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Research Article

Weighted Differentiation Composition Operator from Logarithmic Bloch Spaces to Zygmund-Type Spaces

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Let $H(\mathbb{D})$ denote the space of all holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$, $u \in H(\mathbb{D})$ and let n be a positive integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. In this paper, we investigate the boundedness and compactness of a weighted differentiation composition operator $\mathscr{D}^n_{\varphi,u} f(z) = u(z) f^{(n)}(\varphi(z))$, $f \in H(\mathbb{D})$, from the logarithmic Bloch spaces to the Zygmund-type spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$.

The logarithmic Bloch space is defined as follows:

$$\mathscr{B}_{\log} = \left\{ f \in H(\mathbb{D}) : \|f\| = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left(\log \frac{2}{1 - |z|}\right) \left|f'(z)\right| < \infty \right\}. \quad (1)$$

The space $\mathscr{B}_{\log}$ is a Banach space under the norm $\|f\|_{\mathscr{B}_{\log}} = |f(0)| + \|f\|$. Let $\mathscr{B}_{\log,0}$ denote the subspace of $\mathscr{B}_{\log}$ consisting of those $f \in \mathscr{B}_{\log}$ such that

$$\lim_{|z|\to 1} \left(1 - |z|^2\right) \left(\log \frac{2}{1 - |z|}\right) \left|f'(z)\right| = 0. \quad (2)$$

It is obvious that there are unbounded $\mathscr{B}_{\log}$ functions. For example, consider the function $f(z) = \log\log(e/(1 - z))$. There are also bounded functions that do not belong to $\mathscr{B}_{\log}$. In fact, the interpolating Blaschke products do not belong to $\mathscr{B}_{\log}$. It is easily proved that, for $0 < \alpha < 1$, $\mathscr{B}^\alpha \subsetneq \mathscr{B}_{\log} \subsetneq \mathscr{B}$. $\mathscr{B}_{\log}$ first appeared in the study of boundedness of the Hankel operators on the Bergman space. Attele in [1] proved that, for $f \in L^2_a(\mathscr{D})$, the Hankel operator $H_f : L^1_a(\mathscr{D}) \to L^1(\mathscr{D})$ is bounded if and only if $\|f\|_{\mathscr{B}_{\log}} < \infty$, thus giving one reason, and not the only reason, why log-Bloch-type spaces are of interest. Ye in [2] proved that $\mathscr{B}_{\log,0}$ is a closed subspace of $\mathscr{B}_{\log}$. Galanopoulos in [3] characterized the boundedness and compactness of the composition operator $C_\varphi : \mathscr{B}_{\log} \to Q^p_{\log}$ and the boundedness and compactness of the weighted composition operator $uC_\varphi : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$. Ye in [4] characterized the boundedness and compactness of the weighted composition operator uC_φ between the logarithmic Bloch space $\mathscr{B}_{\log}$ and the α-Bloch spaces $\mathscr{B}^\alpha$ on the unit disk and the boundedness and compactness of the weighted composition operator uC_φ between the little logarithmic Bloch space $\mathscr{B}^0_{\log}$ and the little α-Bloch spaces $\mathscr{B}^\alpha_0$ on the unit disk. Li in [5] characterized the boundedness and compactness of the weighted composition operator uC_φ from Bergman spaces A^p_β into the logarithmic Bloch space $\mathscr{B}_{\log}$ on the unit disk. Ye in [6] characterized the boundedness and compactness of the weighted composition operator uC_φ from the general function space $F(p, q, s)$ into the logarithmic Bloch space $\mathscr{B}_{\log}$ on the unit disk. Colonna and Li in [7] studied the boundedness and compactness of the weighted composition operators from Hardy space into the logarithmic Bloch space and the little logarithmic Bloch space. Petrov in [8] obtains sharp reverse estimates for the logarithmic Bloch spaces on the unit disk. Castillo et al. in [9] characterized the boundedness and compactness of the composition operator from the logarithmic Bloch spaces into weighted Bloch spaces. García Ortiz and Ramos-Fernández in [10] characterized the boundedness and compactness of the composition operators from logarithmic Bloch spaces into Bloch-type spaces.

Let μ be a weight; that is, μ is a positive continuous function on $\mathbb{D}$. The Zygmund-type space $\mathcal{Z}_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z) \left| f''(z) \right| < \infty. \tag{3}$$

With the norm $\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$, it becomes a Banach space. The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ is a subspace of $\mathcal{Z}_\mu$ consisting of those $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z) \left| f''(z) \right| = 0. \tag{4}$$

When $\mu(z) = 1 - |z|^2$, the Zygmund-type space $\mathcal{Z}_\mu$ becomes the Zygmund space $\mathcal{Z}$ [11], while the little Zygmund-type space $\mathcal{Z}_{\mu,0}$ becomes the little Zygmund space $\mathcal{Z}_0$.

Let $\mathcal{D} = \mathcal{D}^1$ be the differentiation operator; that is, $\mathcal{D}f = f'$. If $n \in \mathbb{N}_0$, then the operator $\mathcal{D}^n$ is defined by $\mathcal{D}^0 f = f, \mathcal{D}^n f = f^{(n)}, f \in H(\mathbb{D})$.

The weighted differentiation composition operator, denoted by $\mathcal{D}^n_{\varphi,u}$, is defined as follows [12, 13]:

$$\mathcal{D}^n_{\varphi,u} f(z) = u(z) f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \tag{5}$$

where $u \in H(\mathbb{D})$ and φ is a nonconstant holomorphic self-map of $\mathbb{D}$.

If $n = 0$, then $\mathcal{D}^n_{\varphi,u}$ becomes the weighted composition operator uC_φ, defined by

$$uC_\varphi f(z) = u(z) f(\varphi(z)), \quad z \in \mathbb{D}, \tag{6}$$

which, for $u(z) \equiv 1$, is reduced to the composition operator C_φ for some recent articles on weighted composition operators on some H^∞-type spaces, for example, [14–16] and references therein. If $n = 1, u(z) = \varphi'(z)$, then $\mathcal{D}^n_{\varphi,u} = \mathcal{D}C_\varphi$, which was studied in [17–21]. When $n = 1$, $u(z) \equiv 1$, then $\mathcal{D}^n_{\varphi,u} = C_\varphi\mathcal{D}$, which was studied in [17, 19]. If $n = 1$, $\varphi(z) = z$, then $\mathcal{D}^n_{\varphi,u} = M_u\mathcal{D}$, that is, the product of differentiation operator and multiplication operator M_u defined by $M_u f = uf$. Zhu in [13] completely characterized the boundedness and compactness of linear operators which are obtained by taking products of differentiation, composition, and multiplication operators from Bergman type spaces to Bers spaces. Stević in [12] studied the boundedness and compactness of the weighted differentiation composition operator $\mathcal{D}^n_{\varphi,u}$ from mixed-norm spaces to weighted-type spaces or the little weighted-type space (see also [22–24]). Zhu in [25] studied the boundedness and compactness of the generalized weighted composition operator on weighted Bergman spaces. Yang in [21] studied the boundedness and compactness of the operator $C_\varphi\mathcal{D}$ and $\mathcal{D}C_\varphi$ from $Q_K(p,q)$ to $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$ spaces. Liu and Yu in [18] studied the boundedness and compactness of the operator $\mathcal{D}C_\varphi$ between H^∞ and Zygmund spaces. Ye and Zhou in [26] studied the boundedness and compactness of the weighted composition operators from Hardy to Zygmund type spaces. Stević in [27] studied the boundedness and compactness of the generalized composition operator from mixed-norm space to the Bloch-type space, the little Bloch-type space, the Zygmund space, and the little Zygmund space. For other recently introduced products of operators on spaces of holomorphic functions see [13, 16]. Motivated by the results [12, 18, 23, 24, 27], we consider the boundedness and compactness of the operators $\mathcal{D}^n_{\varphi,u}$ from the logarithmic Bloch spaces to the Zygmund-type spaces and the little Zygmund-type spaces. For the proof, we need different test functions and some complex calculations kills.

Throughout this paper, we will use the letter C to denote a positive constant that can change its value at each occurrence.

2. Auxiliary Results

Here we prove and quote some auxiliary results which will be used in the proofs of the main results in this paper.

Lemma 1. *Let n be a positive integer. Suppose $f \in \mathcal{B}_{\log}$; there exists a constant C such that*

$$\left| f^{(n)}(z) \right| \le \frac{C\|f\|_{\mathcal{B}_{\log}}}{\left(1-|z|^2\right)^n \log(2/(1-|z|))}, \quad z \in \mathbb{D}, \tag{7}$$

Proof. We use induction on n. Using the definition of the logarithmic Bloch spaces we have

$$\left| f'(z) \right| \le \frac{C\|f\|_{\mathcal{B}_{\log}}}{\left(1-|z|^2\right) \log(2/(1-|z|))}; \tag{8}$$

the case holds for $n = 1$. Assume the case $n = k$ holds; since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left|e^{i\theta}-z\right|^2} d\theta = \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{9}$$

let $\rho = (1+|z|)/2 < 1$; then we have $|z/\rho| = 2|z|/(1+|z|) < 1$, so

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left|\rho e^{i\theta}-z\right|^2} d\theta = \frac{1}{\rho^2-|z|^2}, \quad z \in \mathbb{D}. \tag{10}$$

By the Cauchy integral formula we obtain

$$\left| f^{(k+1)}(z) \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f^{(k)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}-z\right)^2} \rho e^{i\theta} d\theta \right|$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f^{(k)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}-z\right)^2}\rho e^{i\theta}\right|d\theta \\
&\leq \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)} \\
&\quad\times\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho}{\left|\rho e^{i\theta}-z\right|^2}d\theta \\
&= \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)}\frac{\rho}{\rho^2-|z|^2} \\
&= \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)}\frac{\rho}{\left(\rho+|z|\right)\left(\rho-|z|\right)} \\
&\leq \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)}\frac{1}{\rho-|z|} \\
&= \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)}\frac{1}{1-\rho} \\
&\leq \frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^k\left(\log\left(2/\left(1-\rho\right)\right)\right)}\frac{2}{\left(1-\rho\right)\left(1+\rho\right)} \\
&\leq \frac{2C\|f\|_{\mathscr{B}_{\log}}}{\left(1-\rho^2\right)^{k+1}\left(\log\left(2/\left(1-\rho\right)\right)\right)}.
\end{aligned}
\tag{11}
$$

Note that

$$\frac{1}{4}\left(1-|z|\right)\leq 1-\rho=\frac{1}{2}\left(1-|z|\right)\leq 1-|z|, \tag{12}$$

$$\frac{1}{2}\left(1-|z|\right)\leq 1-\rho^2=\left(1+\rho\right)\left(1-\rho\right)\leq 1-|z|; \tag{13}$$

we have

$$\left|f^{(k+1)}(z)\right|\leq\frac{C\|f\|_{\mathscr{B}_{\log}}}{\left(1-|z|^2\right)^{k+1}\log\left(2/\left(1-|z|\right)\right)}, \tag{14}$$

for every $z\in\mathbb{D}$. Hence the case $n=k+1$ holds. The desired result follows. The proof of this lemma is complete. □

Lemma 2 (see [4, 28]). *Let*

$$g_t(z)=\frac{\left(1-|z|\right)\log\left(2/\left(1-|z|\right)\right)}{\left(1-|tz|\right)\log\left(2/\left(1-|tz|\right)\right)},\quad t\in[0,1],\ z\in\mathbb{D}; \tag{15}$$

then $|g_t(z)|<2$.

The following criterion for the compactness is a useful tool and it follows from standard arguments (e.g., [29, Proposition 3.11] or [30, Lemma 2.10]).

Lemma 3. *Let $u\in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}(\mathscr{B}_{\log,0})\to\mathscr{Z}_\mu$ is compact if and only if $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}(\mathscr{B}_{\log,0})\to\mathscr{Z}_\mu$ is bounded and, for any bounded sequence $\{f_k\}$ in $\mathscr{B}_{\log}(\mathscr{B}_{\log,0})$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, we have $\|\mathscr{D}^n_{\varphi,u}f_k\|_{\mathscr{Z}_\mu}\to 0$ as $k\to\infty$.*

Lemma 4. *A closed set K in $\mathscr{Z}_{\mu,0}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)\left|f''(z)\right|=0. \tag{16}$$

The proof is similar to that of Lemma 1 in [31]; hence we omit it.

3. Boundedness and Compactness of $\mathscr{D}^n_{\varphi,u}$ from $\mathscr{B}_{\log}(\mathscr{B}_{\log,0})$ to $\mathscr{Z}_\mu(\mathscr{Z}_{\mu,0})$ Spaces

In this section, we study the boundedness and compactness of $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}(\mathscr{B}_{\log,0})\to\mathscr{Z}_\mu(\mathscr{Z}_{\mu,0})$.

Theorem 5. *Let $u\in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then the following statements are equivalent:*

(1) *$\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}\to\mathscr{Z}_\mu$ is bounded;*

(2) *$\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log,0}\to\mathscr{Z}_\mu$ is bounded;*

(3)

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|u''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^n\log\left(2/\left(1-\left|\varphi(z)\right|\right)\right)}<\infty, \tag{17}$$

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+1}\log\left(2/\left(1-\left|\varphi(z)\right|\right)\right)}<\infty, \tag{18}$$

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+2}\log\left(2/\left(1-\left|\varphi(z)\right|\right)\right)}<\infty. \tag{19}$$

Proof. (3) ⇒ (1). Suppose that (17), (18), and (19) hold. Then, for every $z\in\mathbb{D}$ and $f\in\mathscr{B}_{\log}$, by Lemma 1, we have

$$
\begin{aligned}
&\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u}f\right)''(z)\right| \\
&\quad=\mu(z)\left|u''(z)f^{(n)}(\varphi(z))\right. \\
&\qquad\quad+\left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)f^{(n+1)}(\varphi(z)) \\
&\qquad\quad\left.+u(z)\left(\varphi'(z)\right)^2f^{(n+2)}(\varphi(z))\right|
\end{aligned}
$$

$$\leq \mu(z)\left|u''(z)\right|\left|f^{(n)}(\varphi(z))\right| + \mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|\left|f^{(n+1)}(\varphi(z))\right| + \mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|\left|f^{(n+2)}(\varphi(z))\right|$$
$$\leq C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log(2/(1-|\varphi(z)|))} + C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}\log(2/(1-|\varphi(z)|))} + C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|}{\left(1-|\varphi(z)|^2\right)^{n+2}\log(2/(1-|\varphi(z)|))} + C\|f\|_{\mathscr{B}_{\log}}. \tag{20}$$

On the other hand, we have

$$\left|\left(\mathscr{D}^n_{\varphi,u}f\right)(0)\right| = \left|u(0)f^{(n)}(\varphi(0))\right| \leq C\frac{|u(0)|}{\left(1-|\varphi(0)|^2\right)^n\log(2/(1-|\varphi(0)|))}\|f\|_{\mathscr{B}_{\log}},$$
$$\left|\left(\mathscr{D}^n_{\varphi,u}f\right)'(0)\right| = \left|u(0)'f^{(n)}(\varphi(0))+u(0)f^{(n+1)}(\varphi(0))\varphi'(0)\right| \leq C\frac{\left|u'(0)\right|}{\left(1-|\varphi(0)|^2\right)^n\log(2/(1-|\varphi(0)|))}\|f\|_{\mathscr{B}_{\log}} + C\frac{\left|u(0)\varphi'(0)\right|}{\left(1-|\varphi(0)|^2\right)^{n+1}\log(2/(1-|\varphi(0)|))}\|f\|_{\mathscr{B}_{\log}}. \tag{21}$$

Applying conditions (20) and (21), we deduce that the operator $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}\to\mathscr{Z}_\mu$ is bounded.

(1) ⇒ (2). This implication is clear.

(2) ⇒ (3). Assume that $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log,0}\to\mathscr{Z}_\mu$ is bounded; that is, there exists a constant C, such that

$$\left\|\mathscr{D}^n_{\varphi,u}f\right\|_{\mathscr{Z}_\mu}\leq C\|f\|_{\mathscr{B}_{\log}}, \tag{22}$$

for all $f\in\mathscr{B}_{\log,0}$. For $f(z)=z^n/n!\in\mathscr{B}_{\log,0}$, we have that

$$K_1 := \sup_{z\in\mathbb{D}}\mu(z)\left|u''(z)\right|<\infty. \tag{23}$$

Taking $f(z)=z^{n+1}/(n+1)!\in\mathscr{B}_{\log,0}$; we have that

$$\sup_{z\in\mathbb{D}}\mu(z)\left|u''(z)\varphi(z)+2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|<\infty. \tag{24}$$

By (23), (24), and the boundedness of the function $\varphi(z)$, we get

$$K_2 := \sup_{z\in\mathbb{D}}\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|<\infty. \tag{25}$$

In the same way, taking $f(z)=z^{n+2}/(n+2)!\in\mathscr{B}_{\log,0}$, we have that

$$\sup_{z\in\mathbb{D}}\mu(z)\left|u''(z)(\varphi(z))^2 + 2\left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)\varphi(z) + 2u(z)\left(\varphi'(z)\right)^2\right|<\infty. \tag{26}$$

By (23), (25), (26), and the boundedness of the function $\varphi(z)$, we have that

$$K_3 := \sup_{z\in\mathbb{D}}\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|<\infty. \tag{27}$$

For a fixed $\omega\in\mathbb{D}$, set

$$f_\omega(z) = (n+2)(n+3)\frac{1-|\varphi(\omega)|^2}{\left(1-z\overline{\varphi(\omega)}\right)\log(2/(1-|\varphi(\omega)|))} - 2(n+3)\frac{\left(1-|\varphi(\omega)|^2\right)^2}{\left(1-z\overline{\varphi(\omega)}\right)^2\log(2/(1-|\varphi(\omega)|))} + 2\frac{\left(1-|\varphi(\omega)|^2\right)^3}{\left(1-z\overline{\varphi(\omega)}\right)^3\log(2/(1-|\varphi(\omega)|))}. \tag{28}$$

We get that

$$f_\omega^{(n)}(z) = \frac{(n+3)!}{n+1}\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+1}\log(2/(1-|\varphi(\omega)|))} - 2(n+3)\cdot(n+1)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log(2/(1-|\varphi(\omega)|))} + (n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log(2/(1-|\varphi(\omega)|))};$$

$$f_\omega^{(n+1)}(z) = (n+3)!\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log(2/(1-|\varphi(\omega)|))}$$

$$- 2 \cdot (n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)^2 \left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ (n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)^3 \left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)};$$

$$f_\omega^{(n+2)}(z)$$

$$= (n+2) \cdot (n+3)! \frac{\left(1-|\varphi(\omega)|^2\right) \left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$- 2(n+3) \cdot (n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)^2 \left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ (n+4)! \frac{\left(1-|\varphi(\omega)|^2\right)^3 \left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \quad (29)$$

By Lemma 2 we have

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left(\log \frac{2}{1-|z|}\right) \left|f_\omega'(z)\right|$$

$$\leq \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left(\log \frac{2}{1-|z|}\right) \times \left| \frac{(n+2)(n+3)\left(1-|\varphi(\omega)|^2\right)\overline{\varphi(\omega)}}{\left(1-z\overline{\varphi(\omega)}\right)^2 \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \right|$$

$$+ \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left(\log \frac{2}{1-|z|}\right) \times \left| \frac{4(n+3)\left(1-|\varphi(\omega)|^2\right)^2\overline{\varphi(\omega)}}{\left(1-z\overline{\varphi(\omega)}\right)^3 \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \right|$$

$$+ \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left(\log \frac{2}{1-|z|}\right) \times \left| \frac{6\left(1-|\varphi(\omega)|^2\right)^3\overline{\varphi(\omega)}}{\left(1-z\overline{\varphi(\omega)}\right)^4 \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \right|$$

$$\leq 4(n+2)(n+3) \sup_{z\in\mathbb{D}} (1-|z|) \left(\log \frac{2}{1-|z|}\right) \times \frac{\left(1-|\varphi(\omega)|\right)}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)\left(1-\left|z\overline{\varphi(\omega)}\right|\right) \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 32(n+3) \sup_{z\in\mathbb{D}} (1-|z|) \left(\log \frac{2}{1-|z|}\right) \times \frac{\left(1-|\varphi(\omega)|\right)^2}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)^2\left(1-\left|z\overline{\varphi(\omega)}\right|\right) \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 96 \sup_{z\in\mathbb{D}} (1-|z|) \left(\log \frac{2}{1-|z|}\right) \times \frac{\left(1-|\varphi(\omega)|\right)^3}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)^3\left(1-\left|z\overline{\varphi(\omega)}\right|\right) \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$= 4(n+2)(n+3) \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 32(n+3) \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 96 \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$\leq 4(n+2)(n+3) \times \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 32(n+3) \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$+ 96 \sup_{z\in\mathbb{D}} \frac{(1-|z|)\left(\log\left(2/\left(1-|z|\right)\right)\right)}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\left(\log\left(2/\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\right)\right)} \times \frac{\log\left(2/\left(1-\left|\overline{\varphi(\omega)}\right|\right)\right)}{\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}$$

$$\leq 8(n+2)(n+3) + 64(n+3) + 192$$

$$= 84n^2 + 104n + 432. \quad (30)$$

Hence, $f_\omega \in \mathscr{B}_{\log}$ and $\sup_{\omega\in\mathbb{D}} \|f_\omega\|_{\mathscr{B}_{\log}} \leq C$.

On the other hand for each fix $\omega \in \mathbb{D}$, by (30), we obtain that

$$\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\left|f'_\omega(z)\right| \longrightarrow 0, \quad (\text{as } |z| \longrightarrow 1); \tag{31}$$

it follows that $f_\omega \in \mathscr{B}_{\log,0}$ for each fix $\omega \in \mathbb{D}$. From (29), we have $f_\omega^{(n+1)}(\varphi(\omega)) = f_\omega^{(n+2)}(\varphi(\omega)) = 0$ and

$$f_\omega^{(n)}(\varphi(\omega)) = 2\cdot n!\frac{\left(\overline{\varphi(\omega)}\right)^n}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \tag{32}$$

Hence

$$\begin{aligned} C &\geq \left\|\mathscr{D}^n_{\varphi,u}f_\omega\right\|_{\mathscr{Z}_\mu} \\ &\geq \sup_{z\in\mathbb{D}}\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u}f_\omega\right)''(z)\right| \\ &= \sup_{z\in\mathbb{D}}\mu(z)\left|u''(z)f_\omega^{(n)}(\varphi(z))\right. \\ &\qquad + \left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)f_\omega^{(n+1)}(\varphi(z)) \\ &\qquad \left. + u(z)\left(\varphi'(z)\right)^2 f_\omega^{(n+2)}(\varphi(z))\right| \\ &\geq \mu(\omega)\left|u''(\omega)f_\omega^{(n)}(\varphi(\omega))\right. \\ &\qquad + \left(2u'(\omega)\varphi'(\omega)+u(\omega)\varphi''(\omega)\right)f_\omega^{(n+1)}(\varphi(\omega)) \\ &\qquad \left. + u(\omega)\left(\varphi'(\omega)\right)^2 f_\omega^{(n+2)}(\varphi(\omega))\right| \\ &= 2\cdot n!\frac{\mu(\omega)\left|u''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \tag{33}$$

By (33), we obtain that

$$\begin{aligned} &\sup_{1/2<|\varphi(\omega)|<1}\frac{\mu(\omega)\left|u''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad\leq 2^{n+1}\cdot n!\sup_{1/2<|\varphi(\omega)|<1}\frac{\mu(\omega)\left|u''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad\leq 2^{n+1}\cdot n!\sup_{\omega\in\mathbb{D}}\frac{\mu(\omega)\left|u''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad\leq C2^n < \infty. \end{aligned} \tag{34}$$

And from (23), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\leq 1/2}\frac{\mu(\omega)\left|u''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^n\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad\leq \sup_{|\varphi(\omega)|\leq 1/2}\frac{\mu(\omega)\left|u''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^n\log 2} \\ &\quad\leq \left(\frac{4}{3}\right)^n\frac{1}{\log 2}\sup_{|\varphi(\omega)|\leq 1/2}\mu(\omega)\left|u''(\omega)\right| \\ &\quad\leq \left(\frac{4}{3}\right)^n\frac{K_1}{\log 2} < \infty. \end{aligned} \tag{35}$$

Thus combining (35) with (34) we get the condition (17).

For a fixed $\omega \in \mathbb{D}$, set

$$\begin{aligned} g_\omega(z) &= (n+1)(n+3)\frac{1-|\varphi(\omega)|^2}{\left(1-z\overline{\varphi(\omega)}\right)\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad - (2n+5)\frac{\left(1-|\varphi(\omega)|^2\right)^2}{\left(1-z\overline{\varphi(\omega)}\right)^2\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad + 2\frac{\left(1-|\varphi(\omega)|^2\right)^3}{\left(1-z\overline{\varphi(\omega)}\right)^3\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \tag{36}$$

It is easy to see that

$$\begin{aligned} &g_\omega^{(n)}(z) \\ &\quad= (n+3)\cdot(n+1)!\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+1}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\qquad - (2n+5)\cdot(n+1)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\qquad + (n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}; \\ &g_\omega^{(n+1)}(z) \\ &\quad= \frac{(n+1)\cdot(n+3)!}{(n+2)}\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\qquad - (2n+5)\cdot(n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \end{aligned}$$

$$+ (n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)^3 \left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)};$$

$$\begin{aligned} g_\omega^{(n+2)}(z) &= (n+1)\cdot(n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad - (2n+5)\cdot(n+3)! \frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad + (n+4)! \frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \quad (37)$$

Using Lemma 2, we easily get that $g_\omega \in \mathscr{B}_{\log,0}$ and $\sup_{\omega\in\mathbb{D}}\|g_\omega\|_{\mathscr{B}_{\log}} \le C$ with a direct calculation. From (37), we have $g_\omega^{(n)}(\varphi(\omega)) = g_\omega^{(n+2)}(\varphi(\omega)) = 0$,

$$g_\omega^{(n+1)}(\varphi(\omega)) = -(n+1)! \frac{\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \quad (38)$$

Hence

$$\begin{aligned} C &\ge \left\|\mathscr{D}_{\varphi,u}^n g_\omega\right\|_{\mathscr{Z}_\mu} \\ &\ge \sup_{z\in\mathbb{D}} \mu(z)\left|\left(\mathscr{D}_{\varphi,u}^n g_\omega\right)''(z)\right| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\Big|u''(z)\, g_\omega^{(n)}(\varphi(z)) \\ &\qquad + \left(2u'(z)\varphi'(z) + u(z)\varphi''(z)\right) g_\omega^{(n+1)}(\varphi(z)) \\ &\qquad + u(z)\left(\varphi'(z)\right)^2 g_\omega^{(n+2)}(\varphi(z))\Big| \\ &\ge \mu(\omega)\Big|u''(\omega)\, g_\omega^{(n)}(\varphi(\omega)) \\ &\qquad + \left(2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right) g_\omega^{(n+1)}(\varphi(\omega)) \\ &\qquad + u(\omega)\left(\varphi'(\omega)\right)^2 g_\omega^{(n+2)}(\varphi(\omega))\Big| \\ &= (n+1)! \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \quad (39)$$

From (39), we obtain that

$$\begin{aligned} &\sup_{1/2<|\varphi(\omega)|<1} \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad \le 2^{n+1} \sup_{1/2<|\varphi(\omega)|<1} (n+1)! \\ &\qquad \times \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad \le 2^{n+1}(n+1)! \\ &\qquad \times \sup_{\omega\in\mathbb{D}} \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad \le C2^{n+1} < \infty. \end{aligned} \quad (40)$$

By (25), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\le 1/2} \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad \le \sup_{|\varphi(\omega)|\le 1/2} \frac{\mu(\omega)\left|2u'(\omega)\varphi'(\omega) + u(\omega)\varphi''(\omega)\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+1} \log 2} \\ &\quad \le \left(\frac{4}{3}\right)^{n+1} \frac{1}{\log 2} \sup_{|\varphi(\omega)|\le 1/2} \mu(\omega)\Big|2u'(\omega)\varphi'(\omega) \\ &\qquad + u(\omega)\varphi''(\omega)\Big| \\ &\quad \le \left(\frac{4}{3}\right)^{n+1} \frac{K_2}{\log 2} < \infty. \end{aligned} \quad (41)$$

Thus combining (40) with (41) we get the condition (18).

Next, we prove (19). To see this, for a fixed $\omega \in \mathbb{D}$, put

$$\begin{aligned} h_\omega(z) &= (n+1)(n+2) \frac{1-|\varphi(\omega)|^2}{\left(1-z\overline{\varphi(\omega)}\right)\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad - 2(n+2) \frac{\left(1-|\varphi(\omega)|^2\right)^2}{\left(1-z\overline{\varphi(\omega)}\right)^2 \log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\quad + 2\frac{\left(1-|\varphi(\omega)|^2\right)^3}{\left(1-z\overline{\varphi(\omega)}\right)^3 \log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \quad (42)$$

It is easy to see that

$$h_\omega^{(n)}(z) = (n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+1}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} - 2\cdot(n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} + (n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)};$$

$$h_\omega^{(n+1)}(z) = (n+1)\cdot(n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} - 2(n+2)\cdot(n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} + (n+3)!\frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)};$$

$$h_\omega^{(n+2)}(z) = (n+1)(n+2)\cdot(n+2)!\frac{\left(1-|\varphi(\omega)|^2\right)\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} - 2(n+2)\cdot(n+3)!\frac{\left(1-|\varphi(\omega)|^2\right)^2\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} + (n+4)!\frac{\left(1-|\varphi(\omega)|^2\right)^3\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \quad (43)$$

From Lemma 2 we obtain that $h_\omega \in \mathscr{B}_{\log,0}$ and $\sup_{\omega\in\mathbb{D}}\|h_\omega\|_{\mathscr{B}_{\log}} \le C$ with a direct calculation. From (43), we have $h_\omega^{(n)}(\varphi(\omega)) = h_\omega^{(n+1)}(\varphi(\omega)) = 0$,

$$h_\omega^{(n+2)}(\varphi(\omega)) = 2\cdot(n+2)!\frac{\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \quad (44)$$

Hence

$$\begin{aligned} C &\ge \left\|\mathscr{D}_{\varphi,u}^n h_\omega\right\|_{\mathscr{Z}_\mu} \\ &\ge \sup_{z\in\mathbb{D}}\mu(z)\left|\left(\mathscr{D}_{\varphi,u}^n h_\omega\right)''(z)\right| \\ &= \sup_{z\in\mathbb{D}}\mu(z)\Big|u''(z)h_\omega^{(n)}(\varphi(z)) + \left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)h_\omega^{(n+1)}(\varphi(z)) + u(z)\left(\varphi'(z)\right)^2 h_\omega^{(n+2)}(\varphi(z))\Big| \\ &\ge \mu(\omega)\Big|u''(\omega)h_\omega^{(n)}(\varphi(\omega)) + \left(2u'(\omega)\varphi'(\omega)+u(\omega)\varphi''(\omega)\right)h_\omega^{(n+1)}(\varphi(\omega)) + u(\omega)\left(\varphi'(\omega)\right)^2 h_\omega^{(n+2)}(\varphi(\omega))\Big| \\ &= 2\cdot(n+2)!\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)}. \end{aligned} \quad (45)$$

By (45), we obtain that

$$\begin{aligned} &\sup_{1/2<|\varphi(\omega)|<1}\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\le 2^{n+2}\sup_{1/2<|\varphi(\omega)|<1} 2\cdot(n+2)!\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\le 2^{n+2}\sup_{\omega\in\mathbb{D}} 2\cdot(n+2)!\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\le C2^{n+2} < \infty. \end{aligned} \quad (46)$$

By (27), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\le 1/2}\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(\omega)|\right)\right)} \\ &\le \sup_{|\varphi(\omega)|\le 1/2}\frac{\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right|}{\left(1-|\varphi(\omega)|^2\right)^{n+2}\log 2} \\ &\le \left(\frac{4}{3}\right)^{n+2}\frac{1}{\log 2}\sup_{|\varphi(\omega)|\le 1/2}\mu(\omega)\left|u(\omega)\left(\varphi'(\omega)\right)^2\right| \\ &\le \left(\frac{4}{3}\right)^{n+2}\frac{K_3}{\log 2} < \infty. \end{aligned} \quad (47)$$

Thus combining (47) with (46) we get the condition (19), finishing the proof of the theorem. □

Theorem 6. *Let $u \in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then the following statements are equivalent:*

(1) $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_\mu$ *is compact;*

(2) $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_\mu$ *is compact;*

(3) $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_\mu$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^n \log(2/(1-|\varphi(z)|))} = 0, \quad (48)$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1} \log(2/(1-|\varphi(z)|))} = 0, \quad (49)$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{n+2} \log(2/(1-|\varphi(z)|))} = 0. \quad (50)$$

Proof. (3) $\Rightarrow$ (1). Assume that $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_\mu$ is bounded and that conditions (48), (49), and (50) hold. For any bounded sequence $\{f_k\}$ in $\mathscr{B}_{\log}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 3, to show that

$$\|\mathscr{D}^n_{\varphi,u} f_k\|_{\mathscr{Z}_\mu} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \quad (51)$$

We assume that $\|f_k\|_{\mathscr{B}_{\log}} \le 1$. From (48), (49), and (50) we have that, for any $\varepsilon > 0$, there exists $\rho \in (0,1)$; when $\rho < |\varphi(z)| < 1$, we have

$$\begin{gathered}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^n \log(2/(1-|\varphi(z)|))} < \varepsilon, \\ \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1} \log(2/(1-|\varphi(z)|))} < \varepsilon, \\ \frac{\mu(z)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{n+2} \log(2/(1-|\varphi(z)|))} < \varepsilon.\end{gathered} \quad (52)$$

From the boundedness of $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_\mu$ by Theorem 5, we see that (23), (25), and (27) hold. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(n)}$, $f_k^{(n+1)}$, and $f_k^{(n+2)}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$; there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\begin{aligned}&\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)(0)\right| + \left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)'(0)\right| \\ &\quad + \sup_{|\varphi(z)|\le\rho} \mu(z)\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)''(z)\right| \\ &\le |u(0)|\left|f_k^{(n)}(\varphi(0))\right| + |u'(0)|\left|f_k^{(n)}(\varphi(0))\right| \\ &\quad + |u(0)|\left|f_k^{(n+1)}(\varphi(0))\right||\varphi'(0)| \\ &\quad + \sup_{|\varphi(z)|\le\rho} \mu(z)|u''(z)|\left|f_k^{(n)}(\varphi(z))\right| \\ &\quad + \sup_{|\varphi(z)|\le\rho} \mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|\left|f_k^{(n+1)}(\varphi(z))\right| \\ &\quad + \sup_{|\varphi(z)|\le\rho} \mu(z)\left|u(z)(\varphi'(z))^2\right|\left|f_k^{(n+2)}(\varphi(z))\right| \\ &\le |u(0)|\left|f_k^{(n)}(\varphi(0))\right| + |u'(0)|\left|f_k^{(n)}(\varphi(0))\right| \\ &\quad + |u(0)||\varphi'(0)|\left|f_k^{(n+1)}(\varphi(0))\right| \\ &\quad + K_1 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n)}(\varphi(z))\right| + K_2 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n+1)}(\varphi(z))\right| \\ &\quad + K_3 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n+2)}(\varphi(z))\right| < C\varepsilon.\end{aligned} \quad (53)$$

From (52) and (53) and Lemma 1 we have

$$\begin{aligned}&\|\mathscr{D}^n_{\varphi,u} f_k\|_{\mathscr{Z}_\mu} \\ &= \left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)(0)\right| + \left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)'(0)\right| + \sup_{z\in\mathbb{D}}\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)''(z)\right| \\ &\le \Bigg(\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)(0)\right| + \left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)'(0)\right| \\ &\quad + \sup_{|\varphi(z)|\le\rho}\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)''(z)\right|\Bigg) \\ &\quad + \sup_{\rho<|\varphi(z)|<1}\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u} f_k\right)''(z)\right| \\ &< C\varepsilon + \sup_{\rho<|\varphi(z)|<1}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^n\log(2/(1-|\varphi(z)|))}C\|f\|_{\mathscr{B}_{\log}} \\ &\quad + \sup_{\rho<|\varphi(z)|<1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1}\log(2/(1-|\varphi(z)|))}C\|f\|_{\mathscr{B}_{\log}}\end{aligned}$$

$$+\sup_{\rho<|\varphi(z)|<1}\frac{\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|}{\left(1-|\varphi(z)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(z)|\right)\right)}C\|f\|_{\mathcal{B}_{\log}}$$
$$<4C\varepsilon, \tag{54}$$

when $K>K_0$. It follows that the operator $\mathcal{D}^n_{\varphi,u}:\mathcal{B}_{\log}\to\mathcal{Z}_\mu$ is compact.

(1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Assume that $\mathcal{D}^n_{\varphi,u}:\mathcal{B}_{\log,0}\to\mathcal{Z}_\mu$ is compact. Then it is clear that $\mathcal{D}^n_{\varphi,u}:\mathcal{B}_{\log,0}\to\mathcal{Z}_\mu$ is bounded. By Theorem 5 we get that $\mathcal{D}^n_{\varphi,u}:\mathcal{B}_{\log}\to\mathcal{Z}_\mu$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to1$ as $k\to\infty$. We can use the test functions

$$\begin{aligned}f_k(z)&=f_{z_k}(z)\\&=(n+2)(n+3)\frac{1-|\varphi(z_k)|^2}{\left(1-z\overline{\varphi(z_k)}\right)\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad-2(n+3)\frac{\left(1-|\varphi(z_k)|^2\right)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad+2\frac{\left(1-|\varphi(z_k)|^2\right)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}.\end{aligned}\tag{55}$$

Note that

$$\begin{aligned}|f_k(z)|&\le\left|\frac{(n+2)(n+3)\left(1-|\varphi(z_k)|^2\right)}{\left(1-z\overline{\varphi(z_k)}\right)\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\right|\\&\quad+\left|\frac{2(n+3)\left(1-|\varphi(z_k)|^2\right)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\right|\\&\quad+\left|\frac{2\left(1-|\varphi(z_k)|^2\right)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\right|\\&\le\frac{(n+2)(n+3)\left(1+|\varphi(z_k)|\right)\left(1-|\varphi(z_k)|\right)}{\left(1-|\varphi(z_k)|\right)\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad+\frac{2(n+3)\left(1+|\varphi(z_k)|\right)^2\left(1-|\varphi(z_k)|\right)^2}{\left(1-|\varphi(z_k)|\right)^2\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad+\frac{2\left(1+|\varphi(z_k)|\right)^3\left(1-|\varphi(z_k)|\right)^3}{\left(1-|\varphi(z_k)|\right)^3\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\le\frac{2(n+2)(n+3)}{\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}+\frac{8(n+3)}{\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad+\frac{16}{\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&=\frac{2n^2+18n+52}{\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\longrightarrow0,\quad(k\longrightarrow\infty),\end{aligned}\tag{56}$$

for $|z|<1$. We see that f_k converges to 0 uniformly on $\mathbb{D}$; hence, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ and from (30) and (33) we have $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}_{\log}}\le C$, $f_k\in\mathcal{B}_{\log,0}$. Then f_k is a bounded sequence in $\mathcal{B}_{\log,0}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 3, we have

$$\lim_{k\to\infty}\left\|\mathcal{D}^n_{\varphi,u}f_k\right\|_{\mathcal{Z}_\mu}=0.\tag{57}$$

Note that

$$\begin{aligned}f_k^{(n+1)}\left(\varphi(z_k)\right)&=f_k^{(n+2)}\left(\varphi(z_k)\right)=0,\\f_k^{(n)}\left(\varphi(z_k)\right)&=2\cdot n!\frac{\left(\overline{\varphi(z_k)}\right)^n}{\left(1-|\varphi(z_k)|^2\right)^n\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}.\end{aligned}\tag{58}$$

From (33) and using the compactness of $\mathcal{D}^n_{\varphi,u}:\mathcal{B}_{\log,0}\to\mathcal{Z}_\mu$ we obtain

$$\begin{aligned}&2\cdot n!\frac{\mu(z_k)\left|u''(z_k)\right|\left|\overline{\varphi(z_k)}\right|^n}{\left(1-|\varphi(z_k)|^2\right)^n\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad\le\left\|\mathcal{D}^n_{\varphi,u}f_k\right\|_{\mathcal{Z}_\mu}\longrightarrow0,\quad\text{as }k\longrightarrow\infty.\end{aligned}\tag{59}$$

From (59) and $|\varphi(z_k)|\to1$, it follows that

$$\lim_{k\to\infty}\frac{\mu(z_k)\left|u''(z_k)\right|}{\left(1-|\varphi(z_k)|^2\right)^n\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}=0\tag{60}$$

and consequently (48) holds.

Next, let

$$\begin{aligned}g_k(z)&=g_{z_k}(z)\\&=(n+1)(n+3)\frac{1-|\varphi(z_k)|^2}{\left(1-z\overline{\varphi(z_k)}\right)\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad-(2n+5)\frac{\left(1-|\varphi(z_k)|^2\right)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}\\&\quad+2\frac{\left(1-|\varphi(z_k)|^2\right)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\left(2/\left(1-|\varphi(z_k)|\right)\right)}.\end{aligned}\tag{61}$$

By a direct calculation, we obtain that g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, $g_k\in\mathcal{B}_{\log,0}$, and $\sup_{k\in\mathbb{N}}\|g_k\|_{\mathcal{B}_{\log}}\le C$. By Lemma 3, we have

$$\lim_{k\to\infty}\left\|\mathcal{D}^n_{\varphi,u}g_k\right\|_{\mathcal{Z}_\mu}=0.\tag{62}$$

Note that

$$g_k^{(n+1)}(\varphi(z_k)) = -(n+1)!\frac{\left(\overline{\varphi(z_k)}\right)^{n+1}}{\left(1-|\varphi(z_k)|^2\right)^{n+1}\log(2/(1-|\varphi(z_k)|))},$$
$$g_k^{(n)}(\varphi(z_k)) = g_\omega^{(n+2)}(\varphi(z_k)) = 0. \quad (63)$$

From (39) and using the compactness of $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log,0}\to\mathscr{Z}_\mu$ we obtain

$$(n+1)!\frac{\mu(z_k)\left|2u'(z_k)\varphi'(z_k)+u(z_k)\varphi''(z_k)\right|\left|\overline{\varphi(z_k)}\right|^{n+1}}{\left(1-|\varphi(z_k)|^2\right)^{n+1}\log(2/(1-|\varphi(z_k)|))} \le \left\|\mathscr{D}^n_{\varphi,u}g_k\right\|_{\mathscr{Z}_\mu}\longrightarrow 0, \quad \text{as } k\longrightarrow\infty. \quad (64)$$

From (64) and $|\varphi(z_k)|\to 1$, we have

$$\lim_{k\to\infty}\frac{\mu(z_k)\left|2u'(z_k)\varphi'(z_k)+u(z_k)\varphi''(z_k)\right|}{\left(1-|\varphi(z_k)|^2\right)^{n+1}\log(2/(1-|\varphi(z_k)|))}=0; \quad (65)$$

it implies that (49) holds.

In order to prove (50), choose

$$\begin{aligned} h_k(z) &= h_{z_k}(z) \\ &= (n+1)(n+2)\frac{1-|\varphi(z_k)|^2}{\left(1-z\overline{\varphi(z_k)}\right)\log(2/(1-|\varphi(z_k)|))} \\ &\quad -2(n+2)\frac{\left(1-|\varphi(z_k)|^2\right)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log(2/(1-|\varphi(z_k)|))} \\ &\quad +2\frac{\left(1-|\varphi(z_k)|^2\right)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log(2/(1-|\varphi(z_k)|))}. \end{aligned} \quad (66)$$

By a direct calculation, we may easily prove that h_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, $h_k\in\mathscr{B}_{\log,0}$, and $\sup_{k\in\mathbb{N}}\|h_k\|_{\mathscr{B}_{\log}}\le C$. By Lemma 3, we have

$$\lim_{k\to\infty}\left\|\mathscr{D}^n_{\varphi,u}h_k\right\|_{\mathscr{Z}_\mu}=0. \quad (67)$$

Note that

$$h_k^{(n+2)}(\varphi(z_k)) = 2\cdot(n+2)!\frac{\left(\overline{\varphi(z_k)}\right)^{n+2}}{\left(1-|\varphi(z_k)|^2\right)^{n+2}\log(2/(1-|\varphi(z_k)|))},$$
$$h_k^{(n)}(\varphi(z_k)) = h_k^{(n+1)}(\varphi(z_k)) = 0. \quad (68)$$

From (45) and using the compactness of $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log,0}\to\mathscr{Z}_\mu$ we obtain

$$2\cdot(n+2)!\frac{\mu(z_k)\left|u(z_k)(\varphi'(z_k))^2\right|\left|\overline{\varphi(z_k)}\right|^{n+2}}{\left(1-|\varphi(z_k)|^2\right)^{n+2}\log(2/(1-|\varphi(z_k)|))} \le \left\|\mathscr{D}^n_{\varphi,u}h_k\right\|_{\mathscr{Z}_\mu}\longrightarrow 0, \quad \text{as } k\longrightarrow\infty. \quad (69)$$

From (69) and $|\varphi(z_k)|\to 1$, it follows that

$$\lim_{k\to\infty}\frac{\mu(z_k)\left|u(z_k)(\varphi'(z_k))^2\right|}{\left(1-|\varphi(z_k)|^2\right)^{n+2}\log(2/(1-|\varphi(z_k)|))}=0, \quad (70)$$

and consequently (50) holds, finishing the proof of the theorem. □

Theorem 7. *Let $u\in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}\to\mathscr{Z}_{\mu,0}$ is a bounded operator provided that the following conditions are satisfied:*

$$\lim_{|z|\to 1}\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log(2/(1-|\varphi(z)|))}=0, \quad (71)$$

$$\lim_{|z|\to 1}\frac{\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}\log(2/(1-|\varphi(z)|))}=0, \quad (72)$$

$$\lim_{|z|\to 1}\frac{\mu(z)\left|u(z)(\varphi'(z))^2\right|}{\left(1-|\varphi(z)|^2\right)^{n+2}\log(2/(1-|\varphi(z)|))}=0. \quad (73)$$

Proof. Suppose that (71), (72), and (73) hold. It is clear that (17), (18), and (19) hold. By Theorem 5 we have that $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}\to\mathscr{Z}_\mu$ is bounded. In order to prove $\mathscr{D}^n_{\varphi,u}:\mathscr{B}_{\log}\to\mathscr{Z}_{\mu,0}$ is bounded, it is enough to show that, for any $f\in\mathscr{B}_{\log}$, $\mathscr{D}^n_{\varphi,u}f\in\mathscr{Z}_{\mu,0}$. Using (71), (72), and (73) we have that, for any $\varepsilon>0$, there is a constant $0<\eta<1$, such that $\eta<|z|<1$ implies

$$\begin{gathered} \frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log(2/(1-|\varphi(z)|))}<\varepsilon, \\ \frac{\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}\log(2/(1-|\varphi(z)|))}<\varepsilon, \\ \frac{\mu(z)\left|u(z)(\varphi'(z))^2\right|}{\left(1-|\varphi(z)|^2\right)^{n+2}\log(2/(1-|\varphi(z)|))}<\varepsilon. \end{gathered} \quad (74)$$

Then, for any $f \in \mathscr{B}_{\log}$, from Lemma 1 we obtain that

$$\begin{aligned}
&\mu(z)\left|\left(\mathscr{D}_{\varphi,u}^{n}f\right)''(z)\right| \\
&= \mu(z)\Big|u''(z)f^{(n)}(\varphi(z)) \\
&\qquad + \left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)f^{(n+1)}(\varphi(z)) \\
&\qquad + u(z)\left(\varphi'(z)\right)^2 f^{(n+2)}(\varphi(z))\Big| \\
&\le \mu(z)\left|u''(z)\right|\left|f^{(n)}(\varphi(z))\right| \\
&\quad + \mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|\left|f^{(n+1)}(\varphi(z))\right| \\
&\quad + \mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|\left|f^{(n+2)}(\varphi(z))\right| \\
&\le C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log\left(2/\left(1-|\varphi(z)|\right)\right)} \\
&\quad + C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}\log\left(2/\left(1-|\varphi(z)|\right)\right)} \\
&\quad + C\|f\|_{\mathscr{B}_{\log}}\frac{\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|}{\left(1-|\varphi(z)|^2\right)^{n+2}\log\left(2/\left(1-|\varphi(z)|\right)\right)} \\
&< 3C\|f\|_{\mathscr{B}_{\log}}\varepsilon,
\end{aligned} \tag{75}$$

when $\eta < |z| < 1$. Hence $\mathscr{D}_{\varphi,u}^{n}f \in \mathscr{Z}_{\mu,0}$ for all $f \in \mathscr{B}_{\log}$, completing the proof of the theorem. □

Theorem 8. *Let $u \in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. If $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu,0}$ is a bounded operator, then* (17), (18), *and* (19) *hold and the following conditions are satisfied:*

$$\lim_{|z|\to 1}\mu(z)\left|u''(z)\right| = 0, \tag{76}$$

$$\lim_{|z|\to 1}\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right| = 0, \tag{77}$$

$$\lim_{|z|\to 1}\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right| = 0. \tag{78}$$

Proof. Assume that $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu,0}$ is bounded; it is clear that $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu}$ is bounded. By Theorem 5 we have that (17), (18), and (19) hold. On the other hand, for all $f \in \mathscr{B}_{\log}$, $\mathscr{D}_{\varphi,u}^{n}f \in \mathscr{Z}_{\mu,0}$. Take $f(z) = z^n/n! \in \mathscr{B}_{\log}$; we have that

$$\lim_{|z|\to 1}\mu(z)\left|u''(z)\right| = 0; \tag{79}$$

then (76) holds. Let $f(z) = z^{n+1}/(n+1)! \in \mathscr{B}_{\log}$; we have that

$$\lim_{|z|\to 1}\mu(z)\left|u''(z)\varphi(z)+2u'(z)\varphi'(z)+u(z)\varphi''(z)\right| = 0. \tag{80}$$

By (80), (76), and the boundedness of the function $\varphi(z)$, we get

$$\lim_{|z|\to 1}\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right| = 0. \tag{81}$$

Hence, (77) holds. In the same way, let $f(z) = z^{n+2}/(n+2)! \in \mathscr{B}_{\log}$; we have that

$$\begin{aligned}
\lim_{|z|\to 1}\mu(z)\Big|&u''(z)(\varphi(z))^2 \\
&+ 2\left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)\varphi(z) \\
&+2u(z)\left(\varphi'(z)\right)^2\Big| = 0.
\end{aligned} \tag{82}$$

By (76), (77), (82), and the boundedness of the function $\varphi(z)$, we have that

$$\lim_{|z|\to 1}\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right| = 0. \tag{83}$$

That is, (78) holds. The proof is completed. □

Theorem 9. *Let $u \in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ is a bounded operator if and only if $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu}$ is a bounded operator and* (76), (77), *and* (78) *hold.*

Proof. Assume that $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ is a bounded operator; it is clear that $\mathscr{D}_{\varphi,u}^{n} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu}$ is a bounded operator. On the other hand, for all $f \in \mathscr{B}_{\log,0}$, $\mathscr{D}_{\varphi,u}^{n}f \in \mathscr{Z}_{\mu,0}$. Taking $f(z) = z^n/n! \in \mathscr{B}_{\log,0}$, we have that

$$\lim_{|z|\to 1}\mu(z)\left|u''(z)\right| = 0; \tag{84}$$

then (76) holds. Let $f(z) = z^{n+1}/(n+1)! \in \mathscr{B}_{\log,0}$; we have that

$$\lim_{|z|\to 1}\mu(z)\left|u''(z)\varphi(z)+2u'(z)\varphi'(z)+u(z)\varphi''(z)\right| = 0. \tag{85}$$

By (85), (76), and the boundedness of the function $\varphi(z)$, we get

$$\lim_{|z|\to 1}\mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right| = 0. \tag{86}$$

Hence, (77) holds. In the same way, take $f(z) = z^{n+2}/(n+2)! \in \mathscr{B}_{\log,0}$; we have that

$$\begin{aligned}
\lim_{|z|\to 1}\mu(z)\Big|&u''(z)(\varphi(z))^2 \\
&+ 2\left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)\varphi(z) \\
&+ 2u(z)\left(\varphi'(z)\right)^2\Big| = 0.
\end{aligned} \tag{87}$$

By (76), (77), (87), and the boundedness of the function $\varphi(z)$, we have that

$$\lim_{|z|\to 1}\mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right| = 0. \tag{88}$$

That is, (78) holds.

Conversely, suppose that $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_\mu$ is a bounded operator and (76), (77), and (78) hold. For each polynomial $p(z)$ we get that

$$\begin{aligned}
&\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u}p\right)''(z)\right| \\
&\quad= \mu(z)\left|u''(z)\,p^{(n)}(\varphi(z)) + \left(2u'(z)\varphi'(z)+u(z)\varphi''(z)\right)p^{(n+1)}(\varphi(z)) + u(z)\left(\varphi'(z)\right)^2 p^{(n+2)}(\varphi(z))\right| \\
&\quad\le \mu(z)\left|u''(z)\right|\left|p^{(n)}(\varphi(z))\right| + \mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|\left|p^{(n+1)}(\varphi(z))\right| + \mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|\left|p^{(n+2)}(\varphi(z))\right| \\
&\quad\le \mu(z)\left|u''(z)\right|\left\|p^{(n)}\right\|_\infty + \mu(z)\left|2u'(z)\varphi'(z)+u(z)\varphi''(z)\right|\left\|p^{(n+1)}\right\|_\infty + \mu(z)\left|u(z)\left(\varphi'(z)\right)^2\right|\left\|p^{(n+2)}\right\|_\infty \longrightarrow 0 \\
&\qquad (\text{as } |z|\longrightarrow 1).
\end{aligned} \tag{89}$$

Hence, $\mathscr{D}^n_{\varphi,u}p \in \mathscr{Z}_{\mu,0}$. On the other hand, since polynomials are dense in $\mathscr{B}_{\log,0}$, thus, for each $f \in \mathscr{B}_{\log,0}$, there is a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\left\|p_k - f\right\|_{\mathscr{B}_{\log}} = 0. \tag{90}$$

Since $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_\mu$ is a bounded operator, by Theorem 5 we have $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_\mu$ is a bounded operator. Since

$$\left\|\mathscr{D}^n_{\varphi,u}p_k - \mathscr{D}^n_{\varphi,u}f\right\|_{\mathscr{Z}_\mu} \le \left\|\mathscr{D}^n_{\varphi,u}\right\|\left\|p_k - f\right\|_{\mathscr{B}_{\log}} \tag{91}$$

and $\mathscr{Z}_{\mu,0}$ is the closed subset of $\mathscr{Z}_\mu$, we see that $\mathscr{D}^n_{\varphi,u}f \in \mathscr{Z}_{\mu,0}$; thus $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ is a bounded operator. The proof is completed. □

Theorem 10. *Let $u \in H(\mathbb{D})$, and let n be a nonnegative integer, φ a holomorphic self-map of $\mathbb{D}$, and μ a weight. Then the following statements are equivalent:*

(1) $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu,0}$ *is compact;*

(2) $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ *is compact;*

(3) (71), (72), *and* (73) *hold.*

Proof. (3) $\Rightarrow$ (1). Suppose that (71) (72), and (73) hold. By Theorem 7 we know that $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu,0}$ is bounded. Taking the supremum in inequality (20) over all $f \in \mathscr{B}_{\log}$ such that $\|f\|_{\mathscr{B}_{\log}} \le 1$ and letting $|z| \to 1$ yields

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathscr{B}_{\log}}\le 1}\mu(z)\left|\left(\mathscr{D}^n_{\varphi,u}f\right)''(z)\right| = 0. \tag{92}$$

Hence, by Lemma 4, we see that the operator $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log} \to \mathscr{Z}_{\mu,0}$ is compact.

(1) $\Rightarrow$ (2). This implication is clear.

(2) $\Rightarrow$ (3). Assume that $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ is compact. Firstly, it is obvious $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_{\mu,0}$ is bounded. By Theorem 9 we have that (76), (77), and (78) hold. On the other hand, we have that $\mathscr{D}^n_{\varphi,u} : \mathscr{B}_{\log,0} \to \mathscr{Z}_\mu$ is compact. By Theorem 6 we have that (48), (49), and (50) hold. We prove that (76) and (48) imply (71). The proof of (72) and (73) is similar; hence, it will be omitted. From (48), it follows that, for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log\left(2/\left(1-|\varphi(z)|\right)\right)} < \varepsilon, \tag{93}$$

when $\delta < |\varphi(z)| < 1$. Using (76) we see that there exists $\tau \in (0,1)$ such that

$$\mu(z)\left|u''(z)\right| \le \varepsilon\inf_{t\in[0,\delta]}\left(1-t^2\right)^n\log 2, \tag{94}$$

when $\tau < |z| < 1$. Therefore when $\tau < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (93), we have

$$\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log\left(2/\left(1-|\varphi(z)|\right)\right)} < \varepsilon. \tag{95}$$

On the other hand, when $\tau < |z| < 1$ and $|\varphi(z)| \le \delta$, by (94), we obtain

$$\begin{aligned}
&\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log\left(2/\left(1-|\varphi(z)|\right)\right)} \\
&\quad\le \frac{\mu(z)\left|u''(z)\right|}{\inf_{t\in[0,\delta]}\left(1-t^2\right)^n\log 2} < \varepsilon.
\end{aligned} \tag{96}$$

From (95) and (96) we have

$$\lim_{|z|\to 1}\frac{\mu(z)\left|u''(z)\right|}{\left(1-|\varphi(z)|^2\right)^n\log\left(2/\left(1-|\varphi(z)|\right)\right)} = 0; \tag{97}$$

we obtain that (71) holds, as desired. The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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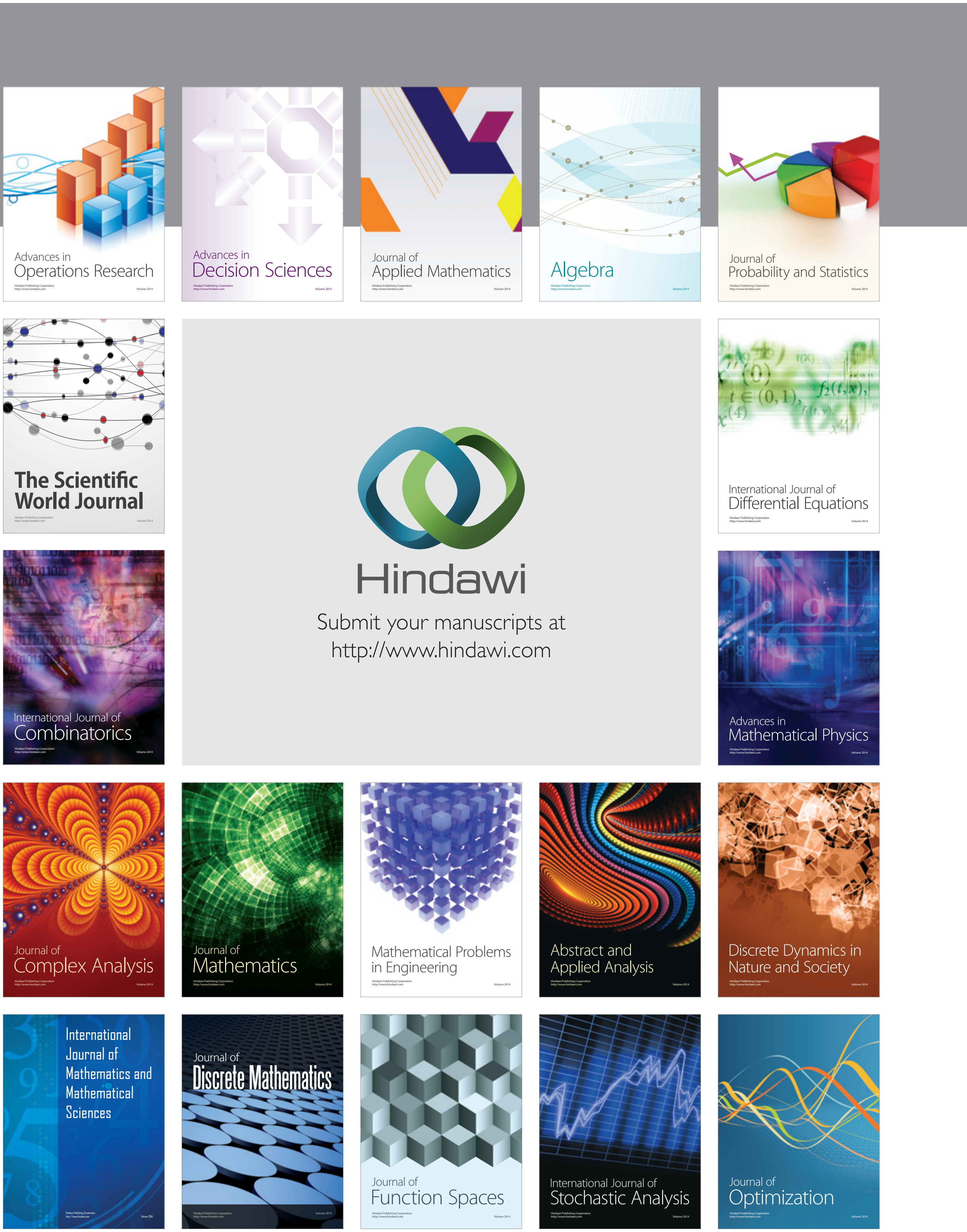
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