Distribution Regression - the Set Kernel Heuristic is Consistent

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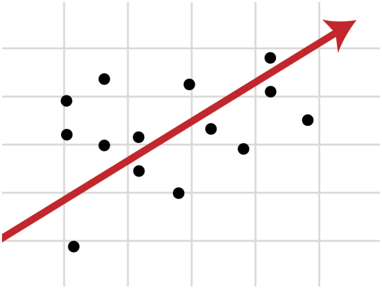
Outline

- Motivation, examples.
- Algorithm, consistency result.
- Numerical illustration.

Regression

- Given: $\{(x_i, y_i)\}_{i=1}^{l}$ samples $\mathcal{H} \ni f =?$ such that $f(x_i) \approx y_i$.



- Typically: $x_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^q$.
- Our interest: x_i-s are distributions (∞-dimensional objects).

Distribution regression: two-stage sampling difficulty

In practise:

- x_i-s are only observable via samples: $x_i \approx \{x_{i,n}\}_{n=1}^{N} \Rightarrow$
- an x_i is represented as a *bag*:
  - image = set of patches,
  - document = bag of words,
  - video = collection of images,
  - different configurations of a molecule = bag of shapes.

Set kernels: consistency?

- Given (2 bags):

$$B_i := \{x_{i,n}\}_{n=1}^{N_i} \sim x_i, \tag{1}$$

$$B_j := \{x_{j,m}\}_{m=1}^{N_j} \sim x_j. \tag{2}$$

- Similarity of the bags (set/multi-instance/ensemble-, convolution kernel; Gärtner'02, Haussler'99):

$$K(B_i, B_j) = \frac{1}{N_i N_j} \sum_{n=1}^{N_i} \sum_{m=1}^{N_j} k(x_{i,n}, x_{j,m}). \tag{3}$$

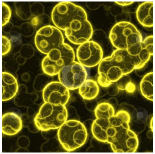
- Many successful applications:
  - classification, regression, clustering.
  - fundamental question: consistency?

Example: supervised entropy learning

- Entropy of $x \sim f$: $-\int f(u)\log[f(u)]\mathrm{d}u$.
- Training: samples from distributions, entropy values.
- Task: estimate the entropy of a new sample set.

Example: toxic level estimation from tissues

- Toxin alters the properties/causes mutations in cells.
- Training data:
  - bag = tissue,
  - samples in the bag = cells described by some simple features,
  - output label = toxic level.
- Task: predict the toxic level given a new tissue.



Example: aerosol prediction using satellite images

- Aerosol = floating particles in the air; climate research.
- Multispectral satellite images: 1 pixel $= 200 \times 200m^2 \in$ bag.
- Bag label: ground-based (expensive) sensor.
- Task: satellite image $\rightarrow$ aerosol density.

Towards problem formulation: kernel, RKHS

- $k : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ kernel on $\mathcal{D}$, if
  - $\exists\varphi : \mathcal{D} \to H$(ilbert space) feature map,
  - $k(a, b) = \langle\varphi(a), \varphi(b)\rangle_H$ $(\forall a, b \in \mathcal{D})$.
- Kernel examples: $\mathcal{D} = \mathbb{R}^d$ $(p > 0,\ \theta > 0)$
  - $k(a, b) = (\langle a, b\rangle + \theta)^p$: polynomial,
  - $k(a, b) = e^{-\|a-b\|_2^2/(2\theta^2)}$: Gaussian,
  - $k(a, b) = e^{-\theta\|a-b\|_1}$: Laplacian.
- In the $H = H(k)$ RKHS $(\exists!)$: $\varphi(u) = k(\cdot, u)$.

Some example domains ($\mathcal{D}$), where kernels exist

- Euclidean spaces: $\mathcal{D} = \mathbb{R}^d$.
- Strings, time series, graphs, dynamical systems.

- Distributions.

Distribution kernel: example (used in our work)

- Given: $(\mathcal{D}, k)$; we saw that $u \to \varphi(u) = k(\cdot, u) \in H(k)$.
- Let x be a distribution on $\mathcal{D}$ ($x \in \mathcal{M}_1^+(\mathcal{D})$); the previous construction can be extended:

$$\mu_x = \int_{\mathcal{D}} k(\cdot, u) \mathrm{d}x(u) \in H(k). \tag{4}$$

- If k is bounded: μ_x is well-defined for *any* distribution x.

Mean embedding based distribution kernel

Simple estimation of $\mu_x = \int_{\mathcal{D}} k(\cdot, u)\mathrm{d}x(u)$:

- Empirical distribution: having samples $\{x_n\}_{n=1}^N$

$$\hat{x} = \frac{1}{N}\sum_{n=1}^{N}\delta_{x_n}. \tag{5}$$

- Mean embedding, inner product – empirically (set kernels!):

$$\mu_{\hat{x}} = \int_{\mathcal{D}} k(\cdot, u)\mathrm{d}\hat{x}(u) = \frac{1}{N}\sum_{n=1}^{N} k(\cdot, x_n), \tag{6}$$

$$K\left(\mu_{\hat{x}_i}, \mu_{\hat{x}_j}\right) = \left\langle \mu_{\hat{x}_i}, \mu_{\hat{x}_j} \right\rangle_{H(k)} = \frac{1}{N_i N_j}\sum_{n=1}^{N_i}\sum_{m=1}^{N_j} k(x_{i,n}, x_{j,m}).$$

- Until now
  - If we are given a domain ($\mathcal{D}$) with kernel k, then
  - one can easily define/estimate the similarity of distributions on $\mathcal{D}$.
- Prototype example: $\mathcal{D} = \mathbb{R}^d$, $k =$ Gaussian, $K =$ lin. kernel.

- The *real* conditions:
  - $\mathcal{D}$: locally compact, Polish. k: c_0-universal.
  - K: Hölder continuous, i.e. $\exists L > 0$, $h \in (0,1]$

$$\|K(\cdot,\mu_a) - K(\cdot,\mu_b)\|_{\mathcal{H}(K)} \leq L\,\|\mu_a - \mu_b\|_{H(k)}^h \quad (\forall \mu_a,\mu_b).$$

- $\mathbf{z} = \{(x_i, y_i)\}_{i=1}^{l}$: $x_i \in M_1^+(\mathcal{D})$, $y_i \in \mathbb{R}$.
- $\hat{\mathbf{z}} = \left\{\left(\{x_{i,n}\}_{n=1}^{N}, y_i\right)\right\}_{i=1}^{l}$: $x_{i,1}, \ldots, x_{i,N} \stackrel{i.i.d.}{\sim} x_i$.
- Goal: learn the relation between x and y based on $\hat{\mathbf{z}}$.
- Idea: embed the distributions (μ) + apply ridge regression

$$M_1^+(\mathcal{D}) \xrightarrow{\mu} X(\subseteq H = H(k)) \xrightarrow{f \in \mathcal{H} = \mathcal{H}(K)} \mathbb{R}.$$

Objective function

- $f_{\mathcal{H}} \in \mathcal{H} = \mathcal{H}(K)$: ideal/optimal in expected risk sense ($\mathcal{E}$):

$$\mathcal{E}\left[f_{\mathcal{H}}\right] = \inf_{f \in \mathcal{H}} \mathcal{E}[f] = \inf_{f \in \mathcal{H}} \int_{X \times \mathbb{R}} [f(\mu_a) - y]^2 \mathrm{d}\rho(\mu_a, y). \qquad (7)$$

- One-stage difficulty ($\int \to \mathbf{z}$):

$$f_{\mathbf{z}}^{\lambda} = \arg\min_{f \in \mathcal{H}} \left(\frac{1}{l} \sum_{i=1}^{l} [f(\mu_{x_i}) - y_i]^2 + \lambda \|f\|_{\mathcal{H}}^2 \right). \qquad (8)$$

- Two-stage difficulty ($\mathbf{z} \to \hat{\mathbf{z}}$):

$$f_{\hat{\mathbf{z}}}^{\lambda} = \arg\min_{f \in \mathcal{H}} \left(\frac{1}{l} \sum_{i=1}^{l} [f(\mu_{\hat{x}_i}) - y_i]^2 + \lambda \|f\|_{\mathcal{H}}^2 \right). \qquad (9)$$

Algorithmically: ridge regression ⇒ simple solution

- Given:
  - training sample: $\hat{\mathbf{z}}$,
  - test distribution: t.
- Prediction:

$$(f_{\hat{\mathbf{z}}}^{\lambda} \circ \mu)(t) = [y_1, \dots, y_l](\mathbf{K} + l\lambda \mathbf{I}_l)^{-1} \begin{bmatrix} K(\mu_{\hat{x}_1}, \mu_t) \\ \vdots \\ K(\mu_{\hat{x}_l}, \mu_t) \end{bmatrix}, \tag{10}$$

$$\mathbf{K} = [K_{ij}] = [K(\mu_{\hat{x}_i}, \mu_{\hat{x}_j})] \in \mathbb{R}^{l \times l}. \tag{11}$$

Consistency result

- We studied
  - the excess error: $\mathcal{E}\left[f_{\hat{\mathbf{z}}}^{\lambda}\right] - \mathcal{E}\left[f_{\mathcal{H}}\right]$, i.e,
  - the goodness compared to the best function from $\mathcal{H}$.
- Result: with high probability

$$\mathcal{E}\left[f_{\hat{\mathbf{z}}}^{\lambda}\right] - \mathcal{E}\left[f_{\mathcal{H}}\right] \to 0, \tag{12}$$

if we appropriately choose the (l, N, λ) triplet.

Consistency result: $\mathcal{P}(b,c)$ class

- Let the $T : \mathcal{H} \to \mathcal{H}$ covariance operator be

$$T = \int_X K(\cdot,\mu_a)K^*(\cdot,\mu_a)\mathrm{d}\rho_X(\mu_a) = \int_X K(\cdot,\mu_a)\delta_{\mu_a}\mathrm{d}\rho_X(\mu_a)$$

  with eigenvalues t_n $(n = 1, 2, \ldots)$.
- Let $\rho \in \mathcal{P}(b,c)$ be the set of distributions on $X \times \mathbb{R}$:
  - $\alpha \le n^b t_n \le \beta \quad (\forall n \ge 1; \alpha > 0, \beta > 0)$,
  - $\exists g \in \mathcal{H}$ such that $f_{\mathcal{H}} = T^{\frac{c-1}{2}} g$ with $\|g\|_{\mathcal{H}}^2 \le R$ $(R > 0)$,

  where $b \in (1,\infty)$, $c \in [1,2]$.

Consistency result: convergence rates

High-level idea:

- The excess error can be upper bounded on $\mathcal{P}(b,c)$ as:

$$g(l,N,\lambda) = \mathcal{E}\left[f_{\hat{\mathbf{z}}}^{\lambda}\right] - \mathcal{E}\left[f_{\mathcal{H}}\right] \leq \frac{\log^h(l)}{N^h\lambda^3} + \lambda^c + \frac{1}{l^2\lambda} + \frac{1}{l\lambda^{\frac{1}{b}}}.$$

- We choose ($h = 1$, i.e., K is Lipschitz)
  - $\lambda = \lambda_{l,N} \to 0$:
    - by matching two terms,
    - $g(l,N,\lambda) \to 0$; moreover, make the 2 equal terms dominant.
  - $l = N^a$ ($a > 0$).

Convergence rate: results

- $\boxed{1} = \boxed{2}$: If $\lambda = \left[\frac{\log(N)}{N}\right]^{\frac{1}{c+3}}$, $\frac{\frac{1}{b}+c}{c+3} \leq a$, then

$$g(N) = \mathcal{O}\left(\left[\frac{\log(N)}{N}\right]^{\frac{c}{c+3}}\right) \to 0. \tag{13}$$

Convergence rate: results

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$$g(N) = \mathcal{O}\left(\left[\frac{\log(N)}{N}\right]^{\frac{c}{c+3}}\right) \to 0. \tag{13}$$
- $\boxed{1} = \boxed{3}$: If $\lambda = N^{a-\frac{1}{2}} \log^{\frac{1}{2}}(N)$, $\frac{1}{6} \leq a < \min\left(\frac{1}{2} - \frac{1}{c+3}, \frac{\frac{1}{2}\left(\frac{1}{b}-1\right)}{\frac{1}{b}-2}\right)$,
$$g(N) = \mathcal{O}\left(\frac{1}{N^{3a-\frac{1}{2}} \log^{\frac{1}{2}}(N)}\right) \to 0. \tag{14}$$
- $\boxed{1} = \boxed{4}$: If $\lambda = \left[N^{a-1} \log(N)\right]^{\frac{b}{3b-1}}$, $\max\left(\frac{b-1}{4b-2}, \frac{1}{3b}\right) \leq a < \frac{bc+1}{3b+bc}$,
$$g(N) = \mathcal{O}\left(\frac{1}{N^{a+\frac{a}{3b-1}-\frac{1}{3b-1}} \log^{\frac{1}{3b-1}}(N)}\right) \to 0. \tag{15}$$

Convergence rate: results

- $\boxed{2} = \boxed{3}$: $\emptyset$ (the matched terms can not be made dominant).
- $\boxed{2} = \boxed{4}$: If $\lambda = \frac{1}{N^{\frac{ab}{bc+1}}}$, $a < \frac{bc+1}{3b+bc}$, then

$$g(N) = \mathcal{O}\left(\frac{1}{N^{\frac{abc}{bc+1}}}\right) \to 0. \tag{16}$$

- $\boxed{3} = \boxed{4}$: If $\lambda = \frac{1}{N^{\frac{ab}{b-1}}}$, $2 < b$, $a < \frac{b-1}{2(2b-1)}$, then

$$g(N) = \mathcal{O}\left(\frac{1}{N^{2a-\frac{ab}{b-1}}}\right) \to 0. \tag{17}$$

Numerical illustration: supervised entropy learning

- Problem: learn the entropy of Gaussians in a supervised manner.
- Formally:
  - $A = [A_{i,j}] \in \mathbb{R}^{2\times 2}$, $A_{ij} \sim U[0,1]$.
  - 100 sample sets: $\{N(0,\Sigma_u)\}_{u=1}^{100}$, where
    - $100 = 25(\text{training}) + 25(\text{validation}) + 50(\text{testing})$.
    - one set $=$ 500 i.i.d. 2D points,
    - $\Sigma_u = R(\beta_u)AA^T R(\beta_u)^T$,
    - $R(\beta_u)$: 2d rotation,
    - angle $\beta_u \sim U[0,\pi]$.

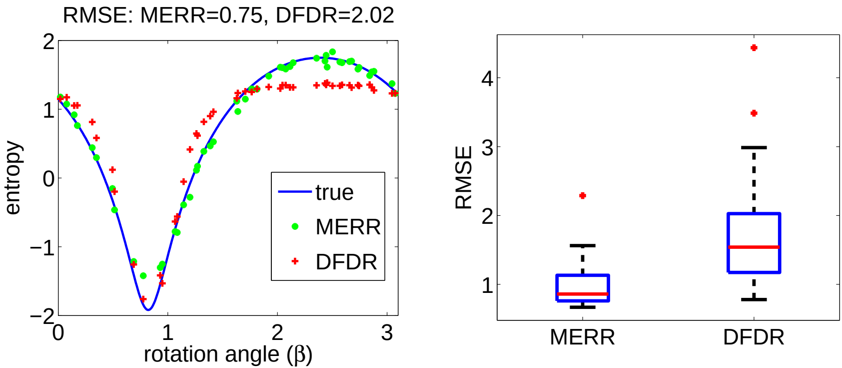
Supervised entropy learning: goal, performance measure

- Goal: learn the entropy of the first marginal

$$H = \frac{1}{2} \ln \left(2\pi e \sigma^2\right), \quad \sigma^2 = M_{1,1}, \quad M = \Sigma_u \in \mathbb{R}^{2\times 2}. \quad (18)$$

- Baseline: kernel smoothing based distribution regression (applying density estimation) =: DFDR.
- Performance: RMSE boxplot over 25 random experiments.

Supervised entropy learning: results



Numerical illustration: aerosol prediction

- Bags:
  - randomly selected pixels,
  - within a $20km$ radius around an AOD sensor.
- 800 bags, 100 instances/bag.
- Instances: $x_{i,n} \in \mathbb{R}^{16}$.

Aerosol prediction - baseline

- Baseline: state-of-the-art mixture model
  - EM optimization,
  - $800 = 4 \times 160$(training) $+ 160$(test); 5-fold CV, 10 times.
  - Accuracy: $100 \times RMSE(\pm \text{ std}) = 7.5 - 8.5$ $(\pm 0.1 - 0.6)$.
- Ridge regression:
  - $800 = 3 \times 160$(training) $+ 160$(validation) $+ 160$(test),
  - 5-fold CV, 10 times,
  - validation: λ regularization, θ kernel parameter.

Aerosol prediction: kernel k

- We picked 10 kernels (k): Gaussian, exponential, Cauchy, generalized t-student, polynomial kernel of order 2 and 3 ($p = 2$ and 3), rational quadratic, inverse multiquadratic kernel, Matérn kernel (with $\frac{3}{2}$ and $\frac{5}{2}$ smoothness parameters).
- We also studied their ensembles.
- Explored parameter domain:

$$(\lambda, \theta) \in \left\{2^{-65}, 2^{-64}, \ldots, 2^{-3}\right\} \times \left\{2^{-15}, 2^{-14}, \ldots, 2^{10}\right\}.$$

- First, K was linear.

Aerosol prediction: kernel definitions

Kernel definitions ($p = 2, 3$):

$$k_G(a,b) = e^{-\frac{\|a-b\|_2^2}{2\theta^2}}, \qquad k_e(a,b) = e^{-\frac{\|a-b\|_2}{2\theta^2}}, \tag{19}$$

$$k_C(a,b) = \frac{1}{1+\frac{\|a-b\|_2^2}{\theta^2}}, \quad k_t(a,b) = \frac{1}{1+\|a-b\|_2^\theta}, \tag{20}$$

$$k_p(a,b) = (\langle a,b\rangle + \theta)^p, \; k_r(a,b) = 1 - \frac{\|a-b\|_2^2}{\|a-b\|_2^2+\theta}, \tag{21}$$

$$k_i(a,b) = \frac{1}{\sqrt{\|a-b\|_2^2+\theta^2}}, \tag{22}$$

$$k_{M,\frac{3}{2}}(a,b) = \left(1+\frac{\sqrt{3}\,\|a-b\|_2}{\theta}\right) e^{-\frac{\sqrt{3}\|a-b\|_2}{\theta}}, \tag{23}$$

$$k_{M,\frac{5}{2}}(a,b) = \left(1+\frac{\sqrt{5}\,\|a-b\|_2}{\theta}+\frac{5\,\|a-b\|_2^2}{3\theta^2}\right) e^{-\frac{\sqrt{5}\|a-b\|_2}{\theta}}. \tag{24}$$

Aerosol prediction: results (K: linear)

$100 \times RMSE(\pm std)$ [baseline: $7.5 - 8.5$ $(\pm 0.1 - 0.6)$]:

k_G 7.97 ($\pm$1.81)	k_e 8.25 ($\pm$1.92)	k_C 7.92 ($\pm$1.69)	k_t 8.73 ($\pm$2.18)
$k_p(p=2)$ 12.5 ($\pm$2.63)	$k_p(p=3)$ 171.24 ($\pm$56.66)	k_r 9.66 ($\pm$2.68)	k_i **7.91** ($\pm$**1.61**)
$k_{M,\frac{3}{2}}$ 8.05 ($\pm$1.83)	$k_{M,\frac{5}{2}}$ 7.98 ($\pm$1.75)	ensemble **7.86** ($\pm$**1.71**)	

Aerosol prediction: nonlinear K

- We fed the mean embedding distance ($\|\mu_x - \mu_y\|_{H(k)}$) to the previous kernels.
- Example (RBF on mean embeddings – valid kernel):

$$K(\mu_a, \mu_b) = e^{-\frac{\|\mu_a - \mu_b\|^2_{H(k)}}{2\theta_K^2}} \quad (\mu_a, \mu_b \in X). \tag{25}$$

- We studied the efficiency of (i) single, (ii) ensembles of kernels [(k, K) pairs].

Aerosol prediction: nonlinear K, results

- Baseline:
  - Mixture model (EM): $7.5 - 8.5$ ($\pm 0.1 - 0.6$),
  - Linear K (single): 7.91 ($\pm$1.61).
  - Linear K (ensemble): **7.86** ($\pm$**1.71**).
- Nonlinear K:
  - Single: 7.90 ($\pm$1.63),
  - Ensemble: **7.81** ($\pm$**1.64**).

Summary

- Problem:
  - consistency of set kernels in regression,
  - open for 15 years.
- Examined solution: ridge regression; simple alg.!
- Contribution (on arXiv: 1402.1754):
  - consistency; convergence rate.
- Code: in ITE (`https://bitbucket.org/szzoli/ite/`).

Thank you for the attention!

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Topological space, open sets

- Given: $\mathcal{X} \neq \emptyset$ set.
- $\tau \subseteq 2^{\mathcal{X}}$ is called a *topology* on $\mathcal{X}$ if:
  1. $\emptyset \in \tau$, $\mathcal{X} \in \tau$.
  2. Finite intersection: $O_1 \in \tau$, $O_2 \in \tau \Rightarrow O_1 \cap O_2 \in \tau$.
  3. Arbitrary union: $O_i \in \tau$ $(i \in I) \Rightarrow \cup_{i \in I} O_i \in \tau$.

Then, $(\mathcal{X}, \tau)$ is called a *topological space*; $O \in \tau$: *open* sets.

Topology: examples

- $\tau = \{\emptyset, \mathcal{X}\}$: indiscrete topology.
- $\tau = 2^{\mathcal{X}}$: discrete topology.
- $(\mathcal{X}, d)$ metric space:
  - Open ball: $B_\epsilon(x) = \{y \in \mathcal{X} : d(x, y) < \epsilon\}$.
  - $O \subseteq \mathcal{X}$ is open if for $\forall x \in O$ $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq O$.
  - $\tau := \{O \subseteq \mathcal{X} : O$ is an open subset of $\mathcal{X}\}$.

Closed set, compact set, closure, subspace topology

Given: $(\mathcal{X}, \tau)$. $A \subseteq \mathcal{X}$ is

- *closed* if $\mathcal{X} \backslash A \in \tau$ (i.e., its complement is open),
- *compact* if for any family $(O_i)_{i \in I}$ of open sets with $A \subseteq \cup_{i \in I} O_i$, $\exists i_1, \ldots, i_n \in I$ with $A \subseteq \cup_{j=1}^n O_{i_j}$.

Closure of $A \subseteq \mathcal{X}$:

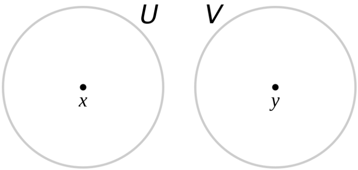
$$\bar{A} := \bigcap_{A \subseteq C \text{ closed in } \mathcal{X}} C. \tag{26}$$

For $A \subseteq \mathcal{X}$ the *subspace topology* on A: $\tau_A = \{O \cap A : O \in \tau\}$.

Hausdorff space

$(\mathcal{X}, \tau)$ is a *Hausdorff space*, if

- for any $x \neq y \in \mathcal{X}$ $\exists U, V \in \tau$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.
- In other words, disjunct points have disjunct open environments.
- Example: metric spaces.



Dense subset, separability, basis of a topology, Polish

- $A \subseteq \mathcal{X}$ is *dense* if $\bar{A} = \mathcal{X}$.
- $(\mathcal{X}, \tau)$ is *separable* if $\exists$ countable, dense subset of $\mathcal{X}$. Counterexample: l^∞/L^∞.
- $\tau_1 \subseteq \tau$ is a *basis* of τ if every open set is the union of sets in τ_1. Example: open balls in a metric space.
- $(\mathcal{X}, \tau)$ is *Polish* if τ has a countable basis and $\exists$ metric defining τ. Example: complete separable metric spaces.

$(\mathcal{X}, \tau)$:

- $V \subseteq \mathcal{X}$ is a *neighborhood* of $x \in \mathcal{X}$ if $\exists O \in \tau$ such that $x \in O \subseteq V$.
- is called *locally compact* if for $\forall x \in \mathcal{X}$ $\exists$ compact neighborhood of x. Example: $\mathbb{R}^d$; not compact.

Examples: LCH, but not (necessarily) compact

- Euclidean spaces: $\mathbb{R}^d$, not compact.
- Discrete spaces: LCH. Compact $\Leftrightarrow |\mathcal{X}| < \infty$.
- Open/closed subsets of an LCH: LC in subspace topology. Example: unit ball (open/closed).

Examples: Hausdorff, but not locally compact

- ($\mathbb{Q}$, topology inherited from $\mathbb{R}$).
    - In other words, not every *subset* of an LCH is LC.
- Infinite dimensional Hilbert spaces.
    - Example: complex $L^2([0,1])$; $\{f_n(x) = e^{2\pi inx}, n \in \mathbb{Z}\}$: ONB.

The discrete space

- $(\mathcal{X}, 2^{\mathcal{X}})$: complete metric space.
- Discrete metric (inducing the discrete topology):

$$d(x,y) = \begin{Bmatrix} 0, \text{ if } x = y \\ 1, \text{ if } x \neq y \end{Bmatrix}. \tag{27}$$

- Discrete space: separable $\Leftrightarrow$ $|\mathcal{X}|$ is countable.

Kernel k: c_0-universal

Let $C_0(\mathcal{D}) = \mathcal{D} \to \mathbb{R}$ continuous functions vanishing at infinity, i.e.,

$$\{u \in \mathcal{D} : |g(u)| \geq \epsilon\} \tag{28}$$

is compact for $g \in C_0(\mathcal{D})$, $\forall \epsilon > 0$. $k : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ is c_0-universal if

- $\|k\|_\infty := \sup_{u \in \mathcal{D}} \sqrt{k(u,u)} < \infty$,
- $k(\cdot, u) \in C_0(\mathcal{D})$ $(\forall u \in \mathcal{D})$.
- $H = H(k)$ is dense in $C_0(\mathcal{D})$ w.r.t. the uniform norm.

ITE: covered quantities

- **entropy**: Shannon entropy, Rényi entropy, Tsallis entropy (Havrda and Charvát entropy), complex entropy, Φ-entropy (f-entropy), Sharma-Mittal entropy,
- **mutual information**: generalized variance, kernel canonical correlation analysis, kernel generalized variance, Hilbert-Schmidt independence criterion, Shannon mutual information (total correlation, multi-information), L_2 mutual information, Rényi mutual information, Tsallis mutual information, copula-based kernel dependency, multivariate version of Hoeffding's Φ, Schweizer-Wolff's σ and κ, complex mutual information, Cauchy-Schwartz quadratic mutual information (QMI), Euclidean distance based QMI, distance covariance, distance correlation, approximate correntropy independence measure, χ^2 mutual information (Hilbert-Schmidt norm of the normalized cross-covariance operator, squared-loss mutual information, mean square contingency),
- **divergence**: Kullback-Leibler divergence (relative entropy, I directed divergence), L_2 divergence, Rényi divergence, Tsallis divergence Hellinger distance, Bhattacharyya distance, maximum mean discrepancy (kernel distance), J-distance (symmetrised Kullback-Leibler divergence, J divergence), Cauchy-Schwartz divergence, Euclidean distance based divergence, energy distance (specially the Cramer-Von Mises distance), Jensen-Shannon divergence, Jensen-Rényi divergence, K divergence, L divergence, f-divergence (Csiszár-Morimoto divergence, Ali-Silvey distance), non-symmetric Bregman distance (Bregman divergence), Jensen-Tsallis divergence, symmetric Bregman distance, Pearson χ^2 divergence (χ^2 distance), Sharma-Mittal divergence,
- **association measures**: multivariate extensions of Spearman's ρ (Spearman's rank correlation coefficient, grade correlation coefficient), correntropy, centered correntropy, correntropy coefficient, correntropy induced metric, centered correntropy induced metric, multivariate extension of Blomqvist's β (medial correlation coefficient), multivariate conditional version of Spearman's ρ, lower/upper tail dependence via conditional Spearman's ρ,
- **cross quantities**: cross-entropy,
- **kernels on distributions**: expected kernel (summation kernel, mean map kernel), Bhattacharyya kernel, probability product kernel, Jensen-Shannon kernel, exponentiated Jensen-Shannon kernel, Jensen-Tsallis kernel, exponentiated Jensen-Rényi kernel(s), exponentiated Jensen-Tsallis kernel(s),
- **+some auxiliary quantities**: Bhattacharyya coefficient (Hellinger affinity), α-divergence.

ITE: summary

- Matlab/Octave (first release).
- Multi-platform.
- GPLv3($\geq$).
- Appeared in JMLR, 2014.
- Homepage: `https://bitbucket.org/szzoli/ite/`

ITE: built-in tests/applications

- Consistency tests.
- Prototype: independent subspace analysis, its extensions.

- Image registration $\rightarrow$ outlier robustness.
- Distribution regression.

K = RBF: Lipschitz on compact $\mathcal{D}$ domains

Let

$$K(\mu_a, \mu_b) = e^{-\frac{\|\mu_a - \mu_b\|_H^2}{2\sigma^2}}. \tag{29}$$

Needed: $\exists\ L > 0$ such that

$$\|K(\cdot, \mu_a) - K(\cdot, \mu_b)\|_{\mathcal{H}} \leq L \|\mu_a - \mu_b\|_H \quad (\forall \mu_a, \mu_b \in X). \tag{30}$$

L.h.s.:

$$\|K(\cdot, \mu_a) - K(\cdot, \mu_b)\|_{\mathcal{H}}^2 = K(\mu_a, \mu_a) + K(\mu_b, \mu_b) - 2K(\mu_a, \mu_b) \tag{31}$$

$$= 2\left[1 - e^{-\frac{\|\mu_a - \mu_b\|_H^2}{2\sigma^2}}\right]. \tag{32}$$

$K = \text{RBF}$: Lipschitz on compact $\mathcal{D}$ domains

The statement is equivalent to $\exists\ L' > 0$:

$$u(\mu_a, \mu_v) := \frac{1 - e^{-\frac{\|\mu_a - \mu_b\|_H^2}{2\sigma^2}}}{\|\mu_a - \mu_b\|_H^2} \leq L'. \tag{33}$$

Idea:

- $\mathcal{D}$ compact $\Rightarrow$ X compact; $\Rightarrow$ $X \times X$ compact (Tychonoff T.).
- $u = u_2 \circ u_1$ continuous (continuity of u_i-s):

$$u_1 : X \times X \to \mathbb{R}^{\geq 0}, \quad u_1(\mu_a, \mu_b) = \|\mu_a - \mu_b\|_H^2, \tag{34}$$

$$u_2 : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}, \quad u_2(v) = \frac{1 - e^{-\frac{v}{2\sigma^2}}}{v}. \tag{35}$$

- Continuous image (u) of a compact set ($X \times X$) is compact.