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Existence of solutions for a class of porous medium type equations with lower order terms

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Abstract

This paper deals with a class of degenerate quasilinear elliptic equations of the form $-\operatorname{div}(a(x,u,\nabla u)) + F(x,u,\nabla u) = f$, where $a(x,u,\nabla u)$ is allowed to degenerate with the unknown u. Under some hypothesis on a, F, and f, we obtain the existence of bounded solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. For the case $f \in L^1(\Omega)$, we also prove that there exists at least one renormalized solution.

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Keywords: degenerate equations; weak and renormalized solutions; L^∞ estimate; natural growth

1 Introduction

This paper concerns the following degenerate problem:

$$(\mathscr{P}) \quad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + F(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $f \in L^q(\Omega)$ with $q \geq 1$ and $a(x,s,\xi)$ is a Carathéodory function. Furthermore, we assume that there exists a continuous function α from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\alpha(0) = 0$ and $a(x,s,\xi)\xi \geq \alpha(|s|)|\xi|^p$ for any $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, and almost every x in Ω. Thus problem $(\mathscr{P})$ degenerates for the subset $\{x \in \Omega : u(x) = 0\}$.

Problem $(\mathscr{P})$ has important and extensive applications to the fluid dynamics in porous media, in hydrology and in petroleum engineering (see [1, 2]). The simplest model is the stationary case of the porous media equation with zero Dirichlet boundary condition:

$$(\mathrm{P}_0) \quad \begin{cases} -\Delta(|u|^{m-1}u) + F(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which has been widely studied in the literature (see [3–6] and references therein).

For the case $\alpha \equiv \text{constant} > 0$, the existence of bounded solutions to problem $(\mathscr{P})$ is proved in [7], when the data f is small in a suitable norm.

Concerning the case that α is a positive function, Porretta and Segura de León investigated the existence results to problem $(\mathscr{P})$; see [8]. We remark that in [8], no sign condition is imposed on F, but the growth of F at infinity need to be controlled. We also point out that a variational inequality related to problem ($\mathscr{P}$) was studied in [9], and similar results can be found in [10] and [11].

In the case $\alpha(0) = 0$, $f \in W^{-1,r}(\Omega) \cap L^1(\Omega)$ with $r \geq p'$, $r > \frac{N}{p-1}$, Rakotoson proved the existence of a bounded weak solution to problem ($\mathscr{P}$) (see [12]), provided that F satisfies a sign condition. As $F = 0$ and $f \in W^{-1,r}(\Omega)$, the existence of solutions to problem ($\mathscr{P}$) has been discussed in [13]. We point out that the parabolic version of [13] has been studied in [14].

As $f \in L^q(\Omega)$ with $q \geq \max\{1, \frac{N}{p}\}$, we shall give a direct method to prove the existence of bounded weak solutions to problem ($\mathscr{P}$) in the standard sense, *i.e.* $u \in W_0^{1,p}(\Omega)$. The main difficulty comes from the facts that its modulus of ellipticity vanishes when the solution u vanishes. To overcome this difficulty, we shall firstly establish the L^∞ estimate for solution u, by the technique of rearrangement which is differs from the usual Stampacchia L^∞ regularity procedure. Then, by constructing suitable approximate problems, and using *a priori* estimates and a test function method, we shall finish the proof of this existence results.

Furthermore, we will study the case when $f \in L^1(\Omega)$. Since no growth conditions are required for ω and β (see (H_2)), it is not obvious that the term $-\operatorname{div}(a(x,u,\nabla u))$ makes sense even as a distribution. To overcome this difficulty, we shall use the concept of renormalized solutions, which is introduced by Diperna and Lions (see [15]). This notion was adapted by many authors to study partial differential equations with measurable data, especially for L^1 data (see [16–18] for example). We remark that an equivalent notion called entropy solutions, was introduced independently by Bénilan *et al.* [19].

The main ideas and methods come from [8, 10, 12, 20]. This paper is organized as follows: in Section 2 we give some preliminaries and state the main results; in Section 3, we study the existence of bounded solution to problem ($\mathscr{P}$); in Section 4, we prove the existence of renormalized solution.

2 Some preliminaries and the main results

2.1 Properties of the relative rearrangement

Let Ω be a bounded open subsets of $\mathbb{R}^N$, we denote by $|E|$ the Lebesgue measure of a set E. Assume that $u : \Omega \to \mathbb{R}$ be a measurable function, we define the distribution function $\mu_u(t)$ of u as follows:

$$\mu_u(t) = \bigl|\bigl\{x \in \Omega : u(x) > t\bigr\}\bigr|, \quad \forall t \in \mathbb{R}.$$

The decreasing rearrangement u_* of u is defined as the generalized inverse function of $\mu_u(t)$, *i.e.*

$$u_*(s) = \inf\bigl\{t \in R : \mu_u(t) \leq s\bigr\}, \quad s \in \Omega^* = \bigl[0, |\Omega|\bigr].$$

We recall also that u and u_* are equi-measurable, *i.e.*

$$\mu_u(t) = \mu_{u_*}(t), \quad t \in \mathbb{R},$$

which implies that for any non-negative Borel function ψ we have

$$\int_\Omega \psi\bigl(u(x)\bigr)\,\mathrm{d}x = \int_0^{|\Omega|} \psi\bigl(u_*(s)\bigr)\,\mathrm{d}s,$$

and if $E \subset \Omega$ be a measurable subset, then

$$\int_E u(x)\,dx \le \int_0^{|E|} u_*(s)\,ds.$$

Using the Fleming-Rishel formula, Hölder's inequality, and the isoperimetric inequality, we can get the following result (see [7, 9, 12]).

Lemma 2.1 *For any non-negative function $u \in W_0^{1,1}(\Omega)$, the following chain of inequalities holds*:

$$NC_N^{1/N}\mu_u(t)^{1-1/N} \le -\frac{d}{dt}\int_{u>t}|\nabla u|\,\mathrm{d}x \le \left(-\mu_u'(t)\right)^{1/p'}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^p\,\mathrm{d}x\right)^{1/p},$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$.

For more details as regards the theory of rearrangement, we just refer to [21] and the references therein.

2.2 Assumptions and the main results

Let Ω be an open bounded set of $\mathbb{R}^N$ ($N \ge 2$) and $p > 1$, we make the following assumptions.

(H_1) $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector function satisfying: there exists a continuous function α from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ if $s > 0$ and

$$a(x,s,\xi)\xi \ge \alpha\left(|s|\right)|\xi|^p, \quad \forall s \in R, \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N,$$
$$\int_0^{+\infty} \alpha^{\frac{1}{p-1}}(s)\,\mathrm{d}s = \int_0^{+\infty} \frac{1}{\alpha(s)}\,\mathrm{d}s = +\infty$$

and

$$\frac{1}{\alpha} \in L^1(0,b) \quad \text{for any given } b > 0.$$

(H_2) There exists a Carathéodory vector function $\bar{a}$ such that for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$:

(i) $a(x,s,\xi) = \alpha(|s|)\bar{a}(x,s,\xi)$.

(ii) $[\bar{a}(x,s,\xi) - \bar{a}(x,s,\xi')][\xi - \xi'] > 0$.

(iii) There exist an increasing function ω from $\mathbb{R}^+$ into $\mathbb{R}^+$ and a non-negative function $\bar{\omega} \in L^{p'}(\Omega)$ such that

$$\left|\bar{a}(x,s,\xi)\right| \le \omega\left(|s|\right)\left[|\xi|^{p-1} + \bar{\omega}(x)\right].$$

(iv) The function $\bar{a}$ is a positively homogeneous of degree $(p-1)$ with respect to the variable ξ, *i.e.*

$$\bar{a}(x,s,t\xi) = t^{p-1}\bar{a}(x,s,\xi), \quad \forall t \ge 0.$$

(H_3) $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function, for which there exists an increasing function β from $[0,+\infty)$ into $[0,+\infty)$ vanishing and continuous at zero such that for a.e. $x\in\Omega$, $\forall s\in\mathbb{R}$ and $\forall\xi\in\mathbb{R}^N$:

$$\left|F(x,s,\xi)\right|\le\beta\left(|s|\right)|\xi|^p.$$

(H_4) $f\in L^q(\Omega)$ with $q>\max\{1,\frac{N}{p}\}$.

(H_5) $\lim_{s\to\infty}\frac{e^{\gamma(|s|)}}{(1+\phi(|s|))^{p-1}}=0$, where γ and ϕ are defined as follows:

$$\gamma(s)=\int_0^s\frac{\beta(|\sigma|)}{\alpha(|\sigma|)}\,\mathrm{d}\sigma;\qquad\phi(s)=\int_0^s\left(\alpha\left(|\sigma|\right)\right)^{\frac{1}{p-1}}e^{\frac{\gamma(|s|)}{p-1}}\,\mathrm{d}\sigma.\tag{2.1}$$

Remark 2.1 Assumption (H_1) allows us to consider the porous medium operators $\triangle(|u|^{m-1}u)=\operatorname{div}(m|u|^{m-1}\nabla u)$. In this case, it yields $\alpha(|s|)=|s|^{m-1}$, so that the conditions $\alpha(0)=0$ and $\frac{1}{\alpha}\in L^1(0,b)$ indicate $1<m<2$. Thus, in this case, the porous medium equation becomes a slow diffusion equation.

We now introduce several auxiliary functions by

$$\tilde{\alpha}(s)=\int_0^s\alpha^{\frac{1}{p-1}}\left(|t|\right)\mathrm{d}t,\tag{2.2}$$

$$\gamma_\theta(s)=\int_0^s\frac{\beta(|\sigma|)}{\alpha(|\sigma|)+\theta}\,\mathrm{d}\sigma\quad\text{for any fixed }\theta>0,\tag{2.3}$$

$$\tilde{\gamma}_\theta(s)=\int_0^s\frac{\beta(|g(t)|)}{\alpha(|g(t)|)+\theta}\,\mathrm{d}t\quad\text{and}\quad\tilde{\gamma}(s)=\int_0^s\frac{\beta(|g(t)|)}{\alpha(|g(t)|)}\,\mathrm{d}t.\tag{2.4}$$

As usual, the usual truncation function T_θ at level $\pm\theta$ is defined as $T_\theta(s)=\max\{-\theta,\min\{\theta,s\}\}$. Throughout this paper, we use $C(\theta_1,\theta_2,\ldots,\theta_m)$ to denote positive constants depending only on specified quantities $\theta_1,\theta_2,\ldots,\theta_m$.

Now we give the definition of weak solutions of problem ($\mathscr{P}$).

Definition 2.1 A measurable function $u\in W_0^{1,p}(\Omega)$ is called a weak solution to problem ($\mathscr{P}$), if $a(\cdot,u,\nabla u)\in L^{p'}(\Omega)$ and $F(\cdot,u,\nabla u)\in L^1(\Omega)$ such that

$$\int_\Omega a(x,u,\nabla u)\nabla v\,\mathrm{d}x+\int_\Omega F(x,u,\nabla u)v\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x,\quad\forall v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).\tag{2.5}$$

For the existence of weak solutions, our result is stated as follows.

Theorem 2.1 *If assumptions* (H_1)-(H_5) *hold, then there exists at least one bounded weak solution $u\in L^\infty(\Omega)$ to problem* ($\mathscr{P}$) *in the sense of Definition* 2.1.

As we have said before, when dealing with the case $f\in L^1(\Omega)$, we shall use the notion of renormalized solution.

Definition 2.2 A measurable function $u:\Omega\to\mathbb{R}$ is a renormalized solution of problem ($\mathscr{P}$) if

$$T_k(u)\in W_0^{1,p}(\Omega)\quad\text{for any }k\ge0,\tag{2.6}$$

$$\lim_{m\to\infty}\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u)\nabla u\,\mathrm{d}x = 0 \tag{2.7}$$

and if for any $h \in W^{1,\infty}(\Omega)$ with compact support and $\upsilon \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, u satisfies

$$\int_\Omega a(x,u,\nabla u)\nabla\bigl(h(u)\upsilon\bigr)\,\mathrm{d}x + \int_\Omega F(x,u,\nabla u)h(u)\upsilon\,\mathrm{d}x = \int_\Omega fh(u)\upsilon\,\mathrm{d}x. \tag{2.8}$$

The existence result for L^1 data is stated as follows.

Theorem 2.2 *Assume that* (H_1) *to* (H_3) *hold and* $\frac{\beta}{\alpha}\in L^1(\mathbb{R}_+)$. *If* $f\in L^1(\Omega)$, *then problem* $(\mathscr{P})$ *admits at least one renormalized solution.*

Remark 2.2 In Theorem 2.1, the conditions (H_4) and (H_5) are only needed in proving the $L^\infty(\Omega)$ estimate of u. Therefore in Theorem 2.2, we do not need these assumptions. But instead, we need the condition $\frac{\beta}{\alpha}\in L^1(\mathbb{R}_+)$ as in [11]. Moreover, by the result of [22], the solution obtained in Theorem 2.2 belongs to $W_0^{1,r}(\Omega)$, provided $2-\frac{1}{N}<p<N$.

3 Existence of weak solution to problem ($\mathscr{P}$)

To prove Theorem 2.1, we first establish the L^∞ estimate of solutions to problem $(\mathscr{P})$.

Lemma 3.1 *Assume that* (H_1) *to* (H_5) *hold. If* $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *is a weak solution to problem* $(\mathscr{P})$, *then u satisfies the following estimate*:

$$\|u\|_{L^\infty(\Omega)} \le M, \tag{3.1}$$

where M is a constant which depends only on $N, p, q, \alpha, \beta, \|f\|_{L^q(\Omega)}$.

Proof of Lemma 3.1 For $t>0$, $h>0$, let $S_{t,h}$ be a real function defined by

$$S_{t,h}(\eta) = \begin{cases} 1, & \eta > t+h,\\ \frac{\eta-t}{h}, & t\le\eta\le t+h,\\ 0, & |\eta|\le t,\\ \frac{\eta+t}{h}, & -t-h\le\eta\le -t,\\ -1, & \eta\le -t-h. \end{cases} \tag{3.2}$$

It is easy to see that $S_{t,h}(\phi(u))\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and so $S_{t,h}(\phi(u))e^{\gamma_\theta(|u|)}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, where ϕ and γ_θ are defined as in (2.1) and (2.3). Taking $v = e^{\gamma_\theta(|u|)}S_{t,h}(\phi(u))$ as a test function in (2.5), we have

$$\begin{aligned} &\frac{1}{h}\int_{\{t<|\phi(u)|\le t+h\}} \phi'(u)e^{\gamma_\theta(|u|)}a(x,u,\nabla u)\nabla u\,\mathrm{d}x\\ &\quad+\int_{\{|\phi(u)|>t\}}\bigl|S_{t,h}\bigl(\phi(u)\bigr)\bigr|\frac{\beta(|u|)}{\alpha(|u|)+\theta}e^{\gamma_\theta(|u|)}a(x,u,\nabla u)\nabla u\,\mathrm{d}x\\ &\quad+\int_{\{|\phi(u)|>t\}} F(x,u,\nabla u)e^{\gamma_\theta(|u|)}S_{t,h}\bigl(\phi(u)\bigr)\,\mathrm{d}x\\ &=\int_{\{|\phi(u)|>t\}} fe^{\gamma_\theta(|u|)}S_{t,h}\bigl(\phi(u)\bigr)\,\mathrm{d}x. \end{aligned}$$

Then letting $\theta \to 0$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<|\phi(u)|\le t+h\}} \phi'(u)e^{\gamma(|u|)}a(x,u,\nabla u)\nabla u\,\mathrm{d}x \\
&\qquad + \int_{\{|\phi(u)|>t\}} \bigl|S_{t,h}\bigl(\phi(u)\bigr)\bigr|\frac{\beta(|u|)}{\alpha(|u|)}e^{\gamma(|u|)}a(x,u,\nabla u)\nabla u\,\mathrm{d}x \\
&\qquad + \int_{\{|\phi(u)|>t\}} F(x,u,\nabla u)e^{\gamma(|u|)}S_{t,h}\bigl(\phi(u)\bigr)\,\mathrm{d}x \\
&\quad = \int_{\{|\phi(u)|>t\}} fe^{\gamma(|u|)}S_{t,h}\bigl(\phi(u)\bigr)\,\mathrm{d}x,
\end{aligned} \tag{3.3}$$

where γ is defined as in (2.1). Notice that $|S_{t,h}(\phi(u))| \le 1$, by (H_1), (H_3), and applying Hölder's inequality, we deduce from (3.3) that

$$\frac{1}{h}\int_{\{t<\omega\le t+h\}} |\nabla\omega|^p\,\mathrm{d}x \le \int_{\{\omega>t\}} |f|e^{\gamma(|u|)}\,\mathrm{d}x \le \|f\|_{L^q(\Omega)}\biggl(\int_{\{\omega>t\}}\bigl|e^{\gamma(|u|)}\bigr|^{q'}\,\mathrm{d}x\biggr)^{\frac{1}{q'}},$$

where $\omega = |\phi(u)| = \phi(|u|)$. Let h tend to zero, we find that

$$-\frac{d}{dt}\int_{\{\omega>t\}} |\nabla\omega|^p\,\mathrm{d}x \le \int_{\{\omega>t\}} |f|e^{\gamma(|u|)}\,\mathrm{d}x \le \|f\|_{L^q(\Omega)}\biggl(\int_{\{\omega>t\}}\bigl|e^{\gamma(|u|)}\bigr|^{q'}\,\mathrm{d}x\biggr)^{\frac{1}{q'}}. \tag{3.4}$$

Setting

$$z(t) = \sup_{\{|s|>\phi^{-1}(t)\}} \frac{e^{\gamma(|s|)}}{(1+\phi(|s|))^{p-1}},$$

since ϕ is strictly increasing and $\lim_{s\to\pm\infty}\phi(s) = 0$, we have

$$\lim_{t\to+\infty} z(t) = 0. \tag{3.5}$$

Concerning the term $(\int_{\{\omega>t\}} |e^{\gamma(|u|)}|^{q'}\,\mathrm{d}x)^{\frac{1}{q}}$, we have

$$\begin{aligned}
\biggl(\int_{\{\omega>t\}}\bigl|e^{\gamma(|u|)}\bigr|^{q'}\,\mathrm{d}x\biggr)^{\frac{1}{q}} &= \biggl(\int_{\{\omega>t\}}\biggl(\frac{e^{\gamma(|u|)}}{(1+\omega)^{p-1}}\biggr)^{q'}(1+\omega)^{q'(p-1)}\,\mathrm{d}x\biggr)^{\frac{1}{q'}} \\
&\le C(p,q)z(t)\biggl[\biggl(\int_{\{\omega>t\}}\omega^{q'(p-1)}\,\mathrm{d}x\biggr)^{\frac{1}{q'}} + \bigl(\mu_\omega(t)\bigr)^{\frac{1}{q'}}\biggr] \\
&\le C(p,q)z(t)\biggl[\biggl(\int_0^{\mu_\omega(t)}\omega_*^{q'(p-1)}\,\mathrm{d}s\biggr)^{\frac{1}{q'}} + \bigl(\mu_\omega(t)\bigr)^{\frac{1}{q'}}\biggr].
\end{aligned} \tag{3.6}$$

By (3.4), (3.6), and Lemma 2.1, it follows that

$$\begin{aligned}
&NC_N^{1/N}\mu_\omega(t)^{1-1/N} \\
&\quad \le \bigl(-\mu_\omega'(t)\bigr)^{1/p'}\biggl(-\frac{d}{dt}\int_{\{u>t\}}|\nabla\omega|^p\,\mathrm{d}x\biggr)^{\frac{1}{p}} \\
&\quad \le \bigl(-\mu_\omega'(t)\bigr)^{1/p'}C(p,q)z^{\frac{1}{p}}(t)\biggl[\biggl(\int_0^{\mu_\omega(t)}\omega_*^{q'(p-1)}\,\mathrm{d}s\biggr)^{\frac{1}{pq'}} + \bigl(\mu_\omega(t)\bigr)^{\frac{1}{pq'}}\biggr],
\end{aligned} \tag{3.7}$$

which indicates that, for $0 < \theta < \theta + h < |\Omega|$,

$$\begin{aligned}\frac{\omega_*(\theta)-\omega_*(\theta+h)}{h} &\le \frac{C(p,q)}{hNC_N^{1/N}}\int_{\omega_*(\theta+h)}^{\omega_*(\theta)} z^{\frac{1}{p}}(t)\frac{(-\mu_\omega'(t))^{1/p'}}{\mu_\omega(t)^{1-1/N}}\\ &\quad\times\left[\left(\int_0^{\mu_\omega(t)}\omega_*^{q'(p-1)}\,\mathrm{d}s\right)^{\frac{1}{pq'}}+\left(\mu_\omega(t)\right)^{\frac{1}{pq'}}\right]\mathrm{d}t\\ &< \frac{C(p,q,N)}{h}\sup_{s\in[\omega_*(\theta+h),+\infty]} z^{\frac{1}{p}}(s)\int_{\omega_*(\theta+h)}^{\omega_*(\theta)}\frac{(-\mu_\omega'(t))^{1/p'}}{\mu_\omega(t)^{1-1/N}}\\ &\quad\times\left[\left(\int_0^{\mu_\omega(t)}\omega_*^{q'(p-1)}\,\mathrm{d}s\right)^{\frac{1}{pq'}}+\left(\mu_\omega(t)\right)^{\frac{1}{pq'}}\right]\mathrm{d}t.\end{aligned}$$

Then we employ (1.15) of [9] to get

$$\begin{aligned}\frac{\omega_*(\theta)-\omega_*(\theta+h)}{h} &< \frac{C(p,q,N)}{h}\sup_{s\in[\omega_*(\theta+h),+\infty]} z^{\frac{1}{p}}(s)\int_\theta^{\theta+h}\frac{(-\omega_*'(\sigma))^{1/p}}{\sigma^{1-\frac{1}{N}}}\\ &\quad\times\left[\left(\int_0^{\sigma}\omega_*^{q'(p-1)}\,\mathrm{d}s\right)^{\frac{1}{pq'}}+\sigma^{\frac{1}{pq'}}\right]\mathrm{d}\sigma.\end{aligned}$$

Then letting h tend to zero, we deduce that, for almost $\theta \in [0,|\Omega|]$,

$$-\omega_*'(\theta) < C(p,q,N)\sup_{s\in[\omega_*(\theta),+\infty]} z^{\frac{1}{p}}(s)\frac{(-\omega_*'(\theta))^{1/p}}{\theta^{1-\frac{1}{N}}}\left[\left(\int_0^{\theta}\omega_*^{q'(p-1)}\,\mathrm{d}s\right)^{\frac{1}{pq'}}+\theta^{\frac{1}{pq'}}\right],$$

which leads, after applying Young's inequality, to

$$\begin{aligned}-\omega_*'(\theta) &< C(p,q,N)\Big[\sup_{s\in[\omega_*(\theta),+\infty]} z^{\frac{1}{p}}(s)\Big]^{p'}\frac{1}{\theta^{(1-\frac{1}{N})p'}}\left[\left(\int_0^{\theta}\omega_*^{q'(p-1)}\,\mathrm{d}s\right)^{\frac{p'}{pq'}}+\theta^{\frac{p'}{pq'}}\right]\\ &\le C(p,q,N)\sup_{s\in[\omega_*(\theta),+\infty]} z^{\frac{p'}{p}}(s)\frac{1}{\theta^{(1-\frac{1}{N})p'}}\left[\omega_*(0)\theta^{\frac{p'}{pq'}}+\theta^{\frac{p'}{pq'}}\right].\end{aligned}\tag{3.8}$$

Since $q > \frac{N}{p}$, we have $q_0 = \frac{p'}{pq'} + \frac{p'}{N} - p' + 1 > 0$. From (3.5), we deduce that there exists $t_0 > 0$ such that

$$C(p,q,N)z^{\frac{p'}{p}}(s)|\Omega|^{q_0} \le \frac{1}{2} \quad \text{for all } s \ge t_0.$$

Hence, upon integration over $[0,\mu_\omega(t_0)]$, inequality (3.8) gives

$$\omega_*(0) \le 1 + 2t_0,$$

which implies that $\|u\|_{L^\infty(\Omega)} \le \phi^{-1}(1+2t_0)$. We observe that t_0 only depends on p, q, N, $|\Omega|$, α, β, thus the proof of Lemma 3.1 is finished. □

To prove Theorem 2.1, we shall consider suitable approximate problems. First of all, we recall the following lemma, proved in [12].

Lemma 3.2 *There exists a function $g \in C^1(\mathbb{R})$ such that g is odd, strictly increasing, and*

$$g'(s) = \alpha\big(|g(s)|\big) \geq 0 \quad \text{in } \mathbb{R}, \tag{3.9}$$

$$g(0) = 0, \qquad \lim_{s\to+\infty} g(s) = +\infty. \tag{3.10}$$

For a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, and $\forall \xi \in \mathbb{R}^N$, we define for fixed $\varepsilon > 0$:

$$F_\varepsilon(x,s,\xi) = \frac{F(x,s,\xi)}{1+\varepsilon|F(x,s,\xi)|},$$
$$a_\varepsilon(x,s,\xi) = \varepsilon|\xi|^{p-2}\xi + a\big(x, g(s), g'(s)\xi\big),$$
$$a_{\varepsilon l}(x,s,\xi) = \varepsilon|\xi|^{p-2}\xi + a\big(x, g\big(T_l(s)\big), g'\big(T_l(s)\big)T_l'(s)\xi\big).$$

For any fixed $\varepsilon > 0$, we introduce the approximate problem

$$(\mathscr{P}_\varepsilon) \quad \begin{cases} -\operatorname{div}(a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)) + F_\varepsilon(x, g(u_\varepsilon), g'(u_\varepsilon)\nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\{f_\varepsilon\}$ satisfy

$$f_\varepsilon \in C_0^\infty(\Omega) \quad \text{such that } f_\varepsilon \to f \text{ strongly in } L^q(\Omega) \text{ as } \varepsilon \to 0.$$

The existence result to problem $(\mathscr{P}_\varepsilon)$ is stated as follows.

Theorem 3.1 *Problem $(\mathscr{P}_\varepsilon)$ admits at least a solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|g(u_\varepsilon)\|_{L^\infty(\Omega)} \leq M_0$, where M_0 is a positive constant depending on M (see Lemma 3.1) and the behavior of function g.*

Proof of Theorem 3.1 For any $l > 0$, let us consider the following truncated problem:

$$(\mathscr{P}_{\varepsilon l}) \quad \begin{cases} -\operatorname{div}(a_{\varepsilon l}(x,u_\varepsilon,\nabla u_\varepsilon)) + F_\varepsilon(x, g(T_l(u_\varepsilon)), g'(T_l(u_\varepsilon))\nabla T_l(u_\varepsilon)) = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

By the classic result (see [23]), problem $(\mathscr{P}_{\varepsilon l})$ admits a solution $u_\varepsilon \in W_0^{1,p}(\Omega) \in L^\infty(\Omega)$. Then using the same argument of Lemma 3.1, we conclude

$$\big\|g\big(T_l(u_\varepsilon)\big)\big\|_{L^\infty(\Omega)} \leq M.$$

In view of Lemma 3.2, it is easy to see that g^{-1} is defined well and strictly increasing in $\mathbb{R}$.

Now choosing $l > g^{-1}(M)$, we obtain

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq g^{-1}(M). \tag{3.11}$$

Thus we have $T_l(u_\varepsilon) = u_\varepsilon$, which implies that u_ε is a weak solution of $(\mathscr{P}_\varepsilon)$. The proof is finished. □

Proof of Theorem 2.1 Taking $e^{\tilde{\gamma}_\theta(|u_\varepsilon|)}u_\varepsilon$ as a test function in problem $(\mathscr{P}_\varepsilon)$, we have

$$\begin{aligned}&\int_\Omega e^{\tilde{\gamma}_\theta(|u_\varepsilon|)}a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,\mathrm{d}x\\&\quad+\int_\Omega |u_\varepsilon|\frac{\beta(|g(u_\varepsilon)|)}{\alpha(|g(u_\varepsilon)|)+\theta}e^{\tilde{\gamma}_\theta(|u_\varepsilon|)}a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,\mathrm{d}x\\&\quad+\int_\Omega F_\varepsilon\bigl(x,g(u_\varepsilon),g'(u_\varepsilon)\nabla u_\varepsilon\bigr)e^{\tilde{\gamma}_\theta(|u_\varepsilon|)}u_\varepsilon\,\mathrm{d}x\\&=\int_\Omega f_\varepsilon e^{\tilde{\gamma}_\theta(|u_\varepsilon|)}u_\varepsilon\,\mathrm{d}x,\end{aligned}$$

where $\tilde{\gamma}_\theta$ is defined as in (2.4), and g is defined as in Lemma 3.2. Then letting θ tend to zero, using assumptions (H_1)-(H_4) and Theorem 3.1 we get

$$\int_\Omega e^{\tilde{\gamma}(|u_\varepsilon|)}a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,\mathrm{d}x\le\int_\Omega f_\varepsilon e^{\tilde{\gamma}(|u_\varepsilon|)}u_\varepsilon\,\mathrm{d}x,$$

where $\tilde{\gamma}$ is defined as in (2.4).

In view of Theorem 3.1, (H_1), and (H_2), the above estimate gives

$$\varepsilon\int_\Omega|\nabla u_\varepsilon|^p+\int_\Omega\bigl|\nabla g(u_\varepsilon)\bigr|^p\,\mathrm{d}x\le e^{\tilde{\gamma}(g^{-1}(M))}g^{-1}(M_0)\|f\|_{L^1(\Omega)}.\tag{3.12}$$

Now denoting $\bar{u}_\varepsilon=g(u_\varepsilon)$, estimates (3.11) and (3.12) imply that $\bar{u}_\varepsilon$ is bounded uniformly in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. As a consequence, there exist a subsequence (still denoted by $\{\bar{u}_\varepsilon\}$) and a measurable function $\bar{u}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that

$$\bar{u}_\varepsilon\rightharpoonup\bar{u}\quad\text{weakly in } W_0^{1,p}(\Omega)\text{ and weakly}^*\text{ in } L^\infty(\Omega),\tag{3.13}$$

$$\bar{u}_\varepsilon\to\bar{u}\quad\text{a.e. in }\Omega.\tag{3.14}$$

In the following, the rest of the proof is divided into several steps.

Step 1: To deal with the difficulty that α vanishes at zero, we define the following truncation function near the origin:

$$\zeta_k(s)=\max\{s,k\}=k+(s-k)_+,\quad\forall s\in\mathbb{R},\tag{3.15}$$

where $k>0$ is a fixed constant. Then we easily get

$$\zeta_k(\bar{u}_\varepsilon)\rightharpoonup\zeta_k(\bar{u})\quad\text{weakly in } W_0^{1,p}(\Omega)\text{ and weakly}^*\text{ in } L^\infty(\Omega).\tag{3.16}$$

Now taking $\rho_\theta^\varepsilon=e^{\gamma_\theta(\bar{u}_\varepsilon)}[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})]_+$ as a test function in problem $(\mathscr{P}_\varepsilon)$, by (H_1) we have

$$\begin{aligned}&\int_\Omega e^{\gamma_\theta(\bar{u}_\varepsilon)}a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x\\&\quad+\varepsilon\int_\Omega e^{\gamma_\theta(\bar{u}_\varepsilon)}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x\\&\quad+\int_\Omega\frac{\beta(|\bar{u}_\varepsilon|)}{\alpha(|\bar{u}_\varepsilon|)+\theta}e^{\gamma_\theta(\bar{u}_\varepsilon)}\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\alpha\bigl(|\bar{u}_\varepsilon|\bigr)|\nabla\bar{u}_\varepsilon|^p\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
&+ \varepsilon \int_\Omega \frac{\beta(|\bar{u}_\varepsilon|)}{\alpha(|\bar{u}_\varepsilon|)+\theta} e^{\gamma_\theta(\bar{u}_\varepsilon)} \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \bar{u}_\varepsilon \, \mathrm{d}x \\
&+ \int_\Omega F_\varepsilon(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) e^{\gamma_\theta(\bar{u}_\varepsilon)} \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x \\
\le{}& \int_\Omega f_\varepsilon e^{\gamma_\theta(\bar{u}_\varepsilon)} \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x. \qquad (3.17)
\end{aligned}$$

It is easy to see that the fourth term of (3.17) is non-negative. So letting θ tend to zero, the above inequality leads to

$$I_1(\varepsilon) + I_2(\varepsilon) \le I_3(\varepsilon), \qquad (3.18)$$

where

$$\begin{aligned}
I_1(\varepsilon) &= \int_\Omega e^{\gamma(\bar{u}_\varepsilon)} a(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \nabla \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x, \\
I_2(\varepsilon) &= \varepsilon \int_\Omega e^{\gamma(\bar{u}_\varepsilon)} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x, \\
I_3(\varepsilon) &= \int_\Omega f_\varepsilon e^{\gamma(\bar{u}_\varepsilon)} \big[\zeta_k(\bar{u}_\varepsilon) - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x.
\end{aligned}$$

Now we estimate all the terms of (3.18).

Estimate of $I_2(\varepsilon)$. Using (3.11), (3.13), and the Hölder inequality, we conclude that

$$\big|I_2(\varepsilon)\big| \le \varepsilon e^{\gamma(M_0)} \left(\int_\Omega |\nabla u_\varepsilon|^p \, \mathrm{d}x\right)^{\frac{p-1}{p}} \left[\left(\int_\Omega \big|\nabla \zeta_k(\bar{u}_\varepsilon)\big|^p \, \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega \big|\nabla \zeta_k(\bar{u})\big|^p \, \mathrm{d}x\right)^{\frac{1}{p}}\right].$$

Hence, by (3.12) we easily get

$$\lim_{\varepsilon \to 0} I_2(\varepsilon) = 0. \qquad (3.19)$$

Estimate of $I_3(\varepsilon)$. By (3.11), (3.14), and the Lebesgue dominated convergence theorem, we infer that

$$\lim_{\varepsilon \to 0} I_3(\varepsilon) = 0. \qquad (3.20)$$

Estimate of $I_1(\varepsilon)$. Since $a(x,s,0) = 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, we obtain

$$\begin{aligned}
I_1(\varepsilon) &= \int_{\Omega^k_{\varepsilon 1}} e^{\gamma(\bar{u}_\varepsilon)} a(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \cdot \nabla \big[\bar{u}_\varepsilon - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x \\
&\quad + \int_{\Omega^k_{\varepsilon 2}} e^{\gamma(\bar{u}_\varepsilon)} a(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \cdot \nabla \big[-k - \zeta_k(\bar{u})\big]_+ \, \mathrm{d}x \\
&= \bar{I}_{11}(\varepsilon) + \bar{I}_{12}(\varepsilon), \qquad (3.21)
\end{aligned}$$

where

$$\Omega^k_{\varepsilon 1} = \{x \in \Omega : \bar{u}_\varepsilon < k\}, \qquad \Omega^k_{\varepsilon 2} = \{x \in \Omega : \bar{u}_\varepsilon \ge k\}.$$

For the term $\bar{I}_{11}(\varepsilon)$, we can write

$$\bar{I}_{11}(\varepsilon) = \int_{\Omega_{\varepsilon 1}^k} e^{\gamma(\bar{u}_\varepsilon)}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x$$
$$+\int_{\Omega_{\varepsilon 1}^k} e^{\gamma(\bar{u}_\varepsilon)}a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x. \tag{3.22}$$

Collecting (3.11), (3.13), (3.14), and (3.16), it is easy to verify that

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon 1}^k} e^{\gamma(\bar{u}_\varepsilon)}a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x = 0. \tag{3.23}$$

Using (3.22), (3.23), (H_1), and (H_2), we find that

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\,\bar{I}_{11}(\varepsilon) &\geq \overline{\lim_{\varepsilon\to 0}}\int_{\Omega_{\varepsilon 1}^k}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\\ &\quad\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x\\ &= \overline{\lim_{\varepsilon\to 0}}\int_{\Omega}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\\ &\quad\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x,\end{aligned}$$

where we have used the fact $a(x,s,0)=0$ for a.e. $x\in\Omega$.

For the term $\bar{I}_{12}(\varepsilon)$, it is easy to get

$$\lim_{\varepsilon\to 0}\bar{I}_{12}(\varepsilon) = 0.$$

The above two convergence results show that

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\,I_1(\varepsilon) &\geq \overline{\lim_{\varepsilon\to 0}}\int_{\Omega}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\\ &\quad\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x.\end{aligned} \tag{3.24}$$

Substituting (3.19), (3.20), and (3.24) into (3.18), we conclude

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}}\int_{\Omega}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\\ &\quad\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_+\,\mathrm{d}x\leq 0.\end{aligned} \tag{3.25}$$

Now choosing $\rho_\theta^\varepsilon = -e^{\gamma_\theta(\bar{u}_\varepsilon)}[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})]_+$ as a test function in problem ($\mathscr{P}_\varepsilon$), by the same arguments as in the proof of (3.25) we arrive at

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}}\int_{\Omega}-\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\\ &\quad\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]_-\,\mathrm{d}x\leq 0.\end{aligned} \tag{3.26}$$

As a consequence of (3.25) and (3.26), we have

$$\overline{\lim_{\varepsilon\to 0}}\int_{\Omega}\bigl[a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u}_\varepsilon)\bigr) - a\bigl(x,\zeta_k(\bar{u}_\varepsilon),\nabla\zeta_k(\bar{u})\bigr)\bigr]\cdot\nabla\bigl[\zeta_k(\bar{u}_\varepsilon)-\zeta_k(\bar{u})\bigr]\,\mathrm{d}x\leq 0.$$

Then, arguing as in [24], we derive that

$$\nabla\zeta_k(\bar{u}_\varepsilon) \to \nabla\zeta_k(\bar{u}) \quad \text{strongly in } \big(L^p(\Omega)\big)^N \text{ and a.e. in } \Omega. \tag{3.27}$$

Step 2: For any fixed $k > 0$, let us define

$$\bar{\zeta}_k(s) = \min\{s, -k\} = -k + (s + k)_-, \quad \forall s \in \mathbb{R}.$$

Proceeding as in Step 1, taking $\rho_\theta^\varepsilon = e^{\gamma_\theta(\bar{u}_\varepsilon)}[\bar{\zeta}_k(\bar{u}_\varepsilon) - \bar{\zeta}_k(\bar{u})]_+$ and $\rho_\theta^\varepsilon = -e^{-\gamma_\theta(\bar{u}_\varepsilon)}[\bar{\zeta}_k(\bar{u}_\varepsilon) - \bar{\zeta}_k(\bar{u})]_-$ as two test functions in problem $(\mathscr{P}_\varepsilon)$, we obtain

$$\nabla\bar{\zeta}_k(\bar{u}_\varepsilon) \to \nabla\bar{\zeta}_k(\bar{u}) \quad \text{strongly in } \big(L^p(\Omega)\big)^N \text{ and a.e. in } \Omega. \tag{3.28}$$

By (3.27) and (3.28), it follows that

$$\chi_{\{|\bar{u}_\varepsilon|\geq k\}}\nabla\bar{u}_\varepsilon \to \chi_{\{|\bar{u}|\geq k\}}\nabla\bar{u} \quad \text{strongly in } \big(L^p(\Omega)\big)^N \text{ and a.e. in } \Omega. \tag{3.29}$$

In the following, we prove that u is a weak solution to problem $(\mathscr{P})$.

Since u_ε is a weak solution to problem $(\mathscr{P})$, it follows that

$$\begin{aligned} &\int_\Omega a(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\nabla v \,\mathrm{d}x + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla v \,\mathrm{d}x + \int_\Omega F_\varepsilon(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)v \,\mathrm{d}x \\ &\quad = \int_\Omega f_\varepsilon v \,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{3.30}$$

Concerning the third term on the left-hand side of (3.30), we rewrite it as

$$\begin{aligned} &\int_\Omega F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\upsilon \,\mathrm{d}x \\ &\quad = \int_{\{x\in\Omega:|\bar{u}_\varepsilon|>k\}} F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\upsilon \,\mathrm{d}x + \int_{\{x\in\Omega:|\bar{u}_\varepsilon|\leq k\}} F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\upsilon \,\mathrm{d}x \\ &\quad = I_{1\varepsilon} + I_{2\varepsilon} \quad \text{for any fixed } k > 0. \end{aligned} \tag{3.31}$$

To take the limits in $I_{1\varepsilon}$, we next show that

$$F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\chi_{\{|\bar{u}_\varepsilon|>k\}} \to F(x, \bar{u}, \nabla\bar{u})\chi_{\{|\bar{u}|>k\}} \quad \text{strongly in } L^1(\Omega). \tag{3.32}$$

Indeed, by (3.14) and (3.29), we already know that $F(x, t, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\chi_{\{|\bar{u}_\varepsilon|>k\}} \to F(x, t, \bar{u}, \nabla\bar{u})\chi_{\{|\bar{u}|>k\}}$ almost everywhere in Ω, it suffices to prove the equi-integrability of this sequence and then apply Vitali's convergence theorem. Using Theorem 3.1 and (H_3), we get

$$\big|F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\chi_{\{|\bar{u}_\varepsilon|>k\}}\big| \leq C_0|\nabla\bar{u}_\varepsilon|^p\chi_{\{|\bar{u}_\varepsilon|>k\}},$$

where C_0 is a positive constant independent of ε and k. Then the equi-integrability of $|\nabla\bar{u}_\varepsilon|^p\chi_{\{|\bar{u}_\varepsilon|>k\}}$, which follows from (3.29), indicates that of $F(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\chi_{\{|\bar{u}_\varepsilon|>k\}}$. Therefore, (3.32) is proved.

As a conclusion, we have

$$\lim_{\varepsilon\to 0} I_{1\varepsilon} = \int_{\{x\in\Omega:|\bar{u}|>k\}} F(x,\bar{u},\nabla\bar{u})\upsilon\,\mathrm{d}x,$$

so that

$$\lim_{k\to 0}\lim_{\varepsilon\to 0} I_{1\varepsilon} = \int_\Omega F(x,\bar{u},\nabla\bar{u})\upsilon\,\mathrm{d}x. \tag{3.33}$$

Moreover, by assumption (H_3) and (3.12) we get

$$|I_{2\varepsilon}| \le \max_{0\le s\le k}\beta(s)\int\int_{\{(x,t)\in Q_\tau:|\bar{u}_\varepsilon(x,t)|\le k\}}\bigl[|\nabla\bar{u}_\varepsilon|^p + h(x,t)\bigr]|\upsilon|\,\mathrm{d}x\,\mathrm{d}t \le C_1\max_{0\le s\le k}\beta(s),$$

where C_1 is a positive constant independent of ε and k. Therefore,

$$\lim_{k\to 0}\lim_{\varepsilon\to 0} I_{2\varepsilon} = 0, \tag{3.34}$$

since β is a continuous function from $[0,+\infty)$ into $[0,+\infty)$ and $\beta(0)=0$.

It follows from (3.31), (3.33), and (3.34) that

$$\lim_{\varepsilon\to 0}\int_\Omega F(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\upsilon\,\mathrm{d}x = \int_\Omega F(x,\bar{u},\nabla\bar{u})\upsilon\,\mathrm{d}x. \tag{3.35}$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla v\,\mathrm{d}x = \int_\Omega a(x,\bar{u},\nabla\bar{u})\nabla v\,\mathrm{d}x. \tag{3.36}$$

Furthermore, the same argument as (3.19) shows that

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v\,\mathrm{d}x = 0. \tag{3.37}$$

Finally, it is easy to see that

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x. \tag{3.38}$$

Now letting ε tend to zero, from (3.36)-(3.38), we deduce that $\bar{u}$ satisfies (2.5), with u replaced by $\bar{u}$. Thus, the proof is finished. □

4 Existence of renormalized solution to problem ($\mathscr{P}$)

Proof of Theorem 2.2 By the proof of Theorem 3.1, we deduce that there exists at least one weak solution u_ε satisfying $u_\varepsilon\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that

$$\begin{aligned}&\varepsilon\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v\,\mathrm{d}x + \int_\Omega a\bigl(x,g(u_\varepsilon),\nabla g(u_\varepsilon)\bigr)\nabla v\,\mathrm{d}x\\ &\quad+\int_\Omega F_\varepsilon\bigl(x,g(u_\varepsilon),\nabla g(u_\varepsilon)\bigr)v\,\mathrm{d}x = \int_\Omega f_\varepsilon v\,\mathrm{d}x,\quad \forall v\in W_0^{1,p}(\Omega),\end{aligned} \tag{4.1}$$

where f_ε satisfy

$$f_\varepsilon \in C_0^\infty(\Omega) \quad \text{such that } f_\varepsilon \to f \text{ strongly in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

As before, set $\bar{u}_\varepsilon = g(u_\varepsilon)$. For any given $l > s_0$ and $\bar{l} = g^{-1}(l)$, let us take $v = e^{\tilde{\gamma}_\theta(|u_\varepsilon|)} T_{\bar{l}}(u_\varepsilon)$ in (4.1), where s_0 is defined as in the proof of Theorem 3.1. Then sending θ tend to zero, using (H_1)-(H_3) and the fact $\frac{\beta}{\alpha} \in L^1(0, +\infty)$, it follows that

$$\varepsilon \int_\Omega \left|\nabla T_{\bar{l}}(u_\varepsilon)\right|^p \,\mathrm{d}x + \int_\Omega \left|\nabla T_l(\bar{u}_\varepsilon)\right|^p \,\mathrm{d}x \le C, \tag{4.2}$$

where C is a positive constant independent of ε.

Hence, by the Sobolev space embedding theorem, there exist a measurable function $\bar{u}$ and a subsequence (still denoted by $\{\bar{u}_\varepsilon\}$), such that

$$\bar{u}_\varepsilon \to \bar{u} \quad \text{a.e. in } \Omega \tag{4.3}$$

and

$$T_l(\bar{u}_\varepsilon) \rightharpoonup T_l(\bar{u}) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{4.4}$$

Step 4.1. In this step, we prove the following result:

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \int_{\{x\in\Omega: n\le|\bar{u}_\varepsilon(x)|\le n+1\}} a(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon \,\mathrm{d}x = 0. \tag{4.5}$$

For any integer $n > 1$, define ρ_n by

$$\rho_n(r) = T_{n+1}(r) - T_n(r), \quad \forall r \in \mathbb{R}.$$

Obviously, we have

$$0 < |\rho_n| \le 1 \quad \text{and} \quad \rho_n(r) \to 0 \quad \text{for any } r \text{ as } n \to \infty. \tag{4.6}$$

Taking $v = e^{\gamma_\theta(|\bar{u}_\varepsilon|)}\rho_n(\bar{u}_\varepsilon)$ in (4.1), we get

$$\begin{aligned}
&\int_\Omega e^{\gamma_\theta(|\bar{u}_\varepsilon|)} a(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\nabla\rho_n(\bar{u}_\varepsilon)\,\mathrm{d}x + \int_\Omega \rho_n(\bar{u}_\varepsilon) e^{\gamma_\theta(|\bar{u}_\varepsilon|)} \frac{\beta(|\bar{u}_\varepsilon|)}{\alpha(|\bar{u}_\varepsilon|)+\theta} a(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon \,\mathrm{d}x \\
&\quad + \int_\Omega \varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\bigl(e^{\gamma_\theta(|\bar{u}_\varepsilon|)}\rho_n(\bar{u}_\varepsilon)\bigr)\,\mathrm{d}x + \int_\Omega F_\varepsilon(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon) e^{\gamma_\theta(|\bar{u}_\varepsilon|)}\rho_n(\bar{u}_\varepsilon)\,\mathrm{d}x \\
&= \int_\Omega f_\varepsilon e^{\gamma_\theta(|\bar{u}_\varepsilon|)}\rho_n(\bar{u}_\varepsilon)\,\mathrm{d}x.
\end{aligned} \tag{4.7}$$

Passing to the limit as θ tend to zero in (4.7), it follows from (H_1) and (H_3) that

$$\int_{\{x\in\Omega: n\le|\bar{u}_\varepsilon(x)|\le n+1\}} e^{\gamma(|\bar{u}_\varepsilon|)} a(x, \bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon \,\mathrm{d}x \le \int_\Omega f_\varepsilon e^{\gamma(|\bar{u}_\varepsilon|)}\rho_n(\bar{u}_\varepsilon)\,\mathrm{d}x. \tag{4.8}$$

Let $\varepsilon \to 0$ and then $n \to \infty$ in (4.8). Recalling that $\frac{\beta}{\alpha} \in L^1(\mathbb{R}_+)$, using (4.6) we get

$$\overline{\lim_{\varepsilon\to 0}} \int_{\{x\in\Omega:n\le|\bar{u}_\varepsilon(x)|\le n+1\}} a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon \,\mathrm{d}x \le \int_\Omega f e^{\gamma(|\bar{u}|)}\rho_n(\bar{u})\,\mathrm{d}x. \tag{4.9}$$

It is easy to check that $\lim_{n\to\infty}\int_\Omega f e^{\gamma(|\bar{u}|)}\rho_n(\bar{u})\,\mathrm{d}x = 0$. Thus, passing to the limit as $n \to \infty$ in (4.9), the desired result (4.5) follows immediately.

Step 4.2. For any fixed $k > 0$ and $l > \max\{k, s_0\}$, we denote

$$\zeta_k^l(s) = \max\{T_l(s), k\} = k + \bigl(T_l(s) - k\bigr)_+, \quad \forall s \in \mathbb{R}.$$

Then we have, in view of (4.3) and (4.4),

$$\zeta_k^l(\bar{u}_\varepsilon) \rightharpoonup \zeta_k^l(\bar{u}) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{4.10}$$

Let λ be a positive number to be determined, denote

$$\varphi(s) = e^{\lambda s} - 1, \quad \forall s \in \mathbb{R}$$

and

$$\rho_\theta^\varepsilon = e^{\gamma_\theta(\bar{u}_\varepsilon)}\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon) - \zeta_k^l(\bar{u})\bigr)_+\bigr)e^{-\gamma_\theta(\zeta_k^l(\bar{u}_\varepsilon))},$$

where γ_θ is defined as in (2.3). We now choose a sequence of increasing function $S_n \in C^\infty(\mathbb{R})$ such that

$$S_n(r) = 1 \quad \text{for } |r| \le n; \qquad \operatorname{supp} S_n \subset [-n-1, n+1]; \qquad \|S_n'\|_{L^\infty(\mathbb{R})} \le 1. \tag{4.11}$$

Taking $v = S_n(\bar{u}_\varepsilon)\rho_\theta^\varepsilon$ in (4.1), we obtain

$$\begin{aligned}
&\hat{I}_1(\theta,\varepsilon,n) + \hat{I}_2(\theta,\varepsilon,n) + \hat{I}_3(\theta,\varepsilon,n) + \hat{I}_4(\theta,\varepsilon,n) + \hat{I}_5(\theta,\varepsilon,n) \\
&\quad \le \hat{I}_6(\theta,\varepsilon,n) + \hat{I}_7(\theta,\varepsilon,n) + \hat{I}_8(\theta,\varepsilon,n) + \hat{I}_9(\theta,\varepsilon,n),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\hat{I}_1(\theta,\varepsilon,n) &= \int_\Omega S_n(\bar{u}_\varepsilon)e^{\gamma_\theta(\bar{u}_\varepsilon)-\gamma_\theta(\zeta_k^l(\bar{u}_\varepsilon))}\varphi'\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon) - \zeta_k^l(\bar{u})\bigr)_+\bigr)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon) \\
&\quad \cdot \nabla\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon) - \zeta_k^l(\bar{u})\bigr)_+\bigr)\,\mathrm{d}x, \\
\hat{I}_2(\theta,\varepsilon,n) &= \varepsilon\int_\Omega S_n(\bar{u}_\varepsilon)e^{\gamma_\theta(\bar{u}_\varepsilon)-\gamma_\theta(\zeta_k^l(\bar{u}_\varepsilon))}\varphi'\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon) - \zeta_k^l(\bar{u})\bigr)_+\bigr)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \\
&\quad \cdot \nabla\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon) - \zeta_k^l(\bar{u})\bigr)_+\bigr)\,\mathrm{d}x, \\
\hat{I}_3(\theta,\varepsilon,n) &= \int_\Omega S_n(\bar{u}_\varepsilon)\alpha\bigl(|\bar{u}_\varepsilon|\bigr)\frac{\beta(|\bar{u}_\varepsilon|)}{\alpha(|\bar{u}_\varepsilon|)+\theta}|\nabla\bar{u}_\varepsilon|^p\rho_\theta^\varepsilon\,\mathrm{d}x, \\
\hat{I}_4(\theta,\varepsilon,n) &= \int_\Omega S_n'(\bar{u}_\varepsilon)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon\rho_\theta^\varepsilon\,\mathrm{d}x,
\end{aligned}$$

$$\hat{I}_5(\theta,\varepsilon,n) = \varepsilon \int_\Omega S_n'(\bar{u}_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \bar{u}_\varepsilon \rho_\theta^\varepsilon \,\mathrm{d}x,$$

$$\hat{I}_6(\theta,\varepsilon,n) = \int_\Omega S_n(\bar{u}_\varepsilon)\beta\big(|\bar{u}_\varepsilon|\big)|\nabla \bar{u}_\varepsilon|^p \rho_\theta^\varepsilon \,\mathrm{d}x,$$

$$\hat{I}_7(\theta,\varepsilon,n) = \int_\Omega S_n(\bar{u}_\varepsilon)\frac{\beta(|\zeta_k^l(\bar{u}_\varepsilon)|)}{\alpha(|\zeta_k^l(\bar{u}_\varepsilon)|)+\theta}\varphi\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\zeta_k^l(\bar{u}_\varepsilon)\,\mathrm{d}x,$$

$$\hat{I}_8(\theta,\varepsilon,n) = \varepsilon\int_\Omega S_n(\bar{u}_\varepsilon)\frac{\beta(|\zeta_k^l(\bar{u}_\varepsilon)|)}{\alpha(|\zeta_k^l(\bar{u}_\varepsilon)|)+\theta}\varphi\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\zeta_k^l(\bar{u}_\varepsilon)\,\mathrm{d}x,$$

$$\hat{I}_9(\theta,\varepsilon,n) = \int_\Omega S_n(\bar{u}_\varepsilon)|f_\varepsilon|\rho_\theta^\varepsilon\,\mathrm{d}x.$$

Limit behaviors of $\hat{I}_2(\theta,\varepsilon,n)$, $\hat{I}_5(\theta,\varepsilon,n)$, and $\hat{I}_8(\theta,\varepsilon,n)$. Thanks to (4.11), we have

$$\lim_{\theta\to 0}\hat{I}_2(\theta,\varepsilon,n) = \varepsilon\int_\Omega S_n'(\bar{u}_\varepsilon)e^{\gamma(T_{n+1}(\bar{u}_\varepsilon))-\gamma(\zeta_k^l(\bar{u}_\varepsilon))}\varphi'\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)$$
$$\times\big|\nabla T_{n+1}(u_\varepsilon)\big|^{p-2}\nabla T_{n+1}(u_\varepsilon)\cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x,$$

and thus

$$\Big|\lim_{\theta\to 0}\hat{I}_2(\theta,\varepsilon,n)\Big| \le \varepsilon C_1\int_\Omega\big|\nabla T_{n+1}(u_\varepsilon)\big|^{p-1}\big(\big|\nabla\zeta_k^l(\bar{u}_\varepsilon)\big|+\big|\nabla\zeta_k^l(\bar{u})\big|\big)\,\mathrm{d}x$$
$$\le \varepsilon C_1\big\|\nabla T_{n+1}(u_\varepsilon)\big\|_{L^p(\Omega)}^{p-1}\big[\big\|\nabla\zeta_k^l(\bar{u}_\varepsilon)\big\|_{L^p(\Omega)}+\big\|\nabla\zeta_k^l(\bar{u})\big\|_{L^p(\Omega)}\big],$$

where C_1 is a positive constant independent of ε. Therefore, using (4.2) we get

$$\lim_{\varepsilon\to 0}\lim_{\theta\to 0}\hat{I}_2(\theta,\varepsilon,n) = 0. \tag{4.13}$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\lim_{\theta\to 0}\hat{I}_5(\theta,\varepsilon,n) = 0 \tag{4.14}$$

and

$$\lim_{\varepsilon\to 0}\lim_{\theta\to 0}\hat{I}_8(\theta,\varepsilon,n) = 0. \tag{4.15}$$

Limit behaviors of $\hat{I}_3(\theta,\varepsilon,n)$ and $\hat{I}_6(\theta,\varepsilon,n)$. Since

$$\hat{I}_3(\theta,\varepsilon,n) = \int_{\{x\in\Omega:\bar{u}_\varepsilon(x)\neq 0\}} S_n'(\bar{u}_\varepsilon)\alpha\big(\big|T_{n+1}(\bar{u}_\varepsilon)\big|\big)\frac{\beta(|T_{n+1}(\bar{u}_\varepsilon)|)}{\alpha(|T_{n+1}(\bar{u}_\varepsilon)|)+\theta}$$
$$\times\big|\nabla T_{n+1}(\bar{u}_\varepsilon)\big|^p\rho_\theta^\varepsilon\,\mathrm{d}x,$$

we get

$$\lim_{\theta\to 0}\hat{I}_3(\theta,\varepsilon,n) = \int_\Omega S_n'(\bar{u}_\varepsilon)\varphi\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)e^{\gamma(\bar{u}_\varepsilon)-\gamma(\zeta_k^l(\bar{u}_\varepsilon))}\beta\big(|\bar{u}_\varepsilon|\big)|\nabla\bar{u}_\varepsilon|^p\,\mathrm{d}x. \tag{4.16}$$

As far as $\hat{I}_6(\theta,\varepsilon,n)$ is concerned, we have

$$\lim_{\theta\to 0}\hat{I}_6(\theta,\varepsilon,n)=\int_\Omega S_n'(\bar{u}_\varepsilon)\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\bigr)e^{\gamma(\bar{u}_\varepsilon)-\gamma(\zeta_k^l(\bar{u}_\varepsilon))}\beta\bigl(|\bar{u}_\varepsilon|\bigr)|\nabla\bar{u}_\varepsilon|^p\,\mathrm{d}x. \tag{4.17}$$

Limit behavior of $\hat{I}_4(\theta,\varepsilon,n)$. From (4.5) and (4.11), it follows that

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\lim_{\theta\to 0}\bigl|\hat{I}_4(\theta,\varepsilon,n)\bigr|=0. \tag{4.18}$$

Limit behavior of $\hat{I}_7(\theta,\varepsilon,n)$. For the term $\hat{I}_7(\theta,\varepsilon,n)$, we have

$$\begin{aligned}\lim_{\theta\to 0}\hat{I}_7(\theta,\varepsilon,n)&=\int_\Omega S_n'(\bar{u}_\varepsilon)\frac{\beta(|\zeta_k^l(\bar{u}_\varepsilon)|)}{\alpha(|\zeta_k^l(\bar{u}_\varepsilon)|)}\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\bigr)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\zeta_k^l(\bar{u}_\varepsilon)\,\mathrm{d}x\\&\le I_{71}(\varepsilon,n)+I_{72}(\varepsilon,n)+I_{73}(\varepsilon,n),\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}I_{71}(\varepsilon,n)&=\max_{s\in[k,l]}\frac{\beta(|s|)}{\alpha(|s|)}\int_\Omega\bigl[a\bigl(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\bigr)-a\bigl(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\bigr)\bigr]\\&\quad\cdot\nabla\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\bigr)S_n'(\bar{u}_\varepsilon)\,\mathrm{d}x,\\I_{72}(\varepsilon,n)&=\int_\Omega\frac{\beta(|\zeta_k^l(\bar{u}_\varepsilon)|)}{\alpha(|\zeta_k^l(\bar{u}_\varepsilon)|)}a\bigl(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\bigr)\\&\quad\cdot\nabla\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\bigr)S_n'(\bar{u}_\varepsilon)\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}I_{73}(\varepsilon,n)&=\int_\Omega\frac{\beta(|\zeta_k^l(\bar{u}_\varepsilon)|)}{\alpha(|\zeta_k^l(\bar{u}_\varepsilon)|)}a\bigl(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\bigr)\nabla\zeta_k^l(\bar{u})\\&\quad\times\varphi\bigl(\bigl(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\bigr)_+\bigr)S_n'(\bar{u}_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Combining (4.3) with (4.4), we infer that

$$\lim_{\varepsilon\to 0}I_{72}(\varepsilon,n)=0 \tag{4.20}$$

and

$$\lim_{\varepsilon\to 0}I_{73}(\varepsilon,n)=0. \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.19), we obtain

$$\overline{\lim_{\varepsilon\to 0}}\lim_{\theta\to 0}\hat{I}_7(\theta,\varepsilon,n)\le\overline{\lim_{\varepsilon\to 0}}I_{71}(\varepsilon,n). \tag{4.22}$$

Limit behavior of $\hat{I}_9(\theta,\varepsilon,n)$. It is straightforward that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\lim_{\theta\to 0}\hat{I}_9(\theta,\varepsilon,n)=0. \tag{4.23}$$

Limit behavior of $\hat{I}_1(\theta,\varepsilon,n)$. Note that $a(x,s,0)=0$ for a.e. $x\in\Omega$ and every $s\in\mathbb{R}$, and we get

$$\begin{aligned}\lim_{\theta\to 0}\hat{I}_1(\theta,\varepsilon,n) &= \int_{\Omega^k_{\varepsilon 1}} S_n'(\bar{u}_\varepsilon)\varphi'\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\cdot\nabla\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\,\mathrm{d}x\\ &\quad+\int_{\Omega^k_{\varepsilon 2}} S_n'(\bar{u}_\varepsilon)e^{\gamma(\bar{u}_\varepsilon)-\gamma(l)}\varphi'\big(\big(l-\zeta_k^l(\bar{u})\big)_+\big)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\cdot\nabla\big(l-\zeta_k^l(\bar{u})\big)_+\,\mathrm{d}x\\ &\quad+\int_{\Omega^k_{\varepsilon 3}} S_n'(\bar{u}_\varepsilon)e^{\gamma(\bar{u}_\varepsilon)-\gamma(k)}\varphi'\big(\big(k-\zeta_k^l(\bar{u})\big)_+\big)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\cdot\nabla\big(k-\zeta_k^l(\bar{u})\big)_+\,\mathrm{d}x\\ &=\hat{I}_{21}(\varepsilon)+\hat{I}_{22}(\varepsilon)+\hat{I}_{23}(\varepsilon),\end{aligned}\tag{4.24}$$

where

$$\begin{aligned}\Omega^k_{\varepsilon 1}&=\{x\in\Omega: k<\bar{u}_\varepsilon<l\},\\ \Omega^k_{\varepsilon 2}&=\{x\in\Omega:\bar{u}_\varepsilon\geq l\},\\ \Omega^k_{\varepsilon 3}&=\{x\in\Omega:\bar{u}_\varepsilon\leq k\}.\end{aligned}$$

Using (4.3), (4.4), and (4.11), it is clear that

$$\lim_{\varepsilon\to 0}\hat{I}_{22}(\varepsilon)=0\tag{4.25}$$

and

$$\lim_{\varepsilon\to 0}\hat{I}_{23}(\varepsilon)=0.\tag{4.26}$$

Note that $a(x,s,0)=0$ for a.e. $x\in\Omega$ and every $s\in\mathbb{R}$, the term $\hat{I}_{21}(\varepsilon)$ can be rewritten as follows:

$$\hat{I}_{21}(\varepsilon)=J_1(\varepsilon)+J_2(\varepsilon),$$

where

$$\begin{aligned}J_1(\varepsilon)&=\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\\ &\quad\cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\varphi'\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x,\\ J_2(\varepsilon)&=\int_\Omega S_n'(\bar{u}_\varepsilon)a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\\ &\quad\cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\varphi'\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x.\end{aligned}$$

By (4.3), (4.4), and (4.10), we find that

$$\lim_{\varepsilon\to 0}J_2(\varepsilon)=0.\tag{4.27}$$

As a direct consequence of (4.24)-(4.27), we have

$$\overline{\lim_{\varepsilon\to 0}}\lim_{\theta\to 0}\hat{I}_1(\theta,\varepsilon,n)=\overline{\lim_{\varepsilon\to 0}}J_1(\varepsilon). \tag{4.28}$$

Choosing $\lambda = 2\max_{s\in[k,l]}\frac{\beta(|s|)}{\alpha(|s|)}$ in the definition of φ, and then combining the limit behaviors of $\hat{I}_1(\theta,\varepsilon,n)$-$\hat{I}_9(\theta,\varepsilon,n)$, we get

$$\begin{aligned}\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\\ \cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\varphi'\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x\le 0,\end{aligned}$$

which yields

$$\begin{aligned}\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\\ \cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x\le 0.\end{aligned} \tag{4.29}$$

Step 4.3. Choosing $v=-S_n(\bar{u}_\varepsilon)e^{-\gamma_\theta(\bar{u}_\varepsilon)+\gamma_\theta(\zeta_k^l(\bar{u}_\varepsilon))}\varphi((\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u}))_-)$ as a test function in (4.1), then arguing as before, we have

$$\begin{aligned}\lim_{n\to\infty}\underline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\\ \cdot\nabla\big(\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)_-\big)\,\mathrm{d}x\ge 0.\end{aligned} \tag{4.30}$$

It follows from (4.29) and (4.30) that

$$\begin{aligned}\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\\ \cdot\nabla\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)\,\mathrm{d}x\le 0.\end{aligned} \tag{4.31}$$

Taking into account that $S_n'(\bar{u}_\varepsilon)a(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon))=a(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon))$ for $n>l$, using (4.31) we get

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)\cdot\nabla\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)\,\mathrm{d}x\le 0,$$

which yields

$$\lim_{\varepsilon\to 0}\int_\Omega\big[a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u}_\varepsilon)\big)-a\big(x,\zeta_k^l(\bar{u}_\varepsilon),\nabla\zeta_k^l(\bar{u})\big)\big]\cdot\nabla\big(\zeta_k^l(\bar{u}_\varepsilon)-\zeta_k^l(\bar{u})\big)\,\mathrm{d}x=0. \tag{4.32}$$

Then, arguing as in [24], we derive

$$\nabla\zeta_k^l(\bar{u}_\varepsilon)\to\nabla\zeta_k^l(\bar{u})\quad\text{strongly in }\big(L^p(\Omega)\big)^N\text{ and a.e. in }\Omega. \tag{4.33}$$

Step 4.4. For any fixed $l>k>0$, we denote

$$\bar{\zeta}_k^l(s)=\min\big\{T_l(s),-k\big\}=-k-\big(T_l(s)+k\big)_-,\quad\forall s\in\mathbb{R}.$$

Choosing $v = S_n(\bar{u}_\varepsilon)e^{\gamma_\theta(\bar{u}_\varepsilon)-\gamma_\theta(\bar{\zeta}_k^l(\bar{u}_\varepsilon))}\varphi((\bar{\zeta}_k^l(\bar{u}_\varepsilon) - \bar{\zeta}_k^l(\bar{u}))_+)$ as a test function in (4.1), arguing as before we obtain

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\bar{\zeta}_k^l(\bar{u}_\varepsilon),\nabla\bar{\zeta}_k^l(\bar{u}_\varepsilon)\big) - a\big(x,\bar{\zeta}_k^l(\bar{u}_\varepsilon),\nabla\bar{\zeta}_k^l(\bar{u})\big)\big] \cdot\nabla\big(\big(\bar{\zeta}_k^l(\bar{u}_\varepsilon) - \bar{\zeta}_k^l(\bar{u})\big)_+\big)\,\mathrm{d}x \le 0.$$

Next choosing $v = -S_n(\bar{u}_\varepsilon)e^{\gamma_\theta(\bar{\zeta}_k^l(\bar{u}_\varepsilon))-\gamma_\theta(\bar{u}_\varepsilon)}\varphi((\bar{\zeta}_k^l(\bar{u}_\varepsilon) - \bar{\zeta}_k^l(\bar{u}))_-)$ as a test function in (4.1), applying the same argument we get

$$\lim_{n\to\infty}\underline{\lim_{\varepsilon\to 0}}\int_\Omega S_n'(\bar{u}_\varepsilon)\big[a\big(x,\bar{\zeta}_k^l(\bar{u}_\varepsilon),\nabla\bar{\zeta}_k^l(\bar{u}_\varepsilon)\big) - a\big(x,\bar{\zeta}_k^l(\bar{u}_\varepsilon),\nabla\bar{\zeta}_k^l(\bar{u})\big)\big] \cdot\nabla\big(\big(\bar{\zeta}_k^l(\bar{u}_\varepsilon) - \bar{\zeta}_k^l(\bar{u})\big)_-\big)\,\mathrm{d}x \ge 0.$$

Proceeding as in Step 4.3, we infer that

$$\nabla\bar{\zeta}_k^l(\bar{u}_\varepsilon) \to \nabla\bar{\zeta}_k^l(\bar{u}) \quad \text{strongly in } \big(L^p(\Omega)\big)^N \text{ and a.e. in } \Omega. \tag{4.34}$$

As a consequence of (4.33) and (4.34), we have

$$\chi_{\{|\bar{u}_\varepsilon|>k\}}\nabla T_l(\bar{u}_\varepsilon) \to \chi_{\{|\bar{u}|>k\}}\nabla T_l(\bar{u}) \quad \text{strongly in } \big(L^p(\Omega)\big)^N \text{ and a.e. in } \Omega. \tag{4.35}$$

Step 4.5. In this step we prove that $\bar{u}$ satisfies (2.7), where u is replaced by $\bar{u}$.

For any fixed $m > k$, one has

$$\int_{\{x\in\Omega:m\le|\bar{u}_\varepsilon(x)|\le m+1\}} a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon\,\mathrm{d}x = \int_\Omega a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\big[\nabla T_{m+1}(\bar{u}_\varepsilon) - \nabla T_m(\bar{u}_\varepsilon)\big]\,\mathrm{d}x. \tag{4.36}$$

Thus, passing to the limit as ε tends to zero in (4.36), we deduce that, for fixed $m > k \ge 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\{x\in\Omega:m\le|\bar{u}_\varepsilon(x)|\le m+1\}} a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon\,\mathrm{d}x &= \int_\Omega a(x,\bar{u},\nabla\bar{u})\big[\nabla T_{m+1}(\bar{u}) - \nabla T_m(\bar{u})\big]\,\mathrm{d}x \\ &= \int_{\{x\in\Omega:m\le|\bar{u}|\le m+1\}} a(x,\bar{u},\nabla\bar{u})\nabla\bar{u}\,\mathrm{d}x.\end{aligned} \tag{4.37}$$

Taking the limit as m tends to $+\infty$ in (4.37) and using (4.5), we conclude that $\bar{u}$ satisfies (2.7).

In the following, we prove that $\bar{u}$ satisfies (2.8). Indeed, by (4.1), we have

$$\int_\Omega h(\bar{u}_\varepsilon)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\upsilon\,\mathrm{d}x + \int_\Omega \varepsilon h(\bar{u}_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\upsilon\,\mathrm{d}x + \int_\Omega h'(\bar{u}_\varepsilon)a(x,\bar{u}_\varepsilon,\nabla\bar{u}_\varepsilon)\nabla\bar{u}_\varepsilon\upsilon\,\mathrm{d}x$$

$$
\begin{aligned}
&+ \int_\Omega \varepsilon h'(\bar{u}_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \bar{u}_\varepsilon \upsilon \,\mathrm{d}x \\
&+ \int_\Omega h(\bar{u}_\varepsilon)F_\varepsilon(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon)\upsilon \,\mathrm{d}x \\
&= \int_\Omega h(\bar{u}_\varepsilon)f_\varepsilon \upsilon \,\mathrm{d}x
\end{aligned} \tag{4.38}
$$

for any given $\upsilon \in W^{1,\infty}(\Omega)$ and $h \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} h \subseteq [-l,l]$ for some $l > 0$.

Now we first analyze the fifth term on the left-hand side of (4.38). Recall that $\operatorname{supp} h \subseteq [-l,l]$, we get

$$h(\bar{u}_\varepsilon)F(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon) = h(\bar{u}_\varepsilon)F\big(x, T_l(\bar{u}_\varepsilon), \nabla T_l(\bar{u}_\varepsilon)\big).$$

Therefore, for any k satisfying $0 < k < l$, one has

$$
\begin{aligned}
&\int_\Omega h(\bar{u}_\varepsilon)F(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon)\upsilon \,\mathrm{d}x \\
&\quad = \int_{\{x\in\Omega:|\bar{u}_\varepsilon|>k\}} h(\bar{u}_\varepsilon)F\big(x, T_l(\bar{u}_\varepsilon), \nabla T_l(\bar{u}_\varepsilon)\big)\upsilon \,\mathrm{d}x \\
&\qquad + \int_{\{x\in\Omega:|\bar{u}_\varepsilon|\le k\}} h(\bar{u}_\varepsilon)F\big(x, T_l(\bar{u}_\varepsilon), \nabla T_l(\bar{u}_\varepsilon)\big)\upsilon \,\mathrm{d}x \\
&\quad = J_{1\varepsilon} + J_{2\varepsilon}.
\end{aligned} \tag{4.39}
$$

Similarly to the proof of (3.33) and (3.34), using (4.3) and (4.35) we obtain

$$
\begin{aligned}
\lim_{k\to 0}\lim_{\varepsilon\to 0} J_{1\varepsilon} &= \int_\Omega h(\bar{u})F\big(x, T_l(\bar{u}), \nabla T_l(\bar{u})\big)\upsilon \,\mathrm{d}x \\
&= \int_\Omega h(\bar{u})F(x,\bar{u},\nabla \bar{u})\upsilon \,\mathrm{d}x
\end{aligned} \tag{4.40}
$$

and

$$\lim_{k\to 0}\lim_{\varepsilon\to 0} J_{2\varepsilon} = 0, \tag{4.41}$$

which imply that

$$\lim_{\varepsilon\to 0}\int_\Omega h(\bar{u}_\varepsilon)F(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon)\upsilon \,\mathrm{d}x = \int_\Omega h(\bar{u})F(x,\bar{u},\nabla \bar{u})\upsilon \,\mathrm{d}x. \tag{4.42}$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\int_\Omega h'(\bar{u}_\varepsilon)a(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon)\nabla \bar{u}_\varepsilon \upsilon \,\mathrm{d}x = \int_\Omega h'(\bar{u})a(x,\bar{u},\nabla \bar{u})\nabla \bar{u}\upsilon \,\mathrm{d}x \tag{4.43}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega h(\bar{u}_\varepsilon)a_\varepsilon(x,\bar{u}_\varepsilon,\nabla \bar{u}_\varepsilon)\nabla \upsilon \,\mathrm{d}x = \int_\Omega h(\bar{u})a(x,\bar{u},\nabla \bar{u})\nabla \upsilon \,\mathrm{d}x. \tag{4.44}$$

As far as the second term of the left-hand side of (4.38) is concerned, by (4.1) we easily get

$$\begin{aligned}
&\left| \int_{\Omega} \varepsilon h(\bar{u}_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \upsilon \,\mathrm{d}x \right| \\
&\quad = \left| \int_{\Omega} \varepsilon h(\bar{u}_\varepsilon) \left|\nabla T_{\tilde{l}}(u_\varepsilon)\right|^{p-2} \nabla T_{\tilde{l}}(u_\varepsilon) \nabla \upsilon \,\mathrm{d}x \right| \\
&\quad \leq \varepsilon \sup_{\sigma \in [-l,l]} \left|h(\sigma)\right| \left\| \nabla T_{\tilde{l}}(u_\varepsilon) \right\|_{L^p(\Omega)}^{p-1} \|\nabla \upsilon\|_{L^p(\Omega)}, \quad \text{where } \tilde{l} = g^{-1}(l),
\end{aligned}$$

thus

$$\lim_{\varepsilon \to 0} \int_{\Omega} \varepsilon h(\bar{u}_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \upsilon \,\mathrm{d}x = 0. \tag{4.45}$$

Reasoning as in (4.45), one has

$$\lim_{\varepsilon \to 0} \int_{\Omega} \varepsilon h'(\bar{u}_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \bar{u}_\varepsilon \upsilon \,\mathrm{d}x = 0. \tag{4.46}$$

Finally, it is clear that

$$\lim_{\varepsilon \to 0} \int_{\Omega} h(\bar{u}_\varepsilon) f_\varepsilon \upsilon \,\mathrm{d}x = \int_{\Omega} h(\bar{u}) f \upsilon \,\mathrm{d}x. \tag{4.47}$$

Then, letting ε tend to zero in (4.38), we conclude from (4.42)-(4.47) that $\bar{u}$ satisfies (2.8). Hence, $\bar{u}$ is a renormalized solution to problem ($\mathscr{P}$). □

Competing interests
The author declares to have no competing interests.

Author's contributions
The author wrote the manuscript and read and approved the final manuscript.

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