Research Article

Minimizing the Makespan for Scheduling Problems with General Deterioration Effects

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This paper investigates the scheduling problems with general deterioration models. By the deterioration models, the actual processing time functions of jobs depend not only on the scheduled position in the job sequence but also on the total weighted normal processing times of the jobs already processed. In this paper, the objective is to minimize the makespan. For the single-machine scheduling problems with general deterioration effects, we show that the considered problems are polynomially solvable. For the flow shop scheduling problems with general deterioration effects, we also show that the problems can be optimally solved in polynomial time under the proposed conditions.

1. Introduction

In the last decade, there is a growing interest in the literatures to study the scheduling problems where the actual processing time of a job may be subject to change due to various possible changes of its starting time and/or its position in the scheduling sequence. In general, there are two main types of effects on actual processing time of jobs in the scheduling literature: position-based effect and sum-of-processing-time-based effect.

Many studies modeled the actual processing times of jobs as specific exponential functions of the scheduled positions of jobs. Biskup [1] considered a scheduling problem with an exponential model of position-dependent effect where the actual processing time of job J_i is

$$p_{ir} = p_i r^a \tag{1}$$

if job J_i is scheduled in position r of the job sequence, where p_i and a $(a < 0)$ are the normal processing time and the

learning rate of job J_i, respectively. The model in Biskup [1] was extended to be

$$p_{ir} = p_i r^{a_i}, \tag{2}$$

where the learning rates of the jobs are different; that is, the learning effects are job dependent in Mosheiov and Sidney [2]. Some recent studies (see, e.g., Kuo and Yang [3], Cheng et al. [4], Chang et al. [5], Yang et al. [6], and Browne and Yechiali [7]) have investigated this issue with exponential processing time models under different machine environments. Some other studies proposed different versions of linear processing time models of jobs. Zhao and Tang [8] considered a linear deterioration model in which the actual processing time of job J_j is

$$p_j^A = p_j + a_j t, \tag{3}$$

where p_j, a_j, and t are the normal processing time, deterioration rate, and starting time of job J_i, respectively. Cheng and

Wang [9] introduced an actual processing time model of J_i as follows:

$$p_j = a_i - v_i \min\{n_i, n_{0i}\}, \quad i = 1, 2, \ldots, n, \tag{4}$$

where p_j and v_i are the normal processing time of job J_i and the learning effect coefficient indicating the relative degree of learning, respectively. The notation n_i is a nonnegative integer such that $0 \le n_i \le n - 1$ which indicates the number of jobs processed before job J_i in the schedule (i.e., $n_i + 1$ is the position of job J_i in the schedule), and the notation n_{0i} is a nonnegative integer such that $n_{0i} \le n - 1$ indicating a threshold value. The scheduling problems with linear actual processing times of jobs were considered in many recent articles, which include Mosheiov [10], Bachman and Janiak [11], Li et al. [12], Cheng et al. [13], and Yin et al. [14].

In the last decade, many researchers have devoted themselves to address different combinations of nonlinear models and linear models for actual processing times of jobs in scheduling studies. Alidaee and Womer [15] proposed a simple processing time model of job J_j as follows:

$$p_{jr}(t) = p_j + \alpha t^a, \tag{5}$$

where $a > 0$ and $\alpha > 0$ are both parameters of the nonlinear deterioration effect. Lee et al. [16] considered the bicriterion single-machine scheduling problem with two job processing time models as follows:

$$\begin{aligned} p_{jr}(t) &= \alpha_j t r^a, \\ p_{jr}(t) &= (p_0 + \alpha_j t) r^a, \end{aligned} \tag{6}$$

where α_j and t are the deterioration rate of job J_j and the starting time of job J_j, respectively.

Several researchers addressed the combined effect of position-based effect and sum-of-processing-time-based effect into various scheduling problems. S.-J. Yang and D.-L. Yang [17] proposed a model in which the actual processing time of job J_j scheduled in position r of a sequence is

$$p_{ir} = p_i \left(1 + \frac{\sum_{k=1}^{r-1} p_{[k]}}{\sum_{k=1}^{n} p_k} \right)^{a_1} r^{a_2}, \tag{7}$$

where $a_1 < 0$ and $a_2 < 0$ denote the learning rates for sum-of-processing-times-based and position-based learning, respectively. In the model, the subscript $[k]$ denotes the job in position k of a sequence, and $p_{[i]}$ denotes the normal processing time of job i. Yin et al. [18] extended the model in S.-J. Yang and D.-L. Yang [17] and proposed another learning model in which the actual processing time of job J_j scheduled in position r of a sequence is

$$p_{ir} = p_i f \left(\sum_{k=1}^{r-1} p_{[k]} \right) g(r), \tag{8}$$

where $\sum_{k=1}^{0} p_{[k]} = 0$, $f : [0, +\infty) \to (0, 1]$ is a differentiable nonincreasing function where f is nondecreasing on $[0, +\infty)$, $f(0) = 1$, and $g : [0, +\infty) \to (0, 1]$ is

a nonincreasing function with $g(1) = 1$. Many recent studies (see, e.g., Wang [19], Toksarı and Güner [20], Yang and Kuo [21], Cheng et al. [22, 23], Cheng et al. [24], Kuo [25], Yang [26], and Lee and Wu [27]) have been conducted to address the combined processing time models in scheduling problems.

In a recent paper, J.-B. Wang and J.-J. Wang [28] proposed a new general learning effect model for both position-based effect and sum-of-processing-time-based effect, which can be described as

$$p_{ir} = p_i f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) g(r), \tag{9}$$

where $\sum_{l=1}^{0} \beta_l p_{[l]} := 0$, $f : [0, +\infty) \to (0, 1]$ is a differentiable nonincreasing function where f' is nondecreasing on $[0, +\infty)$, $g : [1, +\infty) \to (0, 1]$ is a nonincreasing function with $g(1) = 1$, and β_l is the weight of position r. They showed that even with the introduction of a general learning effect to job processing times, some single-machine scheduling problems can be optimally solved in polynomial time. They also proved that some special cases of the flow shop scheduling problems are polynomially solvable under the proposed model.

In this paper, we explore to consider scheduling problems with considerations of two general deterioration effects whose actual processing times functions of jobs are modified from the models in Lee and Wu [27] and J.-B. Wang and J.-J. Wang [28] with respect to deterioration effects. The motivation for our study stems from the hardware production line that cuts a class of products to proper sizes and shapes. The actual processing time of a product depends on the quality of the cutting tool in the manufacturing process. Due to tool blunting, the actual processing time of a product increases with respect to the sum of the processing time of products already processed by the cutting tool and the number of products already processed. The objective of our study is to find the optimal job sequence to minimize the makespan of all the jobs.

The remaining part of this paper is organized as follows. In Section 2, we consider two single-machine scheduling problems. In Section 3, we investigate the flow shop scheduling problems under proper given conditions. We conclude this paper and describe the possible extensions in the last section.

2. Single-Machine Scheduling

The formulation of the considered single-machine scheduling problems can be described as follows. At time zero, there are n independent jobs ready to be processed on a single-machine. Preemption is not allowed and a machine is only able to process one job at a time. Job j has a normal processing time p_j ($j = 1, 2, \ldots, n$).

In this paper, we propose two general deterioration models with respect to both position-based effect and sum-of-processing-time-based effect. By the first model, if job j

is scheduled in the r position of the sequence, the actual processing time of job j is

$$p_{jr} = p_j \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right], \qquad (10)$$

where $\sum_{l=1}^{0} \beta_l p_{[i,l]} := 0$, $f : [0, +\infty) \to [1, +\infty)$ is a differentiable nondecreasing function and $f(0) = 1$, $g : [1, +\infty) \to [1, +\infty)$ is a nondecreasing function with $g(1) = 1$, β_l is the weight of position l, $p_{[l]}$ denotes the normal processing time of job i, and the subscript $[l]$ denotes the job in position l of a sequence. In some actual processes, different jobs always have different complexities which lead to different experience to the worker in production line; thus, we assume that $0 \le \beta_1 \le \beta_2 \le \cdots \beta_n$.

By the second model, if job j is scheduled in the r position, its actual processing time is

$$p_{jr} = p_j f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) g(r), \qquad (11)$$

where all the notations have the same means as those of the first model. Note that the function used in the model (11) is the same as the function used by J.-B. Wang and J.-J. Wang [28]. The difference between this model and the model in J.-B. Wang and J.-J. Wang [28] is that the former aims to study the deterioration effect while the latter (J.-B. Wang and J.-J. Wang [28]) considers learning effect.

By using the three-field notation scheme for the scheduling problem introduced by Graham et al. [29], the single-machine scheduling problem with respect to the first model can be denoted as $1 \mid p_{jr} = p_j[f(\sum_{l=1}^{r-1} \beta_l p_{[l]}) + g(r)]$, DE $\mid C_{\max}$, where DE denotes the deterioration effect. Similarly, the single-machine scheduling problem with respect to the second model can be denoted as $1 \mid p_{jr} = p_j f(\sum_{l=1}^{r-1} \beta_l p_{[l]}) g(r)$, DE $\mid C_{\max}$.

In the following, we first state two lemmas that will be used in the proofs of the following theorems in this section.

Lemma 1. *If $f : [0, +\infty) \to [1, +\infty)$ is nondecreasing and its derived function f' is nonincreasing on $[0, +\infty)$, $\lambda \ge 1$, $\delta_1 \ge 1$, and $x \ge 0$, then $(1 - \lambda)(f(A)\delta_2 + 1) + \lambda(f(A + x)\delta_2 + \delta_1) - (f(a + \lambda x)\delta_2 + \delta_1) \ge 0$.*

Proof. We let $h(x) = (1 - \lambda)(f(A)\delta_2 + 1) + \lambda(f(A + x)\delta_2 + \delta_1) - (f(A + \lambda x)\delta_2 + \delta_1)$. The first derivation of $h(x)$ with respect to x can be obtained that $h'(x) = \lambda \delta_2 [f'(A + x) - f'(A + \lambda x)] \ge 0$ under the condition that $f : [0, +\infty) \to [1, +\infty)$ is a differentiable nondecreasing function with f' being nonincreasing on $[0, +\infty)$, $\lambda \ge 1$, and $\delta_1 \ge 1$. Then, it is obvious that $h(x)$ is an increasing function on x. Therefore, we have $h(x) \ge h(0) = (1 - \lambda)(1 - \delta_1) \ge 0$ for $\lambda \ge 1, \delta_1 \ge 1$. $\square$

With a similar differentiation method, it is easy to obtain the following lemma.

Lemma 2. *If $f : [0, +\infty) \to [1, +\infty)$ is nondecreasing and its derived function f' is nonincreasing on $[0, +\infty)$, $\lambda \ge 1$, $\delta \ge 1$, and $x \ge 0$, then $(1 - \lambda)f(A) + \lambda \delta f(A + x) - \delta f(A + \lambda x) \ge 0$.*

Theorem 3. *The problem $1 \mid p_{jr} = p_j(f(\sum_{l=1}^{r-1} \beta_l p_{[l]}) + g(r))$, DE $\mid C_{\max}$ can be optimally solved by the Longest Processing Time first rule of p_j (the LPT rule) if f' is nonincreasing on $[0, +\infty)$.*

Proof. To prove this theorem, it is sufficient to prove that a schedule can be enhanced by scheduling the job with longer processing time into the earlier position of the schedule under the given condition in the theorem. Let the schedule σ be $[\sigma_1, J_j, J_k, \sigma_{i2}, \sigma_2]$ and let the schedule σ' be $[\sigma_1, J_k, J_j, \sigma_2]$, where σ_1 and σ_2 are partial job sequences, and the difference between σ and σ' is a pairwise interchange of two adjacent jobs J_j and J_k in the schedule. Concretely, J_j and J_k are assumed to be assigned in the rth and $r + 1$th position, respectively, in the schedule σ. In σ', J_j and J_k are assigned in the $r + 1$th and rth position, respectively. Without loss of generality, the normal processing times of J_j and J_k are assumed to satisfy $p_j \le p_k$. To show that σ' dominates σ, it suffices to show that $C_k(\sigma) \ge C_j(\sigma')$ and $C_u(\sigma) \ge C_u(\sigma')$ for any job J_u in σ_2. Let the completion time of the last job in partial schedule σ_1 be C_{σ_1}. In the schedule σ, the completion times of J_j and J_k are

$$C_j(\sigma) = C_{\sigma_1} + p_j \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right], \qquad (12)$$

$$C_k(\sigma) = C_{\sigma_1} + p_j \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right]$$
$$+ p_k \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_j \right) + g(r+1) \right], \qquad (13)$$

respectively. In the schedule σ', the completion times of J_k and J_j are

$$C_k(\sigma') = C_{\sigma_1} + p_k \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right] \qquad (14)$$

$$C_j(\sigma') = C_{\sigma_1} + p_k \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right]$$
$$+ p_j \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_k \right) + g(r+1) \right], \qquad (15)$$

respectively. Combining (13) and (15), we have

$$C_k(\sigma) - C_j(\sigma')$$

$$= (p_j - p_k) \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) + g(r) \right]$$

$$+ p_k \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_j \right) + g(r+1) \right]$$

$$- p_j \left[f \left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_k \right) + g(r+1) \right]$$

$$
\times \frac{p_k}{p_j} \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) \frac{1}{g(r)} + 1 \right]
$$

$$
= p_j g(r) \left\{ \left(1 - \frac{p_k}{p_j} \right) \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[l]} \right) \frac{1}{g(r)} + 1 \right] \right.
$$

$$
+ \frac{p_k}{p_j} \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_j \right) \right.
$$

$$
\times \left. \frac{1}{g(r)} + \frac{g(r+1)}{g(r)} \right]
$$

$$
- \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[l]} + \beta_r p_j \frac{p_k}{p_j} \right) \right.
$$

$$
\left. \left. \times \frac{1}{g(r)} + \frac{g(r+1)}{g(r)} \right] \right\}.
$$

$$
(16)
$$

Let $A = \sum_{l=1}^{r-1} \beta_l p_{[l]}, \lambda = p_k/p_j, \delta_1 = (g(r+1)/g(r)), \delta_2 = (1/g(r))$, and $x = \beta_r p_j$. Based on the formulation of the considered problem, it is easy to obtain that $A \geq 0, \lambda \geq 1, \delta_1 \geq 1, 0 < \delta_2 \leq 1$, and $x \geq 0$. Based on Lemma 1 and (16), we have

$$
C_k(\sigma) - C_j(\sigma')
$$

$$
= p_j g(r) \left[(1 - \lambda)\left(f(A)\delta_2 + 1 \right) \right.
$$

$$
+ \lambda \left(f(A + x)\delta_2 + \delta_1 \right)
$$

$$
\left. - \left(f(a + \lambda x)\delta_2 + \delta_1 \right) \right] \geq 0.
$$

$$
(17)
$$

Assuming that J_s is the job scheduled in the $r + 2$ position of the job sequence, then the completion times of J_s in σ and σ' are

$$
C_s(\sigma) = C_k(\sigma)
$$

$$
+ p_s \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[i,l]} + \beta_r p_j + \beta_{r+1} p_k \right) \right.
$$

$$
\left. + g(r+2) \right],
$$

$$
(18)
$$

$$
C_s(\sigma') = C_j(\sigma')
$$

$$
+ p_s \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[i,l]} + \beta_r p_k + \beta_{r+1} p_j \right) \right.
$$

$$
\left. + g(r+2) \right],
$$

$$
(19)
$$

respectively. Since $p_j \leq p_k$ and $\beta_r \leq \beta_{r+1}$, it can be obtained that

$$
\left(\beta_r p_j + \beta_{r+1} p_k \right) - \left(\beta_r p_k + \beta_{r+1} p_j \right)
$$

$$
= \left(\beta_r - \beta_{r+1} \right)\left(p_j - p_k \right) \geq 0.
$$

$$
(20)
$$

Since f is a nondecreasing function, (20) is equivalent to

$$
f\left(\sum_{l=1}^{r-1} \beta_l p_{[i,l]} + \beta_r p_j + \beta_{r+1} p_k \right)
$$

$$
\geq f\left(\sum_{l=1}^{r-1} \beta_l p_{[i,l]} + \beta_r p_k + \beta_{r+1} p_j \right).
$$

$$
(21)
$$

Let $C_s(\sigma)$ and $C_s(\sigma')$ denote the completion time of job J_s in the schedule σ and σ', respectively. From (17)–(19) and (21), we have $C_s(\sigma) \geq C_s(\sigma')$. Thus, we show that J_s starts earlier in σ' than in σ and completes earlier in σ' than in σ. With a similar proof method, it can be shown that $C_u(\sigma) \geq C_u(\sigma')$ for any job J_u in σ_2. By repeating the above argument, it is obvious to lead to the conclusion that a schedule can be enhanced by scheduling the job with longer processing time into the earlier position of the job sequence. $\square$

Based on Lemma 2, the following theorem can be obtained by using a similar proof method to that of Theorem 3.

Theorem 4. *The problem* $1 \mid p_{jr} = p_j(f(\sum_{l=1}^{r-1} \beta_l p_{[l]})g(r))$, *DE* $\mid C_{\max}$ *can be optimally solved by the Longest Processing Time first rule (the* LPT *rule) if* f' *is nonincreasing on* $[0, +\infty)$.

If $g(r)$ *is unchanged with respect to* r, *then two actual processing time models can be both simplified as* $p_{jr} = p_j f$ $(\sum_{l=1}^{r-1} \beta_l p_{[l]})$. *Similarly, if* $f(\sum_{l=1}^{r-1} \beta_l p_{[l]})$ *is unchanged with respect to* $\sum_{l=1}^{r-1} \beta_l p_{[l]}$, *then two actual processing time models can be both simplified as* $p_{jr} = p_j g(r)$. *It is obvious that the single-machine problem with the above special models can be also optimally solved by the* LPT *rule.*

3. Flow Shop Scheduling

The formulation of the flow shop scheduling problem can be described as follows. There are n independent jobs $J_1, J_2, \ldots, J_n$ ready to be processed on m machines $M_1, M_2, \ldots, M_m$ at time zero. Job J_j consists of m operations $O_{1j}, O_{2j}, \ldots, O_{mj}$. Preemption is not allowed and a machine is only able to process one operation at a time. All machines should process the operations of the jobs in the same order. Operation $O_{i+1,j}$ may start only after operation O_{ij} has been completed. Operation O_{ij} has a normal processing time p_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, and the actual processing time of operation O_{ij} is p_{ijr} if it is scheduled in position r $(r = 1, 2, \ldots, n)$ in a sequence. In this paper, we propose two general deterioration models. By the first model, if operation O_{ij} is scheduled in the r position of the sequence, the actual processing time of operation O_{ij} is

$$
p_{ijr} = p_{ij} \left[f\left(\sum_{l=1}^{r-1} \beta_l p_{[il]} \right) + g(r) \right],
$$

$$
(22)
$$

where $\sum_{l=1}^{0} \beta_l p_{[i,l]} := 0, f : [0, +\infty) \rightarrow [1, +\infty)$ is a differentiable nondecreasing function, and $f(0) = 1, g : [1, +\infty) \rightarrow [1, +\infty)$ is a nonincreasing function with

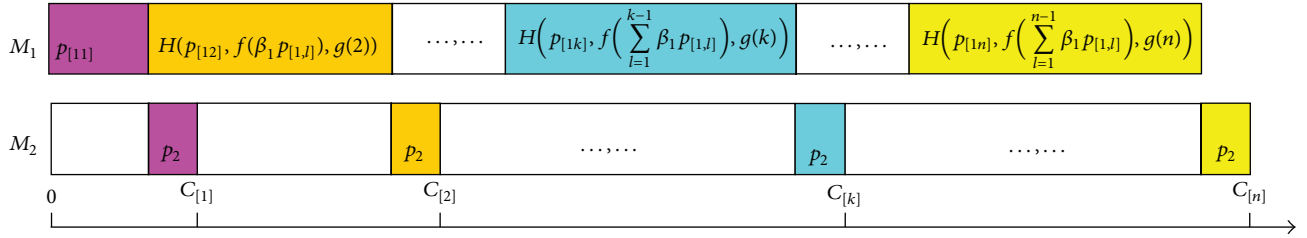FIGURE 1: The illustration of the problem $F \mid p_{ijr} = H(p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]}), g(r)), p_{2j} = p_2, \text{DE} \mid C_{\max}$.

$g(1) = 1$, and β_l is the weight of position l; $p_{[il]}$ denotes the normal processing time of the operation scheduled in the rth position on machine M_i, where the subscript $[i, l]$ denotes the job in position l of the sequence on M_i. In many real life scenarios, positions are characterized by a different grade of complexity; thus, each position can provide different complexity to the manufacturing process; hence, we assume that $0 \le \beta_1 \le \beta_2 \cdots \beta_n$.

By the second model, if job j is scheduled on the r position of machine i, its actual processing time is

$$p_{ijr} = p_{ij} f \left(\sum_{l=1}^{r-1} \beta_l p_{[il]} \right) g(r), \tag{23}$$

where all the notations have the same means as those of the first model.

In the following, we first introduce a new general deterioration model. The actual processing time of job j on the r position of machine i is

$$p_{ijr} = H \left[p_{ij}, f \left(\sum_{l=1}^{r-1} \beta_l p_{[il]} \right), g(r) \right], \tag{24}$$

where $p_{ij1} = H(p_{ij}, f(0), g(1)) = p_{ij}$ and H is nondecreasing in $p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]})$, and $g(r)$. Obviously, model (22) and model (23) are both special cases of new model (24).

Lemma 5. *The problem* $F \mid p_{ijr} = H[p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]}), g(r)], p_{2j} = p_2, \text{DE} \mid C_{\max}$ *has the same optimal schedule as the problem* $1 \mid p_{1jr} = H[p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}), g(r)], \text{DE} \mid C_{\max}$ *if* $p_{1j} \ge p_2$.

Proof. Let $C_{[j]}$ denote the completion time of the job scheduled in jth position of the job sequence. For $j = 1$, we have $C_{[1]} = p_{[11]} + p_2$.

Because $p_{2j} = p_2$ and $p_{1j} \ge p_2$, it can be obtained that $p_{1jr} = H[p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}), g(r)] \ge p_2$ for $j = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, n$. The schedule is illustrated in Figure 1.

Then, we have the completion times for jobs $j = 2, 3, \ldots, n$ in the following:

$$
\begin{aligned}
&C_{[2]} \\
&= \max \left\{ C_{[1]}, p_{[11]} + H\left[p_{[12]}, f(\beta_1 p_{[1,l]}), g(2)\right] \right\} p_{[11]} \\
&\quad + p_2 \\
&= p_{[11]} + H\left(p_{[12]}, f[\beta_1 p_{[1,l]}], g(2)\right) + p_2,
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
&C_{[k+1]} \\
&= \max \left\{ C_{[k]}, \sum_{r=1}^{k} H\left[p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]} \right), g(r) \right] \right\} \\
&\quad \times p_{[11]} + p_2 \\
&= \sum_{r=1}^{k} H\left[p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]} \right), g(r) \right] + p_2,
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
&C_{\max} \\
&= C_{[n]} \\
&= \max \left\{ C_{[k]}, \sum_{r=1}^{k} H\left[p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[i,l]} \right), g(r) \right] \right\} \\
&\quad \times p_{[11]} + p_2 \\
&= \sum_{r=1}^{n} H\left[p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]} \right), g(r) \right] + p_2.
\end{aligned}
\tag{25}
$$

Since p_2 is constant, an optimal schedule of minimizing the makespan for the problem $F2 \mid p_{1jr} = H[p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}, g(r)], \text{DE} \mid C_{\max}$ must be the optimal schedule of minimizing for the problem $F2 \mid p_{ijr} = H[p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}), g(r)], p_{2j} = p_2, \text{DE} \mid C_{\max}$. □

With a similar proof method of Lemma 5, we can obtain the following theorem.

Theorem 6. *The problem* $Fm \mid p_{ijr} = H[p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]}), g(r)], p_{2jr} = p_{3jr} = \cdots = p_{mjr} = p, DE \mid C_{\max}$ *has the same optimal schedule as the problem* $1 \mid p_{1jr} = H(p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}, g(r)), DE \mid C_{\max}$ *if* $p_{1j} \geq p$.

Combining Theorems 3 and 6, we have the following corollary.

Corollary 7. *The problem* $Fm \mid p_{ijr} = p_{ij}[f(\sum_{l=1}^{r-1} \beta_l p_{[il]}) + g(r)], p_{2jr} = p_{3jr} = \cdots = p_{mjr} = p, DE \mid C_{\max}$ *can be optimally solved by the Longest Processing Time first rule of* p_j *(the* LPT *rule) if* f' *is nonincreasing on* $[0, +\infty)$ *and* $p_{1j} \geq p$.

Similarly, based on Theorems 4 and 6, it is easy to get the following result.

Corollary 8. *The problem* $Fm \mid p_{ijr} = p_{ij}f(\sum_{l=1}^{r-1} \beta_l p_{[il]})g(r), p_{2jr} = p_{3jr} = \cdots = p_{mjr} = p, DE \mid C_{\max}$ *can be optimally solved by the Longest Processing Time first rule of* p_j *(the* LPT *rule) if* f' *is nonincreasing on* $[0, +\infty)$ *and* $p_{1j} \geq p$.

Theorem 9. *If a schedule simultaneously minimizes* $H(p_{[2n]}, f(\sum_{l=1}^{n-1} \beta_l p_{[2,l]}), g(n))$ *and the makespan of the problem* $1 \mid p_{1jr} = H(p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}, g(r)), DE \mid C_{\max}$, *it must be the optimal schedule of the problem* $F2 \mid p_{ijr} = H(p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]}, g(r)), \min(p_{1.}) \geq \max(p_{2.}), DE \mid C_{\max}$.

Proof. By using the same notation as that in the proof of Theorem 6, we let $C_{[j]}$ denote the completion time of the job scheduled in jth position of the job sequence. For $j = 1$, we have $C_{[1]} = p_{[11]} + p_2$.

Because $\min(p_{1.}) \geq \max(p_{2.})$, it can be obtained that

$$
H\left(p_{1j}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1l]}\right), g(r)\right)
$$
$$
\geq H\left(p_{2j}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[2l]}\right), g(r)\right)
\tag{26}
$$

for $j = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, n$. Then, we have the completion times for jobs $j = 2, 3, \ldots, n$ in the following:

$$
C_{[2]} = \max \{C_{[1]}, p_{[11]} + H(p_{[12]}, f(\beta_1 p_{[1,l]}), g(2))\}
$$
$$
+ H(p_{[22]}, f(\beta_1 p_{[2,1]}), g(2))
$$
$$
= p_{[11]} + H(p_{[12]}, f(\beta_1 p_{[1,l]}), g(2))
$$
$$
+ H(p_{[22]}, f(\beta_1 p_{[2,1]}), g(2)),
$$
$$
\vdots
$$

$$
C_{[k+1]} = \max \left\{ C_{[k]}, \sum_{r=1}^{k+1} H\left(p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]}\right), g(r)\right) \right\}
$$
$$
+ H\left(p_{[2,k+1]}, f\left(\sum_{l=1}^{k} \beta_l p_{[2,l]}\right), g(k+1)\right)
$$
$$
= \sum_{r=1}^{k+1} H\left(p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]}\right), g(r)\right)
$$
$$
+ H\left(p_{[2,k+1]}, f\left(\sum_{l=1}^{k} \beta_l p_{[2,l]}\right), g(k+1)\right),
$$
$$
\vdots
$$
$$
C_{\max} = C_{[n]}
$$
$$
= \max \left\{ C_{[n-1]}, \right.
$$
$$
\left. \sum_{r=1}^{n} H\left(p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]}\right), g(r)\right) \right\}
$$
$$
+ H\left(p_{[2n]}, f\left(\sum_{l=1}^{n-1} \beta_l p_{[2,l]}\right), g(n)\right)
$$
$$
= \sum_{r=1}^{n} H\left(p_{[1r]}, f\left(\sum_{l=1}^{r-1} \beta_l p_{[1,l]}\right), g(r)\right)
$$
$$
+ H\left(p_{[2n]}, f\left(\sum_{l=1}^{n-1} \beta_l p_{[2,l]}\right), g(n)\right).
\tag{27}
$$

Since $\sum_{r=1}^{n} H(p_{[1r]}, f(\sum_{l=1}^{r-1} \beta_l p_{[1,l]}), g(r))$ can be explained as the makespan of the problem $1 \mid p_{1jr} = H(p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}, g(r)), DE \mid C_{\max}$, the schedule minimizes $H(p_{[2n]}, f(\sum_{l=1}^{n-1} \beta_l p_{[2,l]}), g(n))$ simultaneously and the makespan of the problem $F2 \mid p_{1jr} = H(p_{1j}, f(\sum_{l=1}^{r-1} \beta_l p_{[1l]}, g(r)), DE \mid C_{\max}$ is the optimal schedule of the problem $F2 \mid p_{ijr} = H(p_{ij}, f(\sum_{l=1}^{r-1} \beta_l p_{[il]}, g(r)), \min(p_{1.}) \geq \max(p_{2.}), DE \mid C_{\max}$. $\square$

Combining Theorems 3 and 9, we have the following corollary.

Corollary 10. *The problem* $F2 \mid p_{ijr} = p_{ij}[f(\sum_{l=1}^{r-1} \beta_l p_{[il]}) + g(r)], \min(p_{1.}) \geq \max(p_{2.}), DE \mid C_{\max}$ *can be optimally solved by the Longest Processing Time first rule of* p_j *(the LPT rule) if* f' *is nonincreasing on* $[0, +\infty)$.

Similarly, combining Theorems 4 and 9, we can obtain the following result.

Corollary 11. *The problem* $F2 \mid p_{ijr} = p_{ij}f(\sum_{l=1}^{r-1} \beta_l p_{[il]})g(r), \min(p_{1.}) \geq \max(p_{2.}), DE \mid C_{\max}$ *can be optimally solved by*

the Longest Processing Time first rule of p_j (the LPT rule) if f' is nonincreasing on $[0, +\infty)$.

4. Conclusions

This paper considered three scheduling problems with general deterioration models, where the actual processing time function of a job depends not only on the total weighted normal processing times of jobs already processed but also on the scheduled position in the job sequence. Firstly, for two single-machine scheduling problems with general deterioration models associated with both the total weighted normal processing times of the jobs and the scheduled position in the job sequence, we proved that two single-machine scheduling problems are both polynomially solvable under the condition that the derived actual processing time functions are nonincreasing, respectively. Secondly, based on the results of the single-machine problem, we also showed that four flow shop scheduling problems can be optimally solved in polynomial time under proposed conditions.

Future works may focus on developing randomized algorithms for the scheduling problem without pregiven conditions. The variability of tool changes or maintenance activities is also worth considering in scheduling problems with general deterioration effects.

Acknowledgments

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