WHEN IS AMBIGUITY ATTITUDE CONSTANT?*

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Abstract

This paper studies how updating affects ambiguity attitude. In particular we focus on generalized Bayesian updating of the Jaffray-Philippe sub-class of Choquet Expected Utility preferences. We find conditions for ambiguity attitude to be the same before and after updating. A necessary and sufficient condition for ambiguity attitude to be unchanged when updated on an arbitrary event is for the capacity to be neo-additive. We find a condition for updating on a given partition to preserve ambiguity attitude. We relate this to necessary and sufficient conditions for dynamic consistency. Finally, we study whether ambiguity increases or decreases after updating.

Keywords, Ambiguity, Generalized Bayesian update, Learning, dynamic consistency, Choquet Expected Utility.

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INTRODUCTION

How should an individual update her ambiguous beliefs as new information arrives? Much of the previous literature on updating ambiguous beliefs has implicitly or explicitly assumed ambiguity aversion (Eichberger & Kelsey (1996); Epstein & Schneider (2003); Sarin & Wakker (1998).) Less attention, however, has been paid to updating preferences which are not necessarily ambiguity averse. However there is substantial experimental evidence that individuals are not uniformly ambiguity averse but also at times display ambiguity seeking. For a survey of the relevant evidence see Wakker (2010), p. 292.

In this paper we analyse attitudes towards ambiguity and the updating of ambiguous beliefs in the context of the Choquet expected utility (henceforth CEU) model of Schmeidler (1989). CEU represents beliefs by a capacity or non-additive belief. Preferences are represented by the Choquet integral of a utility function with respect to this capacity, Choquet (1953-4). When new information is received we assume that the decision maker updates her capacity but does not change the utility function or the form of the CEU functional. We think that this is reasonable because in our opinion the capacity is the only part of the CEU functional which reflects the decision maker's subjective perception of the environment. The other aspects of the representation are personal characteristics of the decision maker.

At present there is no general agreement about how to update a capacity, although a number of methods for updating capacities have been proposed, e.g., Gilboa & Schmeidler (1993). To the best of our knowledge all of these coincide with Bayesian updating when the capacity is additive. We believe that the most promising is the Generalized Bayesian Updating rule (henceforth GBU) suggested in Eichberger, Grant & Kelsey (2007) and Horie (2007).

Chateauneuf, Eichberger & Grant (2007) axiomatized CEU preferences where beliefs are represented by a special class of capacities known as *neo-additive capacities.* They showed that the Choquet integral of a utility function with respect to a neo-additive capacity can be expressed as the weighted average of the best payoff, the worst payoff and the expected payoff taken with respect to a (conventional) probability. They interpreted this probability as the individual's best guess or 'theory' about the data generating process. The weight placed on the associated expected payoff can be viewed as measuring the individual's degree of 'confidence' in her 'theory'. The complementary weight may then be viewed as her lack of confidence in her 'theory' or, equivalently, as a measure of the degree of ambiguity she perceives to be facing. The fraction of the degree of ambiguity placed on the worst payoff may in turn be viewed as a measure of her degree of pessimism about the ambiguity that she faces, thus encoding her attitude towards ambiguity. Henceforth we shall refer to this class of preferences as *neo-additive preferences.*

In Eichberger, Grant & Kelsey (2010), we showed that the GBU update of a neo-additive capacity is also neo-additive. Moreover, the probability used to compute the expected payoff in the updated neo-additive capacity is the standard Bayesian update of the probability associated with the prior neo-additive capacity. The updated degree of confidence (that is, the weight attached to the expected payoff in the updated Choquet integral) is positively related to the prior probability of the conditioning event. There is no change in the fraction of the degree of ambiguity placed on the worst-payoff. Consequently, the individual's attitude towards ambiguity remains unchanged. We find this particularly appealing. Firstly, it implies that the class of neo-additive preferences is closed under GBU updating. Secondly, neo-additive preferences depend upon the individual's beliefs, any ambiguity or lack of confidence she has in these beliefs and, finally, her attitude to that ambiguity. We argue that the beliefs and the ambiguity constitute a subjective description of the environment. As such it seems reasonable that they should be revised when new information is received. In contrast, ambiguity attitude is a characteristic of the individual, as such it should be invariant to the receipt of new information.

It can be argued that neo-additive preferences are restrictive because they only allow the best and worst outcomes to be over-weighted. Though in most circumstances the worst outcome is presumably death, individuals may also be worried by ambiguous risks concerned with other bad outcomes such as injury or losing a large amount of money. Thus there is a case for examining a more general class of capacities. This paper investigates whether there is a larger class of preferences which are closed under GBU updating and for which ambiguity attitude is invariant under GBU updating. In other words, we require the updated preferences to have the same functional form and the same ambiguity attitude as the original preference. This is reminiscent of the notion from statistics of a conjugate family of probability distributions for which a posterior distribution has the same form as the prior from which it was updated. It is desirable that a class of preferences be closed under updating, since the prior may itself be an update of an earlier belief.

On the other hand, the CEU model with a general capacity is too general since the number of decision weights grows exponentially with the number of states. Therefore, to keep the analysis tractable and to maintain a natural notion of ambiguity and ambiguity attitude, we shall restrict attention to CEU preferences in which the capacity takes the form of a Jaffray-Philippe capacity[1] (henceforth JP-capacity). JP-capacities are generalizations of neo-additive capacities that also allow for a clean separation between an individual's perception of the ambiguity she faces and her attitudes towards it. This can most easily be seen in the multiple priors representation that any CEU preference relation with a JP-capacity admits. That is, for any JP-capacity there is a unique (closed and convex) set of probabilities C such that the Choquet integral with respect to the JP-capacity is a weighted average of the expected value with respect to the most favourable probability from C and the expected value with respect to the least favourable of these probabilities. Thus, for a JP-capacity, the preferences may be represented by the function,

$$V(a) = \alpha \min_{\pi \in \mathcal{C}} \mathbf{E}_{\pi} u(a(s)) + (1-\alpha) \max_{\pi \in \mathcal{C}} \mathbf{E}_{\pi} u(a(s)),$$

where a denotes a state-contingent outcome (or 'act'), E_{π} denotes expectation with respect to the probability distribution π, and α is a weight in the unit interval. The set C can be interpreted

[1] Jaffray & Philippe (1997) provide a characterization of JP-capacities and the associated Choquet integral in terms of upper and lower probabilities. To the best of our knowledge there is no behavioural axiomatization of CEU with JP-capacities other than for the special case of neo-additive capacities Chateauneuf et al. (2007).

as a set of 'theories' the individual considers possible for the data generating process. Loosely speaking, the larger the set C (that is, the more probability distributions are contained in it) the greater is the degree of ambiguity the individual perceives she is facing. The weight α given to the expectation with respect to the least favourable probability can be interpreted as a measure of her attitude towards ambiguity. The larger α, the smaller is the minimum certain pay-off she would be willing to accept in exchange for an act with ambiguous pay-offs.[2]

We begin by considering the problem of updating on an arbitrary event. In this case, we require the update of a JP-capacity to have the same ambiguity attitude when updated on any non-trivial event. We find that, under some mild assumptions, this property holds if and only if the original preferences can be represented by a neo-additive capacity.[3] This result provides a new characterization of neo-additive capacities.

It can be argued that it is too strong to require updates on *all* events to preserve ambiguity attitude. It may be sufficient to require that ambiguity attitude is preserved for the events at which an individual actually has to make decisions. To model this, we consider the case where there is a given partition of the state space. Ex-post it will be revealed in which element of the partition the true state lies. Many interesting economic problems can be seen as special cases of this framework. For instance, if an individual faces a fixed decision tree then she will only need to update beliefs on events which may be reached in that tree. It is not necessary to consider updates conditional on other events. Signalling models are also a special case, since information about the true element of a partition can be viewed as a signal. In practice, many experimental tests of updating have this form, Cohen, Gilboa, Jaffray & Schmeidler (2000).

For updating on a partition, we find that a larger subclass of JP-capacities has ambiguity attitude preserved by updating. If the GBU-updates have the same ambiguity attitude as the

[2] At present there are some unresolved issues concerning how to separate perceptions of ambiguity from ambiguity attitude. Ghirardato, Maccheroni & Marinacci (2004) present an alternative way to differentiate between ambiguity and ambiguity attitude. But Eichberger, Grant, Kelsey & Koshevoy (2011) show that for a finite state space, Ghirardato, Macheroni and Marinacci's separation implies that the capacity cannot exhibit a constant attitude towards ambiguity. Klibanoff, Mukerji & Seo (2011) and Wakker (2011) present theories of ambiguity and ambiguity attitude, which are closer to the interpretation in the present paper.

[3] This is close to being a converse to Proposition 1 of Eichberger et al. (2010).

original preferences, the prior beliefs must lie in a sub-class of JP-capacities we refer to as partition-additive JP-capacities (henceforth PAJP-capacities). These capacities are additive over events from the partition, provided their union is not the whole space.[4] However they may be non-additive over other events.

To understand these results it is helpful to think of ex-ante ambiguity as being derived from two sources. There may be ambiguity about which state will be observed and about how informative a signal about the true state is. The information which the signal conveys may itself be more or less ambiguous. We find that an increase in either source of ambiguity can increase ex-post ambiguity. However, a realization of the signal which confirms the prior belief will tend to reduce ambiguity, while an unexpected signal realization will increase ambiguity. Hence, we believe that for this class of capacities, GBU updating has intuitive properties.

Finally, we find necessary and sufficient conditions for JP-capacities to be dynamically consistent under GBU updating. We show these are only slightly stronger than the conditions for ambiguity attitude to be invariant when updating. If an individual's ambiguity attitude changes after updating, then it is very likely that she will be dynamically inconsistent. Our result shows that "most" dynamic inconsistencies have this form.

The appendix contains the proofs of any results that do not appear in the text.

1 FRAMEWORK AND DEFINITIONS

1.1 Choquet Expected Utility

Let S denote a state space, which we take to be finite. The set of consequences X, is assumed to be a convex subset of $\mathbb{R}^n$. An act is a function $a : S \to X$. Let $A(S)$ denote the space of all acts. The decision maker has a preference relation $\succcurlyeq$ defined over $A(S)$. We shall assume that $\succcurlyeq$ satisfies the CEU axioms, Schmeidler (1989), Sarin & Wakker (1992). The CEU model of ambiguity represents beliefs as capacities, which are defined as follows.

[4] As with a number of other decision theories, the certain event has a special status in this model.

Definition 1.1 *A* capacity *on S is a function $\nu : 2^S \to \mathbb{R}$ such that $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$ and $\nu(\varnothing) = 0$, $\nu(S) = 1$, where 2^S denotes the set of all subsets of S. The* dual capacity *$\bar{\nu}$ is defined by $\bar{\nu}(A) = 1 - \nu(A^c)$.*

The capacity and its dual are two alternative ways of representing the same information. A third and sometimes convenient way to encode the information contained in a capacity is by way of its Möbius inverse.

Definition 1.2 *Let ν be a capacity on S. The* Möbius inverse *of ν is a function $\beta : 2^S \to \mathbb{R}$ defined by $\beta_E = \sum_{D \subseteq E} (-1)^{|E|-|D|} \nu(D)$.*

The Möbius inverse has the property that $\nu(A) = \sum_{B \subseteq A} \beta_B$ and $\sum_{B \subseteq S} \beta_B = 1$. In the sequel we shall define some examples of capacities in terms of their Möbius inverses. In addition, some proofs in the appendix proceed by demonstrating the requisite properties of the associated Möbius inverses. In the CEU model, preferences over $A(S)$ are represented by the Choquet expected utility of an act a.

Definition 1.3 *The Choquet expected utility of an act a with respect to the capacity ν is defined as*

$$\int (u \circ a) \ d\nu = \int_0^\infty \nu(\{s \in S| \ u(a(s)) \geqslant t\})dt + \int_{-\infty}^0 [\nu(\{s \in S| \ u(a(s)) \geqslant t\}) - 1] \ dt,$$

where $u : X \to R$ denotes the von Neumann-Morgenstern utility function.

The class of convex capacities is of particular interest for us.

Definition 1.4 *A capacity, μ, is* convex *if $\nu(A \cup B) \geqslant \nu(A) + \nu(B) - \nu(A \cap B)$.*

As is well-known, for any convex capacity there exists at least one probability distribution that dominates it. The set of such dominating probability distributions is referred to as the core of the capacity. More formally, we have:

Definition 1.5 *Let ν be a capacity on S. The* core, $\mathcal{C}(\nu)$, *is defined by*

$$\mathcal{C}(\nu) = \{p \in \Delta(S); \forall A \subseteq S, p(A) \geqslant \nu(A)\}.$$

A special subclass of convex capacities are the belief functions.

Definition 1.6 *A capacity ν on S is a* belief function *if for all $A_1, ..., A_m \subseteq S$;*

$$\nu\left(\bigcup_{i=1}^{m} A_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \varnothing}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right)$$

for all $m, 2 \leqslant m \leqslant \infty$.

Convexity is the special case where this property is only required to hold for $m = 2$. One can show that a capacity is a *belief function* if and only if its Möbius inverse is non-negative, that is, for all $B \subseteq S, \beta_B \geqslant 0$, Dempster (1967), Shafer (1976).

1.2 Jaffray-Philippe Capacities

This section introduces the class of JP-capacities which will prove important in our analysis. Jaffray & Philippe (1997) study capacities which may be written as a convex combination of a convex capacity μ and its dual. We shall restrict attention to JP-capacities since there is a natural way to distinguish between the perception of ambiguity and the attitude towards this ambiguity for such capacities. Note that here we only study deviations from expected utility due to ambiguity. In other words we assume that decision makers use expected utility when probabilities are known. JP-capacities are formally defined as follows.

Definition 1.7 *A capacity ν on S is a* JP-capacity *if there exists a convex capacity μ and $\alpha \in [0,1]$, such that $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$.*

A special class of JP-capacities are the *Hurwicz capacities.*

Definition 1.8 *The Hurwicz capacity with parameter* α *is a JP-capacity* ν^H *with the convex capacity* $\mu^H(A) = 0$*, for all* $A \subsetneqq S$*,* $\mu^H(S) = 1$*; i.e.*

$$\nu^H(A) = \alpha\mu^H(A) + (1-\alpha)\bar{\mu}^H(A) = 1-\alpha \quad \textit{for all } A \subsetneqq S.$$

We take the degree of ambiguity associated with the JP-capacity to correspond to standard measures of ambiguity for convex capacities.

Definition 1.9 *Let* μ *be a convex capacity on* S*. Define the degree of ambiguity of event* A *associated with the capacity* μ *by:* $\chi(\mu, A) = \bar{\mu}(A) - \mu(A)$*, and the maximal degree of ambiguity associated with* μ *by*

$$\lambda(\mu) = \max\{\chi(\mu, A) : \varnothing \subsetneqq A \subsetneqq S\}.$$

This definition is based on one in Dow & Werlang (1992). It provides an upper bound on the amount of ambiguity which the decision maker perceives. The degree of ambiguity measures the deviation from event-wise additivity. For a probability it is equal to zero. Convex capacities have degrees of ambiguity between 0 and 1, with higher values corresponding to more ambiguity. For a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, we apply this definition to the convex part μ.

As the following proposition shows, the CEU of a JP-capacity is a convex combination of the minimum and the maximum expected utility over the set of probabilities in the core of μ.

Proposition 1.1 *(Jaffray & Philippe (1997)) The CEU of an act* a *with respect to a* JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ *on* S *is:*

$$\int u(a(s))\,d\nu(s) = \alpha \min_{\pi \in \mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)) + (1-\alpha) \max_{\pi \in \mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)).$$

If beliefs may be represented by JP-capacities, preferences lie in the intersection of the CEU and multiple priors models. Proposition 1.1 suggests an interpretation of the parameter α as a degree of (relative) pessimism, since it gives a weight to the worst expected utility an individual could expect from the act a. If $\alpha = 1$, then we obtain a special case of the MEU model axiomatized by Gilboa & Schmeidler (1989). On the other hand, the weight $(1 - \alpha)$ given to the best expected utility which an individual can obtain with act a provides a natural measure for her optimism. For $\alpha = 0$ we have a pure optimist, while in general for $\alpha \in (0, 1)$, the individual's preferences have both optimistic and pessimistic features. Ambiguity may be measured by the core of the convex capacity μ. A larger core corresponds to a situation, which is perceived to be more ambiguous. Hence, JP capacities allow a distinction between ambiguity and ambiguity attitude.

The neo-additive capacity defined below is another special class of JP-capacities, which will prove useful in our analysis.

Definition 1.10 *Let α, δ be real numbers such that $0 < \delta < 1, 0 < \alpha < 1$. A neo-additive-capacity ν on S is defined by $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A)$, for $\emptyset \subsetneqq A \subsetneqq S$, where π is an additive probability distribution on S.*

Formally, a neo-additive capacity is formed by taking a $(\delta, 1 - \delta)-$ convex combination of a Hurwicz capacity with parameter α and a probability distribution π. This capacity can be interpreted as describing a situation where the decision maker's 'beliefs' are represented by the probability distribution π. However she may have some doubts about these beliefs. This ambiguity about the true probability distribution is reflected by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The reaction to these doubts is in part pessimistic and in part optimistic. As is the case for JP-capacities in general, the ambiguity attitude associated with the neo-additive capacity may be measured by the parameter α. The Choquet expected utility of an act a with respect to the neo-additive-capacity ν is given by

$$\int u(a(s))\, d\nu(s) = \delta\alpha \min_{s \in S} u(a(s)) + \delta(1 - \alpha) \max_{s \in S} u(a(s)) + (1 - \delta) \cdot \mathbf{E}_\pi u(a(s)). \quad (1)$$

That is, the Choquet integral for a neo-additive capacity is a weighted average of the highest payoff, the lowest payoff and the average payoff with respect to π.

1.3 Generalized Bayesian Updating Rule

The following rule for updating a capacity has been axiomatized in Eichberger et al. (2007) and Horie (2007).

Definition 1.11 *Let ν be a capacity on S and let $E \subseteq S$. The* Generalized Bayesian Update (henceforth GBU) *of ν conditional on E is given by:*

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} = \frac{\nu(A \cap E)}{\nu(A \cap E) + \bar{\nu}(A^c \cap E)}.$$

It is straightforward to check that the GBU rule coincides with Bayesian updating when beliefs are additive. In Eichberger et al. (2010) we show that GBU applied to updating neo-additive capacities leaves the ambiguity attitude parameter α unchanged. In the present paper we investigate when a similar result applies to the larger class of JP-capacities.

2 UPDATING ON AN EVENT

We begin with a necessary and sufficient condition for the update of a JP-capacity to have the JP form with the same ambiguity attitude parameter α.

Lemma 2.1 *Let μ be a given convex capacity on S. Define $\nu^\alpha = \alpha\mu + (1-\alpha)\bar{\mu}$. Consider a given event E. Then a necessary and sufficient condition for the GBU update of ν^α conditional on E to be a JP capacity with the same α, for all α, $0 \leqslant \alpha \leqslant 1$, i.e. $\nu_E^\alpha = \alpha\mu_E + (1-\alpha)\bar{\mu}_E$,[5] is that for all partitions A, B of E, $A \cup B = E$, $A \cap B = \varnothing$:*

$$\mu(A \cup E^c) - \mu(A) = \mu(B \cup E^c) - \mu(B). \tag{2}$$

[5] To clarify we require this equation to hold for *all* $\alpha, 0 \leqslant \alpha \leqslant 1$ but only for the *given* event E. The capacity μ_E depends on E but is independent of α.

Remark 2.1 *A sufficient condition for equation (2) to be satisfied is for all* $F \subseteq E$, $\mu(F \cup E^c) = \mu(F) + \mu(E^c)$. *This condition is not necessary. However in practice it may be easier to check than the necessary and sufficient condition.*

If we strengthen our assumptions by requiring μ to be a belief function then we can show that a necessary and sufficient condition for ambiguity attitude to be the same before and after GBU updating is that the capacity be neo-additive. Thus we provide the converse to Proposition 1 of Eichberger et al. (2010) for the case where μ is a belief function.

Proposition 2.1 *Let* $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ *be a JP-capacity where* μ *is a* belief function *on* S, $0 \leqslant \alpha \leqslant 1$ *and* $|S| \geqslant 4$. *Let* ν_E *denote the GBU update of* ν *conditional on* E. *Then a necessary and sufficient condition for* ν_E *to be a JP-capacity with the same* α *for all* $E \subsetneqq S$ *is that* ν *be neo-additive.*

In practice, the condition $|S| \geqslant 4$ is not restrictive. If there are three or fewer states then after updating at most two states will remain possible. If there are only two states, JP-capacities are over-determined. Thus four states is the minimum needed to have a meaningful updating problem. The following example shows that there exists a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, which is not neo-additive even though μ satisfies equation (2). Since μ is convex but not a belief function this demonstrates that it is not possible to drop that requirement in the above result.

Example 2.1 *Suppose there are* 4 *states. The example is a symmetric capacity on* S, *by which we mean that the capacity of an event only depends on the number of states in the event. We adopt the notation that* $\mu(m)$ *denotes the capacity of an event with* m *states. Choose* $\eta < \frac{1}{4}$ *and* $\epsilon < \frac{1}{4} - \eta$. *Let* $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, *where* μ *is the symmetric capacity given by* $\mu(0) = 0$, $\mu(1) = \eta$, $\mu(2) = 2\eta + \epsilon$, $\mu(3) = 3\eta + 2\epsilon$ *and* $\mu(4) = 1$. *Then as Proposition A.1 shows, the updates of* ν *have the JP-form with the same* α, *however* ν *is not neo-additive. The capacity* μ *is convex but is not a belief function.*

3 LEARNING FROM SIGNALS

In this section, we consider the problem of updating beliefs on the events of a given partition of the state space. As argued in the introduction, the problem of updating on a signal can be interpreted as a special case of this. In this context, we find that a sufficient condition for ambiguity attitude to be the same before and after updating is that the prior capacity lies in a class of capacities we refer to as PAJP-capacities (defined below). Under some assumptions, we show that this condition is also necessary.

Let $E_1, ..., E_K$ be a partition of S. There are two time periods $t = 0$ and $t = 1$. The decision maker has initial beliefs at time $t = 0$. At time $t = 1$, she observes which element of the partition obtains and updates her beliefs. We shall use terminology appropriate to the problem of updating on a signal. Thus we shall refer to $E_1, ..., E_K$ as *signal realizations*. However, the analysis of updating a capacity on a partition is applicable more generally.

3.1 Partition-Additive JP-Capacities

Below we define a subclass of JP-capacities, which we call *partition-additive JP-capacities* (PAJP).

Definition 3.1 *A capacity* ν *is a* partition-additive JP-capacity *(PAJP) if it has the form* $\nu = \alpha\mu + (1 - \alpha)\bar{\mu}$, *where* μ *is a convex capacity defined by:*

$$\mu(D) = (1 - \delta) \sum_{k=1}^{K} q_k \mu_k (D \cap E_k), D \subsetneq S; \qquad \mu(S) = 1, \tag{3}$$

where $0 < \delta < 1, q$ *is a probability distribution over the elements of the partition* $\{E_1, ..., E_K\}$ *and* μ_k *is a convex capacity on* E_k *for all* $1 \leqslant k \leqslant K$.

The capacity value of an event D, $\mu(D)$, may be viewed as the fraction $(1 - \delta)$ of the expected value of the event D according to the capacities μ_k. These are defined on the elements of the partition for which the signal is measurable. Notice that for $\delta = 0$ the capacity $\mu(D)$ is equal to this expectation, i.e., the capacity is additive over the partition. If $\delta \in (0,1)$, then for all non-empty events $D \subsetneqq S$, the dual capacity $\bar{\mu}$ is given by $\bar{\mu}(D) = 1 - (1-\delta)\sum_{k=1}^{K} q_k \mu_k (D^c \cap E_k) = \delta + (1-\delta)\sum_{k=1}^{K} q_k \bar{\mu}_k (D \cup E_k^c)$, since $\bar{\mu}_k(E_k) = 1$. Finally $\bar{\mu}(D) = \delta + (1-\delta)\sum_{k=1}^{K} q_k \bar{\mu}_k (D \cap E_k)$. Thus we obtain for all non-empty events $D \subsetneqq S$,

$$\begin{aligned} \nu(D) &= (1-\alpha)\delta + (1-\delta)\sum_{k=1}^{K} q_k \left[\alpha \mu_k (D \cap E_k) + (1-\alpha)\bar{\mu}_k (D \cap E_k)\right] && (4) \\ &= (1-\delta)\sum_{k=1}^{K} q_k \nu_k (D \cap E_k) + \delta(1-\alpha), && (5) \end{aligned}$$

where $\nu_k = \alpha\mu_k + (1-\alpha)\bar{\mu}_k$ is, by construction, a JP-capacity on E_k. Further straightforward calculation yields, for all non-empty events $D \subsetneqq S$,

$$\bar{\nu}(D) = (1-\delta)\sum_{k=1}^{K} q_k \bar{\nu}_k (D \cap E_k) + \delta\alpha. \quad (6)$$

Expression (5) can be viewed as saying that the weight assigned to event D by ν, is a convex combination of the weight assigned by $K+1$ capacities. The expectation of the K JP-capacities ν_k defined on the elements of the partition on which the signal is measurable is weighted by $(1-\delta)$. The complementary fraction δ is reserved for the Hurwicz capacity of complete uncertainty, i.e. a JP-capacity with $\mu(E) = 0$ for all $E \subsetneqq S$ that puts weight α on the worst outcome and weight $(1-\alpha)$ on the best outcome.[6]

3.2 Updating Partition-Additive JP-Capacities

The following result finds the GBU update of a PAJP-capacity ν. In particular, we see that for each possible realization of the signal the update is a JP capacity with the same α. Thus the effect of updating is to revise μ in the light of the new information, while leaving the ambiguity attitude unchanged.

[6] This is reminiscent of the neo-additive capacities introduced by Chateauneuf et al. (2007). Indeed if all the μ_k's are additive (that is, are conditional probabilities and, hence, $\bar{\mu}_k = \mu_k$, for all k) then using expression (4) we see that such an PAJP capacity belongs to the class of neo-additive capacities in which for all non-empty events $D \subsetneqq S$, $\nu(D) = (1-\delta)p(D) + \delta(1-\alpha)$, where p is an unconditional probability given by $p(D) = \sum_{k=1}^{K} q_k \mu_k (D \cap E_k)$.

Proposition 3.1 *The GBU update of the PAJP-capacity, ν conditional on event E_k is given by:* $\hat{\nu}_k(A) = \left(1 - \hat{\delta}\right) \nu_k(A \cap E_k) + \hat{\delta}(1 - \alpha)$

$$= \alpha \left(\frac{(1-\delta) q_k \mu_k (A \cap E_k)}{\delta + (1-\delta) q_k} \right) + (1-\alpha) \left(1 - \frac{(1-\delta) q_k \mu_k (A^c \cap E_k)}{\delta + (1-\delta) q_k} \right),$$

where

$$\hat{\delta} = \frac{\delta}{\delta + (1-\delta) q_k} \geqslant \delta, \tag{7}$$

with the inequality strict whenever $q_k < 1$. The convex component of the updated JP-capacity is given by

$$\mu'_k(A) = \frac{(1-\delta) q_k \mu_k (A \cap E_k)}{\delta + (1-\delta) q_k}. \tag{8}$$

If we further restrict the model by requiring μ to be a belief function then it follows that a necessary and sufficient condition for the GBU updates on a partition to have the same ambiguity attitude as the prior belief, is that the capacity be a PAJP-capacity.

Proposition 3.2 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where μ is a* belief function *on S and $0 \leqslant \alpha \leqslant 1$. Assume that $|E_k| \geqslant 3$, for $1 \leqslant k \leqslant K$. Let ν_{E_k} denote the GBU update of ν conditional on E_k. Then a necessary and sufficient condition for ν_{E_k} to be a JP-capacity with the same α for $1 \leqslant k \leqslant K$, is that ν be a PAJP capacity, i.e. there exists a belief function μ_k on E_k, an additive probability distribution q on $\{1, ..., K\}$ and a number $\delta, 0 \leqslant \delta \leqslant 1$ such that for $A \subsetneqq S$:*

$$\mu(A) = (1-\delta) \sum_{k=1}^{K} q_k \mu_k (A \cap E_k), \qquad \mu(S) = 1.$$

Example 2.1 in combination with Proposition A.1 Part 3, may serve to show that it is not possible to drop the assumption $|E_k| \geqslant 3$, in Proposition 3.2.

3.3 Ex-Ante and Ex-Post Ambiguity

Let us now consider how updating affects perceived ambiguity. Recall that a PAJP-capacity is a convex combination of the belief part μ^H of a Hurwicz capacity representing ambiguity about the states and K convex capacities which represent ambiguous beliefs about the signals. Before updating, the degree of ambiguity is a similar convex combination of the degree of ambiguity of the Hurwicz capacity, $\lambda(\mu^H) = 1$, receiving weight δ and K degrees of ambiguity of the signals $\lambda\left(\mu_j\right)$, receiving weight $(1-\delta)\,q_j$. Now suppose signal E_k is observed. Ex-post $K-1$ of the signals are no longer possible. Thus the updated beliefs are represented by a capacity which is a convex combination of the belief part of a Hurwicz capacity and the one signal capacity which is realized. Correspondingly, the ex-post degree of ambiguity is a convex combination of the degree of ambiguity of the Hurwicz capacity (i.e. $\lambda(\mu^H) = 1$) and that of the signal actually observed, $\lambda\left(\mu_k\right)$. The following result finds expressions for ex-ante and ex-post ambiguity.

Proposition 3.3 *Let $\nu = \alpha\mu + (1-\alpha)\,\bar{\mu}$ be a PAJP capacity, where*

$$\mu\left(A\right) = (1-\delta)\sum_{j=1}^{K} q_j\mu_j\left(A \cap E_j\right) \ \ \textit{for } A \subsetneqq S.$$

1. *The* ex-ante *degree of ambiguity of ν is $\lambda\left(\mu\right) = \delta + (1-\delta)\sum_{j=1}^{K} q_j\lambda\left(\mu_j\right)$.*

2. *If event E_k is observed then the* ex-post *degree of ambiguity is,*

$$\lambda\left(\mu_k'\right) = \frac{\delta}{\delta + (1-\delta)\,q_k} + \frac{(1-\delta)\,q_k}{\delta + (1-\delta)\,q_k}\lambda\left(\mu_k\right). \tag{9}$$

In the ex-ante degree of ambiguity, $\lambda(\mu^H) = 1$ received weight δ and $\lambda\left(\mu_k\right)$ gets weight $(1-\delta)\,q_k$. For the ex-post degree of ambiguity the weights have been renormalized to ensure that they sum to unity. Ex-post the weight on the Hurwicz capacity is greater. Thus for ex-post ambiguity to be lower it is necessary for the second term to be smaller to off-set this effect.

3.4 Comparative Statics

The comparative statics of updating are intuitive. Consider equation (9). As one would expect, ex-post ambiguity is increasing in the ambiguity of the observed signal, i.e. the greater is $\lambda(\mu_k)$, the higher is ex-post ambiguity.

Ex post, the ambiguity is a convex combination of 1 and $\lambda(\mu_k)$. The weight on 1 is $\frac{\delta}{\delta+(1-\delta)q_k}$ while that on $\lambda(\mu_k)$ is $\frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}$. Note that $1 \geqslant \lambda(\mu_k)$. Increasing δ (resp. decreasing q_k) decreases the weight on $\lambda(\mu_k)$ and increases the weight on 1 in the convex combination. Thus ex-post ambiguity is increasing in δ and decreasing in q_k. This reflects the following intuition: The higher is δ the more ex-ante ambiguity there is over the states. As one would expect this increases ex-post ambiguity. The smaller is q_k the more unlikely is the signal realization k. Seeing an unlikely realization of the signal k increases ambiguity. The result below proves this formally.

Proposition 3.4 *Ex post ambiguity $\lambda(\mu'_k)$ is increasing in δ and decreasing in q_k.*

3.5 Examples

We now wish to investigate the factors which determine whether ambiguity increases or decreases after updating. For illustrative purposes we shall consider three special cases.

Case 1 **The prior over the state space is unambiguous.** This implies that the only source of ambiguity comes from the signals, i.e. $\delta = 0$. Thus, $\lambda(\mu) = \sum_{j=1}^{K} q_j \lambda(\mu_j)$ and $\lambda(\mu'_k) = \lambda(\mu_k)$. The degree of ambiguity increases/decreases as $\lambda(\mu_k) \gtrless \sum_{j=1}^{K} q_j \lambda(\mu_j)$. If the signal realization k is less ambiguous than the ex-ante expected ambiguity over signal realizations, then ambiguity will decrease after updating.

When the only source of ambiguity are the realizations of the signals, observing one of the less ambiguous realizations will reduce ambiguity. This makes intuitive sense since there is no longer the possibility of being exposed to the more ambiguous realizations. By continuity, updating will have similar properties when δ is small, i.e., if there is little ambiguity about the prior over the state space.

Case 2 **The signal realization is unambiguous,** $\lambda(\mu_k) = 0$. Ex-ante ambiguity is given by, $\lambda(\mu) = \delta + (1-\delta)\sum_{j\neq k} q_j \lambda(\mu_j)$. Ex-post ambiguity is given by, $\lambda(\mu'_k) = \frac{\delta}{\delta+(1-\delta)q_k}$. For ambiguity to be lower ex-post, we require $\lambda(\mu) - \lambda(\mu'_k) \geqslant 0$. Now $\lambda(\mu) - \lambda(\mu'_k) = (1-\delta)\frac{\delta q_k - \delta + (\delta+(1-\delta)q_k)\sum_{j\neq k} q_j\lambda(\mu_j)}{\delta+(1-\delta)q_k}$. Hence

$$\lambda(\mu) - \lambda(\mu'_k) = (1-\delta)\frac{(\delta + (1-\delta)q_k)\sum_{j\neq k} q_j\lambda(\mu_j) - \delta(1-q_k)}{\delta + (1-\delta)q_k}. \tag{10}$$

Thus $\lambda(\mu) \gtreqless \lambda(\mu'_k)$ as $\frac{1}{(1-q_k)}\left(\sum_{j\neq k} q_j\lambda(\mu_j)\right) \gtreqless \frac{\delta}{(\delta+(1-\delta)q_k)}$. The left-hand side of this inequality is the ex ante expected ambiguity of the signal realizations and the right-hand side is what the ambiguity of the states ex post would be if none of the realizations were ambiguous.

An interesting sub-case is where the observed signal was unambiguous, while all the other signals have the maximal degree of ambiguity, that is, $\lambda(\mu_k) = 0$, $\lambda(\mu_j) = 1$, $j \neq k$. In this case, we have $\sum_{j\neq k} q_j\lambda(\mu_j) = (1-q_k)$. Hence from equation (10), $\lambda(\mu) - \lambda(\mu'_k) = (1-\delta)(1-q_k)\frac{(\delta+(1-\delta)q_k)-\delta}{\delta+(1-\delta)q_k} = \frac{(1-\delta)^2 q_k(1-q_k)}{\delta+(1-\delta)q_k} > 0$. Thus, in this extreme case, observing the realization of the least ambiguous signal always decreases ambiguity. By continuity, if an individual observes the realization of a signal with much lower ambiguity than the other possible realizations ambiguity will be reduced.

Case 3 **All signal realizations are equally ambiguous.** If all signals have the same degree of ambiguity, then $\lambda(\mu_j) = \lambda$, $1 \leqslant j \leqslant K$ for some λ, $0 \leqslant \lambda \leqslant 1$. One may see that ambiguity always rises in this case. Ex-ante ambiguity is given by $\lambda(\mu) = \delta + (1-\delta)\lambda$, while ex-post ambiguity is given by $\lambda(\mu'_k) = \frac{\delta}{\delta+(1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}\lambda$. Both are convex combinations of 1 and λ. Since $\frac{1}{\delta+(1-\delta)q_k} > 1$, the weight on 1 has increased in the expression for the degree of ex-post ambiguity. Thus, provided $\delta > 0$, ex-post ambiguity is always larger than ex-ante ambiguity when all signals are equally ambiguous.

To summarise, ambiguity is more likely to be lower after updating:

1. the smaller is the ambiguity of the states, i.e. δ;

2. if the observed signal was less ambiguous than average;

3. the more likely, ex-ante, the signal realization was, (i.e. the higher is q_k).

3.6 Dynamic Consistency

In previous sections we have explored the implications of keeping ambiguity attitude the same before and after updating. This can be viewed as a weak form of dynamic consistency. The condition is clearly necessary but not sufficient for dynamic consistency. Here we explore the relation between this condition and full dynamic consistency. We find that the necessary and sufficient conditions for dynamic consistency are only slightly stronger. We view this as intuitive since changes in ambiguity attitude when updating are likely to be a major cause of dynamic inconsistency. First we define dynamic consistency.

Definition 3.2 *Preferences are said to be* dynamically consistent *with respect to a partition,* $E_1, ..., E_K$ *if* $\int u(a_k)\, d\nu_{E_k} \geqslant \int u(b_k)\, d\nu_{E_k}$, *for* $1 \leqslant k \leqslant K$*; implies* $\int u(a)\, d\nu \geqslant \int u(b)\, d\nu$*. Here* $a_k \in A(E_k)$ *denotes the restriction of act* a *to the event* E_k*, where* $A(E_k)$ *denotes the set of all acts on* E_k*, i.e. the set of all functions* $f : E_k \to X$*.*

This says that if a given act is never optimal in the second period the individual will not choose that act in the first period.

We now make some additional assumptions of a technical nature. In particular, we assume that the utility function is continuous and that no state is null in the sense that increasing the utility in that state, while not decreasing it in any other state, will lead to a strictly preferred option.

Assumption 3.1 *The utility function* $u : X \to \mathbb{R}$ *is continuous.*

Assumption 3.2 (Strong Monotonicity) *For two acts $a, b \in A(S)$, if $\exists \hat{s} \in S$, such that $u(a(\hat{s})) > u(b(\hat{s}))$ and $\forall s \in S$, $u(a(s)) \geqslant u(b(s))$ then $a \succ b$.*

We shall also restrict attention to dynamic settings in which uncertainty about the realization of the signal is non-trivial and where no realization of a signal fully resolves all uncertainty. That is, the signal partition consists of at least two non-empty sets and no element of the partition is a singleton.

Definition 3.3 *We say that the partition $E_1, ..., E_K$ is non-trivial, if $K \geqslant 2$ and $|E_k| \geqslant 2$, for $1 \leqslant k \leqslant K$.*

The next result finds a necessary and sufficient condition for dynamic consistency. The convex part of the JP-capacity must be additive over the given partition.

Proposition 3.5 *Let $E_1, ..., E_K$ be a non-trivial partition of S. If a decision maker has CEU preferences, which satisfy Assumptions 3.1 and 3.2 with beliefs represented by a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where $\alpha \neq \frac{1}{2}$, and she updates her preferences with GBU updating then the following conditions are equivalent:*[7]

1. *The decision maker is dynamically consistent.*

2. $\sum_{k=1}^{K} \mu(E_k) = 1$.

Both the result and the proof are extensions of Theorem 2.1 in Eichberger, Grant & Kelsey (2005) who, in turn, extended an earlier result in Sarin & Wakker (1998). The main difference is that we have dropped the assumption of ambiguity aversion made in the earlier paper. In the case where $\alpha = \frac{1}{2}$ the sufficiency proof still holds, however we conjecture that this condition is no longer necessary for dynamic consistency. For related research see Dominiak & Lefort (2011) especially Proposition 5.1 and Theorem 5.1.

[7] We do not use the full strength of Assumption 3.2. In fact we only need it to apply to the events C and D in the proof of Proposition 3.5.

Keeping ambiguity attitude unchanged when updating can be seen as a weak form of dynamic consistency. Propositions 3.2 and 3.5 reveal that $\delta = 0$ is the only additional restriction imposed by full dynamic consistency. This implies that first period beliefs are additive or that there is no ambiguity about which signal we shall see. It is clear that changes in ambiguity attitude could be a source of dynamic inconsistency. For $\delta = 0$, changes in ambiguity attitude are the only reason for violations of dynamic consistency.

4 CONCLUSION

This paper studies learning and ambiguity. We have extended previous work on updating ambiguous beliefs by allowing for the possibility of ambiguity seeking behaviour in some choices and ambiguity aversion in others. The main principle used in this paper is that ambiguity attitude should be preserved by updating, while beliefs and perceptions of ambiguity may be revised when new information is received. We believe the principle that updating should not change ambiguity attitude, may be applicable more generally, for instance to other models of ambiguity or to behaviour in games.

One might note that there is a connection between the results of section 2 and 3. If we consider updates on an arbitrary event, the condition for dynamic consistency is that beliefs be additive and the condition for ambiguity attitude to be constant is that beliefs be neo-additive. If we only consider updates on a given partition, the condition for dynamic consistency is that beliefs be additive over that partition and the condition for ambiguity attitude to be constant is that they be neo-additive over the partition.[8]

A PROOFS

This appendix contains proofs of those results not proved in the text.

Proof of Lemma 2.1 Consider $A \subsetneq E$, then

[8] By neo-additive we mean additive except on events where extreme outcomes occur.

$\nu_E(A) = \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))+1-\alpha\mu(A\cup E^c)-(1-\alpha)(1-\mu((A\cup E^c)^c))}$

$= \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha[\mu(A)-1+\mu(B\cup E^c)-\mu(A\cup E^c)+(1-\mu(B))]+1-\mu(B\cup E^c)+\mu(B)}$

$= \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(B\cup E^c))}{\alpha[\mu(A)-\mu(A\cup E^c)+\mu(B\cup E^c)-\mu(B)]+1-\mu(B\cup E^c)+\mu(B)}.$

Sufficiency If $\mu(A\cup E^c) - \mu(A) = \mu(B\cup E^c) - \mu(B)$, then

$$\nu_E(A) = \frac{\alpha\mu(A)}{1-\mu(B\cup E^c)+\mu(B)} + (1-\alpha)\left(\frac{1-\mu(B\cup E^c)}{1-\mu(B\cup E^c)+\mu(B)}\right) \quad (11)$$
$$= \frac{\alpha\mu(A)}{1-\mu(A\cup E^c)+\mu(A)} + (1-\alpha)\left(1-\frac{\mu(E\backslash A)}{1-\mu(A\cup E^c)+\mu(A)}\right),$$

which has the JP form with the same ambiguity attitude parameter α.

Necessity If $\nu_E^\alpha(A) = \alpha\sigma(A) + (1-\alpha)\bar{\sigma}(A)$, where $0 \leqslant \alpha \leqslant 1$ and σ is a convex capacity on E, then $\frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha[\mu(A)-\mu(A\cup E^c)+\mu(B\cup E^c)-\mu(B)]+1-\mu(B\cup E^c)+\mu(B)} = \alpha\sigma(A) + (1-\alpha)\bar{\sigma}(A)$.

This equation has the form $\frac{a\alpha+b}{c\alpha+d} = e\alpha+f$, where $c = \mu(A) + \mu(A^c) - \mu(A\cup E^c) - \mu((A\cup E^c)^c)$, etc. Cross multiplying, $a\alpha + b = \alpha^2 ce + (fc+de)\alpha + fd$. Equating coefficients we obtain: $ce = 0, a = (fc+de), b = fd$.

Unless σ is the complete uncertainty capacity, there exists A such that $\sigma(A) = e \neq 0$, which implies $c = 0$. (Note one can easily show that the result holds if σ is the complete uncertainty capacity.) Hence $\mu(A) - \mu(A\cup E^c) + \mu(B\cup E^c) - \mu(B)$, holds. ■

Proof of Proposition 2.1 Sufficiency follows from Proposition 1 of Eichberger et al. (2010).

Necessity Suppose that μ is a belief function and let β denote the Möbius inverse of μ. It is sufficient to show $\beta_B = 0$ unless $B = S$ or B is a singleton.

Let $\hat{s}$ denote a given state. Let $E = S\backslash\hat{s}$, then $E^c = \{\hat{s}\}$. Take $\sigma \in E$. Let $A = E\backslash\{\sigma\}$ and $B = \{\sigma\}$. Then by equation (2), $\mu(A\cup E^c) - \mu(A) = \mu(B\cup E^c) - \mu(B)$. Rewriting this in terms of the Möbius inverse we obtain: $\sum_{D\subseteq A\cup E^c}\beta_D - \sum_{D\subseteq A}\beta_D = \sum_{D\subseteq \sigma\cup E^c}\beta_D - \beta_\sigma$.

This may be reorganized as,

$\sum_{D\subseteq A}\beta_D+\beta_{\hat{s}}+\sum_{D\subseteq A}\beta_{D\cup\hat{s}}-\sum_{D\subseteq A}\beta_D=\beta_\sigma+\beta_{\hat{s}}+\beta_{\sigma\hat{s}}-\beta_\sigma.$

Simplifying

$$\beta_{\sigma\hat{s}}=\sum_{D\subseteq A}\beta_{D\cup\hat{s}}. \tag{12}$$

Hence $\beta_{\sigma\hat{s}}\geqslant\sum_{s'\neq\hat{s},\sigma}\beta_{s'\hat{s}}$, since we have deleted some non-negative terms from the rhs.

Summing over σ, $\sum_{\sigma\neq\hat{s}}\beta_{\sigma\hat{s}}\geqslant\sum_{\sigma\neq\hat{s}}\sum_{s'\neq\hat{s},\sigma}\beta_{s'\hat{s}}=(n-2)\sum_{s'\neq\hat{s}}\beta_{s'\hat{s}}$.[9] Note that the two sums are identical. Hence if $n\geqslant 4$ this implies $\beta_{s'\hat{s}}=0$ for all $s',\hat{s}\in S$. Substituting into equation (12), $\sum_{D\subseteq A}\beta_{D\cup\hat{s}}=0$. Since $\beta_{D\cup\hat{s}}\geqslant 0$, this implies $\beta_{D\cup\hat{s}}=0$ for all $D\subseteq A$. If we recall that $\hat{s}$ and A were chosen arbitrarily, this establishes that $\beta_G=0$, for all $G, 2\leqslant|G|\leqslant n-1$. ■

Proposition A.1 *If we define a JP-capacity ν, by $\nu=\alpha\mu+(1-\alpha)\bar{\mu}$, where μ is the capacity from Example 2.1 then for all $E\subseteq S$:*

1. *The GBU update ν_E is a JP-capacity with the same ambiguity attitude parameter α. However ν is not neo-additive and μ is not a belief function;*

2. *If E is any 3-element event. Then the GBU update of ν conditional on E is $\nu_E=\alpha\mu_E+(1-\alpha)\bar{\mu}_E$, where μ_E is the symmetric convex capacity on E defined by $\mu_E(0)=0,\mu_E(1)=\frac{\eta}{1-\eta-\epsilon},\mu_E(2)=\frac{2\eta+\epsilon}{1-\eta-\epsilon}$ and $\mu_E(3)=1$;*

3. *If E is any 2-element event, $\nu_E=\alpha\mu_E+(1-\alpha)\bar{\mu}_E$ where μ_E is the symmetric convex capacity on E given by $\mu_E(0)=0,\mu_E(1)=\frac{\eta}{1-2\eta-2\epsilon},\mu_E(2)=1$.*

Proof. First we shall show that μ is convex and satisfies equation (2), which establishes that the GBU update of ν is a JP capacity with the same α. Equation (2) requires that $\mu(3)-\mu(2)=\mu(2)-\mu(1)$ or $3\eta+2\epsilon-(2\eta+\epsilon)=2\eta+\epsilon-\eta$, which clearly holds.

Convexity is satisfied since:

1. $1\geqslant 2\mu(3)-\mu(2)\Leftrightarrow 1\geqslant 6\eta+4\epsilon-(2\eta+\epsilon)=4\eta+2\epsilon$, which holds since $\eta<\frac{1}{4}$ and $\epsilon<\frac{1}{4}-\eta$;

[9] Since E contains $n-2$ elements other than s'.

2. $\mu(3) \geqslant 2\mu(2) - \mu(1) \Leftrightarrow 3\eta + 2\epsilon \geqslant 2(2\eta + \epsilon) - \eta$, which always holds;

3. $\mu(2) \geqslant 2\mu(1) \Leftrightarrow \mu(2) = 2\eta + \epsilon \geqslant 2\eta$.

The Möbius inverse of μ is: $\beta_1 = \eta, \beta_2 = \epsilon, \beta_3 = -\epsilon$, where β_j denotes the Möbius inverse of a set with j states for $1 \leqslant j \leqslant 3$. Since the Möbius inverse has some negative values, μ is not a belief function.

To show the updates have the given form We only need to consider the updates conditional on 2 and 3-element events, since updating on a 1-element event is trivial. Let E be an arbitrary 3-element event. Let C be a 2-element subset of E. By equation (11), the GBU-update is given by: $\nu_E(C) = \frac{\alpha\mu(C)+(1-\alpha)(1-\mu(C^c))}{1-\mu(C^c)+\mu((C\cup E^c)^c)}$. Thus

$$\nu_E(C) = \frac{\alpha(2\eta+\epsilon)}{1-\eta-\epsilon} + (1-\alpha)\left(1 - \frac{\eta}{1-\eta-\epsilon}\right). \tag{13}$$

Similarly if G is a 1-element subset of E, equation (11) implies that the GBU-update is

$$\nu_E(G) = \alpha\frac{\eta}{1-\eta-\epsilon} + (1-\alpha)\left(1 - \frac{2\eta+\epsilon}{1-\eta-\epsilon}\right). \tag{14}$$

This establishes part (2).

Now consider the updates of ν conditional on a 2-element event. Let E denote an arbitrary 2-element event, let A be a non-trivial subset of E and let $B = E\backslash A$. Then by equation (11) the GBU update is given by $\nu_E(A) = \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{1-\mu(A^c)+\mu((A\cup E^c)^c)}$

$$= \alpha\left(\frac{\eta}{1-2\eta-2\epsilon}\right) + (1-\alpha)\left(1 - \frac{\eta}{1-2\eta-2\epsilon}\right),$$

which establishes part 3. Note that $1 - 2\eta - 2\epsilon \geqslant 1 - 2\eta - \frac{1}{2} + 2\eta = \frac{1}{2}$. Since $\eta < \frac{1}{4}$ this implies that μ_E is convex. ■

The example may be understood by considering the symmetric neo-additive capacity defined by $\kappa(A) = |A|(\eta + \epsilon), A \subsetneqq S, \kappa(S) = 1$. Let a be an act such that $a(s_1) > ... > a(s_4)$. Then the Choquet integral of a with respect to κ is:

$$a(s_1)(\eta+\epsilon)+a(s_2)(\eta+\epsilon)+a(s_3)(\eta+\epsilon)+a(s_4)(1-3\eta-3\epsilon).$$

Compare this with the Choquet integral of a with respect to μ:

$$a(s_1)(\eta)+a(s_2)(\eta+\epsilon)+a(s_3)(\eta+\epsilon)+a(s_4)(1-3\eta-2\epsilon).$$

One can see that μ is similar to κ except that it under-weights the best outcome as well as over-weighting the worst outcome in the Choquet integral.

Proof of Proposition 3.1 Suppose that E_k is observed. Let $\hat{\nu}_k$ denote the GBU update of ν conditional on E_k. By definition, $\hat{\nu}_k(A)=\frac{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)}{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)+\alpha\delta+(1-\delta)q_k\bar{\nu}_k(A^c\cap E_k)}$

$$=\frac{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)}{\delta+(1-\delta)q_k}.$$

Set $\hat{\delta}:=\frac{\delta}{\delta+(1-\delta)q_k}$ and we obtain the right-hand side expression of equation (7). To obtain the expression in equation (8), notice that $\frac{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)}{\delta+(1-\delta)q_k}$

$$\begin{aligned}
&= \frac{(1-\alpha)\delta+(1-\delta)q_k\left[\alpha\mu_k(A\cap E_k)+(1-\alpha)\bar{\mu}_k(A\cap E_k)\right]}{\delta+(1-\delta)q_k}\\
&= \alpha\left(\frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right)+(1-\alpha)\left(\frac{\delta+(1-\delta)q_k\bar{\mu}_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right)
\end{aligned}$$

$$=\alpha\left(\frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right)+(1-\alpha)\left(1-\frac{(1-\delta)q_k\mu_k(A^c\cap E_k)}{\delta+(1-\delta)q_k}\right).$$

■

The following lemma assumes that ambiguity attitude is constant and shows that if a set consists of the union of subsets of two different elements of the partition then its Möbius inverse must be zero.

Lemma A.1 *Let $E_1,...,E_K$ be a partition of S, where $|E_k|\geqslant 3$, for $1\leqslant k\leqslant K$. Consider a*

JP-capacity $\nu = \alpha\mu + (1-\alpha)\,\bar{\mu}$, *where* μ *is a* belief function *on* S *and* $0 \leqslant \alpha \leqslant 1$. *Let* ν_{E_k} *denote the GBU update of* ν *conditional on* E_k. *Then a necessary and sufficient condition for* ν_{E_k} *to be a JP-capacity with the same* α *for* $1 \leqslant k \leqslant K$ *is that for* $A \subseteq E_k$, *and for all non-empty* $F \subseteq E_k^c, \beta_{A \cup F} = 0$, *for* $1 \leqslant k \leqslant K$.

Proof. Sufficiency follows from Proposition 3.1.

Necessity Let μ be a belief function and let β denote the Möbius inverse of μ. By equation (2), for all A, B such that $A \cup B = E_k, A \cap B = \varnothing$.

$$\mu\left(A \cup E_k^c\right) - \mu\left(A\right) = \mu\left(B \cup E_k^c\right) - \mu\left(B\right). \tag{15}$$

Consider a given element of the partition E_k. Assume $E_k = \{\sigma_1, ..., \sigma_L\}$. We claim that for $A \subseteq E_k$, all non-empty $F \subseteq E_k^c, \beta_{A \cup F} = 0$. We shall proceed by induction on the number of states in A.

Step 1 $|A| = 1$. In this case $A = \{\sigma_\ell\}$ for some $\ell, 1 \leqslant \ell \leqslant L$. By equation (15),

$\mu\left(\sigma_\ell \cup E_k^c\right) - \mu\left(\sigma_\ell\right) = \mu\left(\left(E_k \backslash \sigma_\ell\right) \cup E_k^c\right) - \mu\left(E_k \backslash \sigma_\ell\right)$. Rewriting in terms of the Möbius inverse, $\sum_{D \subseteq \left(\sigma_\ell \cup E_k^c\right)} \beta_D - \beta_{\{\sigma_\ell\}} = \sum_{D \subseteq \left(E_k \backslash \sigma_\ell\right) \cup E_k^c} \beta_D - \sum_{D \subseteq \left(E_k \backslash \sigma_\ell\right)} \beta_D$

or $\sum_{D \subseteq E_k^c} \beta_D + \sum_{D \subseteq E_k^c} \beta_{D \cup \sigma_\ell} - \beta_{\{\sigma_\ell\}} = \sum_{D \subseteq E_k^c} \beta_D + \sum_{\substack{D \subseteq \left(E_k \backslash \sigma_\ell\right) \cup E_k^c \\ D \nsubseteq E_k^c}} \beta_D - \sum_{D \subseteq \left(E_k \backslash \sigma_\ell\right)} \beta_D$,

which implies $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} = \sum_{D \subseteq \left(E_k \backslash \sigma_\ell\right) \cup E_k^c, D \nsubseteq E_k^c, D \nsubseteq \left(E_k \backslash \sigma_\ell\right)} \beta_D$.

Hence $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} \geqslant \sum_{j \neq \ell} \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_j}$, since we have deleted some non-negative terms from the rhs. Summing over ℓ, $\sum_{\ell=1}^{L} \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} \geqslant (L-1) \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell}$. Since $L \geqslant 3$, this implies $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} = 0$. Since $\beta_{D \cup \sigma_\ell} \geqslant 0$, for $1 \leqslant \ell \leqslant L$, we may deduce $\beta_{D \cup \sigma_\ell} = 0$, for all non-empty $D \subseteq E_k^c$. This establishes the result in the case where $|A| = 1$.

Inductive step Now take a given set $A \subseteq E_k$. Our inductive hypothesis is that for all strictly smaller subsets B of E_k, $\beta_{B \cup F} = 0$, for all non-empty $F \subseteq E_k^c$. There are two cases to consider.

Case 1 $|A| \leqslant \frac{L}{2}$ In this case we may choose $G \subseteq E_k$ such that $|G| = |A| - 1$ and $G \cap A = \varnothing$. Let $H = E_k \backslash G$. Note that $A \subseteq H$. By equation (2), $\mu(G \cup E_k^c) - \mu(G) = \mu(H \cup E_k^c) - \mu(H)$. Rewriting this in terms of the Möbius inverse we obtain:

$\sum_{D \subseteq (G \cup E_k^c)} \beta_D - \sum_{D \subseteq G} \beta_D = \sum_{D \subseteq (H \cup E_k^c)} \beta_D - \sum_{D \subseteq H} \beta_D.$

Expanding $\sum_{D \subseteq E_k^c} \beta_D + \sum_{D \subseteq G} \beta_D + \sum_{D \subseteq (G \cup E_k^c), D \nsubseteq G, D \nsubseteq E_k^c} \beta_D - \sum_{D \subseteq G} \beta_D$

$= \sum_{D \subseteq E_k^c} \beta_D + \sum_{D \subseteq H} \beta_D + \sum_{D \subseteq (H \cup E_k^c), D \nsubseteq H, D \nsubseteq E_k^c} - \sum_{D \subseteq H} \beta_D.$

This may be simplified to:

$$\sum_{D \subseteq (G \cup E_k^c), D \nsubseteq G, D \nsubseteq E_k^c} \beta_D = \sum_{D \subseteq (H \cup E_k^c), D \nsubseteq H, D \nsubseteq E_k^c} \beta_D. \tag{16}$$

Recall that by the inductive hypothesis $\beta_{B \cup F} = 0$, for subsets B of E_k strictly smaller than A and non-empty $F \subseteq E_k^c$. Thus all terms on the lhs of equation (16) are zero. i.e. $0 = \sum_{D \subseteq (H \cup E_k^c), D \nsubseteq H, D \nsubseteq E_k^c} \beta_D$. Since μ is, by assumption, a belief function, all the β's are non-negative, which implies $\beta_D = 0$ for all $D \subseteq (H \cup E_k^c), D \nsubseteq E_k^c D \nsubseteq H$. In particular $\beta_{A \cup F} = 0$, for all non-empty $F \subseteq E_k^c$. This completes the proof of this case.

Case 2, $|A| > \frac{L}{2}$ Let $Q = E_k \backslash A$. Then $|A| > |Q|$. By equation (2), $\mu(A \cup E_k^c) - \mu(A) = \mu(Q \cup E_k^c) - \mu(Q)$. Rewriting this in terms of the Möbius inverse we obtain:

$\sum_{D \subseteq (A \cup E_k^c)} \beta_D - \sum_{D \subseteq A} \beta_D = \sum_{D \subseteq (Q \cup E_k^c)} \beta_D - \sum_{D \subseteq Q} \beta_D.$

As in case 1 this may be simplified to:

$$\sum_{D \subseteq (A \cup E_k^c), D \nsubseteq A, D \nsubseteq E_k^c} \beta_D = \sum_{D \subseteq (Q \cup E_k^c), D \nsubseteq Q, D \nsubseteq E_k^c} \beta_D. \tag{17}$$

Recall that by the inductive hypothesis $\beta_{B \cup F} = 0$, for subsets B of E_k strictly smaller than A and non-empty $F \subseteq E_k^c$. Thus all terms on the rhs of equation (17) are zero, hence $\sum_{D \subseteq (A \cup E_k^c), D \nsubseteq A, D \nsubseteq E_k^c} \beta_D = 0$. As before, this implies $\beta_{A \cup F} = 0$, for all non-empty $F \subseteq E_k^c$. This completes the proof of the inductive step. The result follows. ■

Proof of Proposition 3.2 **Sufficiency** Proposition 3.1 has already established sufficiency.

Necessity Now assume that ν_{E_k} is a JP-capacity with the same α for $1 \leqslant k \leqslant K$ and μ is a belief function. Let $\delta = \beta_S \geqslant 0$. Then $\sum_{D \subsetneq S} \beta_D = 1 - \delta$.

For $1 \leqslant k \leqslant K$, define $q_k = \frac{1}{1-\delta} \sum_{D \subseteq E_k} \beta_B$. If $q_k \neq 0$ define a capacity μ_k on E_k by $\mu_k(A) = \frac{1}{(1-\delta)q_k} \sum_{D \subseteq A} \beta_B$ for $A \subseteq E_k$. It is clear that μ_k is convex since its Möbius inverse is non-negative. If $q_k = 0$, define μ_k by $\mu_k(A) = 0, A \subsetneq E_k; \mu_k(E_k) = 1$. If B is an arbitrary (proper) subset of S, then

$$\mu(B) = \sum_{D \subseteq B} \beta_D = \sum_{k=1}^{K} \sum_{D \subseteq B \cap E_k} \beta_D + \sum_{\substack{D \subseteq B \\ D \nsubseteq B \cap E_k, 1 \leqslant k \leqslant K}} \beta_D.$$

By Lemma A.1, if $A \subseteq E_k$, for all non-empty $F \subseteq E_k^c, \beta_{A \cup F} = 0$, for $1 \leqslant k \leqslant K$, hence the last sum is zero. Thus $\mu(B) = \sum_{k=1}^{K} \sum_{D \subseteq B \cap E_k} \beta_D = (1-\delta) \sum_{k=1}^{K} q_k \mu_k(B \cap E_k)$. Clearly $\mu(S) = 1$. Thus ν is a PAJP capacity. ■

Proof of Proposition 3.3 For $1 \leqslant k \leqslant K$, let $F_k \subseteq E_k$ be such that $\lambda(\mu_k) = \bar{\mu}_k(F_k) - \mu_k(F_k)$. Define $F = \bigcup_{k=1}^{K} F_k$. Now let A be an arbitrary subset of S. Then $\bar{\mu}(A) - \mu(A)$

$$= \delta + (1-\delta) \sum_{k=1}^{K} q_k [1 - \mu_k(A^c \cap E_k) - \mu_k(A \cap E_k)] \leqslant \delta + (1-\delta) \sum_{k=1}^{K} q_k \lambda(\mu_k), \tag{18}$$

which establishes that $\lambda(\mu) \geqslant \delta + (1-\delta) \sum_{k=1}^{K} q_k \lambda(\mu_k)$. Note also that equation (18) holds with equality if $A = F$, which implies $\lambda(\mu) \leqslant \delta + (1-\delta) \sum_{k=1}^{K} q_k \lambda(\mu_k)$.

From Proposition 3.1, if event E_k is observed, the updated capacity $\nu'(A) = \alpha \mu'_k(A) + (1-\alpha) \bar{\mu}'_k(A)$, where $\mu'_k(A) = \frac{(1-\delta)q_k \mu_k(A \cap E_k)}{\delta + (1-\delta)q_k}$. Thus the ex-post degree of ambiguity is,

$\lambda(\mu'_k) = \max_{A \subseteq E_k} \{\bar{\mu}'_k(A) - \mu'_k(A)\}$

$= \max_{A \subseteq E_k} \left\{1 - \frac{(1-\delta)q_k}{\delta + (1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta + (1-\delta)q_k} [\bar{\mu}_k(A) - \mu_k(A)]\right\}$

$= \frac{\delta}{\delta + (1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta + (1-\delta)q_k} \lambda(\mu_k)$. ■

Proof of Proposition 3.4 The effect of the likelihood of the signal on ex-post ambiguity can be measured by the derivative: $\frac{\partial\lambda\left(\mu_k'\right)}{\partial\delta} = \frac{\delta+q_k-\delta q_k-\delta+\delta q_k}{(\delta+(1-\delta)q_k)^2} - \left(\frac{\delta+q_k-\delta q_k+1-q_k-\delta+\delta q_k}{(\delta+(1-\delta)q_k)^2}\right) q_k\lambda\left(\mu_k\right)$ (by the quotient rule), $= q_k\frac{(1-\lambda(\mu_k))}{(\delta+(1-\delta)q_k)^2} > 0$. Thus an increase in the ex-ante ambiguity over the state space increases ex-post ambiguity. Similarly, $\frac{\partial\lambda\left(\mu_k'\right)}{\partial q_k} = (1-\delta)\frac{(\delta+(1-\delta)q_k)\lambda(\mu_k)-\delta-(1-\delta)\lambda(\mu_k)}{(\delta+(1-\delta)q_k)^2} = -(1-\delta)\frac{\delta(1-\lambda(\mu_k))+(1-\delta)(1-q_k)\lambda(\mu_k)}{(\delta+(1-\delta)q_k)^2} < 0$. Thus an increase in the likelihood of the signal, q_k, decreases ex-post ambiguity. ■

Lemma A.2 *Let $E_1, ..., E_K$ be a partition of S and let σ be a convex or concave capacity on S such that $\sum_{i=1}^{K}\sigma(E_i) = 1$ then for any $B \subseteq S$, $\sigma(B) = \sum_{i=1}^{K}\sigma(B \cap E_i)$.*

Proof. First assume that σ is concave and $K = 2$. Define sets C and D by $C = (B \cap E_1) \cup E_2, D = E_1 \cup (B \cap E_2)$. By concavity, $\sigma(C) \leqslant \sigma(B) + \sigma(E_2) - \sigma(B \cap E_2), \sigma(D) \leqslant \sigma(B) + \sigma(E_1) - \sigma(B \cap E_1)$ and $1 = \sigma(S) \leqslant \sigma(C) + \sigma(D) - \sigma(B)$. Substituting we obtain $1 \leqslant \sigma(B) + \sigma(E_2) - \sigma(B \cap E_2) + \sigma(B) + \sigma(E_1) - \sigma(B \cap E_1) - \sigma(B) = 1 + \sigma(B) - \sigma(B \cap E_2) - \sigma(B \cap E_1)$ or $\sigma(B \cap E_2) + \sigma(B \cap E_1) \leqslant \sigma(B)$. However the opposite inequality follows directly from concavity, which establishes the result in this case. The general result follows by repeated application of the result for $K = 2$. If σ is convex the result can be proved by reversing the inequalities in the above proof. ■

Proof of Proposition 3.5 First note that the case $\alpha = 1$ is proved by Theorem 2.1 in Eichberger et al. (2005). If $\alpha = 0$ a similar argument will establish the result. Thus we may assume $\alpha \neq 0, 1$.

2⇒1 Condition (2) implies that we may define a probability distribution over the partition $E_1, ..., E_K$ by setting $q_k = \mu(E_k)$ for $1 \leqslant k \leqslant K$. Lemma A.2 implies that for $A \subseteq S$, $\mu(A) = \sum_{k=1}^{K}\mu(A \cap E_k) = \sum_{k=1}^{K} q_k\mu_k(A \cap E_k)$, where μ_k is a capacity on E_k defined by $\mu_k(B) = \frac{\mu(B)}{q_k}$ for $B \subseteq E_k$. Thus $\nu = \sum_{k=1}^{K} q_k\left[\alpha\mu_k + (1-\alpha)\bar{\mu}_k\right]$, which implies that ν is an PAJP capacity. Hence we may apply Proposition 3.1 to deduce that the GBU update of ν conditional on E_k is $\nu_k = \alpha\mu_k(A) + (1-\alpha)\bar{\mu}_k(A)$.

Suppose that $b_k \in A(E_k)$ is preferred to a_k conditional on E_k, for $1 \leqslant k \leqslant K$. Then

$$\int u(b_k)\, d\nu_k \geqslant \int u(a_k)\, d\nu_k, \text{ for } 1 \leqslant k \leqslant K, \tag{19}$$

with at least one strict inequality. Define $b \in A(S)$, by $b(s) = b_k(s)$ if $s \in E_k$, for $1 \leqslant k \leqslant K$. We shall show that b is preferred to a in the first period, which implies dynamic consistency. Let the range of a (i.e. the set of outcomes generated by act a) be denoted by $\{x_1, ..., x_m\}$, where the outcomes have been numbered so that, $u(x_1) \geqslant u(x_2) \geqslant ... \geqslant u(x_m)$. Also define $A_i = \{s \in S : a(s) \in \{x_1, ..., x_i\}\}$. From the definition of the Choquet integral:

$\int u(a)\, d\nu = u(x_1)\nu(A_1) + \sum_{i=2}^{m} u(x_i)\left[\nu(A_i) - \nu(A_{i-1})\right]$

$= u(x_1)\left[\alpha\mu(A_1) + (1-\alpha)\bar{\mu}(A_1)\right]$

$+ \sum_{i=2}^{m} u(x_i)\left[\alpha\mu(A_i) + (1-\alpha)\bar{\mu}(A_i) - \alpha\mu(A_{i-1}) - (1-\alpha)\bar{\mu}(A_{i-1})\right].$

By Lemma A.2 this may be rewritten as

$\sum_{k=1}^{K} u(x_1)\left[\alpha\mu(A_1 \cap E_k) + (1-\alpha)\bar{\mu}(A_1 \cap E_k)\right]$

$+ \sum_{k=1}^{K}\sum_{i=2}^{m} u(x_i)\left[\alpha\mu(A_i \cap E_k) + (1-\alpha)\bar{\mu}(A_i \cap E_k) - \alpha\mu(A_{i-1} \cap E_k) - (1-\alpha)\bar{\mu}(A_{i-1} \cap E_k)\right]$

$= \sum_{k=1}^{K} \int u(a_{E_k})\, d\nu_k$. Similarly $\int u(b)\, d\nu = \sum_{k=1}^{K} \int u(b_k)\, d\nu_k$.

Thus $\int u(b)\, d\nu \geqslant \int u(a)\, d\nu$, which implies that act a could not be chosen in the first period. It follows that the decision maker is dynamically consistent.

1⇒2 Suppose that the decision maker is dynamically consistent. Consider first the case $K = 2$. Since the partition is non-trivial, we may find events, $A, B, C,$ and D such that, $E_1 = A \cup B$, $E_2 = C \cup D$, where $A \cap B = C \cap D = \varnothing$. Consider acts a, b, c, e, f and g as described in the following table:

	E_1		E_2	
	A	B	C	D
a	1	1	1	1
b	1	1	β	0
c	0	0	1	1
e	0	0	β	0
f	β	β	1	1
g	β	β	β	0

We can ensure that acts with these values exist by appropriately normalizing the utility function, (recall that X is convex). Note that $\int a_1 d\nu_1 = \int b_1 d\nu_1, \int c_1 d\nu_1 = \int e_1 d\nu_1, \int f_1 d\nu_1 = \int g_1 d\nu_1$; $\int a_2 d\nu_2 = \int c_2 d\nu_2 = \int f_2 d\nu_2$ and $\int b_2 d\nu_2 = \int e_2 d\nu_2 = \int g_2 d\nu_2$. By continuity and strong monotonicity we may choose β so that $\int a_2 d\nu_2 = \int b_2 d\nu_2$. Since $\alpha \neq 1$, $\beta > 1$. Dynamic consistency then implies that $a \sim b$, $c \sim e$ and $f \sim g$. By evaluating the Choquet integrals we find: $1 = (\beta - 1)\nu(C) + \nu(E_1 \cup C)$, $\nu(E_2) = \beta\nu(C)$ and $\beta\nu(E_1 \cup C) = \beta\nu(E_1) + 1 - \nu(E_1)$.

Hence $\nu(E_1 \cup C) = 1 - (\beta - 1)\nu(C) = 1 - \frac{\beta-1}{\beta}\nu(E_2)$,

$\beta\nu(E_1) + 1 - \nu(E_1) = \beta - (\beta - 1)\nu(E_2)$,

$1 - \beta = (1 - \beta)\nu(E_1) + (1 - \beta)\nu(E_2) \Leftrightarrow \nu(E_1) + \nu(E_2) = 1.$

Thus $\alpha\mu(E_1) + (1 - \alpha)\bar{\mu}(E_1) + \alpha\mu(E_2) + (1 - \alpha)\bar{\mu}(E_2) = 1.$

Expanding $\alpha\mu(E_1) + (1 - \alpha) - (1 - \alpha)\mu(E_2) + \alpha\mu(E_2) + (1 - \alpha) - (1 - \alpha)\mu(E_1) = 1$,

or $(1 - 2\alpha) - (1 - 2\alpha)\mu(E_2) - (1 - 2\alpha)\mu(E_1) = 0$. Since $\alpha \neq \frac{1}{2}$, this implies $\mu(E_1) + \mu(E_2) = 1$.

The general case can be established as follows. We can apply the above argument to $F_1 = E_1$ and $F_2 = \bigcup_{k=2}^{K} E_k$ to deduce that dynamic consistency implies $\mu(F_1) + \mu(F_2) = 1$. By repeated application of this result we may deduce that $\sum_{k=1}^{K} \mu(E_k) = 1$.

■

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