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Research Article

Time-Space Fractional Heat Equation in the Unit Disk

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We will study a maximal solution of the time-space fractional heat equation in complex domain. The fractional time is taken in the sense of the Riemann-Liouville operator, while the fractional space is assumed in the Srivastava-Owa operator. Here we employ some properties of the univalent functions in the unit disk to determine the upper bound of this solution. The maximal solution is illustrated in terms of the generalized hypergeometric functions.

1. Introduction

Fractional calculus (integrals and derivatives) of any positive order can be considered a branch of mathematical physics which concerns with differential equations, integral equations, and integrodifferential equations, where integrals are of convolution form with weakly singular kernels of power law type. It has prevailed more and more interest in applications in several fields of applied sciences.

Fractional differential equations (real and complex) are viewed as models for nonlinear differential equations. Varieties of them play important roles and tools not only in mathematics but also in physics, dynamical systems, control systems, and engineering to create the mathematical modeling of many physical phenomena. Furthermore, they are employed in social science such as food supplement, climate, and economics. One of these equations is the heat equation of fractional order. Over the last two decades, many authors studied this equation in fractional type. Recently, Vázquez et al. analyzed a set of fractional generalized heat equations [1], Karatay and Bayramoglu considered the numerical solution of a time-fractional heat equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with the Riemann-Liouville fractional derivative [2], Hu et al. established a version of the Feynman-Kac formula for the multidimensional stochastic heat equation with a multiplicative fractional Brownian sheet [3], Khan and Gondal designed a reliable recipe of homotopy analysis method and Laplace decomposition method, namely, homotopy analysis transform method to solve fuzzy fractional heat and wave equations [4], and Povstenko investigated Caputo time-fractional heat equation where solution is obtained by applying the Laplace and finite Hankel integral transforms [5].

In this paper, we shall study a solution of the time-space fractional heat equation in the unit disk. The fractional time is taken in the sense of the Riemann-Liouville operator while the fractional space is assumed in the Srivastava-Owa operator. Here we shall apply some properties of the univalent functions to determine the upper bound of this solution. The maximal solution is illustrated in terms of the generalized hypergeometric functions.

2. Fractional Calculus

The idea of the fractional calculus (i.e., calculus of integrals and derivatives of any arbitrary real or complex order) was found over 300 years ago. Abel in 1823 scrutinized the generalized tautochrone problem and for the first time applied fractional calculus techniques in a physical problem.

2.1. The Riemann-Liouville Operators. This section concerns with some preliminaries and notations regarding the

Riemann-Liouville operators. The Riemann-Liouville fractional derivative strongly poses the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability, and higher accuracy to derive different types of numerical algorithms [6, 7].

Definition 1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau. \quad (1)$$

When $a = 0$, we write $I_a^{\alpha} f(t) = f(t) * \phi_{\alpha}(t)$, where $(*)$ denoted the convolution product (see [8]), $\phi_{\alpha}(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$ and $\phi_{\alpha}(t) = 0$, $t \le 0$, and $\phi_{\alpha} \to \delta(t)$ as $\alpha \to 0$, where $\delta(t)$ is the delta function.

Definition 2. The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

$$D_a^{\alpha} f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)\, d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t). \quad (2)$$

Remark 3. From Definitions 1 and 2, $a = 0$, we have

$$\begin{aligned} D^{\alpha} t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1;\ 0 < \alpha < 1, \\ I^{\alpha} t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1;\ \alpha > 0. \end{aligned} \quad (3)$$

The Leibniz rule is

$$\begin{aligned} D_a^{\alpha} [f(t) g(t)] &= \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} f(t) D_a^k g(t) \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} g(t) D_a^k f(t), \end{aligned} \quad (4)$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}. \quad (5)$$

2.2. The Srivastava-Owa Operators. In [9], Srivastava and Owa defined and studied fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ for analytic functions.

Definition 4. The fractional derivative of order β is defined, for a function $f(z)$, by

$$D_z^{\beta} f(z) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\beta}} d\zeta; \quad 0 \le \beta < 1, \quad (6)$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-\zeta)^{-\beta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Furthermore, for $n \le \beta < n+1$, the fractional differential operator is defined as

$$D_z^{\beta} f(z) = \frac{d^n}{dz^n} D_z^{\beta-n} f(z), \quad n \in \mathbb{N}. \quad (7)$$

Definition 5. The fractional integral of order β is defined, for a function $f(z)$, by

$$I_z^{\beta} f(z) := \frac{1}{\Gamma(\beta)} \int_0^z f(\zeta)(z-\zeta)^{\beta-1} d\zeta; \quad \beta > 0, \quad (8)$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\beta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 6. Consider,

$$\begin{aligned} D_z^{\beta} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)} z^{\mu-\beta}, \quad \mu > -1;\ 0 \le \beta < 1, \\ I_z^{\beta} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)} z^{\mu+\beta}, \quad \mu > -1;\ \beta > 0. \end{aligned} \quad (9)$$

Recently, the Srivastava-Owa operators are generalized into two fractional parameters [10]. Moreover, stability of admissible functions in the unit disk is defined by using these fractional operators [11–13]. Other studies involving these operators can be found in [8, 14–16].

Note that the Srivastava-Owa operators are the complex version (in a simply-connected region) of the Riemann-Liouville operators.

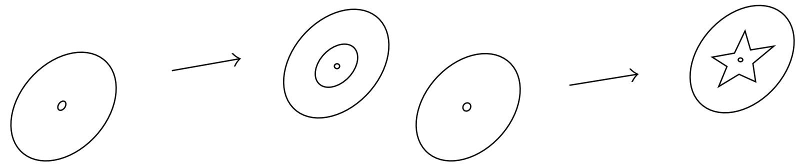
3. The Class of Univalent Functions

One of the major branches of complex analysis is univalent function theory: the study of one-to-one analytic functions. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E. An analytic, univalent function f in the unit disk U mapping the unit disk onto some convex domain is called a convex function. Moreover, A set $D \subset \mathbb{C}$ is said to be starlike with respect to the point $z_0 \in D$ if the line segment joining z_0 to all points $z \in D$ lies in D. A function $f(z)$ which is analytic and univalent in the unit disk U, $f(0) = 0$, and maps U onto a starlike domain with respect to the origin, is called starlike function.

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U := \{z \in \mathbb{C} : |z| < 1\}. \quad (10)$$

Also, let $\mathcal{S}$ and $\mathcal{C}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent and convex in U.



(a) (b)

Figure 1: Distributions of the heat in the unit disk.

It is well known that, if the function $f(z)$ given by (10) is in the class $\mathcal{S}$, then

$$|a_n| \leq n, \quad n \in \mathbb{N} \setminus \{1\}. \tag{11}$$

Equality holds for the Koebe function

$$f(z) = \frac{z}{(1-z)^2}, \quad z \in U. \tag{12}$$

Moreover, if the function $f(z)$ given by (10) is in the class $\mathcal{C}$, then

$$|a_n| \leq 1, \quad n \in \mathbb{N}. \tag{13}$$

Equality holds for the function

$$f(z) = \frac{z}{1-z}, \quad z \in U. \tag{14}$$

A function $f \in \mathcal{A}$ is called starlike of order μ if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \quad (z \in U) \tag{15}$$

for some $0 \leq \mu < 1$. We denoted this class $\mathcal{S}^*(\mu)$.

A function $f \in \mathcal{A}$ is called convex of order μ if it satisfies the following inequality:

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \mu, \quad (z \in U) \tag{16}$$

for some $0 \leq \mu < 1$. We denoted this class $\mathcal{C}(\mu)$. Note that $f \in \mathcal{C}(\mu)$ if and only if $zf' \in \mathcal{S}^*(\mu)$.

For $\alpha_j \in \mathbb{C}$, $j = 1, \ldots, l$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, \ldots, m$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$\begin{gathered} {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}), \end{gathered} \tag{17}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$\begin{aligned} (a)_n &:= \frac{\Gamma(a+n)}{\Gamma(a)} \\ &= \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\cdots(a+n-1), & (n \in \mathbb{N}). \end{cases} \end{aligned} \tag{18}$$

4. Time-Space Fractional Heat Equation

In this section, we will express the maximal solution (a solution which has no extension is called a maximal solution) for the problem

$$\begin{gathered} D^{\alpha} u(t, z) = D_z^{\beta} u(t, z), \quad u(0, z) = 0, \\ (t \in [0, T],\ z \in U,\ \alpha \in (0, 1),\ \beta \in [1, 2)) \end{gathered} \tag{19}$$

and $u : J \times U \to U$. The source of the heat is in the unit disk in neighborhood of the origin. Therefore, the distribution of the heat is either in homogeneous state; in this case we have a convex form

$$\Re\left\{\frac{zu''(t, z)}{u'(t, z)} + 1\right\} > 0, \quad \forall t \in J, \tag{20}$$

(Figure 1(a)); or it is nonhomogeneous then we obtain the starlike (univalent) form

$$\Re\left\{\frac{zu'(t, z)}{u(t, z)}\right\} > 0, \quad \forall t \in J, \tag{21}$$

(Figure 1(b)).

We have the following results.

Theorem 7. *Let $u(t, z)$ be univalent function in the unit disk for all $t \in J$, and let $\beta := 1 + \gamma$, $0 < \gamma < 1$. Then*

$$\begin{gathered} \left|D_z^{\beta} u(t, z)\right| \leq \frac{r^{-\gamma}}{\Gamma(1-\gamma)} \left(rF\left((2)_n, (1)_n; (1-\gamma)_n; rt\right)\right)' \\ \left(' := \frac{d}{dz},\ r = |z|;\ z \in U \setminus \{0\}\right), \end{gathered} \tag{22}$$

where the equality holds true for the Koebe function.

Proof. Since u is univalent, then it can be expressed by

$$u(t, z) = \sum_{n=1}^{\infty} a_n z^n t^{n-1}, \qquad a_1 = 1. \tag{23}$$

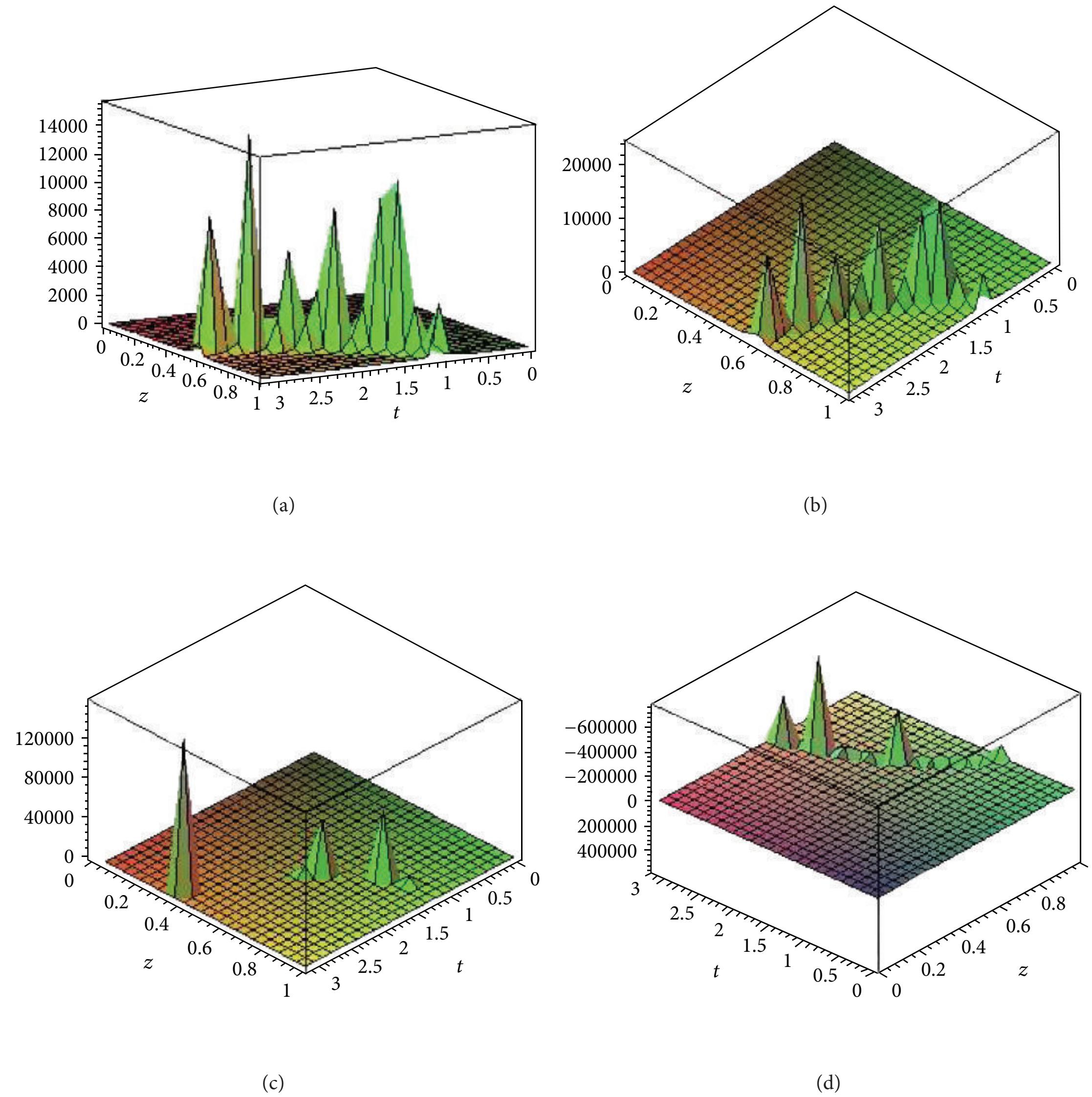


Figure 2: Starlike distribution (nonhomogeneous).

In view of Remark 6, we impose that

$$D_z^{\beta} u(t,z) = z^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} a_n z^n t^{n-1}. \tag{24}$$

Applying (11) implies

$$\begin{aligned} \left| D_z^{\beta} u(t,z) \right| &= \left| z^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} a_n z^n t^{n-1} \right| \\ &\le r^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} n r^n t^{n-1} \\ &= r^{-\gamma} \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{\Gamma(1+n-\gamma)} (n+1) r^n t^n \\ &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)} \sum_{n=0}^{\infty} \frac{(2)_n (1)_n}{(1-\gamma)_n} \frac{n+1}{n!} (rt)^n \\ &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)} \left(rF\left((2)_n, (1)_n; (1-\gamma)_n; rt\right)\right)'. \end{aligned} \tag{25}$$

This completes the proof. □

Theorem 8. *Let $u(t,z)$ be convex function in the unit disk for all $t \in J$, and let β be defined as in Theorem 7. Then*

$$\left| D_z^{\beta} u(t,z) \right| \le \frac{r^{-\gamma}}{\Gamma(1-\gamma)} F\left((2)_n, (1)_n; (1-\gamma)_n; rt\right), \quad (r = |z|;\ z \in U \setminus \{0\}). \tag{26}$$

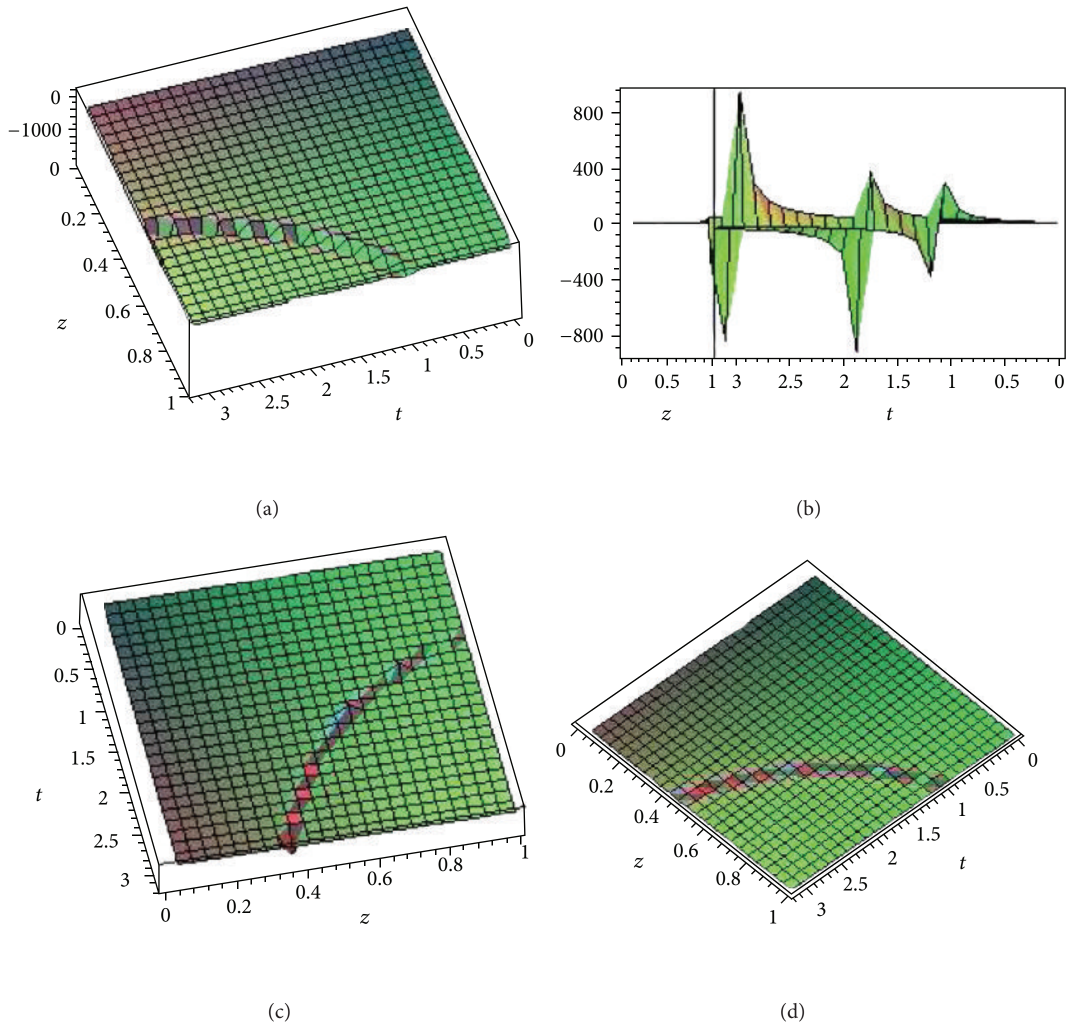


Figure 3: Convex distribution (homogeneous).

Proof. Since u is convex, then it yields

$$D_z^{\beta} u(t,z) = z^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} a_n z^n t^{n-1}. \tag{27}$$

Employing (13) implies

$$\begin{aligned}
\left| D_z^{\beta} u(t,z) \right| &= \left| z^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} a_n z^n t^{n-1} \right| \\
&\le r^{-\gamma-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\gamma)} r^n t^{n-1} \\
&= r^{-\gamma} \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{\Gamma(1+n-\gamma)} r^n t^n \\
&= \frac{r^{-\gamma}}{\Gamma(1-\gamma)} \sum_{n=0}^{\infty} \frac{(2)_n (1)_n}{(1-\gamma)_n} \frac{(rt)^n}{n!} \\
&= \frac{r^{-\gamma}}{\Gamma(1-\gamma)} F\left((2)_n, (1)_n; (1-\gamma)_n; rt\right).
\end{aligned} \tag{28}$$

This completes the proof. □

We proceed to determine the maximal solution of (19) using Theorems 7 and 8.

Theorem 9. *Let $u(t,z)$ be univalent function in the unit disk for all $t \in J$. Then (19) has a maximal solution $u(t,z)$ in the domain $\mathscr{D} := J \times U$ of the form*

$$\begin{aligned}
u(t,z) = {} & \frac{r^{-\gamma}}{\Gamma(1-\gamma)\Gamma(1+\alpha)} \\
& \times \left(rF\left((2)_n, (1)_n, (1)_n; (1-\gamma)_n, (1+\alpha)_n; rt^{\alpha}\right)\right)'.
\end{aligned} \tag{29}$$

Proof. By applying the upper bound of the operator $D_z^{\beta}u(t,z)$ in (19), Theorem 7 implies

$$D^{\alpha}u(t,z) \approx \frac{r^{-\gamma}}{\Gamma(1-\gamma)}\left(rF\left((2)_n,(1)_n;(1-\gamma)_n;rt\right)\right)' = \frac{r^{-\gamma}}{\Gamma(1-\gamma)}\sum_{n=0}^{\infty}\frac{(2)_n(1)_n}{(1-\gamma)_n}\frac{n+1}{n!}(rt)^n \quad (0<|z|=r<1;\ 0<\gamma<1,\ \beta=\gamma+1). \tag{30}$$

Operating (30) by I^{α} and applying Remark 3, we receive

$$\begin{aligned} u(t,z) &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)}\sum_{n=0}^{\infty}\frac{(2)_n(1)_n}{(1-\gamma)_n}\frac{n+1}{n!}r^n\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}t^{n+\alpha} \\ &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\frac{(2)_n(1)_n(1)_n}{(1-\gamma)_n(1+\alpha)_n}\frac{n+1}{n!}(rt^{\alpha})^n \\ &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)\Gamma(1+\alpha)} \\ &\quad\times\left(rF\left((2)_n,(1)_n,(1)_n;(1-\gamma)_n,(1+\alpha)_n;rt^{\alpha}\right)\right)'. \end{aligned} \tag{31}$$

Hence this comletes the proof. □

Note that any extension of the solution u in (31) should be out the unit disk U. Therefore, we obtain some kind of stable solutions. Moreover, the real case can be founded in [1].

In the same manner of Theorem 9, we may obtain the following result which describes the maximal solution for the convex function (homogeneous case).

Theorem 10. *Let $u(t,z)$ be convex function in the unit disk for all $t\in J$. Then (19) has a maximal solution $u(t,z)$ in the domain $\mathscr{D}=J\times U$ of the form*

$$\begin{aligned} u(t,z) &= \frac{r^{-\gamma}}{\Gamma(1-\gamma)\Gamma(1+\alpha)} \\ &\quad\times F\left((2)_n,(1)_n,(1)_n;(1-\gamma)_n,(1+\alpha)_n;rt^{\alpha}\right), \\ &\quad\left(0<|z|=r<1;\ 0<\gamma<1,\ 0<\alpha<1\right). \end{aligned} \tag{32}$$

5. Conclusion and Discussion

We introduced a method depending on some properties of the geometric function theory (univalent, starlike, and convex) for obtaining the maximal solution of heat equation of fractional order in a complex domain. This solution was obtained for two cases: homogeneous and nonhomogeneous distributions of the source. We realized that this solution can be considered as a positive, bounded, and stable solution in the unit disk. We may apply this new method on well-known diffusion equations of fractional order such as the fractional wave equation in complex domain. Furthermore, since these solutions are determined in terms of the hypergeometric function and its generalization, then this leads to have a convergence solution. Note that the hypergeometric function involves the Mittag-Leffler function functions. Figure 2 shows nonhomogeneous distribution (starlike), while Figure 3 shows the homogeneous distribution (convex). The values of (α,γ) are taken as $(0.5, 0.5)$, $(0.19, 0.19)$, $(0.78, 0.78)$, and $(0.99, 0.99)$ for Figures 3(a)–3(d), respectively.

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