Nonlinear Pricing of Storable Goods*

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September 24, 2012

Abstract

This paper develops a model of nonlinear pricing of storable goods. We show that storability imposes novel constraints on a monopolist's ability to extract surplus. We then show that the attempt to relax these constraints can generate cyclical patterns in pricing and sales, even when consumers are homogeneous. Thus, the model provides a novel explanation for sales that does not rely on discrimination motives. Enriching the model to allow for buyer heterogeneity in storage technology, delivers the prediction that larger containers are more likely to be on sale.

1 Introduction

Non-linear pricing is prevalent in many markets, from phone and electricity tariffs to supermarkets items. There is an extensive literature that studies non-linear pricing as a tool for surplus extraction, often as a device for price discrimination in the context of heterogeneous buyers (see Wilson (1997)). However, the contrast with linear pricing is particularly stark when consumers are homogeneous: in this case, the optimal non-linear pricing policy involves a monopolist selling the socially optimal quantity and extracting all the surplus.

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Many of the products sold through non-linear prices are storable. For example, the typical scanner data show quantity discounts in a variety of products ranging from yogurt to detergent (see Hendel and Nevo (2006a) and Hendel and Nevo (2006b)). These products are storable; so are many other products, like intermediate goods, that are also priced non-linearly. Product storability enables consumers to detach the timing of purchase from the timing of consumption. Consumers' ability to store a product may affect sellers capabilities to price non-linearly.

This paper presents a first attempt to understand the consequences of storability on non-linear pricing. Product storability makes the monopolist contend with a new type of constraint that we call *no-skipping* constraint or spot-participation constraint: the consumer may choose to consume out of storage and skip a purchase. In order to satisfy all no-skipping constraints, prices have to be set in such a way that consumers are willing to purchase and consume their intended bundles, as opposed to alternative consumption sequences supported by storage and consumption smoothing.

We show that the impact of storability on non-linear pricing can be severe. In the context of stationary offerings, namely, when a seller offers the same bundle over time, the monopolist can lose all ability to price non-linearly. The monopolist is still able to make a profit, but with stationary policies, non-linear prices do no better than linear prices: by choosing the frequency of purchases and consuming out of storage, consumers fully undo any attempt to extract additional surplus. Storage enables consumers to unbundle non-linear pricing policies. The logic of this result is related to the constraint on non-linear pricing that is imposed by resale: the consumer can purchase bundles cheaply and "resell" them to his future self.

Given the ineffectiveness of stationary policies, we ask whether there are more sophisticated ways for the monopolist to enable surplus extraction via non-linear prices.

We study seller behavior in two set-ups: with homogenous as well as heterogeneous consumers. We start with homogenous buyers so that the only gain from non-linear pricing is surplus extraction (no discrimination motive). We then consider heterogeneity in storage. The analysis is presented through a combination of a two-period model and a particularly stark infinite horizon set-up. The two-period model is a natural first step, since it helps to highlight some key forces. However, the two-period framework introduces asymmetries between periods in that the second period involves no further potential for storage. The infinite horizon model restores symmetry across periods.

We show that the monopolist can partially regain some ability to extract surplus via a suitable cyclical policy that involves the infrequent sale of large bundles. This holds even in an environment with identical consumers, thus, it represents a novel reason for cyclical pricing (or sales) by a monopolist, and provides a stark contrast with models of sales based on price discrimination. In most existing models, sales are a mechanism to discriminate among heterogeneous buyers (like in Salop and Stiglitz (1982), Narasimhan (1988), Sobel (1984), Hong et al. (2002) and Pesendorfer (2002)). In this paper sales arise to enhance non-linear prices.

The seller gains from forcing consumers to fully use their storage capacity as a way of relaxing the "no-skipping" constraint. By offering infrequent purchase opportunities (sales) the seller limits consumers' opportunities to skip purchases, consuming out of storage. At the same time, by making each purchase take up the full storage capacity the seller makes the storage unavailable for skipping, thereby relaxing the no-skipping constraints. In other words, the infrequent sale of large quantities eliminates consumers' ability to get ready to skip, namely, to slowly accumulate inventory in anticipation of skipping a purchase. Notice that the cost of skipping is the utility of foregone consumption. Such cost goes down if consumers could save over many periods, so that the foregone consumption involves a lower utility loss.

While the quantity sold is determined by the storage capacity, consumption and prices are determined by the frequency of purchases. More frequent purchases of a given quantity naturally translate into a higher consumption rate. Prices, the surplus captured by the monopolist, are determined by the following trade-off. In order to make the consumer show up to purchase (namely, prevent skipping), the monopolist must promise him enough surplus so that he does not prefer to consume out of storage. It turns out that sellers can extract at most $V(c) - V(\frac{c}{2})$ (per period). The term $V(c)$ is the surplus at the intended consumption, while $V\left(\frac{c}{2}\right)$ is the surplus of a consumer who skips every second purchase.[1] A higher consumption rate (up to the socially efficient level) increases the target surplus, thus, the amount the seller can potentially capture. On the other hand, a higher consumption flow also increases the utility from skipping a purchase event. The enhanced threat of skipping limits the surplus that can be extracted. We show that consumption is distorted downward, relative to the efficient level (absent storage), but it is larger than consumption under optimal linear prices.

We extend the analysis to introduce heterogeneity in consumers' ability to store, a type of heterogeneity that naturally cannot arise in static models. This heterogeneity allows us to generate more realistic patterns of sales and to generate a novel empirical implication: the model predicts that sales are more important for large packages. While we emphasize that the model is very stark and does not allow for many features that are present in actual markets, it is interesting to note that this prediction is consistent with typical scanner data patterns. For example, Hendel and Nevo (2006b) report that while the small (32 oz.) detergent container is hardly on sale (2% of the time) the larger, most purchased, container size (128 oz.) is on sale 16.6% of the time. In the soda category, twelve and twenty four-packs are on sale twice as often as single soda cans (over 40% of the time vs 19.6%), while six-packs are on sale 34.3% of the time.

2 Related Literature

There is an extensive literature on non-linear pricing (see for instance Wilson (1997) and Rochet and Stole (2002)). The literature has considered many constraints on non-linear pricing, including resale and asymmetric information on consumers' valuations, as well as competition among producers (see for instance Stole (2007)). However, all the theory of non-linear pricing is static, and ignores potential effects that arises from intertemporal substitution in demand.

Several theoretical papers offer models of price dispersion (Varian (1980), Salop and Stiglitz (1982), Narasimhan (1988)), interpreted as sales, however,

[1]Skipping every second purchase is the most tempting deviation: if a consumer does not want to skip once, they also do not want to skip more than once.

these models do not capture the dynamics of demand generated by sales. Hong et al. (2002) closer to our interest, presents one aspect of the dynamics of sales. It is a competitive industry model, where consumers are assumed to chose a store based on the price of a single item and firms are informed about other firms' prices and hence sales. Jeuland and Narasimhan (1985) present a related idea in the context of a monopolist.

Dudine et al. (2006) provide an analysis of the role of commitment in a monopoly market with storable goods. They only consider linear prices and show that, in contrast with the literature on the Coase conjecture that discusses durable goods markets, prices are higher in all periods when the monopolist lacks commitment when goods are storable and demand anticipation motives are present.

Nava and Schiraldi (2012) explain sales, in the absence of consumer heterogeneity or a discrimination motive, based on collusion. Sales, and the induced storage, lower the incentives to deviate from collusion, and lower payoff during punishment periods.

Although it does not involve storage, Sobel (1991) also presents a model of sales (see also Conlisk et al. (1984), Sobel (1984), and Pesendorfer (2002)). The model involves a market with a durable good monopolist; at every date a mass of new consumers enter the market. Consumers have unit demands and two possible valuations for the good. Sobel (1991) characterizes the set of equilibria under the assumption that the monopolist cannot commit. An important feature of the analysis is that there can be price cycles.

Price cycles are also generated in the customer recognition literature (Villas-Boas, 2004) where firms price, non-anonymously, according previous purchasing behavior.

There is an extensive literature on durable goods.[2] The distinction between the products we consider, storables, and durables is tricky. The durable good literature is largely based on unit demand, one-time purchase, and the incentives to postpone such purchase. In contrast, the focus of this paper is on storage, which permits anticipating purchases. Naturally, multiple units of durables, cars or TVs, are often consumed. Buyers of most durables re-

[2]See Waldman (2003) for a survey.

turn to the market, and may do so in anticipation of their needs should prices grant it. We view the distinction between the assumptions in the durable good literature and our paper, as one capturing the frequency of purchase. For infrequently purchased products the one-time, single unit, purchase seems like a reasonable simplification. Instead, for frequently purchased non-perishable products, storage and demand anticipation are important forces to model.

3 Two-Period Model

3.1 Setup

We first consider a 2-period model with homogenous consumers. The simplicity of the two period environment helps to highlight some basic forces created by the interplay between storability and non-linear pricing.

Buyers' per period willingness to pay for consumption C is denoted $V(C)$. The function $V(C)$ is assumed to be increasing and concave, with $V(0) = 0$ and a saturation point C^* so that for all $C \geq C^* : V(C) = V(C^*)$. The latter guarantees bounded monopoly profits.[3] We normalize the marginal cost of production to zero.

As a benchmark, note that absent storability the seller would offer the efficient quantity C^*, with $V'(C^*) = 0$, and extract all the surplus with tariff $P^* = V(C^*)$ each period.

For simplicity we assume that there is no discounting. Thus, if the consumer has Q units to allocate over the two periods, he would like to consume equal amounts in each period. Consumers' payoffs from consumption C_1 and C_2 and payments P_1 and P_2 is given by:

$$V(C_1) + V(C_2) - (P_1 + P_2)$$

We assume that all consumers have a storage capacity of S. Storing quantity s, so that $0 \leq s \leq S$, is free, but it is impossible to store more than S.[4] The feasible consumption of a consumer who purchases bundles Q_1 and Q_2 in

[3] Alternatively, we could assume a positive marginal cost and $\lim_{C \to \infty} V'(C) = 0$.

[4] An earlier version of the paper considered a convex cost of storage and results were qualitatively similar.

period 1 and 2 respectively is given by $C_1 = Q_1 - s$ and $C_2 = Q_2 + s$.

Finally, we assume that the monopolist can commit to future actions: in the first period, the monopolist announces (and commits to) the menu for each period, and then consumers make decisions.[5] All transactions take place on the spot market, so that past history of purchases has no effect on current transactions.[6] The storage capacity is known by the seller. We relax this assumption later, when we allow heterogeneous storage.

3.2 Optimal policy

The monopolist's problem is to choose the sequence of bundles Q_1 and Q_2, and tariffs P_1 and P_2 to maximize the sum of revenues $P_1 + P_2$. The transfers are determined by three constraints.

First, we impose the standard participation constraint modified to take into account the possibility of storage. Transfers must be lower than the sum of utilities:

$$\max_{0 \le s \le S} \{V(Q_1 - s) - P_1 + V(Q_2 + s) - P_2\} \ge 0. \tag{1}$$

Second, consumers must be willing to purchase both bundles, as opposite to just one. Note that in the absence of storage, consumption in period 1 is independent of the consumption in period 2 and hence the participation constraints for period 1 and 2 are independent. By allowing the consumer to store, we introduce an additional obstacle for second period participation.

[5] We briefly discuss the case of no commitment in the two-period model in Section 3.2. Much of of the durable good literature focuses on the case in which the seller lacks commitment, partly because the commitment solution is straightforward. Understanding lack of commitment in the context of our model would be interesting (albeit complex). However, commitment is a useful benchmark to which the case with no commitment must be compared to understand the role of commitment. Furthermore, this is not an implausible assumption for some of the markets we model, in part due to the frequency in which consumers return to the market (in the case of retailers) and the frequency with which manufacturers and retailers contract, say in the case of supermarkets. Dudine, Hendel and Lizzeri (2006) show that in storable goods markets with linear prices the role of commitment can be quite different from markets with durable goods.

[6] If one allows the monopolist to bundle the portions of the good delivered in the future, the problem becomes trivial: the monopolist can offer the contract under which he delivers the efficient level of consumption in each period and the consumer pays his full surplus to the monopolist.

The consumer's outside option in the second period is greater than zero, due to storage.

The constraint is that the consumer should be willing to purchase $\{Q_2, P_2\}$ in the second period, rather than purchase only $\{Q_1, P_1\}$ in the first period and optimally smooth Q_1 over the two periods:

$$\max_{0\leq s\leq S}\{V(Q_1-s)-P_1+V(Q_2+s)-P_2\} \geq \max_{0\leq s\leq S}\{V(Q_1-s)-P_1+V(s)\}. \quad (2)$$

Finally, the consumer should be willing to purchase $\{Q_1, P_1\}$ in the first period:

$$\max_{0\leq s\leq S}\{V(Q_1-s)-P_1+V(Q_2+s)-P_2\} \geq V(Q_2)-P_2, \quad (3)$$

The next Lemma shows that only two of these three constraints are binding.

Lemma 3.1 *Constraints (2) and (3) imply constraint (1).*

For all proofs we refer the reader to the Appendix.

We call constraints (3) and (2) no-skipping constraints. They are participation constraints that assure the consumer does not skip a purchase event and consume out of storage. The simplicity of the two-period set up stems from the limited forms of skipping. In a longer horizon the consumer can prepare herself to skip in numerous ways, namely, the consumer can store out any of the previous purchases as long as she has available storage. Thus, the longer the history the more constraints on participation need to be imposed. As we show below eliminating or limiting the ability to skip plays an important role in shaping optimal offerings by the monopolist.

We now characterize the optimal solution. Denote the optimal sales by X_1 and X_2 and recall that C^* is the efficient consumption obtained by $V'(C^*) = 0$.

Theorem 1 *Assume S is such that $0 < S < C^*$.*[7] *In an optimal policy, the monopolist chooses first period output $X_1 = C^* + S$, and second period output X_2 such that $X_2 + S < C^*$. In this optimal policy*

[7] If $S \geq C^*$ the optimal policy for the monopolist is to set $X_1 = 2C^*$, $X_2 = 0$, and $T_1 = 2V(C^*)$: when storage capacity is very large, the monopolist can extract all the surplus by selling everything in the first period. We have in mind products that are frequently purchased, so that selling once and for all is not relevant nor interesting.

1. *the monopolist induces a binding storage constraint for the consumer*

2. *first period consumption is efficient*

3. *second period consumption is below the efficient level*

4. *consumers enjoy positive surplus.*

Storability induces the monopolist to create distortions in consumption away from the first best, even though consumers are homogenous and the monopolist can choose any non-linear price. Furthermore, the monopolist optimally chooses to allow the consumer to enjoy positive surplus. The two new no-skipping constraints introduced by storability are tighter than the standard participation constraint: if the monopolist targets all the surplus, the consumer always has the option of not purchasing in the second period and smoothing consumption through storage. However, this is not the whole story, since the seller could generate an allocation with fully extractable surplus (for example, by selling only in the first period).

To make the skipping of period-two purchases more difficult, the seller could lower first period quantity.[8] Alternatively, the seller could increase first period quantity beyond the capacity constraint. Indeed the monopolist's optimal policy is designed in such a way, that the consumer fills his storage up to its maximum capacity. Clearly, the storage provides an additional freedom for the consumer to allocate the consumption more efficiently across time, and hence gives him bargaining leverage against the monopolist. However, once the storage is filled, any additional quantity sold in the first period must be consumed immediately and hence the monopolist can extract the full value

[8]If the storage capacity is small enough, the monopolist has an alternative optimal policy, in which the role of the periods is reversed but the same profit is attained. The second period is the one in which the consumption is optimal. Intuitively, the no-skipping constraint becomes less restrictive when the storage capacity is small enough, so that the monopolist can make sure that the consumer is willing to purchase a large bundle (and pay a high tariff) in the second period. In this case it is still true that the first period tariff extracts the full surplus that is generated by the first period bundle. We have downplayed this second implementation for the following reasons. First, we are interested in the impact of storage, so we want to focus on the case where storage is large (bigger threat). Second, as we will see in the next section, the alternative equilibrium is not robust to the introduction of heterogeneity in storage.

of these additional units. This is the underlying reason why the monopolist induces a binding storage constraint for the consumer. Moreover, this also explains why first period consumption is efficient. Since, the seller can extract all surplus (as no first period quantity can be stored, because storage is already filled) from these additional units, it is optimal to expand consumption up to the efficient level.

Thus, the seller offers $X_1 = C^* + S$ and is able to extract the following first period tariff

$$P_1 = V(\underbrace{X_1 - S}_{C^*}) + V(\underbrace{X_2 + S}_{C_2}) - V(X_2).$$

This amount equals the full extra surplus the consumer obtains from the bundle X_1 where: $V(X_1 - S)$ is the surplus of consuming $X_1 - S$ in the first period and $V(X_2 + S) - V(X_2)$ is the surplus of consuming an extra S on top of the bundle X_2 in the second period.

Let us now check the second period offering and why the consumer is left with a positive surplus. The most the seller can extract in the second period is the difference in utility between showing up in both periods and buying only in the first period. Since storage is binding, the latter is $V(X_1 - S) + V(S)$, the efficient consumption plus full storage consumption. The second period tariff is:

$$P_2 = V(\underbrace{X_1 - S}_{C^*}) + V(\underbrace{X_2 + S}_{C_2}) - V(\underbrace{X_1 - S}_{C^*}) - V(S) = V(\underbrace{X_2 + S}_{C_2}) - V(S)$$

We are now ready to discuss the intuition behind the optimal X_2. Second period consumption is not efficient for the following reason. The second period tariff increases with the size of the bundle X_2, so it seems that the seller would have an incentive to sell enough in the second period to lead to a second period consumption $X_2 + S = C^*$. Indeed the latter maximizes P_2. However, these additional second period units affect the amount that can be extracted in the first period. Recall that the first period tariff consists of two parts: the value of consumption in the first period and the additional value of consuming the stored inventories in the second period: $V(X_2 + S) - V(X_2)$. Since the consumer's valuation function is concave, this part of the first period tariff is decreasing in X_2, so that second period consumption is a threat to first period surplus extraction. The monopolist has to balance the two effects, and ends up selling a bundle that leads to less than efficient second period consumption.

Let us return to surplus extraction. Both tariffs are set so that the seller captures the extra surplus generated by each offering. In other words, as we saw above, P_1 captures the additional surplus generated by X_1 relative to the consumer's utility should she only enjoy X_2 in the second period. Similarly for P_2. The seller is able to capture all additional surplus from each bundle. The reason the consumer manages to keep a positive surplus is that the marginal impact of X_1 is evaluated at X_2 and vice versa. Since V is concave, the marginal surplus of each offering is less than the surplus of removing both bundles.

It is easy to see that the optimal policy characterized in Theorem 1 is **not** an equilibrium absent commitment. In any pure strategy equilibrium absent commitment, second period consumption has to be efficient, otherwise the monopolist could increase profits in the second period. It turns out that equilibrium without commitment is quite complicated to characterize even in the two period problem. For instance, there is no pure strategy equilibrium in the two period model without commitment: for a wide range of first period output levels, it cannot be the case that in the second period storage is known and identical for all consumers. This creates many complications but also raises some interesting question for possible follow-up work.

3.3 Heterogeneous storage capacities

We now extend the model by allowing for a limited amount of consumer heterogeneity. The standard second degree price discrimination model (see for instance Tirole 1988, Chapter 1) characterizes optimal non-linear prices when there is heterogeneity in consumer valuations. One could of course perform the same exercise within our environment. However, we think that it is more useful to keep our focus on the effects of storability, so we retain the assumption of homogeneous consumer valuations, and instead allow consumers to have heterogeneous storage capacities. We assume that a fraction α of consumers has no ability to store, and a fraction $1-\alpha$ has storage capacity S as above.

We reserve capital letters for the consumer with storage P_1, P_2, Q_1 and Q_2

and small letters p_1, p_2, q_1 and q_2 for the consumer without one. For brevity we will call the consumers with storage S, and the consumers without storage NS.

The monopolist maximizes profits given by

$$\pi = \alpha(p_1 + p_2) + (1 - \alpha)(P_1 + P_2)$$

subject to the constraints that guarantee consumers choose the bundles that are meant for them. All the constraints that appeared in the case of single consumer carry over to this case, but we now must impose new self-selection constraints.

Neither type of consumer should skip a purchase in either period. For the storing consumer it amounts to constraints (3) and (2) above. For the NS-consumer they amount to the usual static participation constraints:

$$p_1 \leq V(q_1) \tag{4}$$
$$p_2 \leq V(q_2). \tag{5}$$

Furthermore, NS-consumers should not prefer to switch to the bundles P_1, Q_1 or P_2, Q_2:

$$p_1 - P_1 \leq V(q_1) - V(Q_1) \tag{6}$$
$$p_2 - P_2 \leq V(q_2) - V(Q_2) \tag{7}$$

and S-consumers should not switch to the whole bundle meant for NS-consumers:

$$p_1 + p_2 - P_1 - P_2 \geq \max_{0 \leq s \leq S}\{V(q_1 - s) + V(q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\}. \tag{8}$$

S-consumers should not substitute any part of their bundle with the deal that is offered to NS consumers

$$p_1 - P_1 \geq \max_{0 \leq s \leq S}\{V(q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} \tag{9}$$
$$p_2 - P_2 \geq \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} \tag{10}$$

and, finally, S-consumers should not prefer to choose just one period of the bundle intended for NS-consumers

$$p_1 - P_1 - P_2 \geq \max_{0 \leq s \leq S}\{V(q_1 - s) + V(s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} \quad (11)$$
$$p_2 - P_1 - P_2 \geq V(q_2) - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} \quad (12)$$

The next result offers a characterization of the optimal solution in this case with two types of consumers.

Theorem 2 *The optimal bundles with heterogeneous storage are such that $q_1(\alpha) < C^*$, $q_2(\alpha) = C^*$, $Q_1(\alpha) = C^* + S$ and $Q_2(\alpha) < C^* - S$. Under the optimal policy*

1. *the monopolist induces a binding storage constraint for the S-consumers*
2. *S-consumers' consumption is efficient only in the first period*
3. *NS-consumers' consumption is efficient only in the second period*
4. *S-consumers enjoy positive surplus and there is $\widehat{\alpha}$ such NS-consumers enjoy positive surplus if $\alpha < \widehat{\alpha}$.*

The S-consumer's bundles Q_1 and Q_2 are qualitatively similar to those characterized in Theorem 1. The only difference is that the second period quantity is affected by the presence and proportion of non-storers. The presence of S-consumers instead, drastically changes the way NS-consumers' bundles are priced. Both consumption and surplus are affected. First period consumption drops below the efficient level, and even NS-consumers enjoy a positive surplus.

When facing heterogeneous consumers, the monopolist has to worry not only about participation and no-skipping constraints, but also about incentive compatibility constraints. To understand the way incentive compatibility affects the offerings suppose that the bundles for the S-consumer are priced as in Theorem 1. The first period bundle $Q_1 = C^* + S$ is sold. We showed that the S-consumer pays for this bundle more than his first period surplus, since he purchases additional quantity S for second period consumption. Do

NS-consumers prefers to buy the bundle Q_1 instead of q_1? They do not. All consumers have the same valuation for the good, but the NS-consumer unable to store, have to consume the whole bundle in the first period. They are willing to pay at most $V(C^*)$ which is lower than what S-consumer pays, P_1. However, the reverse is possible. If q_1 is large enough, the S-consumer can be tempted to buy the cheaper bundle of the two, which is q_1. The latter might be cheaper as it is priced for current consumption, but can be stored (and smoothed) by S-consumers.

Now consider the second period. NS-consumers are offered bundle $q_2 = C^*$. The S-consumer does not value this bundle as much, because he has S already in storage (recall, that C^* is the saturation point). On the other hand, if the price for the bundle q_2 is close to $V(q_2)$, NS-consumer might prefer the smaller bundle Q_2 that sells for less than $V(Q_2)$, since it is priced for a consumer who has supply in storage.

To summarize, the monopolist only has to make sure S-consumers do not switch to the NS-bundle in the first period, and NS-consumers not switching to S-bundle in the second period. We can now turn to NS-consumer tariffs:

$$\begin{aligned} p_1 &= V(q_1) \\ p_2 &= P_2 + V(q_2) - V(Q_2). \end{aligned}$$

It is easy to see that the optimal policy extracts the full surplus from the NS-consumer in the first period, but the level of consumption is below the efficient one. In the second period the situation is reversed: consumption is at the efficient level, but the tariff is lower than $V(C^*)$.

The monopolist can use two instruments to make sure consumers purchase their intended bundles: the induced consumption levels and the tariffs. It may be surprising that when the monopolist wants to keep the S-consumer from switching to NS-bundle he mostly operates by distorting consumption, and when he wants to prevent the NS-consumer from switching, he operates mostly through lowering the tariff. In the first period, the monopolist sets q_1 low enough, so that it is not attractive for the S-consumer. Once the size of the bundle is determined, the monopolist can set the highest price under which the bundle is still purchased, which is $V(q_1)$. The monopolist cannot achieve the same effect only by using tariffs, because S-consumers' effective willingness to pay in the first period is higher than NS-consumers': this is because S-consumers can smooth consumption of the good across periods, whereas NS-consumers cannot. Naturally, the higher the proportion of NS-consumer the more costly the quantity distortion is, that is why $q_1(\alpha)$ increases in α.

Instead, in the second period the S-consumer receives the bundle Q_2 at price $P_2 = V(Q_2 + S) - V(S) < V(Q_2)$. If the NS-consumer purchases Q_2 instead of q_2 he can get a surplus of $V(Q_2) - P_2$. The monopolist must guarantee the NS-consumer the same surplus from purchasing the bundle q_2. The only way he can eliminate this surplus is to set $Q_2 = 0$, which in turn will result in $P_2 = 0$. In the optimal solution, the monopolist sets the level of NS-consumption to be efficient and extracts all the surplus up to $V(Q_2) - P_2$, by setting a low enough tariff. As the proportion of NS-consumers increases the seller is willing to further distort Q_2 downward, explaining why $Q_2(\alpha)$ is a decreasing function.[9]

The picture that emerges from this analysis is that the skipping constraints can be relaxed by filling the storage capacity of the consumer. Moreover, consumption flows of non-storers are distorted downwards to alleviate storers' incentives constraints.

Finally, recall that in the case of homogeneous consumers if the storage capacity is small enough, there exists an alternative optimal pricing scheme for the monopolist, in which the role of the two periods is switched: the monopolist induces efficient consumption in the second period, and distorts consumption of the good in the first period. When consumers are heterogeneous in storage capacity this alternative pricing scheme is no longer optimal. This policy involves a small bundle in the first period, so that the consumer does not have significant outside options in the second period. However, if the small bundle is sold in the first period for the S-consumer, it must be the case that the small

[9]If the share of the S-consumers is small enough it is optimal to set $Q_2 = 0$ to make sure, that all the surplus of the NS-consumers is extracted. In particular, if $\alpha \geq \frac{V'(S)}{V'(0)}$, it is optimal to set $Q_2 = 0$. However no matter how small or large α, if $\alpha \in (0,1)$ neither S- nor NS-consumers are ignored by the monopolist.

bundle is sold for the NS-consumer as well, which is a significant loss in the revenue collected from NS-consumers.

4 Non-Linear Prices: Infinite Horizon

The two-period model has the advantage that it is a fairly simple way to gain an initial understanding of the constraints imposed by storability, and it could be extended to richer forms of heterogeneity. However, the two-period model also has clear shortcomings. For example, it is unclear if the alternation between different levels of consumption, and the fact that the monopolist induces a binding storage constraint are artifacts of the particular setup or fundamental features of storable goods. To address these questions, we approach the problem from another angle with some advantages and some other limitations. We consider a very stark infinite horizon model, in which the monopolist repeatedly interacts with the same consumers so that the problem has some degree of stationarity. In this longer horizon model consumers can start and end a period with storage, thereby enabling inventories as a tool at the disposal of the consumer in every period; this is not feasible in a two-period world. However, despite its extreme simplicity in some respects, the model is much more difficult to analyze and we are only able to obtain a partial characterization of the monopolist's optimal policies.

We assume that time is continuous. This assumption is mostly made for technical reasons because it helps us to avoid dealing with divisibility issues.[10]

In the infinite horizon model, we reserve small letters for flow variables and capital letters for all other ones. The consumption of an agent is denoted by c_t. The flow value of consumption is given by $V(c_t)$. If the consumer is given a flow c_t for an interval of time $[0, T]$ his average willingness to pay for this flow is given by

$$\frac{1}{T}\int_0^T V(c_t)dt$$

where $V(c)$ is assumed to satisfy the same assumption as in the previous section.

[10] Suppose, that the induced consumption of the consumer in the discrete time model is C^* and the storage is S. If $\frac{S}{C^*}$ is not a natural number, the no-skipping constraints become more cumbersome.

We assume that there is no discounting. This assumption, together with concavity of V imply that an agent endowed with stock Q of the good over time period T, would like to consume Q/T each period. This drastically simplifies the consumer's problem, allowing us to offer a particularly simple exposition of some of the key effects.

The monopolist can choose to sell either a flow or a stock of the consumption good. A flow sale is of the same magnitude as instantaneous consumption, which means that it is of measure zero relative to any interval of periods of positive consumption. This of course does not mean that flow sales cannot be stored. As in the two-period model, we do not allow the monopolist to offer history contingent prices, so all transactions happen on the spot market, hence the flow sales are associated with flow tariffs and stock sales with the stock tariffs.

The monopolist may receive both flow payments p_t (for flow sales) and stock payments P_i (for stock sales). The monopolist maximizes his average profit, i.e.

$$\limsup_{T\to\infty}\frac{1}{T}\left(\int_0^T p_t dt + \sum_{i\in I\subset[0,T]} P_i\right).$$

As in the two-period model we assume that the monopolist can commit to the sequence of bundles.[11] Again, we believe that this is a useful benchmark.

As in the two period model, the consumer has storage of size S. He can fill it in two ways. If he purchases a stock bundle of size Q_t his inventory discontinuously jumps by Q_t:

$$S_t = S_{t-0} + Q_t.$$

If he purchases the flow bundle q_t he can store the leftovers of his consumption. Suppose the consumer purchased only a flow q_t in the interval of time (t_1, t_2),

[11] However, given our assumptions, and in contrast with the two-period model, we believe that the optimal policy under commitment would also be an equilibrium as a limit of a suitably modified model without commitment. Note that in contrast with the extreme models of durable goods monopoly, in our model there are recurring sales which gives ample scope for sustaining non competitive profits in equilibrium when discount factors are high.

then the consumer's inventory at t_2 is given by

$$S_{t_2} = S_{t_1} + \int_{t_1}^{t_2} (q_t - c_t)dt.$$

Observe that if the consumer has no storage capacity, i.e., $S = 0$, then the optimal policy for the monopolist is to sell the efficient flow bundle $c_t = c^*$ for all t, and extract the entire surplus by charging $p_t = V(c^*)$ for all t.

However, we now show that when the consumer can store a monopolist who offers only flow bundles completely loses the power to price non-linearly.

4.1 Storability eliminates non-linear pricing of flow sales

The easiest way to see that no extra surplus can be extracted from non-linear pricing of flow bundles is to consider the case in which the monopolist offers a constant flow $q_t = q$ at flow-bundle price p. Given this policy by the monopolist, consumers' optimal consumption is given by $V'(c) = \frac{p}{q}$ in all periods implying that consumers can fully unbundle the monopolist's flow-bundle price.

More formally, denote by λ the fraction of periods when a consumer purchases the good. Since the consumer can accumulate inventories, and the capacity constraint is not binding in the case of flow sales, perfect smoothing of consumption is feasible and optimal, so that $c_t = \lambda q$ in every period. The cost of this policy on average is λp per period. Thus, the consumer picks λ as follows:

$$\lambda^l = \arg \max_{\lambda \in [0,1]} \{V(\lambda q) - \lambda p\}$$

If the per unit price is not too high (i.e. $\frac{p}{q} < V'(0)$ otherwise there would be no purchases), the optimal λ^l satisfies

$$V'(\lambda^l q) = \frac{p}{q}.$$

Flow profits in this case are given by $c^l V'(c^l)$, where $c^l = \lambda^l q$. Notice that c^l, by the f.o.c. above, is also the optimal consumption of a buyer that faces the linear price $\frac{p}{q}$. Thus, with storage, selling a flow bundle q at bundle price p is equivalent to setting a linear price $\frac{p}{q}$.

The implication is that when the monopolist offers a constant flow of bundles q, the ability of the consumer to store destroys all the monopolist's ability to price non-linearly: the best policy within this class is equivalent to a static linear pricing policy.

This result is an intertemporal parallel to the common wisdom that successful non-linear pricing requires constraints on arbitrage among consumers. In our model, the arbitrage takes place across the different periods for the same consumer.

The next results shows that the difficulty faced by the monopolist is much more pervasive. To start with, we consider what happens for any sequence of flow-bundle sales, not just stationary ones.

Theorem 3 *If $S > 0$ and the monopolist is restricted to sell only flows of the good (i.e., the monopolist can not offer stocks Q_t) then the optimal policy is revenue-equivalent to linear pricing.*

Although the proof is more complex, the logic of this result is similar to the one outlined above for constant flow-bundle sales: with flow sales, the storage constraint is never binding and the consumer can time his purchases to unbundle the monopolist's attempt to price bundles non-linearly.[12]

4.2 Storability and stationary stock sales

We now investigate whether the monopolist can restore some of its ability to extract surplus via non-linear prices by resorting to stock bundle rather than just flow bundles. The next result shows that simply selling stocks instead of flow bundles is not sufficient: intertemporal arbitrage by the consumer does not depend on the monopolist selling a flow of goods, but rather depends on the frequent availability of purchasing opportunities.

[12]There is a connection between this result and results in the dynamic agency literature on the limits on contracting imposed by the agent's ability to time effort or savings decisions to undermine complicated nonlinear incentive schemes. For instance, see Cole and Kocherlakota (2001) where savings impose significant constraints and Holmstrom and Milgrom (1987) where, under exponential utility and Brownian motion, and utility only over final consumption, the optimal contract is linear in aggregate final outcomes.

Theorem 4 *In the class of policies that make available a constant bundle Q at (non-linear) price P at each point in time the optimal policy is revenue-equivalent to linear pricing.*

Proof. The consumer's problem is to choose the intervals, T, at which to purchase in order to maximize flow utility. The consumer then optimally smooths consumption within the period, to get a utility flow $V\left(\frac{Q}{T}\right)$:

$$\max_{T}\left(V\left(\frac{Q}{T}\right)-\frac{P}{T}\right)$$

The optimal T solves:

$$\begin{aligned} -\frac{1}{T^2}\left(QV'\left(\frac{Q}{T}\right)-P\right) &= 0 \quad \text{or} \\ V'\left(\frac{Q}{T}\right) &= \frac{P/T}{Q/T} \end{aligned}$$

The consumer times purchases so that marginal utility equals the unit price of flow consumption $\frac{P/T}{Q/T}$, where the numerator is flow price, and the denominator is flow purchases. Thus, with stationary policies, the monopolist loses all the ability to price non-linearly. Profits are not higher than charging a unit -linear-price.[13] ■

Theorem 4 can be partially extended to a limited class of time-varying bundles Q_t.[14] The main force behind consumers' ability to intertemporally unbundle non-linear prices in the previous results is due the fact that they have ample opportunities to time their purchases to construct their desired sequence of consumption.

We now show that, by choosing cyclical policies the monopolist can do better than the profits from linear prices. It can partially restore its ability to extract surplus via non-linear pricing by limiting the opportunities for

[13] We neglected the constaint $Q \leq S$. Selling more than S would be immaterial since it cannot be carried forward.

[14] The undoing of nonlinear pricing by storage could be obtained much more generally: heterogeneity in consumers' valuation, and convex costs of storage. However, this would require much more elaborate analysis and we believe that the main idea is usefully conveyed in our setup.

consumers to time purchases and unbundle the non-linear prices. The idea is to limit consumers smoothing opportunities by selling only infrequently, and forcing the consumer to buy bundles that use-up all the storage capacity, at each purchase.

We only consider the class of periodic sales where the monopolist offers a stock Q at bundle price P^Q at periods separated by constant time intervals T^Q; in all other periods the monopolist does not sell anything (or sets a price so high that consumers will never purchase).[15]

Theorem 5 *The monopolist can improve on linear pricing profits by using periodic sales.*

In the class of periodic sales, the optimal policy is to only make available a bundle equal to the storage capacity S at periods separated by constant intervals T^S. The optimal price P^S and interval T^S are given by

$$P^S = 2T^S \left(V(c^S) - V\left(\frac{c^S}{2}\right)\right); \quad T^S = \frac{S}{c^S},$$

where c^S is the consumption that solves

$$c^S = \arg\max_{c \geq 0} \{V(c) - V\left(\frac{c}{2}\right)\}.$$

At this optimum, consumption, profits, and welfare are independent of S. In addition, if the function $cV'(c)$ is single-peaked, they are strictly between those obtained under linear pricing and those obtained under non-linear prices absent storage.

Proof. Without loss of generality we can assume that all bundles offered by the monopolist are actually purchased by the consumer at the optimum. Otherwise the monopolist can, at no loss, redesign the policy to get rid of the bundles that are not purchased.

[15] We conjecture that the policy outlined in Theorem 5 is optimal in the class of all policies, but we could not prove this. This result does show that any optimal policy has to have cyclical characteristics because our periodic sales policy does better than any constant stationary policy.

Suppose that the monopolist sells bundle Q every T periods and charges P. Consumption in this case is $c = \frac{Q}{T}$. Since the consumer makes a purchase every T periods, the price must be such, that he is not willing to skip a purchase and consume his inventories. The following inequality guarantees that the consumer does not wish to skip a single purchase:

$$\frac{P}{T} \leq 2V(c) - 2V\left(\frac{c}{2}\right) \tag{13}$$

We now show that the above inequality implies that skipping more than one purchase in a row is not beneficial either. To show that we need to prove that

$$k\frac{P}{T} \leq (k+1)V(c) - (k+1)V\left(\frac{c}{k+1}\right)$$

From (13) we know, that

$$k\frac{P}{T} \leq 2kV(c) - 2kV\left(\frac{c}{2}\right)$$

Note, that

$$2kV(c) - 2kV\left(\frac{c}{2}\right) - \left((k+1)V(c) - (k+1)V\left(\frac{c}{k+1}\right)\right) =$$

$$(k-1)V(c) + (k+1)V\left(\frac{c}{k+1}\right) - 2kV\left(\frac{c}{2}\right)$$

By concavity of $V(\cdot)$ we obtain, that

$$(k-1)V(c) + (k+1)V\left(\frac{c}{k+1}\right) \leq 2kV\left(\frac{c}{2}\right)$$

hence

$$k\frac{P}{T} \leq 2kV(c) - 2kV\left(\frac{c}{2}\right) \leq (k+1)V(c) - (k+1)V\left(\frac{c}{k+1}\right)$$

Of all constraints for this problem, (13) is the tightest. Notice, that (13) does not depend on Q, but only depends on consumption. Hence, setting $Q^S =$

S is weakly optimal for the relaxed problem which only involves the constraint that we discussed above. Also, by setting $Q^S = S$ we make all other constraints obsolete. In fact, if we set $Q^S = S$ consumers can not use inventories that were purchased more than T periods ago for current consumption. Hence, we can restrict our attention to events of skipping consecutive purchases.

Since the monopolist is maximizing the flow of payments, he can set $\frac{P}{T} = 2V(c) - 2V\left(\frac{c}{2}\right)$ and solve for the optimally induced consumption:

$$c^S = \arg\max_{c \geq 0} \left\{ V(c) - V\left(\frac{c}{2}\right) \right\}$$

The rest of the solution is straightforward: $T^S = \frac{S}{c^S}$ and $P^S = 2T^S V(c^S) - 2T^S V\left(\frac{c^S}{2}\right)$.

It remains to show that both profits and the induced consumption at the optimum are strictly between those that arise under linear pricing and in the absence of storage. If the monopolist sells a bundle S at constant intervals T and induces the consumption c, the flow profit is $\pi^S = 2V(c) - 2V(\frac{c}{2})$. If storage is not feasible, the optimal consumption satisfies $V'(c^*) = 0$ and the flow profit is $\pi^* = V(c^*)$. If the monopolist uses the optimal linear pricing that induces the consumption c^l, the flow profit is $\pi^l = c^l V(c^l)$.

First we prove that when storage is unavailable both profits and the consumption are larger then under optimal periodic policy. Note first that $2V(c) - 2V\left(\frac{c}{2}\right) < V(c)$ by the strict concavity of $V(c)$. Since c^* maximizes $V(c)$, we obtain that $V(c^*) \geq V(c) > 2V(c) - 2V(\frac{c}{2})$ for any c including c^S. The first order condition for c^S is $V'(c^S) - \frac{1}{2}V'\left(\frac{c^S}{2}\right) > 0 = V'(c^*)$, hence by strict concavity of $V(c)$, we obtain that $c^* > c^S$.

Under linear pricing, the largest profit the monopolist can obtain is $c^l V'(c^l)$. Here, we argue, that this profit is achievable under the periodic policy as well. Indeed observe, that

$$2V(c^l) - 2V\left(\frac{c^l}{2}\right) = 2\int_{\frac{c^l}{2}}^{c^l} V'(x)dx$$

Since $V'(c^l) < V'(c)$ for any $c < c^l$ we get

$$2\int_{\frac{c^l}{2}}^{c^l} V'(x)dx > 2\int_{\frac{c^l}{2}}^{c^l} V'(c^l)dx = c^l V'(c^l).$$

Finally, $2V(c^S) - 2V\left(\frac{c^S}{2}\right) \geq 2V(c) - 2V\left(\frac{c}{2}\right)$ for any c, hence $2V(c^S) - 2V\left(\frac{c^S}{2}\right) > c^l V'(c^l)$.

To prove that $c^S > c^l$ we use the assumption that $cV'(c)$ is single-peaked. Observe, that the first order condition for c^S is

$$c^S V'(c^S) = \frac{c^S}{2} V'\left(\frac{c^S}{2}\right)$$

By single-peakedness of $cV'(c)$, it must be the case that $c^l \in \left(\frac{c^S}{2}, c^S\right)$. ■

The policy outlined in Theorem 5 is cyclical. This gives us a theory of sales, based on storability, even in a stationary environment with identical consumers. The intuition for the fact that a cyclical policy can restore some of the ability by the monopolist to extract surplus via non-linear prices is the following. The fact that the bundles are only available infrequently implies that there are fewer no-skipping constraints for the monopolist to worry about; it is more costly for the consumer to skip a purchase. By selling bundles that fill up the storage capacity of the consumer, the monopolist makes it harder for the consumer to smooth consumption in the event that he chooses to skip a purchase, thereby enabling the monopolist to extract more surplus.

To gain some additional insight into the role of infrequent sales, let us go back to the no-skipping constraint which determines prices. Any purchase has to give the consumer higher utility than skipping it and smoothing optimally out of storage. The utility from buying for two periods in a row: $2TV\left(\frac{Q}{T}\right) - 2P$ has to exceed the utility from smoothing the first purchase over both periods: $2TV\left(\frac{Q}{2T}\right) - P$. Using both terms we compute the highest P that guarantees participation:

$$P = 2T\left[V\left(\frac{Q}{T}\right) - V\left(\frac{Q}{2T}\right)\right]$$

Skipping saves P but causes the loss in utility derived from consuming $\frac{Q}{2T}$ instead of $\frac{Q}{T}$ over the $2T$ interval. Why is it optimal to fill the storage capacity (namely, setting $Q = S$)? By properly adjusting T, the single-skipping constraint just considered leads to identical profits for any Q as long as $\frac{Q}{T}$ remains unchanged. The advantage from setting $Q = S$ comes from the fact that, when $Q < S$, there is additional storage capacity available to plan ahead of skipping a purchase, thereby smoothing out more evenly the pain of skipping and inducing a tighter no-skipping constraint.

Recall that the first order condition for consumption that maximizes flow revenues is: $V'(c^S) = \frac{1}{2}V'\left(\frac{c^S}{2}\right)$. Note first that $c^S < c^*$. This follows from $V'' < 0$ and the first order conditions for c^S. If $V'(c^S) = 0$ and $V'(\frac{c^S}{2}) > 0$ the first order conditions would not hold. The idea is that by increasing consumption toward the optimal level the seller generates more consumer surplus, but also increases the skipping threat. The higher c^S the higher the utility from skipping the second purchase. This last effect pushes the optimal consumption below the efficient level. The last term is due to storage.

In order to compare c^S to consumption under the optimal linear prices c^l notice that the latter is set where revenue $cV'(c)$ is highest, namely, by $V'(c) + cV''(c) = 0$ or where the inverse demand elasticity $\frac{cV''(c)}{V'(c)} = 1$. In the optimal policy with periodic sales instead, $2V'(c^S) = V'(\frac{c^S}{2})$, which implies that the arc elasticity of V' between c and $\frac{c}{2}$ is 1. Under the standard assumption of decreasing elasticity (which is guaranteed by demand not being too convex) this implies that the elasticity evaluated at c^S is lager than 1, and thus $c^S > c^l$, since the elasticity is 1 at the latter.

4.2.1 Quadratic Example

In order to give some sense of the magnitude of the effects of storability, we now provide an example with quadratic preferences (linear demand). Assume $V(q) = q - \frac{q^2}{2}$. In this case, the optimal solution presented in Theorem 5 involves a frequency of sales: $T^* = \frac{3}{2}S$. This is associated with a flow consumption $\frac{S}{T^*} = \frac{2}{3}$ and average profits $\frac{P^*}{T^*} = \frac{1}{3}$.

Table 1 offers a contrast between this solution and those of non-linear pricing absent storability, and of linear pricing. As one can see, storability generates sizable distortions in consumption and a reduction profits relative

Table 1: Linear vs Non-Linear Pricing

Regime	**Consumption**	**Profits**	**C. Surplus**
Non-Linear no storability	1	$\frac{1}{2}$	0
Non-Linear with storability	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
Linear	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

to static non-linear pricing, but periodic sales allow the monopolist to extract substantially more than via linear pricing.

4.3 Heterogeneous storage capacities

As in Section 3.3, we now consider the possibility that consumers are heterogeneous in their storage capacity. The purpose of this extension is to generate richer testable implications and more realistic pricing patterns. Indeed, an unpalatable feature in the policy outlined in Theorem 5 is that in between stock sales the monopolist does not sell.

We again assume that a fraction $(1-\alpha)$ of consumers have storage capacity S while the rest cannot store at all. All consumers have the same preferences. The presence of no storage consumers (NS-consumers) reintroduces the necessity to offer flow bundles q_t. If this was the only option for the monopolist then it would only capture the surplus from non-linear pricing the non-storers at the expense of reducing the surplus extracted from storers.

However, the monopolist can simultaneously offer a flow-bundle intended for non-storers and an infrequent stock-bundle intended for storers. We now characterize the optimal policy where the monopolist offers a flow bundle q_t at price p_t and a stock-bundle S every T periods at price P.

NS-consumers can only purchase flow q_t, but the S-consumers can purchase both flow and bulk sales. Moreover, since the consumers are anonymous, the S-consumers can purchase multiple flow bundles at a time. So, if at time t, the S-consumer decides to buy b_t flow bundles, he will have to pay $b_t p_t$ and his inventory will go up by $b_t q_t$:

$$s_t = s_{t-0} + b_t q_t.$$

We first note that the price for the flow should be $p_t = V(q_t)$. If it were greater than $V(q_t)$ then consumers who can not store would not buy the good. If it were less than $V(q_t)$ it could be raised up to $V(q_t)$ without violating the participation constraints of those who do not store, and at the same time make skipping bulk sales more difficult for those who have storage.

Recall, that if the monopolist offers a flow bundle q_t to the NS-consumers alone, the flow profit that he collects is

$$\pi^{NS}(q_t) = V(q_t).$$

As we showed in Section 4.2 the policy of selling a stock bundle of size S to S-consumers every T periods generates a flow profit of

$$\pi^{S}(c) = 2V(c) - 2V\left(\frac{c}{2}\right).$$

We also define an auxiliary function that is helpful in stating our next result:

$$\pi^{\Delta}(c,q) \equiv 2\left(V(x) - V\left(\frac{c}{2}\right) - \left(x - \frac{c}{2}\right)V'(x)\right)\mathbf{1}\left\{\frac{c}{2} \leq x\right\},$$

where

$$x(q) = V'^{-1}\left(\frac{V(q)}{q}\right).$$

We are now ready to state a result that characterizes the monopolist's optimal policy. As we mentioned, the monopolist can offer both stock and flow bundles on the market. The only restriction we place on the monopolist, is that if the stock bundles are offered, they have to be sold every T periods and the size of these stock bundles has to be constant. The monopolist is of course free to choose both the period length T and the size of the bundle Q. He is also free to set any flow bundle sequence q_t.

Theorem 6 *In the class of periodic policies, the optimal policy consists of* *both flow and stock bundles. A stock bundle of size S is offered every T^α periods, and a flow bundle of constant size q^α is offered all the time. This policy induces a consumption c^α for the S-consumers and q^α for the NS-consumers. The pair (c^α, q^α) solves*

$$(c^\alpha, q^\alpha) = \arg\max_{c,q}\{\alpha\pi^{NS}(q) + (1-\alpha)\left(\pi^S(c) - \pi^\Delta(c,q)\right)\}.$$

Under the optimal policy:

1. *S-consumers only purchase stock bundles of size S, and NS-consumers only purchase flow bundles of size q^α*
2. *S-consumers pay lower per-unit price than NS-consumers*
3. *the presence of S-consumers lowers the consumption of NS-consumers: $q^\alpha \leq c^*$*
4. *the presence of NS-consumers increases the consumption of S-consumers: $c^\alpha \geq c^S$.*

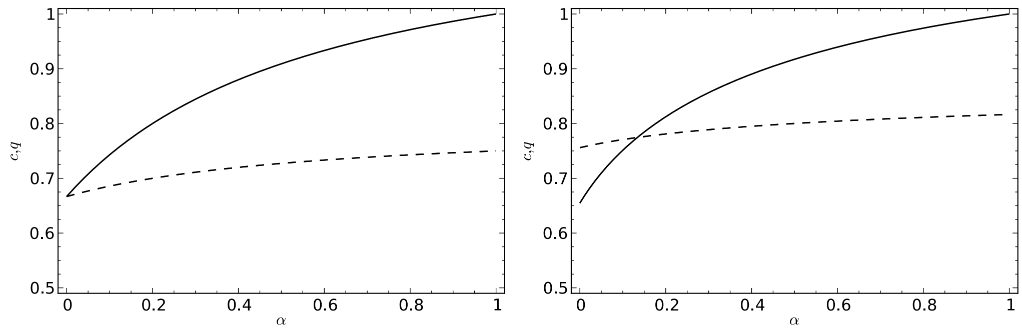
The monopolist's profit consists of two parts. The first part is $\alpha\pi^{NS}(q) + (1-\alpha)\pi^S(c)$, which is the hypothetical profit, the monopolist would collect if he could perfectly identify which consumers can store and which can not. The second part is $(1-\alpha)\pi^\Delta(c,q)$. This is the penalty, due to the fact that the monopolist cannot prevent S-consumers from purchasing the flow bundles that are meant for NS-consumers. Indeed, if $V'\left(\frac{c}{2}\right) \geq \frac{V(q)}{q} = V'(x)$, the S-consumer's bargaining position improves since in the event of skipping a sale of a stock bundle, he can purchase a flow bundle and increase his consumption by $\left(x - \frac{c}{2}\right)$. For the storage consumer, this increase in consumption is effectively priced according to a linear per unit price of $V'(x)$, which is smaller than the average per unit gain in utility $\frac{V(x)-V\left(\frac{c}{2}\right)}{x-\frac{c}{2}}$. The gain from purchasing $\left(x - \frac{c}{2}\right)$ units at linear price $V'(x)$ cannot be extracted by the monopolist, hence must be granted to S-consumers as a discount on the price of a bulk bundle.

According to Theorem 6, S-consumers pay a lower per-unit price than NS-consumers. This result follows from the no-arbitrage condition. Since S-consumers have more freedom in the market (i.e., they can easily mimic the choices of NS-consumers), they pay a lower price.

Note that if $V'\left(\frac{c^S}{2}\right) < \frac{V(c^*)}{c^*}$ the optimal policy is a combination of two optimal policies for for the S- and NS-consumers, as if the types of the consumers were observable. The condition above guarantees, that the flow bundles meant for NS-consumers are not going to be purchased by S-consumers, so the separation of types in this case comes for free.

In light of Theorem 3 it is worth emphasizing that the monopolist's policy involves both flows and stocks. Theorem 3 shows that flow sales can be linearized by the storing consumer, thus failing to deliver profits beyond linear pricing. It is interesting that, when consumers are heterogeneous in their storage capacities Theorem 6 says that the monopolist benefits from offering stocks to storers, even when a flow is available. The monopolist has to leave as much surplus to storers as they would achieve by unbundling the flow offered to non-storers. Actually, the flow is likely to be offered at a low linear price. Recall that the linearized price is $\frac{V(q)}{q}$. Moreover, since the ideal non-storer bundle involves $V'(q) = 0$ the linearized price $\frac{V(q)}{q}$ is quite low, or at least the monopolist would like to target non-storers with a bundle that leads to low linearized prices. The intuition for why it is still possible for the seller to extract more from storers than by just pricing linearly can be seen in a static framework. Can the seller benefit from selling a bundle to a consumer who can also purchase at linear price p? The seller can offer the efficient bundle, and price it to leave buyers with at least the surplus obtained from linear prices. A similar calculation is at work here, with the additional consideration that the seller offers less than optimal consumption to the storer due to the skipping constraint.

The policy outlined in Theorem 6 has some interesting empirical content. It predicts that sales are more relevant for large bundles. As discussed in the Introduction this is a typical pricing pattern found in scanner data. The result is also consistent with smaller, more urban-located, stores to have less prominent promotional activity. Buyers in urban areas have less storage, or at least, a smaller proportion of them are likely to store. Naturally, we have to be careful making strong empirical prediction due to the simplicity of the model,



(a) $V(c) = -\frac{c^2}{2} + c$ (b) $V(c) = -\frac{c^3}{3} + c$

Figure 1: Consumption of S– (dashed line) and NS– (solid line) consumers.

which allows for a very limited form of heterogeneity. It would be natural to assume that customers differ not only in storage but preferences as well, perhaps in a systematic way, with more intense buyers being more likely to store. The positive correlation between storage and usage may reinforce the finding that larger containers are more likely to be promoted.

4.3.1 Example

We now return to the example with linear demand presented in Section 4.2.1. As shown in Fig. 1a (see page 30), consumption of both S– and NS– consumers increases when the share of the latter goes up. In the case of linear demand, NS-consumers always consume more than S–consumers from the same population. However, this property does not hold in general. The example shown on the Fig. 1b illustrates this point. If the share of NS– consumers is small the potential profit that can be extracted from this part of the population is negligible. It is not optimal to offer a large flow for the NS–consumers because it induces a low per unit price of a good and provides S–consumers with additional bargaining leverage against the monopolist. To avoid losses that are caused by the strong bargaining position of S–consumers, the monopolist induces a small consumption (and hence a high per unit price) for NS–consumers.

5 Concluding Remarks

We studied the impact of product storability on non-linear pricing. We showed that storability can enable consumers to undo sellers' attempt to price non-linearly. The constraint is particularly severe under constant offerings, in which case sellers cannot extract more surplus than under linear prices.

Cyclical pricing, in the form of infrequent and bulky sales, constrain buyers ability to undo non-linear prices. Infrequent sales limit skipping opportunities, while bulky sales make harder for consumers to get ready to skip a purchase. Thus, the model delivers a theory of sales. Unlike most explanation of sales, this one is **not** based on a discrimination motive.

Allowing for heterogeneity in storage delivers testable implications consistent with observed patterns. However, this paper has only begun to explore the interactions of heterogeneity and storability. A richer model would be particularly useful for delivering a broader set of empirical predictions. The richer model could allow heterogeneous preference, correlation between preferences and storage, as well as other storage technologies.

This paper has also focused on an environment on a monopolist that can commit to a sequence of bundles. Natural next steps would be to consider seller competition and lack of seller commitment.

6 Appendix

Proof of Theorem 1

Observe that constraints (3) and (2) can be written as

$$\begin{aligned}
P_1 &\leq \max_{0\leq s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-V(Q_2)\\
P_2 &\leq \max_{0\leq s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-\max_{0\leq s\leq S}\{V(Q_1-s)+V(s)\}.
\end{aligned}$$

Since the monopolist maximizes P_1+P_2, both constraints are going to be binding at the optimum. Substituting the constraints into the objective function, the monopolist's problem becomes

$$\max_{Q_1,Q_2}\left\{2\max_{0\leq s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-V(Q_2)-\max_{0\leq s\leq S}\{V(Q_1-s)+V(s)\}\right\}.$$

Suppose that the monopolist induces a binding storage constraint for the consumer. Lemma 6.1 on page 33 proves that under the optimal policy this is indeed the case.

Note that, in our candidate policy, the consumer purchases in both periods. Therefore, the monopolist maximizes

$$\Pi^S = \max_{Q_1,Q_2} \{V(Q_1 - S) + 2V(Q_2 + S) - V(Q_2) - V(S)\}$$

At the optimum we must have

$$V'(Q_1 - S) = 0 \tag{14}$$

and

$$2V'(Q_2 + S) \leq V'(Q_2). \tag{15}$$

Equation (14) implies that the optimal first period output is $X_1 = C^* + S$.

Let us now consider the optimal choice of second period output. If $2V'(S) < V'(0)$ then the solution is $X_2 = 0$. Thus, the optimal solution involves $X_2 = 0$ whenever $S \geq \hat{S}$, where $\hat{S}$ solves

$$2V'(\hat{S}) = V'(0).$$

In this case it is immediate that second period consumption is equal to $S < C^*$. From now on we assume that $S < \hat{S}$ so that X_2 must satisfy (15) as an equality. We now show that $X_2 < C^*$ for every S, despite the fact that $Q_2 \geq C^*$ would satisfy the necessary condition given by equation (15) (since $V'(Q) = 0$ for all $Q \geq C^*$). To see this, assume by way of contradiction that $X_2 \geq C^*$. Note that when $Q_2 \geq C^*$

$$2V(Q_2 + S) - V(Q_2) = V(C^*) \equiv V^*.$$

However, for any $S > 0$,

$$\max_{Q_2} \{2V(Q_2 + S) - V(Q_2)\} \geq 2V^* - V(C^* - S) > V^*$$

which is the desired contradiction.

Because $X_2 < C^*$ the optimality condition (for $S < \hat{S}$) is given by

$$2V'(X_2 + S) = V'(X_2) > 0$$

This implies that $X_2 + S < C^*$, which concludes the proof of parts (ii) and (iii). For future reference, profits are:

$$\Pi^S = V^* + 2V(Q_2 + S) - V(S) - V(Q_2).$$

Finally the following lemma concludes the proof of this theorem.

Lemma 6.1 *Under the optimal policy the storage constraints are binding.*

Proof. See supplementary Appendix. ■

Proof of Theorem 6

We begin this proof with the assumption, that the monopolist sets a constant size of a flow bundle as a part of the optimal policy. Once we find the characterization of the optimal policy, we then prove, that, indeed the monopolist can not do better by allowing the flow bundle to change over time (see Lemma 6.2 on page 37).

Let us look at the problem of the consumer with storage. In principle, S-consumers can purchase share λ of the flow in addition to the the amount S they buy every period. The share λ_1 that they buy if they do not skip any bulk sales is defined by

$$V'(c + \lambda_1 q) \leq \frac{V(q)}{q}$$

and the share λ_2, that they buy if they skipped one bulk sale is defined by

$$V'\left(\frac{c}{2} + \lambda_2 q\right) \leq \frac{V(q)}{q}$$

These conditions hold as strict inequalities if λ is zero, i.e. if the per unit price for the flow is so high that we have a corner solution.

If both λ_1 and λ_2 are strictly positive, given the optimal menu, there should be no bulk sales. To show this, observe that, if both λ_1 and λ_2 are strictly positive, then the inequalities above, should hold as equalities, i.e.

$$\begin{aligned} V'(c + \lambda_1 q) &= \frac{V(q)}{q} \\ V'\left(\frac{c}{2} + \lambda_2 q\right) &= \frac{V(q)}{q} \end{aligned}$$

and, hence $c + \lambda_1 q = \frac{c}{2} + \lambda_2 q$. Denote by x the consumption that is induced by linear pricing:

$$V'(x) = \frac{V(q)}{q}$$

Clearly, x is a function of q, but instead of $x(q)$ we are going to write x wherever it does not lead to the confusion. The no-skipping constraint in this case becomes

$$\frac{P}{T} \leq 2\left(V(c + \lambda_1 q) - V\left(\frac{c}{2} + \lambda_2 q\right) + (\lambda_1 - \lambda_2)V(q)\right) = -\frac{c}{2q}V(q) \leq 0$$

hence setting $P = 0$ and $c = 0$ is constrained optimal. The monopolist's profit under the policy without the stock bundles is

$$\pi^q(q) = \alpha V(q) + (1 - \alpha)xV'(x)$$

Now we proceed to the case where bulk sales are present in optimal menu. We first observe that $\lambda_1 \leq \lambda_2$, hence it must be in this case that $\lambda_1 = 0$.

We have two cases: (i) $\lambda_2 = 0$ and (ii) $\lambda_2 \in (0, 1)$.

Suppose that $\lambda_2 \in (0, 1)$. Then

$$V'\left(\frac{c}{2} + \lambda_2 q\right) = \frac{V(q)}{q}$$

Note that $\frac{c}{2} + \lambda_2 q = x$.

Given this, the monopolist's profit becomes

$$2(1 - \alpha)V(c) + \alpha V(q) - 2(1 - \alpha)\left(V(x) - \lambda_2 V(q)\right)$$

If we substitute $\lambda_2 q = x - \frac{c}{2}$, then the monopolist's profit becomes

$$\alpha V(q) + (1-\alpha)xV'(x) + 2(1-\alpha)\left(V(c) - V(x) - \frac{V'(x)}{2}(c-x)\right)$$

When deriving this expression we assumed that $\lambda_2 > 0$. This assumption is valid only if the following inequality is satisfied

$$V'\left(\frac{c}{2}\right) > \frac{V(q)}{q} \tag{16}$$

If (16) is not satisfied, it must be that $\lambda_2 = 0$, in which case the monopolist's profit is given by

$$\alpha V(q) + 2(1-\alpha)\left(V(c) - V\left(\frac{c}{2}\right)\right)$$

Combining these two expressions, the monopolist's profit is given by

$$\pi(c,q) = \begin{cases} \alpha V(q) + 2(1-\alpha)\left(V(c) - V\left(\frac{c}{2}\right)\right), & \text{if } V'\left(\frac{c}{2}\right) \le \frac{V(q)}{q} \\ \alpha V(q) + (1-\alpha)xV'(x) & \\ \qquad + 2(1-\alpha)\left(V(c) - V(x) - \frac{V'(x)}{2}(c-x)\right), & \text{otherwise.} \end{cases}$$

We can now prove, that if $\alpha < 1$ monopolist always offers a stock bundle for sale. For that we need to show, that $\max\{\pi(c,q)\} \ge \max\{\pi^q(q)\}$. Take $\widehat{q} = \arg\max\{\pi^q(q)\}$. We are going to argue, that we can always find c such, that $\pi(c,\widehat{q}) \ge \pi^q(\widehat{q})$. As we mentioned, x is the function of q, so we introduce $\widehat{x} = x(\widehat{q})$. We assume (and later verify), that c is such that $V'\left(\frac{c}{2}\right) > \frac{V(\widehat{q})}{\widehat{q}}$. Then,

$$\pi(c,\widehat{q}) - \pi^q(\widehat{q}) = 2(1-\alpha)(c-\widehat{x})\left(\frac{V(c)-V(\widehat{x})}{c-\widehat{x}} - \frac{V'(\widehat{x})}{2}\right).$$

Since, $\lim\limits_{c\to\widehat{x}} \frac{V(c)-V(\widehat{x})}{c-\widehat{x}} = V'(\widehat{x})$, we can always find $\widehat{c}$ close enough to $\widehat{x}$ such, that

$$\frac{V(\widehat{c})-V(\widehat{x})}{\widehat{c}-\widehat{x}} > \frac{V'(\widehat{x})}{2}$$

and hence $\pi(\widehat{c},\widehat{q})-\pi^q(\widehat{q})>0$.

Finally, since $\widehat{c}=\widehat{x}+\epsilon$ for small enough $\epsilon>0$, $\frac{\widehat{c}}{2}<\widehat{x}$ and hence

$$V'\left(\frac{\widehat{c}}{2}\right)>V'(x)=\frac{V(\widehat{q})}{\widehat{q}}.$$

The optimal policy must satisfy $V'(c)\leq\frac{V(q)}{q}$ and must solve

$$(c^\alpha,q^\alpha)=\arg\max_{c,q}\{\pi(c,q)\},$$

where

$$\begin{aligned}\pi(c,q)=&\alpha V(q)+2(1-\alpha)\left(V(c)-V\left(\frac{c}{2}\right)\right)\\&-2(1-\alpha)\left(V(x)-V\left(\frac{c}{2}\right)-\left(x-\frac{c}{2}\right)V'(x)\right)\mathbf{1}\left\{V'\left(\frac{c}{2}\right)\geq\frac{V(q)}{q}\right\}\end{aligned}$$

If $V'\left(\frac{c^S}{2}\right)<\frac{V(c^*)}{c^*}$, the consumption induced by optimal pricing policy is $c^\alpha=c^S$ and $q^\alpha=c^*$. If the condition above is violated then either (c^α,q^α) satisfies first order conditions (i)

$$\begin{aligned}2V'(c^\alpha)&=\frac{V(q^\alpha)}{q^\alpha}\\\alpha q^\alpha V'(q^\alpha)&=2(1-\alpha)\left(V'(q^\alpha)-V'(x^\alpha)\right)\left(\frac{c^\alpha}{2}-x^\alpha\right)\end{aligned}$$

or (ii)

$$\begin{aligned}V'\left(\frac{c^\alpha}{2}\right)&=\frac{V(q^\alpha)}{q^\alpha}\\\alpha q^\alpha V'(q^\alpha)&=2(1-\alpha)\frac{\left(2V'(c^\alpha)-V'\left(\frac{c^\alpha}{2}\right)\right)}{\left(-V''\left(\frac{c^\alpha}{2}\right)\right)\left(V'(q^\alpha)-V'(x^\alpha)\right)}\end{aligned}$$

Clearly, in both cases

$$2V'(c^\alpha)-V'\left(\frac{c^\alpha}{2}\right)\leq 0,$$

which leads to the conclusion that $c^\alpha \geq c^S$.

Let us now look at the per-unit prices paid by S- and NS-consumers. The consumers without storage pay $\frac{V(q^\alpha)}{q^\alpha}$ for one unit of the good. The storers pay $\frac{P^\alpha}{c^\alpha}$, where

$$P^\alpha = \begin{cases} 2V(c^\alpha) - 2V\left(\frac{c^\alpha}{2}\right), & \text{if } V'\left(\frac{c^\alpha}{2}\right) \leq \frac{V(q^\alpha)}{q^\alpha} \\ 2V(c^\alpha) - 2V\left(x^\alpha\right) + 2\left(x^\alpha - \frac{c^\alpha}{2}\right)V'(x^\alpha), & \text{if } V'\left(\frac{c^\alpha}{2}\right) > V'\left(x^\alpha\right) = \frac{V(q^\alpha)}{q^\alpha} \end{cases}$$

We show that in both cases $\frac{P^\alpha}{c^\alpha} < \frac{V(q^\alpha)}{q^\alpha}$. Suppose, that $V'\left(\frac{c^\alpha}{2}\right) \leq \frac{V(q^\alpha)}{q^\alpha}$. Then

$$\frac{P^\alpha}{c^\alpha} = \frac{V(c^\alpha) - V\left(\frac{c^\alpha}{2}\right)}{\frac{c^\alpha}{2}} < V'\left(\frac{c^\alpha}{2}\right) \leq \frac{V(q^\alpha)}{q^\alpha}$$

If $V'\left(\frac{c^\alpha}{2}\right) > V'\left(x^\alpha\right) = \frac{V(q^\alpha)}{q^\alpha}$, then

$$\frac{P^\alpha}{c^\alpha} = \frac{V(c^\alpha) - V\left(x^\alpha\right) + \left(x^\alpha - \frac{c^\alpha}{2}\right)V'(x^\alpha)}{\frac{c^\alpha}{2}} < V'(x^\alpha) = \frac{V(q^\alpha)}{q^\alpha}.$$

Now, the following lemma is going to show, that the monopolist can not increase profits by allowing the size of the flow bundle to depend on time.

Lemma 6.2 *Suppose, the monopolist can set the size of the flow bundle at each moment in time. The optimal policy is $q_t^\alpha = q^\alpha$.*

Proof. Take any arbitrary policy q_t. We are going to show, that we can find a policy with a constant size of a flow bundle, that generates weakly higher revenue than q_t.

First, we have to characterize the consumer choice of what bundles to buy and how much to consume. Clearly, consumer will smooth his consumption up to a point, when he faces one of the two storage constraints: when either the storage is empty or it is full. Here we introduce the purely technical assumption, that the consumption as a function of time is right-continuous.

Lemma 6.3 *Suppose, the agent's consumption at time t is x_t. Then,*

1. *if $V'(x_t) > \frac{V(q_t)}{q_t}$, the agent purchases as much flow bundles as his storage allows*

2. *if $V'(x_t) < \frac{V(q_t)}{q_t}$, the agent does not purchase any flow bundles.*

Proof. See supplementary Appendix. ■

Suppose, that the bulk bundle is sold at time $0, T, 2T, 3T$, and so on. Let us first consider the price of a bulk bundle sold at time T. It is, as before, determined by no-skipping constraint. If the S-consumer skips the sale, he will purchase the flow bundle if and only if the per-unit price of the good is lower then the marginal utility of consumption and his storage is not full (see Lemma 6.3). Since we allow for multiple purchases of the flow, we can assume with out loss of generality, that he is going to buy the flow at the countably many points in time. Suppose, in the interval $[0, T]$ he purchases the flow K_T times at $\{t_i^T\}_{i=1}^{K_T}$. The consumption x_t^T is piece-wise constant with discontinuities at t_i^T.

Now let us consider the bulk bundle sold at time 0. Again, if the S-consumer skips the sale at time 0, his consumption x_t^0 is going to be piece-wise constant with discontinuities at t_i^0. Let us take a coarsest refinement of the two sets of intervals that are induced by $\{t_i^T\}_{i=1}^{K_T}$ and by $\{t_i^0\}_{i=1}^{K_0}$. By $K_{0,T}$ we denote the number of intervals and by $\{t_i^{0,T}\}_{i=1}^{K_{0,T}}$ their endpoints. Both x_t^0 and x_t^T are constant within each interval. We denote their value for the interval $\left(t_i^{0,T}, t_{i+1}^{0,T}\right)$ by x_i^0 and x_i^T respectively.

We start with observation, that for any $t \in \left(t_i^{0,T}, t_{i+1}^{0,T}\right)$,

$$\frac{V(q_t)}{q_t} \geq \max\{V'(x_i^0), V'(x_i^T)\}.$$

Indeed, if it were not the case, i.e. the inequality were reversed for some t, this t would be the endpoint of some interval by definition, which would lead to the contradiction, that t belongs to the open interval. Since the S-consumer is not purchasing the flow bundles within the interval $t \in \left(t_i^{0,T}, t_{i+1}^{0,T}\right)$, we can change the size of the flow bundle inside the interval $\left(t_i^{0,T}, t_{i+1}^{0,T}\right)$ to $q_i^{0,T}$. The new size of the bundle is such that, the S-consumer's does not change his decisions on when and how much of the flow bundles to buy:

$$\frac{V(q_i^{0,T})}{q_i^{0,T}} = \max\{V'(x_i^0), V'(x_i^T)\}.$$

Notice, that $q_t \leq q_i^{0,T}$ for any $t \in \left(t_i^{0,T}, t_{i+1}^{0,T}\right)$. This change will not affect the price of the bulk bundle and will weakly increase the profit collected from NS-consumers. From now on we will restrict our attention to the policies that are obtained by this transformation.

If the S-consumer skips a sale, he plans his consumption while having the storage filled up to its maximum capacity. He consumes from the storage when the price of the flow bundle is too high. The utility of the S-consumer in this case is

$$\int_0^{2T} \left[V(x_t) - x_t V'(x_t) + SV'\left(\min_{\tau \in [0,2T]}\{x_\tau\}\right)\right] dt.$$

The part of the tariff that depends on x_i^0 and x_i^T is[16]

$$P(\mathbf{x}^0, \mathbf{x}^T) = \alpha \sum_{i=0}^{K} \tau_i V(q_i) - (1-\alpha)\frac{S}{2}\left(V'\left(\min_i\{x_i^0\}\right) + V'\left(\min_i\{x_i^T\}\right)\right)$$
$$- (1-\alpha)\sum_{i=0}^{K} \tau_i \left(\left(V(x_i^0) + V(x_i^T)\right) - \left(x_i^0 V'(x_i^0) + x_i^T V'(x_i^T)\right)\right).$$

Notice, that by construction $V(q_i) \leq \frac{1}{2}V(q_i^0) + \frac{1}{2}V(q_i^T)$, and hence

$$P(\mathbf{x}^0, \mathbf{x}^T) \leq \sum_{i=0}^{K} \tau_i \left[\frac{\alpha}{2}V(q_i^0) - (1-\alpha)\frac{c}{2}V'\left(\min_i\{x_i^0\}\right) - (1-\alpha)\left(V(x_i^0) - x_i^0 V'(x_i^0)\right)\right]$$
$$+ \sum_{i=0}^{K} \tau_i \left[\frac{\alpha}{2}V(q_i^T) - (1-\alpha)\frac{c}{2}V'\left(\min_i\{x_i^T\}\right) - (1-\alpha)\left(V(x_i^T) - x_i^T V'(x_i^T)\right)\right].$$

Recall, that q_i^0 satisfies the equation $\frac{V(q_i^0)}{q_i^0} = V'(x_i^T)$, and that q_i^0 satisfies

[16] From now on we drop $0, T$ from the upper indexes: $K^{0,T}$ becomes K, $t_i^{0,T}$ becomes t_i and so on.

similar one. Also recall that our solution (q^{α}, x^{α}) maximizes

$$\frac{\alpha}{2}V\left(q(x)\right) - (1-\alpha)\frac{c}{2}V'(x) - (1-\alpha)\left(V(x) - xV'(x)\right),$$

where $q(x)$ is such that $\frac{V(q(x))}{q(x)} = V'(x)$. From this we conclude that $P(\mathbf{x}^0, \mathbf{x}^T)$ is lower, than what we get under the flat q^{α}, hence under the optimal policy the size of the flow bundle is constant. ■

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7 Supplementary Appendix (Not for Publication)

Proof of Lemma 3.1

Combining equations (2) and (3) we obtain:

$$P_1+P_2 \leq 2\max_{s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-\left(\underset{s\leq S}{Max}\{V(Q_1-s)+V(s)\}+V(Q_2)\right).$$

Because V is concave, we have that

$$\max_{s\leq S}\{V(Q_1-s)+V(s)\}+V(Q_2)\geq \underset{s\leq S}{Max}\{V(Q_1-s)+V(Q_2+s)\}.$$

Therefore,

$$P_1+P_2\leq \underset{s\leq S}{Max}\{V(Q_1-s)+V(Q_2+s)\}$$

which, is the same as inequality (1).

Proof of Theorem 2

Monopolist maximizes the sum of the tariffs, subject to all the constraints introduced in Section 3.3. We start the proof by noticing, that some constraints can be omitted from the problem.

Lemma 7.1 *Constraints (3) and (5) are not binding:*

- *(4) and (9) imply (3);*
- *(2) and (7) imply (5).*

Proof. First we prove, that (4) and (9) imply (3). If we add up constraints (4) and (9), we obtain

$$P_1\leq \max_{0\leq s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-\max_{0\leq s\leq S}\{V(q_1-s)+V(Q_2+s)\}+V(q_1),$$

which is a tighter bound on P_1 than (3). Similarly, by adding up constraints (2) and (7) we obtain that

$$p_2\leq V(q_2)+\max_{0\leq s\leq S}\{V(Q_1-s)+V(Q_2+s)\}-\max_{0\leq s\leq S}\{V(Q_1-s)+V(s)+V(Q_2)\},$$

which implies (5). ■

Let us assume (and later prove) that inequalities (2), (4) and (7) are binding, i.e.

$$\begin{aligned} p_1 &= V(q_1) && (17)\\ P_2 &= \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - \max_{0\le s\le S}\{V(Q_1-s)+V(s)\} && (18)\\ p_2 &= P_2 + V(q_2) - V(Q_2). && (19)\end{aligned}$$

Given that, inequalities (9) and (11) provide the condition on P_1:

$$\begin{aligned} P_1 &\le p_1 + \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - \max_{0\le s\le S}\{V(q_1-s)+V(Q_2+s)\}\\ P_1 &\le p_1 + \max_{0\le s\le S}\{V(Q_1-s)+V(s)\} - \max_{0\le s\le S}\{V(q_1-s)+V(s)\}.\end{aligned}$$

We also assume that one of these two inequalities is binding. By $Z(Q_1, q_1, x)$ we denote the following expression:

$$Z(Q_1,q_1,x) = \max_{0\le s\le S}\{V(Q_1-s)+V(x+s)\} - \max_{0\le s\le S}\{V(q_1-s)+V(x+s)\}$$

Using notation we obtain that

$$P_1 = p_1 + \min\{Z(Q_1,q_1,0), Z(Q_1,q_1,Q_2)\}. \qquad (20)$$

Equations (17),(18),(19) and (20) can be used to write down the profit of the monopolist as a function of q_1, q_2, Q_1 and Q_2.

$$\begin{aligned} \pi &= V(q_1) + \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - \max_{0\le s\le S}\{V(Q_1-s)+V(s)\}\\ &+ \alpha(V(q_2)-V(Q_2)) + (1-\alpha)\min\{Z(Q_1,q_1,0), Z(Q_1,q_1,Q_2)\}\end{aligned}$$

Observe, that $q_2^* = C^*$ and $Q_1^* = C^* + S$ maximize profit. To find the rest of the solution we need to maximize

$$\begin{aligned} &V(q_1) + V(Q_2+S) - \alpha V(Q_2) + (1-\alpha)\min\{V(Q_2+S)\\ &-\max_{0\le s\le S}\{V(q_1-s)+V(Q_2+s)\}, V(S) - \max_{0\le s\le S}\{V(q_1-s)+V(s)\}\}\end{aligned}$$

By proving the following lemma, we check, that the set of inequalities, that we assumed to be binding, are actually binding in the optimum.

Lemma 7.2 *Inequalities (2), (4), (7), (9) and (11) define the tariffs that maximize monopolist's profits.*

Proof. To prove this lemma we need to check if the rest of the inequalities are satisfied with the solution of the relaxed problem. In particular we need to check if inequalities (6), (8), (10) and (12). We start with (6)

$$p_1 - P_1 - V(q_1) + V(Q_1) =$$
$$\max\left\{\max_{0\le s\le S}\left\{\begin{matrix} V(q_1-s)+V(Q_2+s) \\ -V(Q_2+S)-V(q_1)\end{matrix}\right\}, \max_{0\le s\le S}\left\{\begin{matrix} V(q_1-s)+V(s) \\ -V(S)-V(q_1)\end{matrix}\right\}\right\} \le 0$$

(8) is equivalent to

$$p_1 + p_2 - P_1 - P_2 + \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - \max_{0\le s\le S}\{V(q_1-s)+V(q_2+s)\} \ge$$
$$\max_{0\le s\le S}\{V(q_1-s)+V(Q_2+s)\} - V(Q_2) - V(q_1) \ge 0$$

(10) is equivalent to

$$p_2 - P_2 + \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - \max_{0\le s\le S}\{V(Q_1-s)+V(q_2+s)\} =$$
$$V(Q_2+s) - V(Q_2) \ge 0$$

and finally (12) is also satisfied:

$$p_2 - P_1 - P_2 + \max_{0\le s\le S}\{V(Q_1-s)+V(Q_2+s)\} - V(q_2) \ge$$
$$\max_{0\le s\le S}\{V(q_1-s)+V(Q_2+s)\} - V(q_1) - V(Q_2) \ge 0.$$

■

Lemma 7.2 ensures, that the solution of a relaxed maximization problem coincides with the solution of the original profit maximization problem. Now that we have this result, we can get back to solving the relaxed problem. Let us consider the following expression:

$$\min\{V(Q_2+S)-\max_{0\le s\le S}\{V(q_1-s)+V(Q_2+s)\}, V(S)-\max_{0\le s\le S}\{V(q_1-s)+V(s)\}\} \quad (21)$$

If $Q_2 \ge q_1$ we have

$$\max_{0\le s\le S}\{V(Q_2+S)+V(q_1-s)+V(s)\} - V(q_1) - V(Q_2) - V(S) \le 0$$

and hence (21) becomes

$$\min\{V(Q_2+S)-V(q_1)-V(Q_2), V(S)-\max_{0\le s\le S}\{V(q_1-s)+V(s)\}\} =$$
$$V(Q_2+S)-V(q_1)-V(Q_2).$$

If $Q_2 + 2S \ge q_1 \ge Q_2$ we observe, that

$$\max_{0\le s\le S}\{V(Q_2+S)+V(q_1-s)+V(s)\} - 2V\left(\frac{q_1+Q_2}{2}\right) - V(S) \le 0$$

and (21) can be rewritten as

$$\min\{V(Q_2+S)-2V\left(\frac{q_1+Q_2}{2}\right), V(S)-\max_{0\le s\le S}\{V(q_1-s)+V(s)\}\} =$$
$$V(Q_2+S)-2V\left(\frac{q_1+Q_2}{2}\right).$$

Finally, if $Q_2 + 2S \le q_1$ (21) becomes

$$\min\{-V(q_1-S), -V(q_1-S)\} = -V(q_1-S).$$

We can rewrite part of maximization problem that solves for Q_2 and q_1 as

$$\max_{Q_2,q_1}\{f(q_1,Q_2)\}$$

where

$$f(q_1, Q_2) = \begin{cases} \alpha V(q_1) + (2-\alpha)V(Q_2+S) - V(Q_2), & \text{if } q_1 \le Q_2 \\ V(q_1) + (2-\alpha)V(Q_2+S) - \alpha V(Q_2) & \\ \qquad - 2(1-\alpha)V\left(\dfrac{q_1+Q_2}{2}\right), & \text{if } Q_2 < q_1 \le Q_2 + 2S \\ V(q_1) + V(Q_2+S) - \alpha V(Q_2) - (1-\alpha)V(q_1 - S), & \text{if } Q_2 + 2S < q_1 \end{cases}$$

First we observe that the solution for maximization problem can never satisfy $q_1 < Q_2$ because function $\alpha V(q_1) + (2-\alpha)V(Q_2+S) - V(Q_2)$ has unique maximum $q_1 = C^*$ and $Q_2 + S < C^*$. It means, that we should look for the solution in the set where $q_1 \ge Q_2$. There, however, observe that first order condition for q_1 and Q_2 suggest that $V'(q_1) > 0$ and $V'(Q_2+S) > 0$, hence $q_1 < C^*$ and $Q_2 < C^* - S$.

First order conditions that define Q_2^* and q_1^* are

$$V'(q_1^*) = \begin{cases} (1-\alpha)V'\left(\frac{q_1^*+Q_2^*}{2}\right) & \text{, if } Q_2^* \le q_1^* < Q_2^* + 2S \\ (1-\alpha)V'(q_1^* - S) & \text{, otherwise} \end{cases}$$

and

$$V'(Q_2^*+S) = \begin{cases} \alpha V'(Q_2^*) + (1-\alpha)V'\left(\dfrac{q_1^*+Q_2^*}{2}\right) & \\ \qquad - (1-\alpha)V'(Q_2^*+S) & \text{, if } Q_2^* \le q_1^* < Q_2^* + 2S \\ \alpha V'(Q_2^*) & \text{, otherwise} \end{cases}$$

Proof of Theorem 3

Fix the time horizon to be T. Suppose the monopolist offers the flow q_t

and charges p_t for it. Then consumer's problem is

$$\begin{aligned}
\max_{c_t \geq 0, b_t \in \{0,1\}} \quad & \{\int_0^T (V(y_t c_t) - x_t b_t p_t)\, dt\} \\
s_t &= \int_0^t (x_\tau b_\tau q_\tau - y_\tau c_\tau) d\tau \\
x_t &\in \begin{cases} 0, \text{ if } s_t = S \text{ and } b_t q_t - y_t c_t > 0 \\ \{0,1\}, \text{ otherwise} \end{cases} \\
y_t &\in \begin{cases} 0, \text{ if } s_t = 0 \text{ and } b_t q_t - y_t c_t < 0 \\ \{0,1\}, \text{ otherwise} \end{cases}
\end{aligned}$$

Where vector (x_t, y_t) denotes so-called regime. The meaning of this regime variables in our problem is the following. By setting variable $x_t = 0$ we make sure, that when the agent's storage is filled up to maximum capacity, the agent does not purchase the flow of good that is larger than his consumption . Similarly, by setting $y_t = 0$, we guarantee, that when the agent's storage is empty, the agent can not consume more than what he purchases. Note, that correspondence that defines the domain of x_t and y_t is right continuous.

The control variables in this problem are c_t and b_t. By b_t we denote a binary decision whether consumer buys a flow at time t or not. Naturally, by c_t we denote consumption at time t.

The state variable for this problem is the amount of good, that is stored in the consumer's inventories at time t, i.e. s_t.

By H we denote Hamiltonian for this problem:

$$H = \Psi_t (x_t b_t q_t - y_t c_t) + V(y_t c_t) - x_t b_t p_t.$$

Following Panteleev et al. (2011)[17], we obtain necessary conditions for this

[17]See Panteleev, A.V., Bortakovskiy, A.S., Letova, T.A., "Optimalnoe Upravlenie V Primerah I Zadachah", Izdatelstvo MAI, 1996.

problem:

$$c_t^* = \begin{cases} 0, \text{ if } y_t^* = 0 \\ V'^{-1}(\Psi_t^*), \text{ otherwise} \end{cases}$$

$$b_t^* \in \begin{cases} 1, \text{ if } x_t^* = 1 \text{ and } \Psi_t^* > \frac{p_t}{q_t} \\ \{0,1\}, \text{ if } x_t^* = 1 \text{ and } \Psi_t^* = \frac{p_t}{q_t} \\ 0, \text{ otherwise} \end{cases}$$

Note, that Ψ_t^* is piecewise constant with jumps at discontinuity points of x_t^* and y_t^*.

These necessary conditions state, that agents smooths his consumption whenever he has some amount of good in the storage. Also, the agent purchases the good only when per unit price is lower than the marginal utility of his current consumption.

Lemma 7.3 *If $c_\tau^* = 0$, it must be that $c_t^* = 0$ and $b_t^* = 0$ for all $t \in [0, \tau]$*

Proof. By contradiction let us assume that $c_t^* > 0$ for all $t \in [t_1, t_2] \subset [0, \tau]$ (since Ψ_t^* is piecewise constant there must exist non-degenerate interval). Then it must be that $\int_0^{t_2} b_t^* dt > 0$. We can always find $\epsilon > 0$ small enough, such that consumer stores ϵ more by the time t_2 and consumes it around time τ. Since $c_\tau^* = 0$ and $V(\cdot)$ is concave, it is an improvement, hence the contradiction. ■

By this Lemma, we can restrict our attention on policies that induce strictly positive consumption everywhere.

Let us partition our time interval $[0, T]$ into intervals $\{[t_{i-1}, t_i]\}_{i=1}^I$ such that $t_0 = 0$, $t_I = T$, and for all $1 \leq i < I$: $t_i = t \iff s_t^* = S$ and for any $\epsilon > 0$ s_τ^* is not constant on $\tau \in (t - \epsilon, t + \epsilon)$. By construction, consumption inside interval i is constant if $\forall t \in [t_{i-1}, t_i] : s_t > 0$ (we denote the consumption in the interval i by c_i in this case). Aggregate amount of good purchased within such interval i is

$$\int_{t\in[t_{i-1},t_i]} x_t^* b_t^* q_t dt = \begin{cases} c_i(t_i - t_{i-1}), \text{ if } 1 < i < I \\ c_1 t_1 + S, \text{ if } i = 1 \\ c_I(T - t_{I-1}) - S, \text{ if } i = I \end{cases}$$

Now let us look at the intervals that have the property that $\exists t \in (t_{i-1}, t_i)$: $s_t^* = 0$. Take such interval $[t_{i-1}, t_i]$ if it exists and partition it further into subintervals, such that consumption is constant within each subinterval (we can do that because Ψ_t^* is piecewise constant). Lets index those subintervals by $j = 1, ..., m_i$. The endpoints of those intervals are $t_i^0 = t_{i-1}$, and $\{t_i^j\}_{j=1}^{m_i}$, where t_i^j is the right endpoint of jth interval. Observe, that consumption is increasing in j i.e. $c_i^j < c_i^{j+1}$ for all $j = 1, ..., m_i - 1$. Also the aggregate amount of good purchased within each subinterval is

$$\int_{t\in[t_i^{j-1},t_i^j]} x_t^* b_t^* q_t dt = \begin{cases} c_i^j(t_i^j - t_i^{j-1}), \text{ if } 1 < j < m_i \\ c_i^1(t_i^0 - t_i^1) - S, \text{ if } j = 1 \\ c_i^{m_i}(t_i - t_i^{m_i-1}) + S, \text{ if } j = m_i \end{cases}$$

From necessary conditions we know that per unit price of a good is bounded from above by $V'(c)$. Also, we know that $V'(c_i^{m_i}) \leq V'(c_i^1)$, hence the profits of the monopolist that are collected from sales in interval i are bounded from above by

$$\sum_{j=1}^{m_i}(t_i^{j-1} - t_i^j)c_i^j V'(c_i^j) + S(V'(c_i^{m_i}) - V'(c_i^1)) \leq \sum_{j=1}^{m_i}(t_i^{j-1} - t_i^j)c_i^j V'(c_i^j)$$

Now let us reindex our partitions by $k \in K$ such that the new partition is the coarsest refinement of the partitions above. Again price of a good is bounded from above by $V'(c)$ so total profits that are bounded from above by

$$\sum_{k\in K} \frac{(t_k - t_{k-1})}{T} c_k V'(c_k) + \frac{S}{T}(V'(c_1) - V'(c_I))$$

In the limit this bound becomes

$$\limsup_{T\to\infty}\left(\sum_{k\in K}\frac{(t_k-t_{k-1})}{T}c_kV'(c_k)+\frac{S}{T}(V'(c_1)-V'(c_I))\right)\leq\max_{c\geq 0}\{cV'(c)\}$$

The expression on the left hand side is the profit from pricing the good linearly.

Proof of Lemma 6.1

We now need to prove that it is indeed optimal to induce binding storage constraints. There are several cases to be considered.

Consider first, the case in which the monopolist sets a policy in which storage is interior: $0 < s < S$. We first discuss the case in which storage does not bind even if the consumer chooses to skip the second period purchase, i.e., $\frac{Q_1}{2} \leq S$. In this case, consumer optimal smoothing behavior implies that profits are:

$$\Pi^I = 4V\left(\frac{Q_1+Q_2}{2}\right)-2V\left(\frac{Q_1}{2}\right)-V(Q_2).$$

The first order conditions can be combined to yield

$$2V'\left(\frac{Q_1^I+Q_2^I}{2}\right)=V'\left(\frac{Q_1^I}{2}\right)=V'(Q_2^I)$$

so that

$$\frac{Q_1^I}{2}=Q_2^I \text{ and } 2V'\left(\frac{3Q_2^I}{2}\right)=V'(Q_2^I).$$

For storage not to be binding even when the consumer chooses to skip the second period purchase, it must be the case that $S \geq Q_2^I$.

We now show that all policies in the interior of this class violate the second order conditions for the monopolist.

Assume by way of contradiction that Q_1^I and Q_2^I satisfy the first and second order conditions, namely

$$\begin{aligned} V''\left(\frac{Q_1^I+Q_2^I}{2}\right)-V''(Q_2^I) &\leq 0 \\ \frac{1}{2}V''(Q_2^I)\left(V''(Q_2^I)-3V''\left(\frac{Q_1^I+Q_2^I}{2}\right)\right) &\geq 0 \end{aligned}$$

These two inequalities imply that

$$2V''\left(\frac{Q_1^I+Q_2^I}{2}\right)\leq V''(Q_2^I)\leq 3V''\left(\frac{Q_1^I+Q_2^I}{2}\right)$$

which can only be true if $V''\left(\frac{Q_1^I+Q_2^I}{2}\right)\geq 0$. This contradicts the assumption that V is strictly concave showing the desired contradiction.

This means that the solution to the maximization problem must be on the boundary of this set: one of the constraints on storage is binding.

We now need to consider the case where capacity binds for skipping the second period purchase but not for smoothing consumption. The reasoning is very similar. In this case, profits are

$$\Pi^{II}=4V\left(\frac{Q_1+Q_2}{2}\right)-V(Q_1-S)-V(S)-V(Q_2).$$

First order conditions for this problem imply that $Q_1^{II}-S=Q_2^{II}$. Second order condition then is

$$\begin{aligned} V''\left(\frac{Q_1^{II}+Q_2^{II}}{2}\right)-V\prime\prime(Q_2^{II}) &\leq 0\\ V''(Q_2^{II})\left(V''(Q_2^{II})-2V''\left(\frac{Q_1^{II}+Q_2^{II}}{2}\right)\right) &\geq 0\end{aligned}$$

By combining two inequalities together we get that

$$V''\left(\frac{Q_1^I+Q_2^I}{2}\right)\leq V''(Q_2^I)\leq 2V''\left(\frac{Q_1^I+Q_2^I}{2}\right)$$

but this contradicts the concavity of V.

Finally, we need to show that the seller does not prefer to sell $Q_1<Q_2$ in which case optimal storage would be zero. When the seller sets $Q_1<Q_2$ (and as long as capacity binds in the event that the consumer skips second period purchases), there are two possibilities: in the first case, when $Q_1\geq 2S$, if the consumer skips the second period purchase, then storage capacity binds. In

this case, profits are

$$\Pi^0 = 2(V(Q_1) + V(Q_2)) - V(Q_1 - S) - V(S) - V(Q_2)$$

which are maximized when

$$\begin{aligned} 2V'(Q_1^0) &= V'(Q_1^0 - S) \\ V'(Q_2^0) &= 0 \end{aligned}$$

thus we can rewrite profits as:

$$2V(Q_1^0) + V^* - V(Q_1^0 - S) - V(S).$$

It is easy to see that these are the same profits as in our candidate optimal policy. The role of Q_1 and Q_2 are now reversed: second period consumption is efficient while first period consumption is inefficiently low.[18] This only happens when capacity is small enough, i.e. $S \leq \tilde{S}$ where $\tilde{S}$ solves

$$2V'(2\tilde{S}) = V'(\tilde{S}).$$

When $S \geq \tilde{S}$, capacity does not bind when the consumer skips second period purchases. Observe that $Q_1 < 2S$ in this case. Monopolist profits are then

$$\max_{Q_1}\{V^* + 2V(Q_1) - 2V\left(\frac{Q_1}{2}\right)\}.$$

We need to show, that

$$\max_{Q_1}\{V^* + 2V(Q_1) - 2V\left(\frac{Q_1}{2}\right)\} < \Pi^S$$

We notice that if $Q_1 \geq S$ we can set $Q_2 = Q_1 - S$ and obtain

$$2V(Q_1) - 2V\left(\frac{Q_1}{2}\right) < 2V(Q_2 + S) - V(Q_2) - V(S) \leq \Pi^S$$

[18] Thus, when storage capacity is low there is another solution. We do not highlight this solution because it is no longer optimal in the cases considered later.

and, if $Q_1 < S$, we obtain that

$$2V(Q_1) - 2V\left(\frac{Q_1}{2}\right) < V(Q_1) < V(S) \leq \Pi^S$$

Proof of Lemma 6.3

This lemma is almost identical to Theorem 3, so we only sketch the proof here.

Suppose by contradiction, that $V'(x_t) > \frac{V(q_t)}{q_t}$, and agent's storage is not full, i.e. $s_t < S$. If agent buys an $\epsilon > 0$ of good (where ϵ is small enough), he is going to pay $\frac{\epsilon V(q_t)}{q_t}$. After the purchase agent can spread this small portion of the good across $\sqrt{\epsilon}$ of time. His consumption is going to go up by $\frac{\epsilon}{\sqrt{\epsilon}} = \sqrt{\epsilon}$. The net gain in utility in this case is

$$\sqrt{\epsilon}\left(\sqrt{\epsilon}V'(x_t)\right) - \frac{\epsilon V(q_t)}{q_t} > 0$$

which is a desired contradiction. The same logic works for the case, when $V'(x_t) < \frac{V(q_t)}{q_t}$.