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Reich type weak contractions on metric spaces endowed with a graph

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Abstract

In this paper, we define a new class of Reich type multi-valued contractions on a complete metric space satisfying the g-graph preserving condition and we study the fixed point theorem for such mappings. In addition, we present the existence and uniqueness of the fixed point for at least one of two multi-valued mappings. The results of this paper extend and generalize several well-known results. Some examples illustrate the usability of our results.

MSC: 47H04; 47H10

Keywords: fixed point theorem; multi-valued mapping; graph preserving; Reich type contraction

1 Introduction

The classical contraction mapping principle of Banach states that if (X,d) is a complete metric space and $f : X \to X$ such that $d(f(x),f(y)) \leq \alpha d(x,y)$ for all $x,y \in X$, where $\alpha \in [0,1)$, then f has a unique fixed point. The Banach fixed point theorem plays an important role in studying the existence of solutions of nonlinear integral equations, system of linear equations, nonlinear differential equations, and proving the convergence of algorithms in computational mathematics. The Banach fixed point theorem has been extended in many directions; see [1–17]. Fixed point theory of multi-valued mappings plays a central role in control theory, optimization, partial differential equations, and economics. For a metric space (X,d), we let $\mathrm{CB}(X)$ be the set of all nonempty, closed, and bounded subsets of X. A point $x \in X$ is a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. Nadler [18] has proved a multi-valued version of the Banach contraction principle which we state as the following theorem.

Theorem 1.1 *Let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$. Assume that there exists $k \in [0,1)$ such that $H(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz$.*

Reich [19] generalized the Banach fixed point theorem for single-valued maps and multi-valued maps as the two following theorems.

Theorem 1.2 *Let (X,d) be a complete metric space and let $f : X \to X$ be a Reich type single-valued (a,b,c)-contraction, that is, there exist nonnegative numbers a, b, c with $a+b+c<1$*

such that

$$d\big(f(x),f(y)\big) \le ad(x,y) + bd\big(x,f(x)\big) + cd\big(y,f(y)\big)$$

for each $x, y \in X$. *Then* f *has a unique fixed point.*

Theorem 1.3 *Let* (X,d) *be a complete metric space and let a mapping* $T : X \to P_{\mathrm{cl}}(X)$, *where* $P_{\mathrm{cl}}(X)$ *is the set of all nonempty closed subsets of* X, *be a Reich type multi-valued* (a,b,c)*-contraction, that is, there exist nonnegative numbers* a, b, c *with* $a + b + c < 1$ *such that*

$$H(Tx, Ty) \le ad(x,y) + bD(x, Tx) + cD(y, Ty)$$

for each $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

In 2008, Jachymski [20] introduced the concept of a contraction concerning a graph, called a G-contraction, and proved some fixed point results of the G-contraction in a complete metric space endowed with a graph and he showed that the results of many authors can be derived by his results.

Definition 1.4 Let (X,d) be a metric space and $G = (V(G),E(G))$ a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\Delta = \{(x,x) \mid x \in X\} \subset E(G)$. We say that a mapping $f : X \to X$ is a *G-contraction* if f preserves edges of G, *i.e.*, for every $x, y \in X$,

$$(x,y) \in E(G) \quad \Rightarrow \quad \big(f(x),f(y)\big) \in E(G)$$

and there exists $\alpha \in (0,1)$ such that, for $x, y \in X$,

$$(x,y) \in E(G) \quad \Rightarrow \quad d\big(f(x),f(y)\big) \le \alpha d(x,y).$$

Jachymski showed in [20] that assuming some properties for X, a G-contraction $f : X \to X$ has a fixed point if and only if there exists $x \in X$ such that $(x,f(x)) \in E(G)$. The results of Jachymski were generalized by several authors (see, for example, Bojor [3]; Chifu and Petrusel [6]; Samreen and Kamran [13]; Asl *et al.* [2]; Abbas and Nazir [1]). Recently, Tiammee and Suantai [21] introduced the concept of g-graph preserving for multi-valued mappings and proved their fixed point theorem in a complete metric space endowed with a graph.

Definition 1.5 [21] Let X be a nonempty set and $G = (V(G),E(G))$ be a graph such that $V(G) = X$, and let $T : X \to \mathrm{CB}(X)$. T is said to be *graph preserving* if it satisfies the following:

$$\text{if } (x,y) \in E(G), \text{then } (u,v) \in E(G) \text{ for all } u \in Tx \text{ and } v \in Ty.$$

Definition 1.6 [21] Let X be a nonempty set and $G = (V(G),E(G))$ be a graph such that $V(G) = X$, and let $T : X \to \mathrm{CB}(X)$, $g : X \to X$. T is said to be *g-graph preserving* if it satisfies the following: for each $x, y \in X$,

$$\text{if } \big(g(x), g(y)\big) \in E(G), \text{then } (u, v) \in E(G) \text{ for all } u \in Tx, v \in Ty.$$

By using the concept of 'g-graph preserving' introduced by Tiammee and Suantai [21] and the concept of a Reich type multi-valued contraction defined by Reich [19], we define a new class of Reich type multi-valued contraction on a complete metric space satisfying the g-graph preserving condition and then we shall study the fixed point theorem for such mappings. Moreover, we establish some results on common fixed points for two multi-valued mappings. The results of this research extend and generalize several well-known results from previous work.

2 Main results

Let (X, d) be a metric space. Denote $\mathrm{CB}(X)$ the set of all nonempty closed and bounded subsets of X. For $a \in X$ and $A, B \in \mathrm{CB}(X)$, define

$$d(x, A) = \inf\big\{d(x, y) \mid y \in A\big\},$$
$$D(A, B) = \inf\big\{d(x, B) \mid x \in A\big\}.$$

Also, define

$$H(A, B) = \max\Big\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\Big\}.$$

The mapping H is said to be a *Hausdorff metric* induced by d. The next lemma will play central roles in our main results.

Lemma 2.1 *Let (X, d) be a metric space. If $A, B \in \mathrm{CB}(X)$ and $x \in A$, then for each $\epsilon > 0$, there is $b \in B$ such that*

$$d(a, b) < H(A, B) + \epsilon.$$

We start with the new class of Reich type multi-valued (α, β, γ)-contraction on a complete metric space.

Definition 2.2 Let (X, d) be a metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, $g : X \to X$, and $T : X \to \mathrm{CB}(X)$. T is said to be a *Reich type weak G-contraction* with respect to g or a $(g, \alpha, \beta, \gamma)$-$G$-contraction provided that

(1) T is g-graph preserving;
(2) there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ and

$$H(Tx, Ty) \le \alpha d\big(g(x), g(y)\big) + \beta D\big(g(x), Tx\big) + \gamma D\big(g(y), Ty\big)$$

for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$.

Example 2.3 Let $\mathbb{N}$ be a metric space with the usual metric. Consider the directed graph defined by $V(G) = X$ and $E(G) = \{(2n-1, 2n+1) : n \in \mathbb{N}\} \cup \{(2n, 2n+2) : n \in \mathbb{N} - \{1\}\} \cup$

$\{(2n, 2n+4) : n \in \mathbb{N} - \{1\}\} \cup \{(2n, 2n) : n \in \mathbb{N} - \{1\}\} \cup \{(1,1), (6,4)\}$. Let $T : X \to \mathrm{CB}(X)$ be defined by

$$Tn = \begin{cases} \{2k, 2k+2\} & \text{if } n = 2k-1, k \in \mathbb{N}, \\ \{1\} & \text{if } n = 2k, k \in \mathbb{N}, \end{cases}$$

and $g : \mathbb{N} \to \mathbb{N}$ be defined by

$$g(n) = \begin{cases} 2k & \text{if } n = 2k+2, k \in \mathbb{N}, \\ 2k-1 & \text{if } n = 2k+1, k \in \mathbb{N}, \\ 2 & \text{if } n = 1, 2. \end{cases}$$

We will show that T is a $(g, \alpha, \beta, \gamma)$-$G$-contraction with $\alpha = 0$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{3}$. Let $(g(x), g(y)) \in E(G)$. If $(g(x), g(y)) = (2k-1, 2k+1)$ for $k \in \mathbb{N}$, then $(x, y) = (2k+1, 2k+3)$, $Tx = \{2k+2, 2k+4\}$, and $Ty = \{2k+4, 2k+6\}$. We obtain $(2k+2, 2k+4), (2k+2, 2k+6), (2k+4, 2k+4), (2k+4, 2k+6) \in E(G)$. Also, $D(g(x), Tx) = 3$, $D(g(y), Ty) = 3$, and

$$\begin{aligned} H(Tx, Ty) &= \max\Big\{\sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty)\Big\} \\ &= \max\Big\{\sup_{a \in Ty} d\big(a, \{2k+2, 2k+4\}\big), \sup_{b \in Tx} d\big(b, \{2k+4, 2k+6\}\big)\Big\} \\ &= \max\{2, 2\} \\ &= 2 \\ &\le 0d\big(g(x), g(y)\big) + \frac{1}{3}(3) + \frac{1}{3}(3) \\ &\le 0d\big(g(x), g(y)\big) + \frac{1}{3}D\big(g(x), Tx\big) + \frac{1}{3}D\big(g(y), Ty\big). \end{aligned}$$

If $(g(x), g(y)) = (2k, 2k+2)$ or $(2k, 2k+4)$ or $(2k, 2k)$ for $k \in \mathbb{N} - \{1\}$, then $Tx = Ty = \{1\}$ and $(1,1) \in E(G)$. It follows that

$$\begin{aligned} H(Tx, Ty) &= \max\Big\{\sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty)\Big\} \\ &= 0 \\ &\le 0d\big(g(x), g(y)\big) + \frac{1}{3}D\big(g(x), Tx\big) + \frac{1}{3}D\big(g(y), Ty\big). \end{aligned}$$

If $(g(x), g(y)) = (1,1)$, then $x = y = 3$, $Tx = Ty = \{4, 6\}$, and $(4,4), (4,6), (6,4), (6,6) \in E(G)$. It follows that

$$\begin{aligned} H(Tx, Ty) &= \max\Big\{\sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty)\Big\} \\ &= 0 \\ &\le 0d\big(g(x), g(y)\big) + \frac{1}{3}D\big(g(x), Tx\big) + \frac{1}{3}D\big(g(y), Ty\big). \end{aligned}$$

If $(g(x),g(y)) = (6,4)$, then $x = 8$, $y = 6$, and $Tx = Ty = \{1\}$ and $(1,1) \in E(G)$. Note that $d(g(x),g(y)) = 2$, $D(g(x),T8) = 5$, $D(g(y),T6) = 3$, and so

$$\begin{aligned} H(Tx,Ty) &= \max\Bigl\{\sup_{a\in Ty} d(a,Tx), \sup_{b\in Tx} d(b,Ty)\Bigr\} \\ &= 0 \\ &\le 0d\bigl(g(x),g(y)\bigr) + \frac{1}{3}D\bigl(g(x),Tx\bigr) + \frac{1}{3}D\bigl(g(y),Ty\bigr). \end{aligned}$$

It follows that T is a $(g,0,\frac{1}{3},\frac{1}{3})$-$G$-contraction.

Property A For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n,x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}}$ such that $(x_{n_k},x) \in E(G)$ for $n_k \in \mathbb{N}$.

Lemma 2.4 *Let (X,d) be a metric space with the directed graph G, $g_1,g_2 : X \to X$ be surjective maps and, for $i = 1,2$, $T_i : X \to \mathrm{CB}(X)$ be g-graph preserving satisfying the following: there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that, for all $x,y \in X$, if $(g_1(x),g_2(y)) \in E(G)$, then*

$$H(T_1x,T_2y) \le \alpha d\bigl(g_1(x),g_2(y)\bigr) + \beta D\bigl(g_1(x),T_1x\bigr) + \gamma D\bigl(g_2(y),T_2y\bigr),$$

and if $(g_2(x),g_1(y)) \in E(G)$, then

$$H(T_2x,T_1y) \le \alpha d\bigl(g_2(x),g_1(y)\bigr) + \beta D\bigl(g_2(x),T_2x\bigr) + \gamma D\bigl(g_1(y),T_1y\bigr).$$

If

(A) *there exists $x_0 \in X$ such that $(g_1(x_0),u) \in E(G)$ for some $u \in T_1x_0$, and*

(B) *if $(g_1(x),g_2(y)) \in E(G)$, then $(z,w) \in E(G)$ for all $z \in T_1x$, $w \in T_2y$, and if $(g_2(x),g_1(y)) \in E(G)$, then $(b,r) \in E(G)$ for all $b \in T_2x$, $r \in T_1y$,*

then there exists a sequence $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$ in X such that, for each $k \in \mathbb{N}$,

$$\begin{aligned} &g_1(x_{2k}) \in T_2x_{2k-1}, \qquad g_2(x_{2k-1}) \in T_1x_{2k-2}, \\ &\bigl(g_1(x_{2k-2}),g_2(x_{2k-1})\bigr), \bigl(g_2(x_{2k-1}),g_1(x_{2k})\bigr) \in E(G), \end{aligned}$$

and $\{S(x_k)\}$ is a Cauchy sequence in X where

$$S(x_k) = \begin{cases} g_1(x_k) & \text{if } k \text{ is even}, \\ g_2(x_k) & \text{if } k \text{ is odd}. \end{cases}$$

Proof Since g_2 is surjective, there exists $x_1 \in X$ such that $g_2(x_1) \in T_1x_0$ and $(g_1(x_0),g_2(x_1)) \in E(G)$. By Lemma 2.1, there exists $x_2 \in X$ such that $g_1(x_2) \in T_2x_1$ and $d(g_2(x_1),g_1(x_2)) \le H(T_1x_0,T_2x_1) + (\alpha + \beta)$.

By (B), it follows that $(g_2(x_1),g_1(x_2)) \in E(G)$. By assumption,

$$\begin{aligned} d\bigl(g_2(x_1),g_1(x_2)\bigr) &\le H(T_1x_0,T_2x_1) + (\alpha + \beta) \\ &\le \alpha d\bigl(g_1(x_0),g_2(x_1)\bigr) + \beta D\bigl(g_1(x_0),T_1x_0\bigr) + \gamma D\bigl(g_2(x_1),T_2x_1\bigr) \end{aligned}$$

$$\begin{aligned}
&\quad + (\alpha + \beta) \\
&\leq \alpha d\big(g_1(x_0), g_2(x_1)\big) + \beta d\big(g_1(x_0), g_2(x_1)\big) + \gamma d\big(g_2(x_1), g_1(x_2)\big) \\
&\quad + (\alpha + \beta).
\end{aligned}$$

Hence,

$$(1-\gamma) d\big(g_2(x_1), g_1(x_2)\big) \leq (\alpha + \beta) d\big(g_1(x_0), g_2(x_1)\big) + (\alpha + \beta).$$

It follows that

$$\begin{aligned}
d\big(g_2(x_1), g_1(x_2)\big) &\leq \frac{(\alpha + \beta)}{1-\gamma} d\big(g_1(x_0), g_2(x_1)\big) + \frac{\alpha + \beta}{1-\gamma} \\
&= \eta d\big(g_1(x_0), g_2(x_1)\big) + \eta,
\end{aligned} \tag{1}$$

where $\eta = \frac{(\alpha+\beta)}{1-\gamma} < 1$.

Next, by Lemma 2.1, we can choose $x_3 \in X$ such that $g_2(x_3) \in T_1 x_2$ and

$$d\big(g_1(x_2), g_2(x_3)\big) \leq H(T_2 x_1, T_1 x_2) + (\alpha + \beta)\eta.$$

By (B) again, it follows that $(g_1(x_2), g_2(x_3)) \in E(G)$. By assumption again, we have

$$\begin{aligned}
d\big(g_1(x_2), g_2(x_3)\big) &\leq H(T_2 x_1, T_1 x_2) + (\alpha + \beta)\eta \\
&\leq \alpha d\big(g_2(x_1), g_1(x_2)\big) + \beta D\big(g_2(x_1), T_2 x_1\big) + \gamma D\big(g_1(x_2), T_1 x_2\big) \\
&\quad + (\alpha + \beta)\eta \\
&\leq \alpha d\big(g_2(x_1), g_1(x_2)\big) + \beta d\big(g_2(x_1), g_1(x_2)\big) + \gamma d\big(g_1(x_2), g_2(x_3)\big) \\
&\quad + (\alpha + \beta)\eta.
\end{aligned}$$

Hence,

$$(1-\gamma) d\big(g_1(x_2), g_2(x_3)\big) \leq (\alpha + \beta) d\big(g_2(x_1), g_1(x_2)\big) + (\alpha + \beta)\eta.$$

It follows that

$$d\big(g_1(x_2), g_2(x_3)\big) \leq \eta d\big(g_2(x_1), g_1(x_2)\big) + \eta^2.$$

By the inequality of (1), we have

$$d\big(g_1(x_2), g_2(x_3)\big) \leq \eta^2 d\big(g_1(x_0), g_2(x_1)\big) + 2\eta^2.$$

Continuing in this fashion, we obtain sequences $\{x_k\}$ and $\{S(x_k)\}$ with the property that

$$S(x_k) = \begin{cases} g_1(x_k) & \text{if } k \text{ is even}, \\ g_2(x_k) & \text{if } k \text{ is odd}, \end{cases}$$

and for each $k \in \mathbb{N}$,

$$g_1(x_{2k}) \in T_2 x_{2k-1}, \qquad g_2(x_{2k-1}) \in T_1 x_{2k-2},$$
$$\big(g_1(x_{2k-2}), g_2(x_{2k-1})\big), \big(g_2(x_{2k-1}), g_1(x_{2k})\big) \in E(G)$$

and

$$d\big(S(x_k), S(x_{k+1})\big) \leq \eta^k d\big(g_1(x_0), g_2(x_1)\big) + k\eta^k.$$

Since $\eta < 1$, we have

$$\sum_{k=0}^{\infty}\big(d\big(S(x_k), S(x_{k+1})\big)\big) \leq d\big(g_1(x_0), g_2(x_1)\big)\sum_{k=0}^{\infty}\eta^k + \sum_{k=0}^{\infty}k\eta^k < \infty.$$

It is straightforward to check that $\{S(x_k)\}$ is a Cauchy sequence in X. □

Theorem 2.5 *Let (X, d) be a complete metric space with the directed graph G, $g_1, g_2 : X \to X$ be surjective maps and, for $i = 1, 2$, $T_i : X \to \mathrm{CB}(X)$ be g-graph preserving satisfying the following: there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that, for all $x, y \in X$, if $(g_1(x), g_2(y)) \in E(G)$, then*

$$H(T_1x, T_2y) \leq \alpha d\big(g_1(x), g_2(y)\big) + \beta D\big(g_1(x), T_1x\big) + \gamma D\big(g_2(y), T_2y\big),$$

and if $(g_2(x), g_1(y)) \in E(G)$, then

$$H(T_2x, T_1y) \leq \alpha d\big(g_2(x), g_1(y)\big) + \beta D\big(g_2(x), T_2x\big) + \gamma D\big(g_1(y), T_1y\big).$$

If the following hold:
1. *there exists $x_0 \in X$ such that $(g_1(x_0), u) \in E(G)$ for some $u \in T_1x_0$;*
2. *if $(g_1(x), g_2(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in T_1x$, $w \in T_2y$ and if $(g_2(x), g_1(y)) \in E(G)$, then $(b, r) \in E(G)$ for all $b \in T_2x$, $r \in T_1y$;*
3. *X has Property* A,

then there exist $u, v \in X$ such that $g_1(u) \in T_1u$ or $g_2(v) \in T_2v$.

Proof By (1), let $x_0 \in X$ be such that $(g_1(x_0), g_2(x_1)) \in E(G)$ for some $g_2(x_1) \in T_1x_0$. By Lemma 2.4, there exists a sequence $\{x_k\}_{k \in \mathbb{N} \cup \{0\}}$ in X such that, for each $k \in \mathbb{N}$,

$$g_1(x_{2k}) \in T_2 x_{2k-1}, \qquad g_2(x_{2k-1}) \in T_1 x_{2k-2},$$
$$\big(g_1(x_{2k-2}), g_2(x_{2k-1})\big), \big(g_2(x_{2k-1}), g_1(x_{2k})\big) \in E(G),$$

and $\{S(x_k)\}$ is a Cauchy sequence in X where

$$S(x_k) = \begin{cases} g_1(x_k) & \text{if } k \text{ is even}, \\ g_2(x_k) & \text{if } k \text{ is odd}. \end{cases}$$

Since X is complete, the sequence $\{S(x_k)\}$ converges to a point w for some $w \in X$. Let $u, v \in X$ be such that $g_1(u) = w = g_2(v)$. By Property A in (3), there is a subsequence $\{S(x_{k_n})\}$ such that $(S(x_{k_n}), g_1(u)) \in E(G)$ for any $n \in \mathbb{N}$. We claim that $g_1(u) \in T_1 u$ or $g_2(v) \in T_2 v$. Let $A = \{k_n \mid k_n \text{ is even}\}$ and $B = \{k_n \mid k_n \text{ is odd}\}$. Since $A \cup B$ is infinite, at least A or B must be infinite. If A is infinite, for each $g_2(x_{k_n+1})$, where $k_n \in A$, we have

$$\begin{aligned} D\big(g_2(v), T_2 v\big) &\leq d\big(g_2(v), g_2(x_{k_n+1})\big) + D\big(g_2(x_{k_n+1}), T_2 v\big) \\ &\leq d\big(g_2(v), g_2(x_{k_n+1})\big) + H(T_1 x_{k_n}, T_2 v) \\ &\leq d\big(g_2(v), g_2(x_{k_n+1})\big) + \alpha d\big(g_1(x_{k_n}), g_2(v)\big) + \beta D\big(g_1(x_{k_n}), T_1 x_{k_n}\big) \\ &\quad + \gamma D\big(g_2(v), T_2 v\big) \\ &\leq d\big(g_2(v), g_2(x_{k_n+1})\big) + \alpha d\big(g_1(x_{k_n}), g_2(v)\big) + \beta d\big(g_1(x_{k_n}), g_2(x_{k_n+1})\big) \\ &\quad + \gamma D\big(g_2(v), T_2 v\big). \end{aligned}$$

We obtain

$$D\big(g_2(v), T_2 v\big) \leq \frac{1}{1-\gamma}\big[d\big(g_2(v), g_2(x_{k_n+1})\big) + \alpha d\big(g_1(x_{k_n}), g_2(v)\big) + \beta d\big(g_1(x_{k_n}), g_2(x_{k_n+1})\big)\big].$$

Since $\{g_1(x_{k_n})\}$ and $\{g_2(x_{k_n+1})\}$ are subsequences of $S(x_m)$, they converge to $g_2(v)$ as $n \to \infty$, and hence $D(g_2(v), T_2 v) = 0$. Since $T_2 v$ is closed, we conclude that $g_2(v) \in T_2 v$. Similarly, if B is infinite, we can show that $g_1(u) \in T_1 u$, completing the proof. Note that if A and B in Theorem 2.5 are both infinite then $g_2(v) \in T_2 v$ and $g_1(u) \in T_1 u$. □

The following example illustrates Theorem 2.5.

Example 2.6 Let (X, d) be a metric space where $X = [0,1]$ and d is a usual metric on $\mathbb{R}$. Consider the directed graph $G = (V(G), E(G))$ defined by $V(G) = X$ and

$$\begin{aligned} E(G) = {} & \left\{(0,0), (1,1), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\} \\ & \cup \left\{\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}\right)\right\}. \end{aligned}$$

Let $T, S : X \to \mathrm{CB}(X)$ and $g_1, g_2 : X \to X$ be defined by

$$Tx = \begin{cases} \{\frac{1}{2}\} & \text{if } x = 1, \\ \{0, \frac{1}{2}\} & \text{if } x \in (0,1) \setminus \{\frac{1}{2}, \frac{1}{\sqrt{2}}\}, \\ \{\frac{1}{4}\} & \text{if } x = 0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \end{cases}$$

$$Sx = \begin{cases} \{\frac{1}{4}\} & \text{if } x = 0, \frac{1}{4}, \frac{1}{2}, 1, \\ \{0, \frac{1}{4}\} & \text{if } x \in (0,1) \setminus \{\frac{1}{2}, \frac{1}{4}\}, \end{cases}$$

$$g_1(x) = x^2 \quad \text{and} \quad g_2(x) = x \quad \text{for all } x \in X.$$

It is clear that S, T are g_1, g_2-graph preserving, respectively. It is straightforward to check that the conditions (1), (2), (3) of Theorem 2.5 are satisfied. Next, we will show that, for all

$x, y \in X$, if $(g_1(x), g_2(y)) \in E(G)$, then

$$H(Tx, Sy) \le \alpha d\big(g_1(x), g_2(y)\big) + \beta D\big(g_1(x), Tx\big) + \gamma D\big(g_2(y), Sy\big),$$

and if $(g_2(x), g_1(y)) \in E(G)$, then

$$H(Sx, Ty) \le \alpha d\big(g_2(x), g_1(y)\big) + \beta D\big(g_2(x), Sx\big) + \gamma D\big(g_1(y), Ty\big),$$

where $\alpha = 0$, $\beta = \gamma = \frac{1}{3}$.

If $(g_1(x), g_2(y)), (g_2(x), g_1(y)) \in E(G) \setminus \{(1,1)\}$, then $H(Tx, Sy) = 0 = H(Sx, Ty)$. So the above inequalities are satisfied.

If $(g_1(x), g_2(y)) = (1,1)$, then $g_1(x) = 1$, $g_2(y) = 1$, which implies that $x = 1$ and $y = 1$. Thus we have $Tx = \frac{1}{2}$ and $Sy = \frac{1}{4}$ and hence

$$\begin{aligned} H(Tx, Sy) &= H\left(\left\{\frac{1}{2}\right\}, \left\{\frac{1}{4}\right\}\right) \\ &= \frac{1}{4} \\ &\le 0(0) + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{3}{4}\right) \\ &= \alpha d\big(g_1(x), g_2(y)\big) + \beta D\big(g_1(x), Tx\big) + \gamma D\big(g_2(y), Sy\big). \end{aligned}$$

If $(g_2(x), g_1(y)) = (1,1)$, then $g_2(x) = 1$ and $g_1(y) = 1$, which yields $x = 1$ and $y = 1$. Thus we have $Sx = \frac{1}{4}$ and $Ty = \frac{1}{2}$ and hence

$$\begin{aligned} H(Sx, Ty) &= H\left(\left\{\frac{1}{4}\right\}, \left\{\frac{1}{2}\right\}\right) \\ &= \frac{1}{4} \\ &\le 0(0) + \frac{1}{3}\left(\frac{3}{4}\right) + \frac{1}{3}\left(\frac{1}{2}\right) \\ &= \alpha d\big(g_2(x), g_1(y)\big) + \beta D\big(g_2(x), Sx\big) + \gamma D\big(g_1(y), Ty\big). \end{aligned}$$

By Theorem 2.5, there are $u, v \in X$ such that $g_1(u) \in Tu$ or $g_2(v) \in Sv$. In this example, choose $u = \frac{1}{4}$. Since $g_2(u) = u$, we have $u \in Su = \{u\}$.

From Theorem 2.5 one deduces the next two corollaries.

Corollary 2.7 *Let (X, d) be a complete metric space with the directed graph G, $g : X \to X$ be a surjective map, and $T_1, T_2 : X \to \mathrm{CB}(X)$ be g-graph preserving satisfying*

$$H(T_1x, T_2y) \le \alpha d\big(g(x), g(y)\big) + \beta D\big(g(x), T_1x\big) + \gamma D\big(g(y), T_2y\big)$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$. If the following hold:

(1) *there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in T_1x_0$;*

(2) *X has Property* A,

then there exists $u \in X$ such that $g(u) \in T_1u$ or $g(u) \in T_2u$.

Proof Set $g_1 = g_2 = g$. Then this corollary follows immediately from Theorem 2.5. □

Corollary 2.8 *Let (X,d) be a complete metric space, $G = (V(G),E(G))$ be a directed graph such that $V(G) = X$, and let $g : X \to X$ be a surjective map. If $T : X \to \mathrm{CB}(X)$ is a multi-valued mapping satisfying the following properties*:

(1) *T is a (g,α,β,γ)-G-contraction*;
(2) *the set $X_T = \{x \in X \mid (g(x),y) \in E(G)$ for some $y \in Tx\} \neq \emptyset$*;
(3) *X has Property* A,

then there exists $u \in X$ such that $g(u) \in Tu$.

Proof Set $T_1 = T_2 = T$ and $g_1 = g_2 = g$. Then the result follows directly from Theorem 2.5. □

A partial order is a binary relation $\leq$ over the set X which satisfies the following conditions:

(1) $x \leq x$ (reflexivity);
(2) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity),

for all $x, y \in X$. A set with a partial order $\leq$ is called a partially ordered set. We write $x < y$ if $x \leq y$ and $x \neq y$.

Definition 2.9 Let $(X, \leq)$ be a partially ordered set. For each $A, B \subset X$, $A \prec B$ if $a \leq b$ for any $a \in A, b \in B$.

Definition 2.10 [21] Let (X,d) be a metric space endowed with a partial order $\leq$, $g : X \to X$ a surjective map, and $T : X \to \mathrm{CB}(X)$. T is said to be *g-increasing* if for any $x, y \in X$,

$$g(x) < g(y) \quad \Rightarrow \quad Tx \prec Ty.$$

In the case g is the identity map, the mapping T is called an increasing mapping.

Theorem 2.11 *Let (X,d) be a metric space endowed with a partial order $\leq$, $g : X \to X$ be a surjective map and $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that*

(1) *T is g-increasing*;
(2) *there exist $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$*;
(3) *for each sequence $\{x_k\}$ such that $g(x_k) < g(x_{k+1})$ for all $k \in \mathbb{N}$ and $g(x_k)$ converges to $g(x)$ for some $x \in X$, then $g(x_k) < g(x)$ for all $k \in \mathbb{N}$*;
(4) *there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that*

$$H(Tx, Ty) \leq \alpha d\bigl(g(x), g(y)\bigr) + \beta D\bigl(g(x), Tx\bigr) + \gamma D\bigl(g(y), Ty\bigr)$$

for all $x, y \in X$ such that $g(x) < g(y)$;
(5) *the metric d is complete.*

Then there exists $u \in X$ such that $g(u) \in Tu$. If g is injective, then there is a unique $u \in X$ such that $g(u) \in Tu$.

Proof Define $G = (V(G),E(G))$, where $V(G) = X$ and $E(G) = \{(x,y) \mid x < y\}$. Let $x, y \in X$ be such that $(g(x), g(y)) \in E(G)$. Then $g(x) < g(y)$ so by (1) it implies that $Tx \prec Ty$. For each

$u \in Tx$, $v \in Ty$, we have $u < v$, thus $(u,v) \in E(G)$. That is, T is g-graph preserving. By assumption (2), there exist $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$. So $(g(x_0), u) \in E(G)$ and hence the property (1) in Corollary 2.7 is satisfied. Moreover, we obtain the property (2) of Corollary 2.7 from the assumption (3). Set $T_1 = T_2 = T$, then the T_1, T_2 are g-graph preserving mappings satisfying

$$H(T_1x, T_2y) \leq \alpha d\bigl(g(x), g(y)\bigr) + \beta D\bigl(g(x), T_1x\bigr) + \gamma D\bigl(g(y), T_2y\bigr)$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$. Therefore, from the result of this theorem follows Corollary 2.7.

Assume that g is injective. Let $u, v \in X$ be such that $g(u) \in Tu$ and $g(v) \in Tv$. Suppose that $g(u) \neq g(v)$. Without loss of generality, assume that $g(u) < g(v)$. Since $g(u) \in Tu$ and $g(v) \in Tv$, it follows that $D(g(u), Tu) = D(g(v), Tv) = 0$ and hence

$$\begin{aligned} d\bigl(g(u), g(v)\bigr) &\leq H(Tu, Tv) \\ &\leq \alpha d\bigl(g(u), g(v)\bigr) + \beta D\bigl(g(u), Tu\bigr) + \gamma D\bigl(g(v), Tv\bigr) \\ &< d\bigl(g(u), g(v)\bigr). \end{aligned}$$

This leads to a contradiction. Thus $g(u) = g(v)$. Since g is injective, we have $u = v$. □

We obtain the following result by considering $g(x) = x$ for all $x \in X$.

Corollary 2.12 *Let (X, d) be a metric space endowed with a partial order $\leq$ and $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that*

(1) *T is increasing;*
(2) *there exist $x_0 \in X$ and $u \in Tx_0$ such that $x_0 < u$;*
(3) *for each sequence $\{x_k\}$ such that $x_k < x_{k+1}$ for all $k \in \mathbb{N}$ and x_k converges to x for some $x \in X$, then $x_k < x$ for all $k \in \mathbb{N}$;*
(4) *there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that*

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta D(x, Tx) + \gamma D(y, Ty)$$

for all $x, y \in X$ such that $x < y$;

(5) *the metric d is complete.*

Then there is a unique $u \in X$ such that $u \in Tu$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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