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INCLUSION RESULTS FOR CONVOLUTION SUBMETHODS

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If B is a summability matrix, then the submethod B_λ is the matrix obtained by deleting a set of rows from the matrix B. Comparisons between Euler-Knopp submethods and the Borel summability method are made. Also, an equivalence result for convolution submethods is established. This result will necessarily apply to the submethods of the Euler-Knopp, Taylor, Meyer-König, and Borel matrix summability methods.

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1. Introduction and notation. Let E be an infinite subset of $\mathbb{N}\cup\{0\}$ and consider E as the range of a strictly increasing sequence of nonnegative integers, say $E := \{\lambda(n)\}_{n=0}^{\infty}$. If $B := (b_{n,k})$ is a summability matrix, then the submethod B_λ is the matrix whose nkth entry is $B_\lambda[n,k] := b_{\lambda(n),k}$. Thus, for a given sequence x, the B_λ-transform of x is the sequence $B_\lambda x$ with

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} := \sum_{k=0}^{\infty} b_{\lambda(n),k}x_k. \tag{1.1}$$

Since B_λ is a row submatrix of B, it is regular (i.e., limit preserving) whenever B is regular.

Row submatrices have appeared throughout the literature [5, 6, 8, 12], but they were first studied as a class unto themselves by Goffman and Petersen [7], and later by Steele [14]. The class of Cesàro submethods has been studied by Armitage and Maddox [1] and Osikiewicz [11].

Let A and B be two summability matrices. If every sequence which is A-summable is also B-summable to the same limit, then B includes A, denoted by $A \subseteq B$. Also, B is called a triangle if $b_{n,k} = 0$ for all $k > n$ and $b_{n,n} \neq 0$ for all n. The following lemma extends [1, Theorem 1].

Lemma 1.1. *Let B be a summability matrix and let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N}\cup\{0\}$.*

(1) *If $F\setminus E$ is finite, then $B_\lambda \subseteq B_\rho$.*

(2) *If B is a triangle and $B_\lambda \subseteq B_\rho$, then $F\setminus E$ is finite.*

(3) *If B is a triangle, then B_λ is equivalent to B_ρ if and only if the symmetric difference $E \triangle F$ is finite.*

In particular, $B \subseteq B_\lambda$ for any λ.

Proof. Assume $F\setminus E$ is finite and let x be a sequence that is B_λ-summable to L. Then there exists an N such that $\{\rho(n) : n \geq N\} \subseteq E$. That is, $\{\rho(n) : n \geq N\}$ is a subsequence of $\{\lambda(n)\}$. Since $\lim_n (B_\lambda x)_n = \lim_n (Bx)_{\lambda(n)} = L$, we have $\lim_n (B_\rho x)_n = \lim_n (Bx)_{\rho(n)} = L$.

Now assume B is a triangle, and hence invertible, and $F \setminus E$ is infinite. Let $F \setminus E := \{\rho(n(j))\}_{j=0}^{\infty}$ with $\rho(n(j)) < \rho(n(j+1))$. Consider the sequence y defined by

$$y_k := \begin{cases} (-1)^j, & \text{if } k = \rho(n(j)) \text{ for some } j, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

and let x be the sequence $B^{-1}y$. Then, for every n,

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} = \left(B\left(B^{-1}y\right)\right)_{\lambda(n)} = y_{\lambda(n)} = 0. \tag{1.3}$$

Hence, $\lim_n (B_\lambda x)_n = 0$. However, for every j,

$$(B_\rho x)_{n(j)} = (Bx)_{\rho(n(j))} = \left(B\left(B^{-1}y\right)\right)_{\rho(n(j))} = y_{\rho(n(j))} = (-1)^j. \tag{1.4}$$

Thus x is not B_ρ-summable. Therefore B_ρ does not include B_λ, which completes the contrapositive of assertion (2). Lastly, assertion (3) follows from (1) and (2) since $E \triangle F := (E \setminus F) \cup (F \setminus E)$. □

To show the reason for the necessity of B being a triangle in assertion (2) of Lemma 1.1, consider the matrix B whose nkth entry is

$$B[n,k] := \begin{cases} 0, & \text{if } n \text{ even and } k \neq \dfrac{n}{2}, \\ 1, & \text{if } n \text{ even and } k = \dfrac{n}{2}, \\ 0, & \text{if } n \text{ odd and } n \neq k, \\ 1, & \text{if } n \text{ odd and } n = k. \end{cases} \tag{1.5}$$

Then if $\lambda(n) := 2n$ and $\rho(n) := 2n+1$, $F \setminus E$ is infinite and $B_\lambda \subseteq B_\rho$.

2. Inclusion results for Euler-Knopp submethods. For $r \in \mathbb{C} \setminus \{0,1\}$, the Euler-Knopp method of order r is given by the matrix E_r whose nkth entry is

$$E_r[n,k] := \begin{cases} \dbinom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.1}$$

For the case $r = 1$, E_1 is the identity matrix, and E_0 is the matrix whose nkth entry is

$$E_0[n,k] := \begin{cases} 1, & \text{if } k = 0,\ n = 0,1,2,\ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

It is well known that E_r is regular if and only if $0 < r \leq 1$ (see [4]).

Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r \in \mathbb{C} \setminus \{0,1\}$. The submethod $E_{r,\lambda}$ is the matrix whose nkth entry is

$$E_{r,\lambda}[n,k] := \begin{cases} \binom{\lambda(n)}{k} r^k (1-r)^{\lambda(n)-k}, & \text{if } k \le \lambda(n), \\ 0, & \text{if } k > \lambda(n). \end{cases} \tag{2.3}$$

Then $E_{r,\lambda}$ is regular if and only if E_r is regular.

By a direct application of Lemma 1.1, we have the following inclusion result for the $E_{r,\lambda}$ methods.

Lemma 2.1. *Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$ and $r \neq 0$.*

(1) *The method $E_{r,\lambda} \subseteq E_{r,\rho}$ if and only if $F \setminus E$ is finite.*

(2) *The method $E_{r,\lambda}$ is equivalent to $E_{r,\rho}$ if and only if the symmetric difference $E \triangle F$ is finite.*

We now examine the relationship between $E_{r,\lambda}$ and the Borel summability method. Recall that a sequence x is Borel summable to L if

$$\lim_{t\to\infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = L. \tag{2.4}$$

Theorem 2.2. *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r > 0$. Then the Borel summability method includes $E_{r,\lambda}$ if and only if $S := (\mathbb{N} \cup \{0\}) \setminus E$ is finite.*

Proof. If S is finite, then by Lemma 2.1, E_r and $E_{r,\lambda}$ are equivalent. But the Borel summability method includes E_r for $r > 0$ (see [4]). Hence, it also includes $E_{r,\lambda}$. If S is infinite, then it may be written as a strictly increasing sequence of nonnegative integers, say $S := \{\rho(m)\}_{m=0}^{\infty}$. If $M_n := \max_{0\le k\le n} |E_r[n,k]|$, consider the sequence y defined by

$$y_n := \begin{cases} (\rho(m)!)^2 (\rho(m)+1) M_{\rho(m)}, & \text{if } n = \rho(m), \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

and let x be the sequence $E_r^{-1} y$; that is, $y = E_r x$ and

$$\lim_{n\to\infty} (E_{r,\lambda} x)_n = \lim_{n\to\infty} (E_r x)_{\lambda(n)} = \lim_{n\to\infty} y_{\lambda(n)} = 0. \tag{2.6}$$

Hence, x is $E_{r,\lambda}$-summable to 0. Now observe that for a given n,

$$|y_n| = |(E_r x)_n| \le \sum_{k=0}^{n} |E_r[n,k]| |x_k| \le M_n \sum_{k=0}^{n} |x_k|. \tag{2.7}$$

Thus, for $n = \rho(m)$, we have

$$(\rho(m)!)^{1/\rho(m)} = \left(\frac{1}{\rho(m)!}\cdot\frac{1}{\rho(m)+1}\cdot\frac{|y_{\rho(m)}|}{M_{\rho(m)}}\right)^{1/\rho(m)} \leq \left(\frac{1}{\rho(m)!}\cdot\frac{1}{\rho(m)+1}\sum_{k=0}^{\rho(m)}|x_k|\right)^{1/\rho(m)}. \tag{2.8}$$

Since $\limsup_m(\rho(m)!)^{1/\rho(m)} = \infty$,

$$\limsup_{m\to\infty}\left(\frac{1}{\rho(m)!}\cdot\frac{1}{\rho(m)+1}\sum_{k=0}^{\rho(m)}|x_k|\right)^{1/\rho(m)} = \infty, \tag{2.9}$$

and it follows that $\limsup_n(|x_n|/n!)^{1/n} = \infty$. Thus, $\sum_{k=0}^{\infty}(x_k/k!)t^k$ diverges for all nonzero t and hence x is not Borel summable. □

Theorem 2.3. *There exists a sequence which is Borel summable but not $E_{r,\lambda}$-summable for any λ and $r > 0$.*

Proof. Let $r > 0$ and consider the sequence x defined by

$$x_n := n\left(-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)^{n-1}. \tag{2.10}$$

Then it can be shown that $(E_{r,\lambda}x)_n = (-1)^{\lambda(n)}\lambda(n)$. Hence x is not $E_{r,\lambda}$-summable for any λ. However,

$$\begin{aligned} e^{-t}\sum_{k=0}^{\infty}x_k\frac{t^k}{k!} &= e^{-t}\sum_{k=1}^{\infty}\left[k\left(-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)^{k-1}\right]\frac{t^k}{k!} \\ &= \left(-\frac{1}{r}\right)e^{-t}\sum_{k=1}^{\infty}\left(1-\frac{2}{r}\right)^{k-1}\frac{t^k}{(k-1)!} \\ &= \left(-\frac{1}{r}\right)te^{-t}\sum_{k=0}^{\infty}\left(1-\frac{2}{r}\right)^{k}\frac{t^k}{k!} \\ &= \left(-\frac{1}{r}\right)te^{-t}e^{(1-2/r)t} \\ &= \left(-\frac{1}{r}\right)te^{-(2/r)t}. \end{aligned} \tag{2.11}$$

Since $r > 0$,

$$\lim_{t\to\infty}e^{-t}\sum_{k=0}^{\infty}x_k\frac{t^k}{k!} = \lim_{t\to\infty}\left(-\frac{1}{r}\right)te^{-(2/r)t} = 0, \tag{2.12}$$

and hence x is Borel summable to 0. □

3. Convolution methods. Let p and q be sequences of real numbers with $p_k \geq 0$, $q_k \geq 0$, $\sum_{k=0}^{\infty} p_k = 1$, and $\sum_{k=0}^{\infty} q_k = 1$. The convolution summability method is given by the matrix $C^* := (c_{n,k})$ whose nkth entry is

$$c_{n,k} := \begin{cases} q_k, & \text{if } n = 0, \\ \sum_{j=0}^{k} c_{n-1,j}\, p_{k-j}, & \text{if } n \geq 1. \end{cases} \tag{3.1}$$

It is clear that C^* is a nonnegative matrix such that for every n, $\sum_{k=0}^{\infty} c_{n,k} = 1$. Some classical summability matrices are examples of the matrix C^*. If $0 \leq r \leq 1$, $p := \{1-r, r, 0, 0, \ldots\}$, and $q := \{1, 0, 0, \ldots\}$, then C^* is the Euler-Knopp method of order r. If $0 \leq r < 1$, $p := \{0, (1-r), (1-r)r, (1-r)r^2, \ldots\}$, and $q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\}$, then C^* is the Taylor method of order r, denoted by T_r. If $0 < r < 1$ and $p := q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\}$, then C^* is the Meyer-König method of order r, denoted by S_r. If $p := q := \{1/k!e\}$, then C^* is the Borel matrix method B^*. Similar forms of the convolution method are known by different names, such as the random-walk method and Sonnenschein method. (Further information on all of these methods may be found in [3, 4, 13].)

If C^* is the convolution method formed from the sequences p and q, then let

$$\mu := \sum_{j=0}^{\infty} j p_j, \qquad \nu := \sum_{j=0}^{\infty} j q_j. \tag{3.2}$$

We note here that for the remainder of this work, p and q are nonnegative sequences whose sums are 1, and μ and ν represent the sums in (3.2). Also, $c_{n,k} := 0$ whenever $k < 0$.

We next present some preliminary results concerning the convolution method.

Lemma 3.1. *The convolution method C^* is regular if and only if $p_0 < 1$.*

Proof. See [9]. □

Lemma 3.2. *If $\mu < \infty$ and $\nu < \infty$, then for every n,*

$$\sum_{k=0}^{\infty} k c_{n,k} = n\mu + \nu. \tag{3.3}$$

Proof. Note that for $n = 0$, the result holds. So assume the result holds for some integer $n > 0$. Then

$$\begin{aligned} \sum_{k=0}^{\infty} k c_{n+1,k} &= \sum_{k=0}^{\infty} k \left(\sum_{j=0}^{k} c_{n,j} p_{k-j} \right) = \sum_{j=0}^{\infty} c_{n,j} \sum_{k=j}^{\infty} k p_{k-j} \\ &= \sum_{j=0}^{\infty} c_{n,j} \left(\sum_{i=0}^{\infty} i p_i + j \sum_{i=0}^{\infty} p_i \right) = \sum_{j=0}^{\infty} \mu c_{n,j} + \sum_{j=0}^{\infty} j c_{n,j} = (n+1)\mu + \nu. \end{aligned} \tag{3.4}$$

By induction, the result follows. □

Lemma 3.3. *Let C^* be the convolution method formed from the sequences p and q and $D^* := (d_{n,k})$ the convolution method formed from the sequences p and $\tilde{q} := \{1,0,0,\dots\}$. Then for nonnegative integers n, k, and j,*

$$c_{n+j,k} = \sum_{i=0}^{k} c_{n,k-i} d_{j,i}. \tag{3.5}$$

The proof of this lemma is a straightforward induction argument left to the reader.

Lemma 3.4. *Let C^* be the convolution method formed from the sequences p and q. If $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty}(j-\mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$, then*

$$\sum_{k=0}^{\infty} \left| c_{n,k+1} - c_{n,k} \right| = O\left(\frac{1}{\sqrt{n}}\right). \tag{3.6}$$

Proof. Let $D^* := (d_{n,k})$ be the convolution method formed from the sequences p and $\tilde{q} := \{1,0,0,\dots\}$. We first prove that the result holds for D^*.

Let $\phi(t) := (\sqrt{2\pi} e^{t^2/2})^{-1}$ and $x_{n,k} := (k - n\mu)/\sigma\sqrt{n}$, where $\sigma^2 := \sum_{j=0}^{\infty}(j-\mu)^2 p_j$. Then

$$\begin{aligned} \sqrt{n} \sum_{k=0}^{\infty} \left| d_{n,k+1} - d_{n,k} \right| \le \sqrt{n} \sum_{k=0}^{\infty} & \left| d_{n,k+1} - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) \right| \\ & + \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) \right| \\ & + \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) - d_{n,k} \right|. \end{aligned} \tag{3.7}$$

The first and the third terms on the right-hand side of the inequality are bounded by a result of Bikjalis and Jasjunas [2]. For the middle term, the mean value theorem yields

$$\begin{aligned} \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) \right| &= \frac{1}{\sigma} \sum_{k=0}^{\infty} \left| \phi'(\xi_{n,k}) \right| (x_{n,k+1} - x_{n,k}) \\ &< \frac{K}{\sigma} \int_{\mathbb{R}} \left| \phi'(t) \right| dt < \infty, \end{aligned} \tag{3.8}$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$ and $K > 0$ is some constant. Thus, the result holds for the convolution method D^*. Then, by Lemma 3.3,

$$\begin{aligned} \sum_{k=0}^{\infty} \left| c_{n,k+1} - c_{n,k} \right| &= \sum_{k=0}^{\infty} \left| \sum_{i=0}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^{k} q_{k-i} d_{n,i} \right| \\ &= \sum_{k=0}^{\infty} \left| q_{k+1} d_{n,0} + \sum_{i=1}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^{k} q_{k-i} d_{n,i} \right| \end{aligned}$$

$$\begin{aligned}&\le p_0^n\sum_{k=0}^{\infty}q_{k+1}+\sum_{k=0}^{\infty}\sum_{i=0}^{k}q_{k-i}\,|d_{n,i+1}-d_{n,i}|\\&\le p_0^n+\sum_{i=0}^{\infty}|d_{n,i+1}-d_{n,i}|\sum_{k=i}^{\infty}q_{k-i}\\&=p_0^n+\sum_{i=0}^{\infty}|d_{n,i+1}-d_{n,i}|=O\Big(\frac{1}{\sqrt{n}}\Big).\end{aligned}\tag{3.9}$$

□

4. Equivalence results for convolution submethods. Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N}\cup\{0\}$. The convolution submethod C_λ^* is the matrix whose nkth entry is

$$C_\lambda^*[n,k] := C^*[\lambda(n),k].\tag{4.1}$$

Lemma 4.1. *The convolution submethod C_λ^* is regular if and only if $p_0 < 1$.*

Proof. If $p_0 < 1$, then C^* is regular and hence C_λ^* is also regular. Conversely, if C_λ^* is regular and $p_0 = 1$, then $C_\lambda^*[n,k] = q_k$ for all n and k. Since $\sum_{k=0}^{\infty} q_k = 1$, there exists a $\hat{k}$ such that $q_{\hat{k}} \ne 0$. Then $\lim_n C_\lambda^*[n,\hat{k}] = q_{\hat{k}} \ne 0$, which contradicts the regularity of C_λ^*. □

The following theorem compares C_λ^* with C^* for bounded sequences.

Theorem 4.2. *Let C^* be the convolution method formed from the sequences p and q with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty}(j-\mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N}\cup\{0\}$. If*

$$\lim_{n\to\infty}\frac{\lambda(n+1)-\lambda(n)}{\sqrt{\lambda(n)}} = 0,\tag{4.2}$$

then C^* and C_λ^* are equivalent for bounded sequences.

Proof. By Lemma 1.1, $C^* \subseteq C_\lambda^*$ for any λ. So assume $\lim_n(\lambda(n+1)-\lambda(n))/\sqrt{\lambda(n)} = 0$ and let x be a bounded sequence that is C_λ^*-summable to L. Consider the set $S := \{\rho(n)\} := (\mathbb{N}\cup\{0\})\setminus E$. If S is finite, then Lemma 1.1 shows that C_λ^* and C^* are equivalent for all sequences. So assume S is infinite. Then there exists an N such that for $n \ge N$, $\rho(n) > \lambda(0)$. Since E and S are disjoint, for $n \ge N$, there exists an integer m such that $\lambda(m) < \rho(n) < \lambda(m+1)$. We write $\rho(n) := \lambda(m)+j$, where $0 < j < \lambda(m+1)-\lambda(m)$. Then, for $n \ge N$,

$$\begin{aligned}|(C_\rho^*x)_n-(C_\lambda^*x)_m| &= \left|\sum_{k=0}^{\infty}c_{\rho(n),k}x_k-\sum_{k=0}^{\infty}c_{\lambda(m),k}x_k\right|\\&=\left|\sum_{k=0}^{\infty}c_{\lambda(m)+j,k}x_k-\sum_{k=0}^{\infty}c_{\lambda(m),k}x_k\right|.\end{aligned}\tag{4.3}$$

By Lemma 3.3, this becomes

$$
\begin{aligned}
\left|(C_\rho^* x)_n-(C_\lambda^* x)_m\right| &= \left|\sum_{k=0}^{\infty}\left(\sum_{i=0}^{\infty}c_{\lambda(m),k-i}d_{j,i}\right)x_k-\sum_{k=0}^{\infty}c_{\lambda(m),k}x_k\right| \\
&= \left|\sum_{k=0}^{\infty}x_k\left[\left(\sum_{i=0}^{\infty}c_{\lambda(m),k-i}d_{j,i}\right)-\left(\sum_{i=0}^{\infty}c_{\lambda(m),k}d_{j,i}\right)\right]\right| \\
&\le \|x\|_\infty \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}d_{j,i}\left|c_{\lambda(m),k-i}-c_{\lambda(m),k}\right| \\
&= \|x\|_\infty \sum_{i=0}^{\infty}d_{j,i}\sum_{k=0}^{\infty}\left|\sum_{l=0}^{i-1}c_{\lambda(m),k-l}-c_{\lambda(m),k-l-1}\right| \\
&\le \|x\|_\infty \sum_{i=0}^{\infty}d_{j,i}\sum_{k=0}^{\infty}\sum_{l=0}^{i-1}\left|c_{\lambda(m),k-l}-c_{\lambda(m),k-l-1}\right| \\
&= \frac{\|x\|_\infty}{\sqrt{\lambda(m)}}\sum_{i=0}^{\infty}d_{j,i}\sum_{l=0}^{i-1}\sqrt{\lambda(m)}\sum_{k=0}^{\infty}\left|c_{\lambda(m),k-l}-c_{\lambda(m),k-l-1}\right|.
\end{aligned}
\tag{4.4}
$$

By Lemma 3.4, there exists an $M>0$ such that

$$
\sqrt{\lambda(m)}\sum_{k=0}^{\infty}\left|c_{\lambda(m),k-l}-c_{\lambda(m),k-l-1}\right|<M. \tag{4.5}
$$

Then, by Lemma 3.2,

$$
\left|(C_\rho^* x)_n-(C_\lambda^* x)_m\right| \le \frac{\|x\|_\infty}{\sqrt{\lambda(m)}}\sum_{i=0}^{\infty}d_{j,i}\sum_{l=0}^{i-1}M=\frac{\|x\|_\infty M}{\sqrt{\lambda(m)}}\sum_{i=0}^{\infty}i d_{j,i}\le\frac{\|x\|_\infty M}{\sqrt{\lambda(m)}}\cdot j\mu. \tag{4.6}
$$

Since $0<j<\lambda(m+1)-\lambda(m)$,

$$
\left|(C_\rho^* x)_n-(C_\lambda^* x)_m\right| < \|x\|_\infty M\mu\cdot\frac{\lambda(m+1)-\lambda(m)}{\sqrt{\lambda(m)}}=o(1). \tag{4.7}
$$

Thus,

$$
0\le\left|(C_\rho^* x)_n-L\right|\le\left|(C_\rho^* x)_n-(C_\lambda^* x)_m\right|+\left|(C_\lambda^* x)_m-L\right|=o(1)+o(1)=o(1). \tag{4.8}
$$

Therefore, the sequence C^*x may be partitioned into two disjoint subsequences, namely $(C_\lambda^* x)_n=(C^*x)_{\lambda(n)}$ and $(C_\rho^* x)_n=(C^*x)_{\rho(n)}$, each having the common limit L. Thus, x must be C^*-summable to L, and hence C^* and C_λ^* are equivalent for bounded sequences. □

The following theorem is a well-known result due to Meyer-König (see [10, Theorem 25]).

Theorem 4.3. *The methods E_r $(0<r<1)$, S_r $(0<r<1)$, T_r $(0<r<1)$, and the Borel method are equivalent for bounded sequences.*

Since the Euler-Knopp methods of order $0 < r < 1$, Taylor methods of order $0 < r < 1$, Meyer-König methods of order $0 < r < 1$, and the Borel matrix method all have generating sequences satisfying the conditions in Theorem 4.2, the following corollary is immediate.

Corollary 4.4. *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $0 < r < 1$. If λ satisfies condition (4.6), then $E_{r,\lambda}$, E_r, $T_{r,\lambda}$, T_r, $S_{r,\lambda}$, S_r, B_λ^*, B^*, and the Borel method are all equivalent for bounded sequences.*

The next theorem presents an equivalence relationship between the C_λ^* submethods.

Theorem 4.5. *Let C^* be the convolution method formed from the sequences p and q with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty}(j-\mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$. If*

$$\lim_{n\to\infty} \frac{\rho(n)-\lambda(n)}{\sqrt{\lambda(n)}} = 0, \tag{4.9}$$

then C_λ^* and C_ρ^* are equivalent for bounded sequences.

Proof. Let x be a bounded sequence and consider the sequences $M(n) := \max\{\lambda(n), \rho(n)\}$ and $m(n) := \min\{\lambda(n), \rho(n)\}$. We write $M(n) := m(n) + j$, where $j := M(n) - m(n)$. For $n \geq 1$, we have

$$\begin{aligned} |(C_\rho^* x)_n - (C_\lambda^* x)_n| &= \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(n),k} x_k \right| \\ &= \left| \sum_{k=0}^{\infty} c_{M(n),k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right| \\ &= \left| \sum_{k=0}^{\infty} c_{m(n)+j,k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right|. \end{aligned} \tag{4.10}$$

Then, as in the proof of Theorem 4.2, we have

$$\begin{aligned} |(C_\rho^* x)_n - (C_\lambda^* x)_n| &\leq O(1)\frac{j}{\sqrt{m(n)}} = O(1)\frac{M(n)-m(n)}{\sqrt{m(n)}} \\ &= O(1)\sqrt{\frac{\lambda(n)}{m(n)}}\,\frac{|\rho(n)-\lambda(n)|}{\sqrt{\lambda(n)}} \\ &= O(1)\cdot O(1)\cdot o(1) = o(1). \end{aligned} \tag{4.11}$$

Then if x is C_λ^*-summable to L,

$$\begin{aligned} 0 \leq |(C_\rho^* x)_n - L| &\leq |(C_\rho^* x)_n - (C_\lambda^* x)_n| + |(C_\lambda^* x)_n - L| \\ &= o(1) + o(1) = o(1). \end{aligned} \tag{4.12}$$

Similarly, if x is C_ρ^*-summable to L, then

$$\begin{aligned} 0 \le \left|(C_\lambda^* x)_n - L\right| &\le \left|(C_\rho^* x)_n - (C_\lambda^* x)_n\right| + \left|(C_\rho^* x)_n - L\right| \\ &= o(1) + o(1) = o(1). \end{aligned} \tag{4.13}$$

Thus, C_λ^* and C_ρ^* are equivalent for bounded sequences. □

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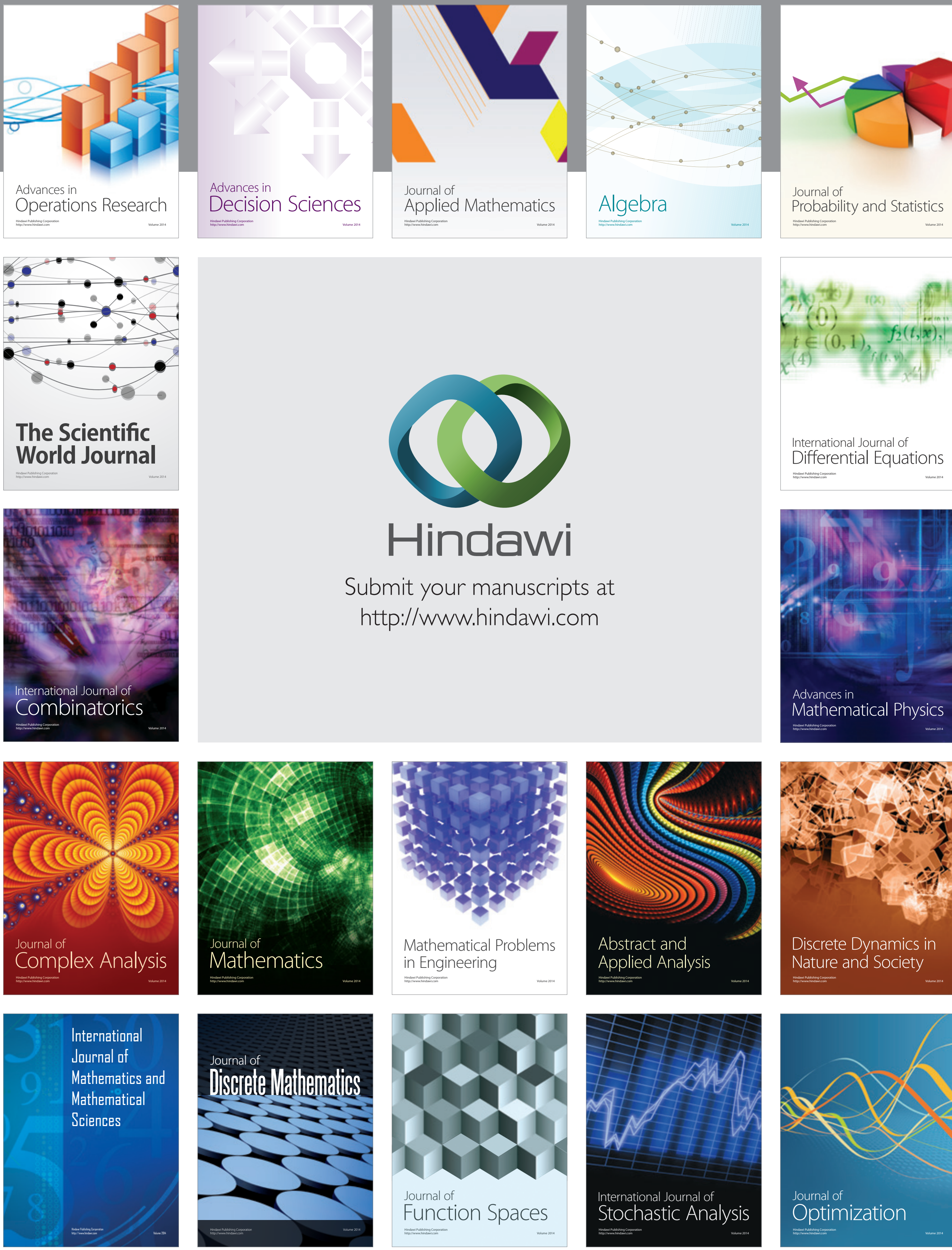
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