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Research Article

Some Generalized Pythagorean Fuzzy Bonferroni Mean Aggregation Operators with Their Application to Multiattribute Group Decision-Making

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The Pythagorean fuzzy set as an extension of the intuitionistic fuzzy set characterized by membership and nonmembership degrees has been introduced recently. Accordingly, the square sum of the membership and nonmembership degrees is a maximum of one. The Pythagorean fuzzy set has been previously applied to multiattribute group decision-making. This study develops a few aggregation operators for fusing the Pythagorean fuzzy information, and a novel approach to decision-making is introduced based on the proposed operators. First, we extend the generalized Bonferroni mean to the Pythagorean fuzzy environment and introduce the generalized Pythagorean fuzzy Bonferroni mean and the generalized Pythagorean fuzzy Bonferroni geometric mean. Second, a new generalization of the Bonferroni mean, namely, the dual generalized Bonferroni mean, is proposed by considering the shortcomings of the generalized Bonferroni mean. Furthermore, we investigate the dual generalized Bonferroni mean in the Pythagorean fuzzy sets and introduce the dual generalized Pythagorean fuzzy Bonferroni mean and dual generalized Pythagorean fuzzy Bonferroni geometric mean. Third, a novel approach to multiattribute group decision-making based on proposed operators is proposed. Lastly, a numerical instance is provided to illustrate the validity of the new approach.

1. Introduction

Decision-making is a common and significant activity in daily life. In the past decades, decision-making problems in real life have become increasingly complicated because of the increasing complexity in economic and social management. Given the fuzziness and vagueness in decision-making, crisp numbers are inadequate and insufficient for managing real decision-making problems. In 1965, Zadeh introduced the concept of fuzzy set (FS) [1], which is an effective tool in handling fuzziness and uncertainty. However, FS only has a membership degree, which is unsuitable in managing several real decision-making problems. Atanassov [2] introduced the intuitionistic fuzzy set (IFS), which simultaneously has membership and nonmembership degrees, due to the shortcomings of FS. A few achievements on IFS have been reported [3, 4]. Mao et al. [5] introduced a few new cross-entropy and entropy measures for IFSs and applied them to decision-making. Liu and Teng [6] introduced the normal intuitionistic fuzzy numbers and several new normal intuitionistic fuzzy aggregation operators and applied them to multiattribute group decision-making (MAGDM). Lakshmana et al. [7] proposed a total order on the entire class of intuitionistic fuzzy numbers using upper, lower dense sequence in the interval $[0,1]$. Lakshmana et al. [8] introduced a new principle for ordering trapezoidal intuitionistic fuzzy numbers. P. Liu and X. Liu [9] introduced the linguistic intuitionistic fuzzy set and a few linguistic intuitionistic fuzzy power Bonferroni mean (BM) aggregation operators by combining IFS and the linguistic terms set and applied them to MAGDM. Liu et al. [10] introduced the interval-valued intuitionistic fuzzy ordered weighted cosine similarity measure by combining the interval-valued intuitionistic fuzzy cosine similarity measure with the generalized ordered weighted averaging operator. Liu [11] used the Hamacher operations as basis to develop several new aggregation operators to fuse the interval-valued intuitionistic fuzzy information.

IFS is a powerful tool in decision-making. An extension of IFS, which is called the neutrosophic set [12], was introduced in 1999 to effectively address several real decision-making problems. In recent years, a few neutrosophic aggregation operators have been introduced [13–18]. A new extension of IFS, namely, the Pythagorean fuzzy set (PFS) [19], has been developed. The difference between PFS and IFS is that the square sum of the membership and nonmembership degrees is a maximum of one in PFS, whereas the sum of the membership and nonmembership degrees is a maximum of one in IFS. Several studies have been conducted on PFSs. Gou et al. [20] developed a few Pythagorean fuzzy functions and studied their fundamental properties. Zhang and Xu [21] introduced several operations for the Pythagorean fuzzy numbers (PFNs) and extended the technique for order preference by similarity to ideal solution (TOPSIS) method to solve MAGDM problems with Pythagorean fuzzy information. Several Pythagorean fuzzy aggregation operators have been introduced because aggregation operators are vital in decision-making [22]. Yager and Abbasov [23] introduced a few Pythagorean fuzzy aggregation operators, such as the Pythagorean fuzzy weighted averaging (PFWA) operator and Pythagorean fuzzy weighted geometric (PFWG) operator. Ma and Xu [24] introduced new score and accuracy functions of PFNs and developed the symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator and the symmetric Pythagorean fuzzy weighted geometric (SPFWG) operator. Zeng et al. [25] introduced the Pythagorean fuzzy ordered weighted averaging weighted average distance (PFOWAWAD) operator, from which a hybrid TOPSIS method was proposed for the Pythagorean fuzzy MAGDM problems. Garg [26] introduced the Pythagorean fuzzy Einstein operations and developed a few new Pythagorean fuzzy aggregation operators. Peng and Yuan [27] developed a series of Pythagorean fuzzy point operators. However, these aggregation operators cannot consider the correlations among PFNs. Therefore, Peng and Yang [28] developed several Choquet integral-based operators for the Pythagorean fuzzy information.

Several aggregation operators, such as the BM [29] and the Heronian mean (HM) [30], can capture the interrelationship between arguments. These operators have been successfully extended to IFSs [31–33] and hesitant FSs [34–36]. However, BM and HM can only consider the interrelationship between any two arguments. Beliakov et al. [37] introduced the generalized Bonferroni mean (GBM) to overcome the drawback of BM; GBM has also been extended to IFSs [38]. However, to the best of our knowledge, no research has been conducted on GBM in the Pythagorean fuzzy environment. Therefore, it is necessary to extend the GBM to the Pythagorean fuzzy environment. The shortcoming of GBM is that it can only consider the interrelationship among any three arguments. However, the correlations are ubiquitous among all arguments. To overcome the shortcoming of GBM, we introduce a few new extensions of BM, which can consider the interrelationship among all arguments. Therefore, the main objective of this study is to investigate GBM in PFSs. This research aims to develop several new GBM aggregation operators for PFNs and a new approach to MAGDM with Pythagorean fuzzy information.

The rest of this paper is organized as follows. Section 2 briefly reviews a few basic concepts. Section 3 extends GBM to PFSs and develops a few generalized Pythagorean fuzzy BM operators. Section 4 proposes and utilizes several new GBM operators to aggregate PFNs. Section 5 introduces a novel approach to MAGDM. Section 6 provides a numerical example to illustrate the approach. The final section summarizes this study.

2. Basic Concepts

This section reviews a few notions, such as IFS, PFS, and GBM.

2.1. IFS and PFS. In 1986, Atanassov [2] introduced IFS, which simultaneously has membership and nonmembership degrees.

Definition 1 (see [2]). Let X be an ordinary fixed set. An IFS A defined on X is expressed as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_A(x)$ and $\nu_A(x)$ represent the membership and nonmembership degrees, respectively, thereby satisfying $0 \le \mu_A(x) \le 1$, $0 \le \nu_A(x) \le 1$, and $\mu_A(x) + \nu_A(x) \le 1$. For convenience, the pair $\langle \mu, \nu\rangle$ is called an intuitionistic fuzzy number (IFN) [39], in which $\mu \in [0,1]$, $\nu \in [0,1]$, and $\mu + \nu \le 1$. The hesitancy degree is denoted by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

In 2014, Yager [19] introduced PFS, which is a generalization of IFS.

Definition 2 (see [19]). Let X be an ordinary fixed set; a PFS P defined on X is expressed as follows:

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle \mid x \in X\}, \tag{2}$$

where $\mu_P(x)$ and $\nu_P(x)$ are the membership and nonmembership degrees, respectively, thereby satisfying $\mu_P(x), \nu_P(x) \in [0,1]$ and $(\mu_P(x))^2 + (\nu_P(x))^2 \le 1$. Thereafter, the indeterminacy degree is expressed by $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_A(x))^2}$. Zhang and Xu [21] called the pair $(\mu_P(x), \nu_p(x))$ a PFN, which can be denoted by $P = (\mu_P, \nu_P)$.

Peng and Yuan [27] introduced the comparison law for PFNs to compare two PFNs.

Definition 3 (see [27]). For any PFN $p = (\mu, \nu)$, the score function of p is defined as $s(p) = \mu^2 - \nu^2$. For any two PFNs, such as $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$, if $s(p_1) > s(p_2)$, then $p_1 > p_2$; if $s(p_1) = s(p_2)$, then $p_1 = p_2$.

Zhang and Xu [21] introduced a few operations for PFNs.

Definition 4 (see [21]). Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be any three PFNs and λ be a positive real number. Thereafter,

(1) $p_1 \oplus p_2 = (\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2)$,

(2) $p_1 \otimes p_2 = (\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2})$,

(3) $\lambda p = (\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda)$, and

(4) $p^\lambda = (\mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda})$.

2.2. GBM. Beliakov et al. [37] introduced GBM, which can consider the correlations of any three aggregated arguments because the traditional BM can only determine the interrelationship between any two arguments. Nevertheless, Xia et al. [38] highlighted that the GBM introduced by Beliakov et al. [37] has a drawback. Therefore, Xia et al. [38] introduced a new form of GBM. In the new GBM, the weights of the arguments are also considered.

Definition 5 (see [38]). Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative crisp numbers with the weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^n w_i = 1$. The generalized weighted BM (GWBM) is defined as follows:

$$\mathrm{GWBM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \left(\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p a_j^q a_k^r\right)^{1/(p+q+r)}. \tag{3}$$

Xia et al. [38] also introduced the generalized weighted Bonferroni geometric mean (GWBGM).

Definition 6 (see [38]). Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative crisp numbers with the weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^n w_i = 1$. If

$$\mathrm{GWBGM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r}\prod_{i,j,k=1}^{n}\left(pa_i + qa_j + ra_k\right)^{w_i w_j w_k}, \tag{4}$$

then $\mathrm{GWBGM}^{p,q,r}$ is called GWBGM.

3. The Generalized Pythagorean Fuzzy Weighted Bonferroni Mean

This section extends GWBM and GWBGM to fuse the Pythagorean fuzzy information and proposes several new Pythagorean fuzzy aggregation operators.

Definition 7. Let $s, t, r > 0$ and $p_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs with their weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. If

$$\mathrm{GPFWBM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \left(\bigoplus_{i,j,k=1}^{n} w_i w_j w_k \left(p_i^s \otimes p_j^t \otimes p_k^r\right)\right)^{1/(s+t+r)}, \tag{5}$$

then $\mathrm{GPFWBM}^{s,t,r}$ is called the generalized Pythagorean fuzzy weighted Bonferroni mean (GPFWBM).

We can obtain the following theorem according to Definition 4.

Theorem 8. *Let $s, t, r > 0$ and $p_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs. The aggregated value by GPFWBM is also a PFN and*

$$GPFWBM^{s,t,r}(p_1, p_2, \ldots, p_n) = \left(\left(\sqrt{1 - \prod_{i,j,k=1}^{n}\left(1 - \mu_i^{2s}\mu_j^{2t}\mu_k^{2r}\right)^{w_i w_j w_k}}\right)^{1/(s+t+r)}, \sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n}\left(1 - (1 - \nu_i^2)^s\left(1 - \nu_j^2\right)^t\left(1 - \nu_k^2\right)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}}\right). \tag{6}$$

Proof. According to Definition 4, we can obtain

$$p_i^s = \left(\mu_i^s, \sqrt{1 - \left(1 - \nu_i^2\right)^s}\right),$$
$$p_j^t = \left(\mu_j^t, \sqrt{1 - \left(1 - \nu_j^2\right)^t}\right),$$
$$p_k^r = \left(\mu_k^r, \sqrt{1 - \left(1 - \nu_k^2\right)^r}\right). \tag{7}$$

Thus,

$$p_i^s \otimes p_j^t \otimes p_k^r$$

$$= \left(\mu_i^s \mu_j^t \mu_k^r, \sqrt{1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r} \right). \tag{8}$$

Thereafter,

$$w_i w_j w_k \left(p_i^s \otimes p_j^t \otimes p_k^r \right) = \left(\sqrt{1 - \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}}, \left(\sqrt{1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r} \right)^{w_i w_j w_k} \right). \tag{9}$$

Furthermore,

$$\bigoplus_{i,j,k=1}^{n} w_i w_j w_k \left(p_i^s \otimes p_j^t \otimes p_k^r \right) = \left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}}, \left(\sqrt{\prod_{i,j,k=1}^{n} \left(1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r\right)} \right)^{w_i w_j w_k} \right). \tag{10}$$

Therefore,

$$\mathrm{GPFWBM}^{s,t,r}\left(p_1, p_2, \ldots, p_n\right) = \left(\left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}} \right)^{1/(s+t+r)}, \sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}} \right). \tag{11}$$

Hence, (6) is maintained.

Thereafter,

$$0 \le \left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}} \right)^{1/(s+t+r)} \le 1, \quad 0 \le \sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}} \le 1. \tag{12}$$

Thereafter,

$$\left(\left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}} \right)^{1/(s+t+r)} \right)^2 + \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}} \right)^2$$
$$= \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \mu_i^{2s} \mu_j^{2t} \mu_k^{2r}\right)^{w_i w_j w_k}\right)^{1/(s+t+r)} + 1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - v_i^2\right)^s \left(1 - v_j^2\right)^t \left(1 - v_k^2\right)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)} \le 1$$

$$+\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\nu_i^2\right)^s\left(1-\nu_j^2\right)^t\left(1-\nu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)}$$
$$-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\nu_i^2\right)^s\left(1-\nu_j^2\right)^t\left(1-\nu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)}=1, \quad (13)$$

thereby completing the proof. □

Moreover, GPFWBM has the following properties.

Theorem 9 (idempotency). *If p_i $(i=1,2,\ldots,n)$ are equal, that is, $p_i=p=(\mu,\nu)$, then*

$$GPFWBM^{s,t,r}\left(p_1,p_2,\ldots,p_n\right)=p. \quad (14)$$

Proof.

$$\begin{aligned}&\mathrm{GPFWBM}^{s,t,r}\left(p_1,p_2,\ldots,p_n\right)\\&=\left(\bigoplus_{i,j,k=1}^{n}w_iw_jw_k\left(p_i^s\otimes p_j^t\otimes p_k^r\right)\right)^{1/(s+t+r)}\\&=\left(\sum_{i,j,k=1}^{n}w_iw_jw_kp\right)^{1/(s+t+r)}=\sum_{i=1}^{n}w_i\sum_{j=1}^{n}w_j\sum_{k=1}^{n}w_kp\\&=p.\end{aligned} \quad (15)$$

□

Theorem 10 (monotonicity). *Let $p_i=(\mu_{p_i},\nu_{p_i})$ $(i=1,2,\ldots,n)$ and $q_i=(\mu_{q_i},\nu_{q_i})$ $(i=1,2,\ldots,n)$ be two collections of PFNs. If $\mu_{p_i}\le\mu_{q_i}$ and $\nu_{p_i}\ge\nu_{q_i}$ holds for all i, then*

$$\begin{aligned}&GPFWBM^{s,t,r}\left(p_1,p_2,\ldots,p_n\right)\\&\le GPFWBM^{s,t,r}\left(q_1,q_2,\ldots,q_n\right).\end{aligned} \quad (16)$$

Proof. Let $\mathrm{GPFWBM}^{s,t,r}(p_1,p_2,\ldots,p_n)=(\mu_p,\nu_p)$ and $\mathrm{GPFWBM}^{s,t,r}(q_1,q_2,\ldots,q_n)=(\mu_q,\nu_q)$. Given that $\mu_{p_i}\le\mu_{q_i}$, we can obtain

$$\begin{aligned}&\mu_{p_i}^{2s}\mu_{p_j}^{2t}\mu_{p_k}^{2r}\le\mu_{q_i}^{2s}\mu_{q_j}^{2t}\mu_{q_k}^{2r},\\&\left(1-\mu_{p_i}^{2s}\mu_{p_j}^{2t}\mu_{p_k}^{2r}\right)^{w_iw_jw_k}\ge\left(1-\mu_{q_i}^{2s}\mu_{q_j}^{2t}\mu_{q_k}^{2r}\right)^{w_iw_jw_k},\\&1-\prod_{i,j,k=1}^{n}\left(1-\mu_{p_i}^{2s}\mu_{p_j}^{2t}\mu_{p_k}^{2r}\right)^{w_iw_jw_k}\\&\quad\le1-\prod_{i,j,k=1}^{n}\left(1-\mu_{q_i}^{2s}\mu_{q_j}^{2t}\mu_{q_k}^{2r}\right)^{w_iw_jw_k}.\end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned}&\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\mu_{p_i}^{2s}\mu_{p_j}^{2t}\mu_{p_k}^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)}\\&\le\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\mu_{q_i}^{2s}\mu_{q_j}^{2t}\mu_{q_k}^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)}.\end{aligned} \quad (18)$$

Thus,

$$\begin{aligned}&\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\mu_{p_i}^{2s}\mu_{p_j}^{2t}\mu_{p_k}^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)}\right)^2\\&\le\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\mu_{q_i}^{2s}\mu_{q_j}^{2t}\mu_{q_k}^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)}\right)^2,\end{aligned} \quad (19)$$

which means $\mu_p^2\le\mu_q^2$. Similarly, we can obtain $\nu_p^2\ge\nu_q^2$.

If $\mu_p^2<\mu_q^2$ because $\nu_p^2\ge\nu_q^2$, then $\mathrm{GPFWBM}^{s,t,r}(p_1,p_2,\ldots,p_n)<\mathrm{GPFWBM}^{s,t,r}(q_1,q_2,\ldots,q_n)$;

If $\mu_p^2=\mu_q^2$ and $\nu_p^2>\nu_q^2$, then $\mathrm{GPFWBM}^{s,t,r}(p_1,p_2,\ldots,p_n)<\mathrm{GPFWBM}^{s,t,r}(q_1,q_2,\ldots,q_n)$;

If $\mu_p^2=\mu_q^2$ and $\nu_p^2=\nu_q^2$, then $\mathrm{GPFWBM}^{s,t,r}(p_1,p_2,\ldots,p_n)=\mathrm{GPFWBM}^{s,t,r}(q_1,q_2,\ldots,q_n)$.

Therefore, the proof of Theorem 10 is completed. □

Theorem 11 (boundedness). *Let $p_i=(\mu_i,\nu_i)$ $(i=1,2,\ldots,n)$ be a collection of PFNs. If $p^+=(\max_i(\mu_i),\min_i(\nu_i))$ and $p^-=(\min_i(\mu_i),\max_i(\nu_i))$, then*

$$p^-\le GPFWBM^{s,t,r}\left(p_1,p_2,\ldots,p_n\right)\le p^+. \quad (20)$$

Proof. From Theorem 9, we can obtain

$$\begin{aligned}&\mathrm{GPFWBM}^{s,t,r}\left(p^-,p^-,\ldots,p^-\right)=p^-,\\&\mathrm{GPFWBM}^{s,t,r}\left(p^+,p^+,\ldots,p^+\right)=p^+.\end{aligned} \quad (21)$$

From Theorem 10, we can obtain

$$\begin{aligned}&\mathrm{GPFWBM}^{s,t,r}\left(p^-,p^-,\ldots,p^-\right)\\&\le\mathrm{GPFWBM}^{s,t,r}\left(p_1,p_2,\ldots,p_n\right)\\&\le\mathrm{GPFWBM}^{s,t,r}\left(p^+,p^+,\ldots,p^+\right).\end{aligned} \quad (22)$$

Therefore, $p^{-} \leq \mathrm{GPFWBM}^{s,t,r}(p_1, p_2, \ldots, p_n) \leq p^{+}$.

Thereafter, we extend GWBGM to PFSs and introduce the generalized Pythagorean fuzzy weighted Bonferroni geometric mean (GPFWBGM). □

Definition 12. Let $s,t,r > 0$ and $p_i = (\mu_i, \nu_i)$ $(i = 1,2,\ldots,n)$ be a collection of PFNs with their weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^{n} \left(sp_i \oplus tp_j \oplus rp_k\right)^{w_i w_j w_k}, \tag{23}$$

then $\mathrm{GPFWBGM}^{s,t,r}$ is called GPFWBGM.

We can obtain the following theorem based on Definition 4.

Theorem 13. *Let $s,t,r > 0$ and $p_i = (\mu_i, \nu_i)$ $(i = 1,2,\ldots,n)$ be a collection of PFNs. The aggregated value by GPFWBGM is also a PFN and*

$$GPFWBGM^{s,t,r}(p_1, p_2, \ldots, p_n) = \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - (1-\mu_i^2)^s \left(1-\mu_j^2\right)^t (1-\mu_k^2)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}}, \right.$$
$$\left. \left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \nu_i^{2s} \nu_j^{2t} \nu_k^{2r}\right)^{w_i w_j w_k}}\right)^{1/(s+t+r)} \right). \tag{24}$$

Proof. Through Definition 4, we can obtain

$$sp_i = \left(\sqrt{1 - (1-\mu_i^2)^s}, \nu_i^s\right),$$
$$tp_j = \left(\sqrt{1 - \left(1-\mu_j^2\right)^t}, \nu_j^t\right), \tag{25}$$
$$rp_k = \left(\sqrt{1 - (1-\mu_k^2)^r}, \nu_k^r\right),$$

$$sp_i \oplus tp_j \oplus rp_k = \left(\sqrt{1 - (1-\mu_i^2)^s \left(1-\mu_j^2\right)^t (1-\mu_k^2)^r}, \nu_i^s \nu_j^t \nu_k^r\right). \tag{26}$$

Thereafter,

$$\left(sp_i \oplus tp_j \oplus rp_k\right)^{w_i w_j w_k} = \left(\left(\sqrt{1 - (1-\mu_i^2)^s \left(1-\mu_j^2\right)^t (1-\mu_k^2)^r}\right)^{w_i w_j w_k}, \sqrt{1 - \left(1 - \nu_i^{2s} \nu_j^{2t} \nu_k^{2r}\right)^{w_i w_j w_k}}\right). \tag{27}$$

Therefore,

$$\bigotimes_{i,j,k=1}^{n} \left(sp_i \oplus tp_j \oplus rp_k\right)^{w_i w_j w_k} = \left(\left(\sqrt{\prod_{i,j,k=1}^{n} \left(1 - (1-\mu_i^2)^s \left(1-\mu_j^2\right)^t (1-\mu_k^2)^r\right)}\right)^{w_i w_j w_k}, \sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \nu_i^{2s} \nu_j^{2t} \nu_k^{2r}\right)^{w_i w_j w_k}}\right); \tag{28}$$

thus,

$$\mathrm{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^{n} \left(sp_i \oplus tp_j \oplus rp_k\right)^{w_i w_j w_k}$$
$$= \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - (1-\mu_i^2)^s \left(1-\mu_j^2\right)^t (1-\mu_k^2)^r\right)^{w_i w_j w_k}\right)^{1/(s+t+r)}}, \right.$$
$$\left. \left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \nu_i^{2s} \nu_j^{2t} \nu_k^{2r}\right)^{w_i w_j w_k}}\right)^{1/(s+t+r)} \right). \tag{29}$$

Hence, (24) is maintained.

Thereafter,

$$0 \leq \sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\mu_i^2\right)^s\left(1-\mu_j^2\right)^t\left(1-\mu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)}} \leq 1, \tag{30}$$

$$0 \leq \left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\nu_i^{2s}\nu_j^{2t}\nu_k^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)} \leq 1. \tag{31}$$

Therefore,

$$\begin{aligned}
&\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\mu_i^2\right)^s\left(1-\mu_j^2\right)^t\left(1-\mu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)}}\right)^2 \\
&+\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\nu_i^{2s}\nu_j^{2t}\nu_k^{2r}\right)^{w_iw_jw_k}}\right)^{1/(s+t+r)}\right)^2 = 1 \\
&-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\mu_i^2\right)^s\left(1-\mu_j^2\right)^t\left(1-\mu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)} + \left(1-\prod_{i,j,k=1}^{n}\left(1-\nu_i^{2s}\nu_j^{2t}\nu_k^{2r}\right)^{w_iw_jw_k}\right)^{1/(s+t+r)} \leq 1 \\
&-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\mu_i^2\right)^s\left(1-\mu_j^2\right)^t\left(1-\mu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)} \\
&+\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\mu_i^2\right)^s\left(1-\mu_j^2\right)^t\left(1-\mu_k^2\right)^r\right)^{w_iw_jw_k}\right)^{1/(s+t+r)} = 1,
\end{aligned} \tag{32}$$

thereby completing the proof. □

Similar to GPFWBM, the GPFWBGM has the same properties. The proofs of these properties are similar to that of the properties of GPFWBM. Accordingly, the proofs are omitted to save space.

Theorem 14. *Let $s,t,r > 0$ and $p_i = (\mu_i, \nu_i)$ $(i = 1,2,\ldots,n)$ be a collection of PFNs.*

(1) Idempotency. If p_i $(i = 1,2,\ldots,n)$ are equal, that is, $p_i = p = (\mu, \nu)$, then

$$\text{GPFWBGM}^{s,t,r}\left(p_1, p_2, \ldots, p_n\right) = p. \tag{33}$$

(2) Monotonicity. Let $q_i = (\mu_{q_i}, \nu_{q_i})$ $(i = 1,2,\ldots,n)$ be two collections of PFNs. If $\mu_{p_i} \leq \mu_{q_i}$ and $\nu_{p_i} \geq \nu_{q_i}$ holds for all i, then

$$\begin{aligned}&\text{GPFWBGM}^{s,t,r}\left(p_1, p_2, \ldots, p_n\right)\\ &\quad\leq \text{GPFWBGM}^{s,t,r}\left(q_1, q_2, \ldots, q_n\right).\end{aligned} \tag{34}$$

(3) Boundedness. If $p^+ = (\max_i(\mu_i), \min_i(\nu_i))$ and $p^- = (\min_i(\mu_i), \max_i(\nu_i))$, then

$$p^- \leq \text{GPFWBGM}^{s,t,r}\left(p_1, p_2, \ldots, p_n\right) \leq p^+. \tag{35}$$

4. Dual Generalized Pythagorean Fuzzy Weighted BM

The primary advantage of BM is that it can determine the interrelationship between arguments. However, the traditional BM can only consider the correlations of any two aggregated arguments. Thereafter, Beliakov et al. [37] extended the traditional BM and introduced GBM, which can determine the correlations between any three aggregated arguments. Xia et al. [38] introduced GBWM and GBWGM given that the GBM introduced by Beliakov et al. [37] still has a few drawbacks. However, GBWM and GBWGM can only consider the interrelationship between any three aggregated arguments. We introduce a new generalization of the traditional BM because the correlations are ubiquitous among all arguments. The new generalization of the traditional BM is called the dual GBM (DGBM) to distinguish the new aggregation operator from the GBM introduced by Beliakov et al. [37] and Xia et al. [38]. Furthermore, we develop the dual generalized weighted BM (DGWBM) and dual generalized weighted Bonferroni geometric mean (DGWBGM) to consider the weights of the arguments.

Definition 15. Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative crisp numbers with the weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{DGWBM}^{R}(a_1, a_2, \ldots, a_n) = \left(\sum_{i_1, i_2, \ldots, i_n = 1}^{n} \left(\prod_{j=1}^{n} w_{i_j} a_{i_j}^{r_j} \right) \right)^{1/\sum_{j=1}^{n} r_j}, \tag{36}$$

where $R = (r_1, r_2, \ldots, r_n)^T$ is the parameter vector with $r_i \geq 0$ $(i = 1, 2, \ldots, n)$, then DGWBM^R is called DGWBM.

Several special cases can be obtained given the change of the parameter vector.

(1) If $R = (\lambda, 0, 0, \ldots, 0)$, then we obtain

$$\mathrm{DGWBM}^{(\lambda,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_i^{\lambda} \right)^{1/\lambda}, \tag{37}$$

which is the generalized weighted averaging operator.

(2) If $R = (s, t, 0, 0, \ldots, 0)$, then we obtain

$$\mathrm{DGWBM}^{(s,t,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \left(\sum_{i,j=1}^{n} w_i w_j a_i^{s} a_j^{t} \right)^{1/(s+t)}, \tag{38}$$

which is the weighted BM.

(3) If $R = (s, t, r, 0, 0, \ldots, 0)$, then we obtain

$$\mathrm{DGWBM}^{(s,t,r,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \left(\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^{s} a_j^{t} a_k^{r} \right)^{1/(s+t+k)}, \tag{39}$$

which is the GWBM.

Definition 16. Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative crisp numbers with the weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{DGWBGM}^{R}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} r_j} \left(\prod_{i_1, i_2, \ldots, i_n = 1}^{n} \left(\sum_{j=1}^{n} \left(r_j p_{i_j} \right) \right)^{\prod_{j=1}^{n} w_{i_j}} \right), \tag{40}$$

where $R = (r_1, r_2, \ldots, r_n)^T$ is the parameter vector with $r_i \geq 0$ $(i = 1, 2, \ldots, n)$, then DGWBGM^R is called DGWBGM.

Similar to the DGWBM, we can consider some special cases given the change of the parameter vector.

(1) If $R = (\lambda, 0, 0, \ldots, 0)$, then we obtain

$$\mathrm{DGWBGM}^{(\lambda,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{\lambda} \left(\prod_{i=1}^{n} \left(\lambda a_i \right)^{w_i} \right), \tag{41}$$

which is the generalized weighted geometric averaging operator.

(2) If $R = (s, t, 0, 0, \ldots, 0)$, then we obtain

$$\mathrm{DGWBGM}^{(s,t,0,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{s+t} \prod_{i,j=1}^{n} \left(s a_i + t a_j \right)^{w_i w_j}, \tag{42}$$

which is the weighted Bonferroni geometric mean.

(3) If $R = (s, t, r, 0, 0, \ldots, 0)$, then

$$\mathrm{DGWBGM}^{(s,t,r,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{s+t+r} \prod_{i,j,k=1}^{n} \left(s a_i + t a_j + r a_k \right)^{w_i w_j w_k}, \tag{43}$$

which is the GWBGM.

We extend DGWBM and DGWBGM to PFSs, as well as introduce several new aggregation operators for fusing the Pythagorean fuzzy information.

Definition 17. Let $p_i = (\mu_i, v_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs with their weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Thereafter, the dual generalized Pythagorean fuzzy weighted Bonferroni mean (DGPFWBM) is defined as

$$\mathrm{DGPFWBM}^{R}(p_1, p_2, \ldots, p_n) = \left(\bigoplus_{i_1, i_2, \ldots, i_n = 1}^{n} \left(\bigotimes_{j=1}^{n} w_{i_j} p_{i_j}^{r_j} \right) \right)^{1/\sum_{i=1}^{n} r_j}, \tag{44}$$

where $R = (r_1, r_2, \ldots, r_n)^T$ is the parameter vector with $r_i \geq 0$ $(i = 1, 2, \ldots, n)$.

We can derive the following theorem based on Definition 4.

Theorem 18. *Let $p_i = (\mu_i, v_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs. Hence, the aggregated value by DGPFWBM is also PFN and*

$$\mathit{DGPFWBM}^{R}(p_1, p_2, \ldots, p_n) = \left(\left(\sqrt{1 - \prod_{i_1, i_2, \ldots, i_n = 1}^{n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{i_j}^{2r_j} \right)^{w_{i_j}} \right) \right)} \right)^{1/\sum_{j=1}^{n} r_j}, \sqrt{1 - \left(1 - \prod_{i_1, i_2, \ldots, i_n = 1}^{n} \left(1 - \left(1 - \left(1 - v_{i_j}^{2} \right)^{r_j} \right)^{w_{i_j}} \right) \right)^{1/\sum_{j=1}^{n} r_j}} \right). \tag{45}$$

Proof. Through Definition 4, we obtain

$$p_{i_j}^{r_j} = \left(\mu_{i_j}^{r_j}, \sqrt{1-\left(1-\nu_{i_j}^2\right)^{r_j}}\right),$$
$$w_{i_j} p_{i_j}^{r_j} = \left(\sqrt{1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}}, \left(\sqrt{1-\left(1-\nu_{i_j}^2\right)^{r_j}}\right)^{w_{i_j}}\right). \quad (46)$$

Therefore,

$$\bigotimes_{j=1}^{n} w_{i_j} p_{i_j}^{r_j} = \left(\prod_{j=1}^{n}\sqrt{1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)}\right). \quad (47)$$

Thus,

$$\bigoplus_{i_1,i_2,\ldots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n} w_{i_j} p_{i_j}^{r_j}\right) = \left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)}, \prod_{i_1,i_2,\ldots,i_n=1}^{n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)}\right). \quad (48)$$

Therefore,

$$\left(\bigoplus_{i_1,i_2,\ldots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n} w_{i_j} p_{i_j}^{r_j}\right)\right)^{1/\sum_{i=1}^{n} r_j} = \left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j}, \sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right). \quad (49)$$

Thus, (45) is maintained.

Thereafter,

$$0 \le \left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j} \le 1,$$
$$0 \le \sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}} \le 1. \quad (50)$$

In addition,

$$\left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j}\right)^2 + \left(\sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right)^2 = \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{j=1}^{n} p_j} + 1 - \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j} \le \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1 - \left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j} + 1 - \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j} = 1, \quad (51)$$

thereby completing the proof. □

Moreover, DGPFWBM has the following properties.

Theorem 19 (monotonicity). *Let* $p_i = (\mu_{p_i}, \nu_{p_i})$ $(i = 1, 2, \ldots, n)$ *and* $q_i = (\mu_{q_i}, \nu_{q_i})$ $(i = 1, 2, \ldots, n)$ *be two collections of PFNs. If* $\mu_{p_i} \le \mu_{q_i}$ *and* $\nu_{p_i} \ge \nu_{q_i}$ *holds for all i, then*

$$DGPFWBM^{R}(p_1, p_2, \ldots, p_n) \le DGPFWBM^{R}(q_1, q_2, \ldots, q_n). \quad (52)$$

Proof. Let $DGPFWBM^R(p_1, p_2, \ldots, p_n) = (\mu_p, \nu_q)$ and $DGPFWBM^R(q_1, q_2, \ldots, q_n) = (\mu_q, \nu_q)$.
Given that $\mu_{p_i} \le \mu_{q_i}$, we obtain

$$\left(1-\mu_{p_{i_i}}^{2r_j}\right)^{w_{i_j}} \ge \left(1-\mu_{q_{i_i}}^{2r_j}\right)^{w_{i_j}},$$

$$1-\left(1-\mu_{p_{i_j}}^{2r_j}\right)^{w_{i_j}} \le 1-\left(1-\mu_{q_{i_j}}^{2r_j}\right)^{w_{i_j}},$$

$$\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{p_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right) \ge \prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{q_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right). \quad (53)$$

Therefore,

$$\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{p_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j} \le \left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{q_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j},$$

$$\left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{p_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j}\right)^2 \quad (54)$$

$$\le \left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{q_{i_j}}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j}\right)^2;$$

thus, $\mu_p^2 \le \mu_q^2$. Similarly, we can obtain $\nu_p^2 \ge \nu_q^2$.

If $\mu_p^2 < \mu_q^2$ because $\nu_p^2 \ge \nu_q^2$, then

$$DGPFWBM^R(p_1, p_2, \ldots, p_n) < DGPFWBM^R(q_1, q_2, \ldots, q_n); \quad (55)$$

If $\mu_p^2 = \mu_q^2$ and $\nu_p^2 > \nu_q^2$, then $DGPFWBM^R(p_1, p_2, \ldots, p_n) < DGPFWBM^R(q_1, q_2, \ldots, q_n)$;

If $\mu_p^2 = \mu_q^2$ and $\nu_p^2 = \nu_q^2$, then $DGPFWBM^R(p_1, p_2, \ldots, p_n) = DGPFWBM^R(q_1, q_2, \ldots, q_n)$.

Therefore, $DGPFWBM^R(p_1, p_2, \ldots, p_n) \le DGPFWBM^R(q_1, q_2, \ldots, q_n)$ and the proof of Theorem 19 is completed. □

Theorem 20 (boundedness). *Let* $p_i = (\mu_{p_i}, \nu_{p_i})$ $(i = 1, 2, \ldots, n)$ *be a collection of PFNs. If* $p^+ = (\max_i(\mu_i), \min_i(\nu_i))$ *and* $p^- = (\min_i(\mu_i), \max_i(\nu_i))$, *then*

$$DGPFWBM^R(p^-, p^-, \ldots, p^-) \le DGPFWBM^R(p_1, p_2, \ldots, p_n) \le DGPFWBM^R(p^+, p^+, \ldots, p^+). \quad (56)$$

Proof. According to Theorem 18, we can obtain

$$DGPFWBM^R(p^-, p^-, \ldots, p^-) = \left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\min \mu_{i_j}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j},\right.$$

$$\left.\sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\left(\max \nu_{i_j}\right)^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right).$$

$$DGPFWBM^R(p_1, p_2, \ldots, p_n) = \left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{i_j}^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j},\right.$$

$$\left.\sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\nu_{i_j}^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right).$$

$$DGPFWBM^R\left(p^+,p^+,\ldots,p^+\right)=\left(\left(\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\max\mu_{i_j}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)}\right)\right)^{1/\sum_{j=1}^{n} r_j},\right.$$

$$\left.\sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\left(1-\left(1-\left(\max\nu_{i_j}\right)^2\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right). \tag{57}$$

According to Theorem 19, we can obtain

$$\begin{aligned} &DGPFWBM^R\left(p^-,p^-,\ldots,p^-\right)\\ &\quad\le DGPFWBM^R\left(p_1,p_2,\ldots,p_n\right)\\ &\quad\le DGPFWBM^R\left(p^+,p^+,\ldots,p^+\right). \end{aligned} \tag{58}$$

Evidently, the DGPFWBM operator lacks the property of idempotency. □

We extend DGBWGM to PFSs and introduce the dual generalized Pythagorean fuzzy weighted Bonferroni geometric mean (DGPFWBGM) operator.

Definition 21. Let $p_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs with their weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\begin{aligned} &\mathrm{DGPFWBGM}^R\left(p_1,p_2,\ldots,p_n\right)\\ &\quad=\frac{1}{\sum_{j=1}^{n} r_j}\left(\bigotimes_{i_1,i_2,\ldots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n}\left(r_j p_{i_j}\right)\right)^{\prod_{j=1}^{n} w_{i_j}}\right), \end{aligned} \tag{59}$$

where $R = (r_1, r_2, \ldots, r_n)^T$ is the parameter vector with $r_i \ge 0$ $(i = 1, 2, \ldots, n)$; then $DGPFWBGM^R$ is called the DGPFWBGM

We obtain the following theorem based on Definition 4.

Theorem 22. *Let $p_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs. The aggregated value by the DGPFWBGM operator is also PFN and*

$$\begin{aligned} &DGPFWBGM^R\left(p_1,p_2,\ldots,p_n\right)\\ &=\left(\sqrt{1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{i_j}^2\right)^{r_j}\right)^{\prod_{j=1}^{n} w_{i_j}}\right)^{1/\sum_{j=1}^{n} r_j}},\left(\sqrt{1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\nu_{i_j}^{2r_j}\right)^{\prod_{j=1}^{n} w_{i_j}}}\right)^{1/\sum_{j=1}^{n} r_j}\right). \end{aligned} \tag{60}$$

The proof of Theorem 22 is similar to that of Theorem 18; thus, such proof is omitted to save space.

Similar to DGPFWBM, we can obtain the following properties of DGPFWBGM. The proofs of these properties are likewise omitted to save space.

Theorem 23. *Let $p_i = (\mu_{p_i}, \nu_{p_i})$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs.*

(1) Monotonicity. Let $q_i = (\mu_{q_i}, \nu_{q_i})$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs. If $\mu_{p_i} \le \mu_{q_i}$ and $\nu_{p_i} \ge \nu_{q_i}$ holds for all i, then

$$\begin{aligned} &DGPFWBGM^R\left(p_1,p_2,\ldots,p_n\right)\\ &\quad\le DGPFWBGM^R\left(q_1,q_2,\ldots,q_n\right). \end{aligned} \tag{61}$$

(2) Boundedness. If $p^+ = (\max_i(\mu_i), \min_i(\nu_i))$ and $p^- = (\min_i(\mu_i), \max_i(\nu_i))$, then

$$\begin{aligned} &DGPFWBGM^R\left(p^-,p^-,\ldots,p^-\right)\\ &\quad\le DGPFWBGM^R\left(p_1,p_2,\ldots,p_n\right)\\ &\quad\le DGPFWBGM^R\left(p^+,p^+,\ldots,p^+\right). \end{aligned} \tag{62}$$

5. Novel Approach to MAGDM with Pythagorean Fuzzy Information

This section introduces a novel approach to MAGDM under the Pythagorean fuzzy environment. A typical MAGDM problem with the Pythagorean fuzzy information can be described as follows. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of alternatives and $G = \{G_1, G_2, \ldots, G_n\}$ be a set of attributes with the weight vector being $w = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Several decision makers are organized to decide over alternatives. For the attribute G_j $(j = 1, 2, \ldots, n)$ of alternative x_i $(i = 1, 2, \ldots, m)$, the decision makers are required to use PFNs to express their preference information, which can be denoted as $p_{ij} = (\mu_{ij}, \nu_{ij})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. Therefore, a Pythagorean fuzzy decision matrix can be obtained by $P = (p_{ij})_{m\times n}$. A novel approach based on the dual generalized Pythagorean fuzzy BM aggregation operators is introduced to solve this problem.

Step 1. The two types of attributes are benefit and cost attributes. Xu and Hu [40] introduced the normalization regulation for intuitionistic fuzzy decision matrix, which can be extended to the Pythagorean fuzzy decision matrix. Therefore, the decision matrix should be normalized by

$$p_{ij} = \begin{cases} \left(\mu_{ij}, \nu_{ij}\right), & G_j \in I_1, \\ \left(\nu_{ij}, \mu_{ij}\right), & G_j \in I_2, \end{cases} \tag{63}$$

where I_1 and I_2 represent the benefit and cost attributes, respectively. Thereafter, a normalized decision matrix can be obtained.

Step 2. For the alternative x_i $(i = 1, 2, \ldots, m)$, we utilize the DGPFWBM operator

$$p_i = \mathrm{DGPFWBM}^R\left(p_{i1}, p_{i2}, \ldots, p_{in}\right) = \left(\left(\sqrt{1 - \prod_{i_{l1}, i_{l2}, \ldots, i_{ln}=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \mu_{i_{lj}}^{2r_j}\right)^{w_{i_{lj}}}\right)\right)}\right)^{1/\sum_{j=1}^{n} r_j}, \sqrt{1 - \left(1 - \prod_{i_{l1}, i_{l2}, \ldots, i_{ln}=1}^{n}\left(1 - \left(1 - \left(1 - \nu_{i_{lj}}^{2}\right)^{r_j}\right)^{w_{i_{lj}}}\right)\right)^{1/\sum_{j=1}^{n} r_j}}\right) \tag{64}$$

or the DGPFWBGM operator

$$p_i = \mathrm{DGPFWBGM}^R\left(p_{i1}, p_{i2}, \ldots, p_{in}\right) = \left(\sqrt{1 - \left(1 - \prod_{i_{l1}, i_{l2}, \ldots, i_{ln}=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \mu_{i_{lj}}^{2}\right)^{r_j}\right)^{\prod_{j=1}^{n} w_{i_{lj}}}\right)^{1/\sum_{j=1}^{n} r_j}}, \left(\sqrt{1 - \prod_{i_{l1}, i_{l2}, \ldots, i_{ln}=1}^{n}\left(1 - \prod_{j=1}^{n}\nu_{i_{lj}}^{2r_j}\right)^{\prod_{j=1}^{n} w_{i_{lj}}}}\right)^{1/\sum_{j=1}^{n} r_j}\right) \tag{65}$$

to aggregate all the attribute values. Therefore, we can obtain a series of overall values p_i $(i = 1, 2, \ldots, m)$ of alternatives.

Step 3. p_i $(i = 1, 2, \ldots, m)$ is ranked based on the score function by Definition 3.

Step 4. The alternative x_i $(i = 1, 2, \ldots, m)$ is ranked based on the rank of the corresponding overall values.

6. Numerical Example

We provide a numerical example adopted from [24] and a few comparative analyses to illustrate the validity of the new approach. The Civil Aviation Administration of Taiwan (CAAT) aims to determine the best airline in Taiwan. Hence, CAAT organizes several experts to form a committee that will evaluate four major domestic airlines, namely, UNI Air (x_1), Transasia (x_2), Mandarin (x_3), and Daily Air (x_4). The experts are required to evaluate the four airlines using the following four aspects: (1) booking and ticketing service (G_1), (2) check-in and boarding process (G_2), (3) cabin service (G_3), and (4) responsiveness (G_4). The weight vector of the attributes is $w = (0.15, 0.25, 0.35, 0.25)^T$. For the attribute G_j $(j = 1, 2, 3, 4)$ of airline x_i $(i = 1, 2, 3, 4)$, the experts are required to utilize the PFN $p_{ij} = (\mu_{ij}, \nu_{ij})$ to express their assessments. Moreover, a Pythagorean fuzzy decision matrix $P = (p_{ij})_{4\times 4}$ $(i, j = 1, 2, 3, 4)$ can be obtained (see Table 1). We utilize the newly introduced decision-making approach to solve this problem.

Table 1: Pythagorean fuzzy decision matrix.

	C_1	C_2	C_3	C_4
x_1	(0.9, 0.3)	(0.7, 0.6)	(0.5, 0.8)	(0.6, 0.3)
x_2	(0.4, 0.7)	(0.9, 0.2)	(0.8, 0.1)	(0.5, 0.3)
x_3	(0.8, 0.4)	(0.7, 0.5)	(0.6, 0.2)	(0.7, 0.4)
x_4	(0.7, 0.2)	(0.8, 0.2)	(0.8, 0.4)	(0.6, 0.6)

6.1. Decision-Making Process

Step 1. The decision matrix does not require being normalized because all the attributes are benefit attributes.

Step 2. For the alternative x_i $(i = 1, 2, 3, 4)$ is utilized to aggregate the attribute values. Therefore, we can obtain a set of overall values. In this step, let $R = (1, 1, 1, 1)$.

$$p_1 = (0.7543, 0.2324),$$
$$p_2 = (0.8424, 0.0004),$$
$$p_3 = (0.7664, 0.0211),$$
$$p_4 = (0.8387, 0.0131). \quad (66)$$

Step 3. The scores of p_i $(i = 1, 2, 3, 4)$ are calculated based on Definition 3 to obtain $s(p_1) = 0.5150$, $s(p_2) = 0.7103$, $s(p_3) = 0.5869$, and $s(p_4) = 0.7032$. Therefore, the rank of the overall values is $p_2 \succ p_4 \succ p_3 \succ p_1$.

Step 4. The alternative x_i $(i = 1, 2, 3, 4)$ is ranked based on the rank of p_i $(i = 1, 2, 3, 4)$ to obtain $x_2 \succ x_4 \succ x_3 \succ x_1$. Therefore, x_2 is the best alternative. That is, Daily Air is the best airline in Taiwan.

6.2. Influence of the Parameter Vector R on the Final Result. The prominent characteristic of the DGPFWBM and the DGPFWBGM operators is that they can consider the interrelationship among all PFNs. We conduct several comparative analyses to demonstrate the advantages of the new operators. Table 2 presents further details.

Table 2 shows that the aggregation operators introduced in [23–26] cannot consider the interrelationship among PFNs. Although PFCIA and PFCIG can capture the interrelationship among all PFNs, they only focus on changing the weight vector of the aggregation operators. In addition, the correlations of the aggregated arguments are measured subjectively by the decision makers. GPFWBM, GPFWBGM, DGPFWBM, and DGPFWBGM focus on the aggregated PFNs. The DGPFWBM and DGPFWBGM operators can consider the interrelationship among all PFNs compared with the GPFWBM and GPFWBGM operators. In addition, the DGPFWBM and DGPFWBGM operators have a parameter vector, thereby enabling the aggregation process to be substantially flexible.

The parameter vector R plays a crucial role in the final result. We may obtain a different ranking result by assigning different values to R. We set a different weight vector R and discuss the ranking results. Tables 3 and 4 provide further details.

Tables 3 and 4 show that the different ranking results can be obtained by assigning different values in the parameter vector R. Therefore, the DGPFWBM and the DGPFWBM operators are considerably flexible by using a parameter vector. Table 3 shows that the best alternatives are consistently the same, although the ranking results are different by using different parameter vectors. That is, the final results become increasingly objective by considering the interrelationship among all the attribute values. The best alternative is consistently x_2 regardless of the parameter vector. Table 4 shows that the ranking results increase and become steady with the increase of values in parameter vector R. These features of the DGPFWBM and DGPFWBM operators are crucial in real decision-making problems. Accordingly, we can assign a weight vector with large values to the DGPFWBM and the DGPFWBM operators for steady and reliable final results.

7. Conclusions

PFS is a powerful tool for expressing the fuzziness, uncertainty, and hesitancy of decision makers. This research extends the GWBM and GWBGM operators to the Pythagorean fuzzy environment, as well as introduces the GPFWBM and the GPFWBGM operators. First, we extend the GWBM and GWBGM operators, as well as develop the DGWBM and DGWBGM operators, because the two operators can only consider the interrelationship between any two IFNs. The prominent advantage of the DGWBM and DGWBGM operators is that they can consider the interrelationship among all the arguments being fused. Moreover, we extend the GWBGM and DGWBM operators to the Pythagorean fuzzy environment, as well as develop the DGPFWBM and DGPFWBGM operators. Thereafter, the new operators are used as bases to propose a novel approach to MAGDM with Pythagorean fuzzy information. We apply the new approach to illustrate its validity to the problem of selecting the best airline. Moreover, we investigate the influence of the parameter vector R on the ranking results to show the advantages of the new approach. The limitation of the DGPFWFBM and DGPFWFBGM operators is that the calculation process maybe more complicated than the existing Pythagorean fuzzy aggregation operators as they can consider the interrelationship between all PFNs. Therefore, the calculation process of the proposed method to MAGDM is little more complicated than existing methods. The focus of future research is to reduce complexity of the calculation of the proposed method.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Table 2: Comparison of the different operators.

Aggregation operators	Whether the operator can capture the interrelationship between any two PFNs	Whether the operator can capture the interrelationship between any three PFNs	Whether the operator can capture the interrelationship among all PFNs	Whether a parameter vector exists to manipulate the ranking results
PFWA [23]	No	No	No	No
PFWG [23]	No	No	No	No
SPFWA [24]	No	No	No	No
SPFWG [24]	No	No	No	No
PFOWAWAD [25]	No	No	No	No
PFEWA [26]	No	No	No	No
PFEOWA [26]	No	No	No	No
GPFEWA [26]	No	No	No	No
GPFEOWA [26]	No	No	No	No
PFCIA [28]	Yes	Yes	Yes	No
PFCIG [28]	Yes	Yes	Yes	No
GPFWBM	Yes	Yes	No	Yes
GPFWBGM	Yes	Yes	No	Yes
DGPFWBM	Yes	Yes	Yes	Yes
DGPFWBGM	Yes	Yes	Yes	Yes

Table 3: Ranking results by assigning different values to parameter vector R in the DGPFWBM operator.

Parameter R	Scores of overall values	Ranking results
$R = (1, 1, 1, 1)$	$s(p_1) = 0.5150$ $s(p_2) = 0.7103$ $s(p_3) = 0.5869$ $s(p_4) = 0.7032$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (2, 2, 2, 2)$	$s(p_1) = 0.3742$ $s(p_2) = 0.6487$ $s(p_3) = 0.4906$ $s(p_4) = 0.5944$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (3, 3, 3, 3)$	$s(p_1) = 0.3657$ $s(p_2) = 0.6517$ $s(p_3) = 0.4572$ $s(p_4) = 0.5584$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (4, 4, 4, 4)$	$s(p_1) = 0.3872$ $s(p_2) = 0.6613$ $s(p_3) = 0.4417$ $s(p_4) = 0.5411$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (5, 5, 5, 5)$	$s(p_1) = 0.4148$ $s(p_2) = 0.6743$ $s(p_3) = 0.4361$ $s(p_4) = 0.5330$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (8, 8, 8, 8)$	$s(p_1) = 0.4870$ $s(p_2) = 0.6904$ $s(p_3) = 0.4435$ $s(p_4) = 0.5303$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$R = (10, 10, 10, 10)$	$s(p_1) = 0.5215$ $s(p_2) = 0.7004$ $s(p_3) = 0.4550$ $s(p_4) = 0.5339$	$x_2 \succ x_4 \succ x_1 \succ x_3$

Table 4: Ranking results by assigning different values to parameter vector R in the DGPFWBGM operator.

Parameter R	Scores of overall values	Ranking results
$R = (1, 1, 1, 1)$	$s(p_1) = -0.9636$ $s(p_2) = -0.6197$ $s(p_3) = -0.5997$ $s(p_4) = -0.5908$	$x_4 \succ x_3 \succ x_2 \succ x_1$
$R = (2, 2, 2, 2)$	$s(p_1) = -0.7322$ $s(p_2) = -0.4985$ $s(p_3) = -0.2989$ $s(p_4) = -0.3209$	$x_3 \succ x_4 \succ x_2 \succ x_1$
$R = (3, 3, 3, 3)$	$s(p_1) = -0.5973$ $s(p_2) = -0.4662$ $s(p_3) = -0.1154$ $s(p_4) = -0.1561$	$x_3 \succ x_4 \succ x_2 \succ x_1$
$R = (4, 4, 4, 4)$	$s(p_1) = -0.5155$ $s(p_2) = -0.4324$ $s(p_3) = -0.0206$ $s(p_4) = -0.0836$	$x_3 \succ x_4 \succ x_2 \succ x_1$
$R = (5, 5, 5, 5)$	$s(p_1) = -0.4705$ $s(p_2) = -0.4068$ $s(p_3) = 0.0285$ $s(p_4) = -0.0491$	$x_3 \succ x_4 \succ x_2 \succ x_1$
$R = (8, 8, 8, 8)$	$s(p_1) = -0.4178$ $s(p_2) = -0.3663$ $s(p_3) = 0.3364$ $s(p_4) = -0.0136$	$x_3 \succ x_4 \succ x_2 \succ x_1$
$R = (10, 10, 10, 10)$	$s(p_1) = -0.4056$ $s(p_2) = -0.3538$ $s(p_3) = 0.3477$ $s(p_4) = 0.3533$	$x_3 \succ x_4 \succ x_2 \succ x_1$

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