



I Congreso de Educación Matemática de América Central y El Caribe

I CEMACYC

6 al 8 noviembre. 2013

i.cemacyc.org

Santo Domingo, República Dominicana



An activity involving geometry, arithmetic and numerical representations

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Summary

We propose the a priori analysis of an activity involving geometry, arithmetic and numerical representations of rational numbers. It starts with a geometric manipulation with GeoGebra and gradually leads students to consider the divisors of 360 and to reflect upon approximations and exact representations of angle measurements. A first 'off-class' experiment is presented and discussed, as a case study. Through it, we acknowledge that it is possible to design suitable teaching situations, so as to make students reflect in a meaningful context on the dualities ideal object / visual representation, exact measurements / approximations, and on the type of certainty intended from a mathematical standpoint. The activity takes place in the 'geometer-physicist' paradigm proposed by Tanguay & Geeraerts (2012, 2013) and instilled by Jahnke (2007, 2010).

Resumen

En este texto presentamos el análisis de una actividad que combina la aritmética y la geometría y suscita un trabajo sobre las representaciones de los números racionales. A través una serie de manipulaciones geométricas con GeoGebra, los alumnos examinan los divisores de 360 y reflexionan sobre las aproximaciones y las representaciones de la medida exacta de los diferentes ángulos creados. Una primera experimentación que tuvo lugar fuera del aula de clase es presentada en este texto y discutida como estudio de caso. El análisis nos permite concluir que es posible elaborar situaciones de enseñanza construidas para que los alumnos reflexionen, en un contexto fructuoso y significativo, sobre las dualidades objeto ideal / representación visual, medidas exactas / aproximaciones, y sobre los tipos de certezas requeridos en las matemáticas. Este actividad se inscribe en el paradigma del geómetra-físico, propuesto por Tanguay & Geeraerts (2012, 2013) bajo la influencia de las investigaciones de Jahnke (2007, 2010).

Résumé

Nous proposons l'analyse a priori d'une activité qui convoque arithmétique et géométrie, et suscite un travail sur les représentations des nombres rationnels. Elle est initiée par des manipulations géométriques avec GeoGebra, et conduit progressivement les élèves à examiner les diviseurs de 360 et à réfléchir au sujet des approximations et des représentations exactes des mesures d'angles en jeu. Une première expérimentation « hors classe » est présentée et discutée, comme étude de cas. Nous en concluons qu'il est possible d'élaborer des situations d'enseignement construites pour que les élèves réfléchissent, en contexte riche et significatif, aux dualités objet idéal / représentation visuelle, mesures exactes / approximations, et aux types de certitude visés par les mathématiques. L'activité a pour cadre le paradigme du géomètre-physicien, proposé par Tanguay & Geeraerts (2012, 2013) sous l'impulsion des travaux de Jahnke (2007, 2010).

Key words : geometry, arithmetic, regular polygons, number representations, dynamic geometry software, measurement, empiricism, geometer-physicist paradigm

Projet CRSH du Canada n° 169912. The authors also express their gratitude to Lourdes Guerrero and Christian Morales, from *Universidad Michoacana de San Nicolás de Hidalgo* (UMSNH), Morelia, Mexico.

Jahnke's framework on empiricism in mathematics

According to Jahnke (2007), dealing with mathematics as in physics would provide a more harmonious transition between everyday life thinking and mathematical thinking, and a stronger epistemological foundation for teaching proof and proving. His argumentation is carried out by considering how *generality* is dealt with :

- general statements from everyday life and from physics share their empirical basis and the fact that the conditions delimiting their scope of validity is (more often) virtually unattainable;
- in *everyday life*, determining all the possible conditions of validity or non-validity is seldom relevant;
- in *physics*, physicists try to relate general statements to the most accurate domain of validity, even if their theory will always remain subject to falsification by new observed phenomena (e.g. Popper, 1963/1991);

- by contrast, determining the (exact) domain of validity of a general statement is intrinsically linked to the way *mathematics* operate : the set of conditions is fixed and closed by the building of a (preferably axiomatized) theory.

As in mathematics, general statements in physics are connected by hypothetico-deductive developments that integrate them into a network and build them up as a theory. Yet in physics, the theory does not disqualify the empirical bases but rather makes them richer : any experimental verification about a statement not only corroborates it, but also increases the conviction that all other statements connected to it in the network are true :

The epistemological motivation of proof is not to be founded on the idea that proofs in contrast to measurement provide absolute certainty, but on the idea that proofs open new and more complex possibilities of empirical corroboration. In short, in an empirical environment *proofs do not replace measurements but make them more intelligent*. (Jahnke, 2007, p. 83, italics from the original)

Doing geometry as would do physicists in their lab

Based on Jahnke's reflections, a precise teaching framework for classroom geometry has been proposed by Tanguay & Geeraerts (2012, 2013), within what they call the *geometer-physicist paradigm*, that they situate between Natural Geometry (G-I) and Natural Axiomatic Geometry (G-II) (Houdement & Kuzniak, 2006). It claims that the class should be considered a researchers' community building a theory (Legrand, 1988), that experimentation and empirical validations should be brought back into the fold and regarded as complementary to proof. It assumes that measurement provides nothing but approximations (Tanguay & Geeraerts, 2012), this assumption being a basis to acknowledge the hypothetical status of empirically validated statements, this status being considered with (greater) care :

It is for us symptomatic that institutional teaching resources rebuke equalities such as $\frac{4}{3} = 1,33$ or $\sqrt{2} = 1,414$, but in the same time agree without a murmur with equalities such as $mes[AB] = 5 \text{ cm}$, in contexts where inferred measures and measures obtained with the geometry tools are blithely combined . (Op. cit., p. 21; our translation)

Accordingly, the issues of exactness and approximation should be fully problematized, and integrated into the quest for certainty pertaining to the statements at stake.

Measurements and numerical representations

Measurement involves going back and forth between the field of (synthetic) geometry and the field of arithmetic, between geometric figures and numbers. In problem-solving contexts, some measures are inferred — e.g. from Pythagoras' or Thales' theorems — and give rise to irrational numbers or rational numbers with infinite decimal expansion. The numerical representations may then come from different registers — e.g. quotients written in the form p/q , finite decimal, repeating decimal with ellipses or with the period overlined... — then calling for a *coordination between registers* (Duval, 1993). In these instances, the duality exactness-approximation should be a teaching goal and issue, and will inevitably echo with the approximation of rational and irrational numbers by finite decimal expansions, and in parallel with the problems of (visually) representing ideal geometrical objects and dealing with measurements from the figural representations.

Now, when measurements are not *performed* with the traditional geometric tools but rather are *red* on a screen via the specific functionalities of dynamic geometry software packages (DGSs),

then students' relationship to exactness changes because they are convinced that what is displayed at the screen is necessarily exact (Kuzniak, 2010, p. 86). So it is important to plan teaching situations where students are brought to reflect on the precision of the displays, on the way DGSs handle internally the numbers and their representations, especially when they come from measurements related to the represented geometric figures. The following activity has been designed regarding these issues, through a collaboration between researchers from Quebec and Mexico.

A GeoGebra activity intertwining geometry and arithmetic

The activity is intended for students from the beginning of secondary school (12-14 years old). They work in teams of two, one team per each computer terminal. The instructions are open :

The triangle ΔABC visible at the screen is isosceles, with $[AC] \cong [BC]$. Form all the regular polygons you can by rotating ΔABC around point C. You may vary $\angle ACB$ either with the slider α or by typing directly α , the measure of $\angle ACB$, in the given box. The slider n allows you to change the number of triangles obtained as images of ΔABC by repeated rotations around center C and of angle α . Do you know the name of each polygon you formed ? As you go, fill in the table. You can add as many lines as you want.

Measure α of $\angle ACB$	Number of triangles, obtained by rotating ΔABC	Name of the regular polygon you formed
	$n =$	
	$n =$	
N		

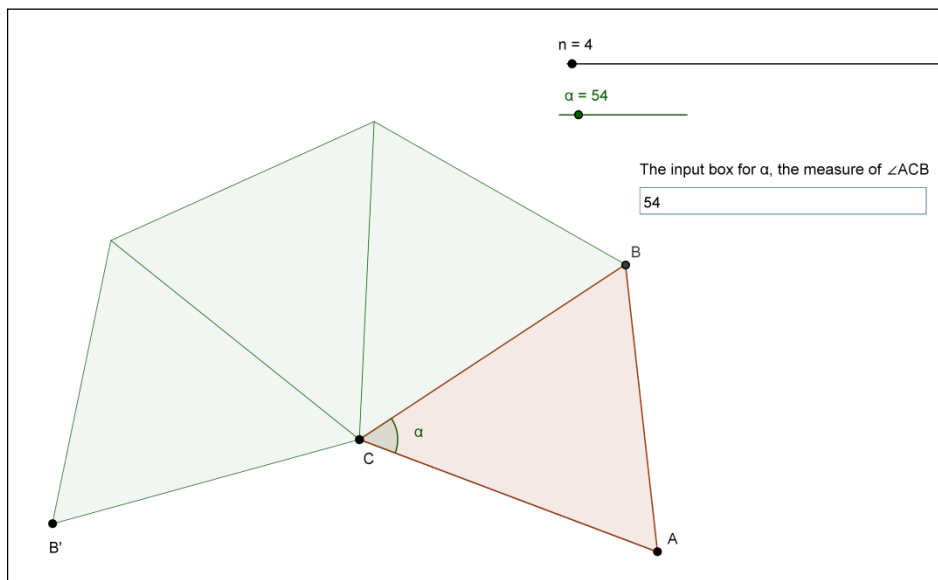


Figure 1. The GeoGebra screen given from the start

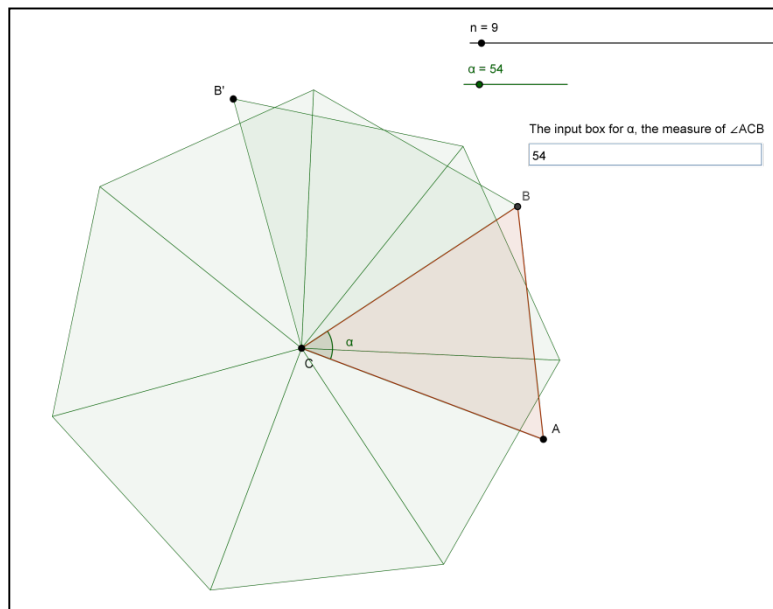


Figure 2. Increasing the number of triangles with the slider n

To do the task, students lean on the fact that any regular polygon can be decomposed in as many isosceles triangles as there are sides, these triangles sharing a common vertex, the center C of the circumscribed circle. Some knowledge elements are thus activated here, that the students could have previously met or that they may be discovering through the activity.

Elements of *a priori* analysis (Artigue, 1988)

We will settle for giving the outline; for a detailed analysis, see Tanguay et al. (2013). The situation can easily fill up two lessons. In the first one, while using the two sliders so as to ‘close’ the polygon without overlaps, students are expected to correlate the decomposition of the regular n -gon into n isosceles triangles with the divisibility of 360 by α , the measure of the central angle. In the institutionalization phase (Brousseau, 1998) concluding the first lesson, a list of the divisors of 360 may be established by the whole group, either along a decreasing list, considering α and starting with $\alpha = 120^\circ$, or along an increasing list for n , starting with $n = 3$. For each couple (α, n) possible in $\mathbb{IN} \times \mathbb{IN}$, the corresponding polygon is produced at the screen. A special discussion should be held to deal with the pairs $(180^\circ, 2)$ and $(360^\circ, 1)$.

In the next lesson, the teacher proposes to investigate the case $n = 7$. We anticipate that a majority of students won’t enter $360/7$ in the input box for α , but will rather enter the approximation they’ll get from their pocket calculator. But when the entered value is not $360/7$ or its approximation with at least 15 decimals¹, then a sufficiently close zoom shows a polygon

¹ Let k denotes the number of decimals selected in the pull-down menu (\rightarrow Options \rightarrow Rounding), $k = 2$ being the default, $k = 15$ being the maximum available from GeoGebra. If the entered number is $360/7$ or an approximation with more than k decimals, then the number displayed by GeoGebra (version 4.0.41.0) in the α box is its approximation rounded to the k -th decimal. The polygon will appear well closed even after zooming only if the entered value is $360/7$ or an approximation with 15 decimals or more, independently of the number displayed. So as regard the measure of $\angle ACB$, the relation between the entered number, the displayed number and the number used internally by the software is not transparent.

which is not well closed. Of course the teacher must show how to use the zoom and ask for a verification that the polygon is well closed.

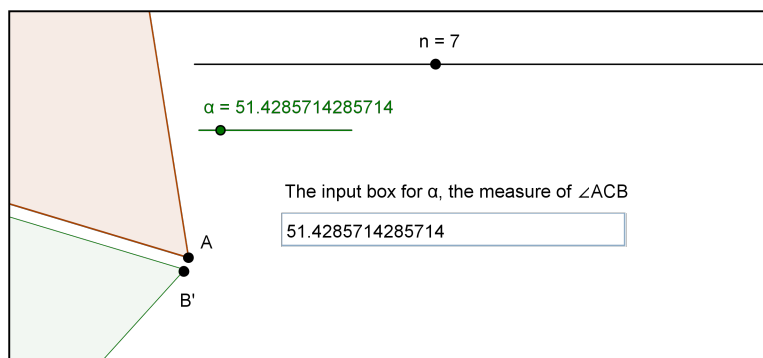


Figure 3. A close zoom, for an approximation of $360/7$ with 13 decimals

The final discussion should concern the ‘exact’ decimal expansion of $360/7$ — the teacher may go as far as explaining the period by carrying out the long division on the blackboard —, the value used internally by GeoGebra to deal with $360/7$, and the one that should be considered in theory when the ideal figure of the heptagon is decomposed.

A first experiment, as a case study (Karsenty & Savoie-Zajc, ch. 9)

A first experiment, conducted ‘off-class’ during a week-end afternoon with four students aged 12 and 13, confirmed, to a great extent, the *a priori* analysis. Student 1 and Student 2 of Team 1 are considered strong in school mathematics. Student 2 is notably at ease with hand computation and appears eager to show this skill off. In Team 2, Student 3 is largely over average and Student 4 is plain average. The experiment consisted in two periods of 75 minutes each, separated by a 20-minute break. Each period was devoted to the corresponding above-mentioned ‘lesson’, and each ended with a 15-minute whole group discussion. Although the concern of this article is mainly related to the second period, we will outline some interesting outcomes from the first as well.

The divisors of 360 and a flat polygon

At the very beginning of their work, having quickly constructed the polygons with central angle 90° , 120° and 72° , Students 1 and 2 seemed to discern the relation $n \times \alpha = 360^\circ$ although they didn’t explicitly state it. Indeed, they started listing the divisors of 360 by increasing the value of n with an increment of 1, Student 2 finding each corresponding value of α by hand calculation. They stopped at $n = 20$, explaining to the observing researcher that “after, it goes the other way round, ... the angle measurement becomes the number of triangles and the number of triangles becomes the angle measurement”. They also explained that the numbers 7, 11, 13, 14, 17, 19 have to be skipped, because the quotient is not a whole number (« la division ne tombe pas juste », in French). When asked what was to be done with these values of n , they answered without hesitation that the corresponding polygon does not exist. During the whole group discussion concluding the first period, they argued, quite surprisingly, that the values $n = 2$ and $\alpha = 180^\circ$ give rise to a two-sided (regular) polygon with a flat central angle, each of the two sides being reduced to a point.

The regular heptagon

In the second period, being asked to explore the case $n = 7$, Team 1 had to accept reconsidering its previous conclusion about it. Student 1 asked for a calculator but Student 2 quickly delved into the long division of 360 by 7, and soon realized that the quotient is periodic. As successive entered values for α , their attempts unfolded the numbers 51.5 – 51.42857143 – 51.428571428571 – 51.428571428572 – 51.4285714285714 – 51.42857142857142 – 51.428571428571428, this last one being displayed as 51.42857142857143 by GeoGebra. Each entered value was followed by zooming-in and only the last one resulted in a seemingly well closed polygon, even after repeated zoom-in. Nevertheless, the two students remained uncomfortable with the idea of a figure constructed from a non exact measurement for the central angle, as would soon be revealed by the concluding whole group discussion.

The following exchanges show how Team 2 became aware of the limits of the software, as much for what concerns the computation and the use of an approximate value for $360/7$ by GeoGebra, as for what concerns the precision of the graphical display where the two sides [AC] and [B'C] — [B'C] being the image of [BC] in the n -th triangle — are supposed to coincide exactly. It is worth noting that Student 4, in principle the weakest of the four, is really the one who first expressed the idea that beyond a certain threshold, more precise numbers induce no visible effects and are thus somewhat pointless.

Team 2 has made attempts to construct the heptagon with 2 and then 10 decimals for α . Seeing with the zoom that the polygon is not well closed, they try to enter more decimals. They ask for 'a bigger calculator, with more digits'. The researcher put on the desktop the computer calculator. Then Student 3 realizes that the number is periodic, and enter four periods in a row into the α box. They zoom and zoom and zoom and are now convinced that the polygon is well closed.

Student 3 : 428571 428571 428571... it goes on like this at infinity.

Student 4 : the more 428571 we put... Here, we can't see it because of this system, but the more we put of 428571, the more exact it gets.

Researcher : it would mean that what you've put there, it is not perfectly exact?

Student 4 : no, but, it's exact for the software.

Student 3 : for the software, we've reached the maximum.

Student 4 : we can add this at infinity but at some point, it will be, ... as an atom.

Researcher : OK, so the software is not able to see... with more precision ... than that?

Student 4 : nothing more to be said, we've got it now! [He refers to their first attempts and their disappointment, having seen the polygon not well closed and invalid]. It's no more human for the eyes...

Researcher : no more human for the eyes??

Student 4 : the eyes can't see that. [that is if the two sides don't coincide exactly, it is not visible.]

Student 3 : it's atomic.

The two speaking at the same time

Student 4 : yes, that's it. Now, we've got to the atomic. [...] We've got it, we've made it, we're Science makers!

Student 3 : it's a more-than-perfect polygon, we are finished!

In the closing discussion between the four students and the researchers, Team 1 argued, quite radically, that since an exact (finite) decimal expansion of $360/7$ is not possible, hence the regular heptagon does not exist. Team 2 didn't agree, arguing that a precision going beyond what is visible is unnecessary. The researchers concluded the debate in terms of *theory* versus *practice* : in theory, the regular heptagon exists as an ideal object, and the exact measure of its

central angle is $360/7$ degrees. In practice, one can always manage to obtain a representation as accurate as needed.

Conclusion

Through this case study, we may acknowledge that despite the difficulties and challenges engendered by DGSs with respect to empiricism, because of their precision and presumed infallibility, it is nevertheless possible, by designing suitable teaching situations, to make students reflect, in a meaningful context, on the dualities *ideal object – visual representation*, *exact measurements – approximations*, on the type of certainty aimed at according to a mathematical perspective and on the strong dependence this perspective nurture with *the theoretical*.

A coordination between numbers and figural representations

Team 2 really gets into the activity in accordance with the *a priori* analysis. The two students follow a trial-and-error approach in which the designed computer tool is used as a construction, exploration and validation tool, to produce representations of the targeted geometrical objects. This approach allows Team 2 to foster a coordination between related systems of *signs*, namely the numbers as measurements and the displayed figural representations, so that in this instance the technological setting can be assessed as a *tool of semiotic mediation*, in Bartolini Bussi & Mariotti's sense (2002). First the possibility of constructing a regular polygon with integers as angle measurements is linked to the notion of divisibility, geometry providing a meaningful context to strengthen the understanding of this notion. Secondly, by linking numerical approximations to geometrical approximations, the relation between the ideal object 'regular heptagon' and its figural representations is conceptualized by Students 3 and 4 as analogous to the relation between the number $360/7$ and its decimal approximations. Their comments in each phase of the second period show their capability of distancing themselves from the technical constraint of the software regarding numbers and geometrical displays, and to consider numbers and figural representations from a more abstract and purely mathematical standpoint. To such an extent that the activity can be said having succeed in "promoting the evolution of signs expressing the relationship between the artifact and tasks into signs expressing the relationship between artifact and [mathematical] knowledge" (op. cit., p. 753).

A gap between the numerical and the geometrical 'worlds'

By contrast, Team 1's approach is further remote from what was expected. It is deeply rooted into the 'numerical'. The relation between the central angle measurement and the number of sides having been rapidly guessed at an intuitive level by Team 1's members, the technological tool is immediately dropped and the quest for regular polygons is reduced to a listing of the divisors of 360. The software is used as a verification tool but not as an experimentation tool; for example the values $n = 7, 11, 13, 14, 17, 19$ are not 'tried', even less 'explored' with the tool during the first period, the numerical conclusions being deprived of a deeper geometrical corroboration. The reaction of Students 1 and 2 about the heptagon in the concluding discussion is also revealing of their difficulty in coordinating the numerical world and the geometrical world. Although they have constructed an heptagon seemingly well closed in the second period, although one of the researcher showed them an heptagon obtained directly from *Regular Polygon* in the GeoGebra pull-down menu, Students 1 and 2 kept arguing that the heptagon doesn't exist because it is impossible to obtain an exact value [*sic*] for $360 \div 7$. Student 1 will eventually nuance her position by saying that the heptagon exists in theory but is not 'drawable'

(« dessinable », in French). Anyhow, we felt until the end an uneasiness with Team 1's idea about the 'real' existence of the regular heptagon, as if its GeoGebra display was something like a fraud.

Institutionalization and teachers' involvement

The activity has nevertheless reached its goal to the extent that it brought the four students to reflect on the status of what is seen at the screen, on the way numbers are possibly processed by the software, and on the necessity of taking a step back when mathematical conclusions are to be drawn from numbers and displays technologically generated. Besides, considering that some of these issues remained 'technologically non transparent' i.e. that some displays and processing kept their 'black-box' feature, and also that not all of the four students were equally at ease with the conclusions, the present case study brings us to stress the importance of the institutionalization phases, of researchers' or teachers' involvement in acting "... both at the cognitive and the metacognitive level, both fostering the evolution of meanings and guiding pupils to be aware of their mathematical status" (op. cit., p. 754). By explaining that in a way, even representing a square entails displaying it with lines of a certain thickness and measuring it with tools the exactness of which can never be ascertain, the researchers brought the students to realize, at a 'meta'-level, that reasoning and theoretical scaffolding in mathematics must be considered apart, in parallel, from the problem of physically representing the objects at stake.

More to come

It also stresses that if the activity instilled an awareness in the students' minds, it is of course far from having settled the above mentioned issues. The study therefore calls for more experiments, especially in more standard classroom contexts, and for development of activities and teaching sequences in which issues pertaining to measurement, empiricism and proof would be further explored. These are among the goals of a larger ongoing research program about the geometer-physicist paradigm.

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