), and 1(n), we see that there are one periodic orbit and an open curve of system

(10) defined by $H(u, y) = h$ if and only if one of the following conditions holds:

(c_1) $c < 0$, $0 < A < -(1/2)c$, $h < h_0$;

(c_2) $c < 0$, $0 < A < -(1/2)c$, $h_2 < h < h_1$;

(c_3) $c < 0$, $A = -(1/2)c$, $h < h_0$;

(c_4) $c < 0$, $A > -(1/2)c$, $h < h_0$;

(c_5) $c > 0$, $0 < A < 4c$, $h_0 < h < h_2$;

(c_6) $c > 0$, $0 < A < 4c$, $h < h_1$;

(c_7) $c > 0$, $A = 4c$, $h < h_1$;

(c_8) $c > 0$, $A > 4c$, $h_2 < h < h_0$;

(c_9) $c > 0$, $A > 4c$, $h < h_1$.

The periodic orbit passes points $(\gamma_1, 0)$ and $(\gamma_2, 0)$, and the open curve passes point $(\gamma_3, 0)$, where $\gamma_1, \gamma_2, \gamma_3$ $(\gamma_3 < \gamma_2 < \gamma_1)$ are three real roots of $Az^3+c(4c-A)z^2+(h-4c^3)z+c(4c^3-h) = 0$. Their expressions are, respectively,

$$y = \pm\frac{\sqrt{A(\gamma_1 - u)(u - \gamma_2)(u - \gamma_3)}}{c - u}, \quad \gamma_2 \le u \le \gamma_1, \tag{26}$$

$$y = \pm\frac{\sqrt{A(\gamma_1 - u)(\gamma_2 - u)(\gamma_3 - u)}}{c - u}, \quad u \le \gamma_3. \tag{27}$$

Substituting (26) into (17) and integrating it along the periodic orbit yield the following:

$$\int_u^{\gamma_1} \frac{ds}{\sqrt{(\gamma_1 - s)(s - \gamma_2)(s - \gamma_3)}} = \sqrt{A}\,|\chi|. \tag{28}$$

Completing (28) and using (8) and (16), we can get a periodic wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= \gamma_1 + (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\omega\chi, k), \\ \rho(x,t) &= \frac{2c^2}{(\gamma_1 - c) + (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\omega\chi, k)}, \\ t &= \frac{1}{c}\Big(x - \Big((c-\gamma_1)\chi + \frac{\gamma_1 - \gamma_2}{\omega k^2} \\ &\quad \times (\omega\chi - \mathrm{E}(\mathrm{am}(\omega\chi, k), k))\Big)\Big), \end{aligned} \tag{29}$$

where $\omega = (1/2)\sqrt{A(\gamma_1 - \gamma_3)}$, $k = \sqrt{(\gamma_1 - \gamma_2)/(\gamma_1 - \gamma_3)}$.

Substituting (27) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u} \frac{ds}{\sqrt{(\gamma_1 - s)(\gamma_2 - s)(\gamma_3 - s)}} = \sqrt{A}\,|\chi|. \tag{30}$$

Completing (30) and using (8) and (16), we can get a periodic blow-up wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= \gamma_1 + (\gamma_3 - \gamma_1)\,\mathrm{ns}^2(\omega\chi, k), \\ \rho(x,t) &= \frac{2c^2}{(\gamma_1 - c) + (\gamma_3 - \gamma_1)\,\mathrm{ns}^2(\omega\chi, k)}, \\ t &= \frac{1}{c}\Big(x - \Big((c-\gamma_3)\chi + \frac{\gamma_3 - \gamma_1}{\omega} \\ &\quad \times (\mathrm{E}(\mathrm{am}(\omega\chi, k), k) + \mathrm{dn}(\omega\chi, k)\,\mathrm{cs}(\omega\chi, k))\Big)\Big), \end{aligned} \tag{31}$$

where $\omega = (1/2)\sqrt{A(\gamma_1 - \gamma_3)}$, $k = \sqrt{(\gamma_1 - \gamma_2)/(\gamma_1 - \gamma_3)}$.

3.2. Solitary Wave and Blow-Up Wave Solutions. From Figure 1(a), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi, y) = h_0$ when $c < 0$, $A < 4c$. The homoclinic orbit connecting with saddle point $(0, 0)$ and passes point $(u_m, 0)$, and the open curve passes saddle point $(0, 0)$, where $u_m = c(A - 4c)/A$. Their expressions are, respectively,

$$y = \pm\frac{u\sqrt{-A(u - u_m)}}{c - u}, \quad u_m \le u < 0, \tag{32}$$

$$y = \pm\frac{u\sqrt{-A(u - u_m)}}{c - u}, \quad u \ge 0. \tag{33}$$

Substituting (32) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_{u_m}^{u} \frac{ds}{s\sqrt{s - u_m}} = -\sqrt{-A}\,|\chi|. \tag{34}$$

Completing (34) and using (8) and (16), we can get a solitary wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= \frac{c(A-4c)}{A}\,\mathrm{sech}^2(\omega\chi), \\ \rho(x,t) &= \frac{2Ac}{(A-4c)\,\mathrm{sech}^2(\omega\chi) - A}, \\ t &= \frac{1}{c}\Big(x - c\Big(\chi - \frac{(A-4c)}{\omega A}\tanh(\omega\chi)\Big)\Big), \end{aligned} \tag{35}$$

where $\omega = (1/2)\sqrt{c(A - 4c)}$.

Substituting (33) into (17) and integrating it along the open curve yield the following:

$$\int_{u}^{+\infty} \frac{ds}{s\sqrt{s - u_m}} = \sqrt{-A}\,|\chi|. \tag{36}$$

Completing (36) and using (8) and (16), we can get a blow-up wave solution of system (4) as follows:

$$u(x,t) = \frac{c(4c-A)}{A}\operatorname{csch}^2(\omega\chi),$$
$$\rho(x,t) = \frac{2Ac}{(4c-A)\operatorname{csch}^2(\omega\chi) - A}, \quad (37)$$
$$t = \frac{1}{c}\left(x - c\left(\chi + \frac{(A-4c)}{\omega A}\coth(\omega\chi)\right)\right),$$

where $\omega = (1/2)\sqrt{c(A-4c)}$.

From Figure 1(n), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi, y) = h_0$ when $c > 0$, $A > 4c$. The homoclinic orbit connects with saddle point $(0,0)$ and passes point $(u_M, 0)$, and the open curve passes saddle point $(0,0)$, where $u_M = c(A-4c)/A$. Their expressions are, respectively,

$$y = \pm\frac{u\sqrt{A(u_M - u)}}{c-u}, \quad 0 < u \le u_M, \quad (38)$$

$$y = \pm\frac{u\sqrt{A(u_M - u)}}{c-u}, \quad u \le 0. \quad (39)$$

Substituting (38) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_u^{u_M}\frac{ds}{s\sqrt{u_M - s}} = \sqrt{A}\,|\chi|. \quad (40)$$

Completing (40) and using (8) and (16), we can get a solitary wave solution of system (4) the same as (35).

Substituting (39) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u}\frac{ds}{s\sqrt{u_M - s}} = \sqrt{A}\,|\chi|. \quad (41)$$

Completing (41) and using (8) and (16), we can get a blow-up wave solution of system (4) same as (37).

From Figure 1(b), we see that there is an open curve of system (10) defined by $H(\phi, y) = h_0$ passing cusp $(0,0)$ when $c < 0$, $A = 4c$. Its expression is

$$y = \pm\frac{2u\sqrt{-cu}}{c-u}, \quad u \ge 0. \quad (42)$$

Substituting (42) into (17) and integrating it along the open curve yield the following:

$$\int_u^{+\infty}\frac{ds}{s\sqrt{-cs}} = 2\,|\chi|. \quad (43)$$

Completing (43) and using (8) and (16), we can get a blow-up wave solution of system (4) as follows:

$$u(x,t) = -\frac{1}{c\chi^2},$$
$$\rho(x,t) = -\frac{2c^3\chi^2}{1+c^2\chi^2}, \quad (44)$$
$$t = \frac{1}{c}\left(x - \left(c\chi - \frac{1}{c\chi}\right)\right).$$

From Figure 1(m), we see that there is an open curve of system (10) defined by $H(\phi, y) = h_0$ passes cusp $(0,0)$ when $c > 0$, $A = 4c$. Its expression is

$$y = \pm\frac{2u\sqrt{-cu}}{c-u}, \quad u \le 0. \quad (45)$$

Substituting (45) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u}\frac{ds}{s\sqrt{-cs}} = -2\,|\chi|. \quad (46)$$

Completing (46) and using (8) and (16), we can get a blow-up wave solution of system (4) the same as (44).

From Figure 1(c), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi, y) = h_1$ when $c < 0$, $4c < A < 0$. The homoclinic orbit connects with saddle point $(u_1, 0)$ and passes point $(u_m, 0)$, and the open curve passes saddle point $(u_1, 0)$, where $u_m = -c((A + 2c) + 2\sqrt{c(2A+c)})/A$. Their expressions are, respectively,

$$y = \pm\frac{(u_1 - u)\sqrt{-A(u-u_m)}}{c-u}, \quad u_m \le u < u_1, \quad (47)$$

$$y = \pm\frac{(u - u_1)\sqrt{-A(u-u_m)}}{c-u}, \quad u \ge u_1. \quad (48)$$

Substituting (47) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_{u_m}^{u}\frac{ds}{(u_1 - s)\sqrt{s - u_m}} = \sqrt{-A}\,|\chi|. \quad (49)$$

Completing (49) and using (8) and (16), we can get a solitary wave solution of system (4) as follows:

$$u(x,t) = \Omega + (u_1 - \Omega)\tanh^2(\omega\chi),$$
$$\rho(x,t) = \frac{2c^2}{(\Omega - c) + (u_1 - \Omega)\tanh^2(\omega\chi)}, \quad (50)$$
$$t = \frac{1}{c}\left(x - \left((c-u_1)\chi + \frac{u_1 - \Omega}{\omega}\tanh(\omega\chi)\right)\right),$$

where $\Omega = -c((A + 2c) + 2\sqrt{c(2A+c)})/A$, $\omega = (1/2)\sqrt{A(\Omega - u_1)}$.

Substituting (48) into (17) and integrating it along the open curve yield the following:

$$\int_{u}^{+\infty} \frac{ds}{(s-u_1)\sqrt{s-u_m}} = \sqrt{-A}\,|\chi|. \tag{51}$$

Completing (51) and using (8) and (16), we can get a blow-up wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= \Omega + (u_1-\Omega)\coth^2(\omega\chi), \\ \rho(x,t) &= \frac{2c^2}{(\Omega-c)+(u_1-\Omega)\coth^2(\omega\chi)}, \\ t &= \frac{1}{c}\left(x-\left((c-u_1)\chi + \frac{u_1-\Omega}{\omega}\coth(\omega\chi)\right)\right), \end{aligned} \tag{52}$$

where $\Omega = -c((A+2c)+2\sqrt{c(2A+c)})/A$, $\omega = (1/2)\sqrt{A(\Omega-u_1)}$.

From Figure 1(e), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi,y) = h_1$ when $c < 0$, $0 < A < -(1/2)c$. The homoclinic orbit connects with saddle point $(u_1,0)$ and passes point $(u_M,0)$, and the open curve passes saddle point $(u_1,0)$, where $u_M = -c((A+2c)+2\sqrt{c(2A+c)})/A$. Their expressions are, respectively,

$$y = \pm\frac{(u-u_1)\sqrt{A(u_M-u)}}{c-u}, \quad u_1 < u \le u_M, \tag{53}$$

$$y = \pm\frac{(u_1-u)\sqrt{A(u_M-u)}}{c-u}, \quad u \le u_1. \tag{54}$$

Substituting (53) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_{u}^{u_M} \frac{ds}{(s-u_1)\sqrt{u_M-s}} = \sqrt{A}\,|\chi|. \tag{55}$$

Completing (55) and using (8) and (16), we can get a solitary wave solution of system (4) the same as (50).

Substituting (54) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u} \frac{ds}{(u_1-s)\sqrt{u_M-s}} = \sqrt{A}\,|\chi|. \tag{56}$$

Completing (56) and using (8) and (16), we can get a blow-up wave solution of system (4) the same as (52).

From Figure 1(f), we see that there is an open curve of system (10) defined by $H(\phi,y) = -(19/2)c^3$ passes cusp $(3c,0)$ when $c<0$, $A=-(1/2)c$. Its expression is

$$y = \pm\frac{(3c-u)\sqrt{-(1/2)c(3c-u)}}{c-u}, \quad u \le 3c. \tag{57}$$

Substituting (57) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u} \frac{ds}{(3c-s)\sqrt{3c-s}} = \sqrt{-\frac{c}{2}}\,|\chi|. \tag{58}$$

Completing (58) and using (8) and (16), we can get a blow-up wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= 3c + \frac{8}{c\chi^2}, \\ \rho(x,t) &= \frac{2c^3\chi^2}{8+2c^2\chi^2}, \\ t &= \frac{1}{c}\left(x+\left(2c\chi - \frac{8}{c\chi}\right)\right). \end{aligned} \tag{59}$$

From Figure 1(i), we see that there is an open curve of system (10) defined by $H(\phi,y) = -(19/2)c^3$ passes cusp $(3c,0)$ when $c>0$, $A=-(1/2)c$. Its expression is

$$y = \pm\frac{(u-3c)\sqrt{(1/2)c(u-3c)}}{c-u}, \quad 3c \le u. \tag{60}$$

Substituting (60) into (17) and integrating it along the open curve yield the following:

$$\int_{u}^{+\infty} \frac{ds}{(s-3c)\sqrt{s-3c}} = \sqrt{\frac{c}{2}}\,|\chi|. \tag{61}$$

Completing (61) and using (8) and (16), we can get a blow-up wave solution of system (4) the same as (59).

From Figure 1(j), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi,y) = h_2$ when $c > 0$, $-(1/2)c < A < 0$. The homoclinic orbit connects with saddle point $(u_2,0)$ and passes point $(u_m,0)$, and the open curve passes saddle point $(u_2,0)$, where $u_m = -c((A+2c)-2\sqrt{c(2A+c)})/A$. Their expressions are, respectively,

$$y = \pm\frac{(u_2-u)\sqrt{-A(u-u_m)}}{c-u}, \quad u_m \le u < u_2, \tag{62}$$

$$y = \pm\frac{(u-u_2)\sqrt{-A(u-u_m)}}{c-u}, \quad u \ge u_2. \tag{63}$$

Substituting (62) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_{u_m}^{u} \frac{ds}{(u_2-s)\sqrt{s-u_m}} = \sqrt{-A}\,|\chi|. \tag{64}$$

Completing (64) and using (8) and (16), we can get a solitary wave solution of system (4) as follows:

$$\begin{aligned} u(x,t) &= \Omega + (u_2-\Omega)\tanh^2(\omega\chi), \\ \rho(x,t) &= \frac{2c^2}{(\Omega-c)+(u_2-\Omega)\tanh^2(\omega\chi)}, \\ t &= \frac{1}{c}\left(x-\left((c-u_2)\chi + \frac{u_2-\Omega}{\omega}\tanh(\omega\chi)\right)\right), \end{aligned} \tag{65}$$

where $\Omega = -c((A+2c)-2\sqrt{c(2A+c)})/A$, $\omega = (1/2)\sqrt{A(\Omega-u_2)}$.

Substituting (63) into (17) and integrating it along the open curve yield the following:

$$\int_{u}^{+\infty} \frac{ds}{(s-u_2)\sqrt{s-u_m}} = \sqrt{-A}\,|\chi|. \quad (66)$$

Completing (66) and using (8) and (16), we can get a blow-up wave solution of system (4) as follows:

$$u(x,t) = \Omega + (u_2 - \Omega)\coth^2(\omega\chi),$$
$$\rho(x,t) = \frac{2c^2}{(\Omega - c) + (u_2 - \Omega)\coth^2(\omega\chi)}, \quad (67)$$
$$t = \frac{1}{c}\left(x - \left((c-u_2)\chi + \frac{u_2-\Omega}{\omega}\coth(\omega\chi)\right)\right),$$

where $\Omega = -c((A+2c) - 2\sqrt{c(2A+c)})/A$, $\omega = (1/2)\sqrt{A(\Omega - u_2)}$.

From Figure 1(l), we see that there are one homoclinic orbit and an open curve of system (10) defined by $H(\phi, y) = h_2$ when $c > 0$, $0 < A < 4c$. The homoclinic orbit connects with saddle point $(u_2, 0)$ and passes point $(u_M, 0)$, and the open curve passes saddle point $(u_2, 0)$, where $u_M = -c((A + 2c) - 2\sqrt{c(2A+c)})/A$. Their expressions are, respectively,

$$y = \pm\frac{(u-u_2)\sqrt{A(u_M - u)}}{c-u}, \quad u_2 < u \le u_M, \quad (68)$$

$$y = \pm\frac{(u_2-u)\sqrt{A(u_M - u)}}{c-u}, \quad u \le u_2. \quad (69)$$

Substituting (68) into (17) and integrating it along the homoclinic orbit yield the following:

$$\int_{u}^{u_M} \frac{ds}{(s-u_2)\sqrt{u_M - s}} = \sqrt{A}\,|\chi|. \quad (70)$$

Completing (70) and using (8) and (16), we can get a solitary wave solution of system (4) the same as (65).

Substituting (69) into (17) and integrating it along the open curve yield the following:

$$\int_{-\infty}^{u} \frac{ds}{(u_2 - s)\sqrt{u_M - s}} = \sqrt{A}\,|\chi|. \quad (71)$$

Completing (71) and using (8) and (16), we can get a blow-up wave solution of system (4) the same as (67).

4. Conclusion

In this paper, we studied the bifurcations of travelling wave solutions of a generalized two-component Hunter-Saxton system and obtained different kinds of periodic wave solutions, which concluded periodic blow-up wave and periodic loop solutions and so forth. Some solitary wave and blow-up wave solutions are also obtained. The results of this paper have enriched the results of [10–12].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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