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Research Article

A New Iterative Scheme of Modified Mann Iteration in Banach Space

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We introduce the modified iterations of Mann's type for nonexpansive mappings and asymptotically nonexpansive mappings to have the strong convergence in a uniformly convex Banach space. We study approximation of common fixed point of asymptotically nonexpansive mappings in Banach space by using a new iterative scheme. Applications to the accretive operators are also included.

1. Introduction

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \rightarrow C$ a mapping. Recall that T is a nonexpansive mapping [1] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all integers $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $\text{Fix}(T)$ the set of fixed points of T; that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$.

Iterative methods are often used to solve the fixed point equation $Tx = x$. One classical iteration process is introduced in 1953 by Mann [3] which is well known as Mann iteration process and is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1}$$

where the sequence $\{\alpha_n\}$ is chosen in $(0, 1)$ and the initial guess $x_0 \in C$ is arbitrarily chosen.

There exists rich literature on the convergence of Mann iteration for different classes of operators considered on various spaces. Mann's iteration method (1) has been proved to be a powerful method for solving nonlinear operator equations involving nonexpansive mapping, asymptotically nonexpansive mapping, and other kinds of nonlinear mapping; see [3–11] and the references therein.

It is known that Mann's iteration method (1) is in general not strongly convergent for nonexpansive mappings. So to get strong convergence, one has to modify the iteration method (1). In this regard, we will show the modified iteration in Section 3.

Motivated and inspired by the research going on in these fields, we suggest and analyze now new modified Mann's iteration for finding the common fixed point of the nonexpansive mappings and asymptotically nonexpansive mappings in Banach space. We propose the modified Mann's iteration and consider the strong convergence of the approximate solutions for nonexpansive and asymptotically nonexpansive in Banach space.

We suggest and analyze the following iterative:

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ &z_n = \gamma_n x_n + (1 - \gamma_n) S x_n, \\ &x_{n+1} = \alpha_n y_n + (1 - \alpha_n) R z_n, \quad n \geq 0, \end{aligned} \tag{2}$$

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x_n + (1 - \beta_n) T^n x_n, \\ &z_n = \gamma_n x_n + (1 - \gamma_n) S^n x_n, \\ &x_{n+1} = \alpha_n y_n + (1 - \alpha_n) z_n, \quad n \geq 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x_n + (1-\beta_n) J_{1,n} x_n, \\ &z_n = \gamma_n x_n + (1-\gamma_n) J_{2,n} x_n, \\ &x_{n+1} = \alpha_n y_n + (1-\alpha_n) J_{3,n} z_n, \quad n \geq 0, \end{aligned} \tag{4}$$

and if there exists two sequences $\{x'_n\}$ and $\{x''_n\}$ generated by

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x'_n + (1-\beta_n) T^n x'_n, \\ &z_n = \gamma_n x''_n + (1-\gamma_n) S^n x''_n, \\ &x'_{n+1} = \alpha_n z_n + (1-\alpha_n) y_n, \\ &x''_{n+1} = \alpha_n y_n + (1-\alpha_n) z_n, \quad n \geq 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_n = \beta_n x'_n + (1-\beta_n) J_{1,n} x'_n, \\ &z_n = \gamma_n x''_n + (1-\gamma_n) J_{2,n} x''_n, \\ &x'_{n+1} = \alpha_n z_n + (1-\alpha_n) y_n, \\ &x''_{n+1} = \alpha_n y_n + (1-\alpha_n) z_n, n \geq 0. \end{aligned} \tag{6}$$

Our second modification of Mann's iteration method (1) is adaption to (2) for finding a zero of an m-accretive operator A, for which we assume that the zero set $A^{-1}(0) \neq \emptyset$. Our iterations process $\{x_n\}$ is given by (4), and sequences $\{x'_n\}$ and $\{x''_n\}$ are as follows (6), where, for each $r > 0$, $J_r = (I + rA)^{-1}$ is the resolvent of A. We prove that not only $\{x_n\}_n^\infty$ defined by (4) but also $\{x'_n\}$ and $\{x''_n\}$ generated by (6) converge strongly to a zero of A under certain assumptions in a uniformly Banach space.

We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Using F is to denote the set of common fixed point of the mappings T, S, and R, and using $\overline{F}$ is to denote the set of common fixed points of the mappings T and S.

2. Preliminaries

This section collects some lemmas, which will be used in the proofs for the main results in the next section.

Lemma 1 (see [8]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad n \geq 1. \tag{7}$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2 (see [12]). *Suppose that E is a uniformly convex Banach space and $0 < t_n < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$; then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3 (see [10]). *A mapping $T: C \to C$ with a nonempty fixed point set F in C will be said to satisfy Condition (I).*

If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 4 (see [13]). *Given a number $r > 0$, let E be a uniformly convex Banach space; then there exists a continuous strictly increasing function $\varphi : [0, \infty] \to [0, \infty]$ with $\varphi(0) = 0$, such that*

$$\begin{aligned} \|\lambda x + \mu y + \gamma z\|^2 \leq{}& \lambda \|x\|^2 + \mu \|y\|^2 \\ &+ \gamma \|z\|^2 - \lambda\mu\varphi(\|x - y\|), \end{aligned} \tag{8}$$

for all $\lambda, \mu, \gamma \in [0, 1]$ and $x, y, z \in E$ such that $\|x\| \leq r$, $\|y\| \leq r$ and $\|z\| \leq r$.

Lemma 5 (see [14]). *Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^\infty \alpha_n = \infty$. If $\sum_{n=1}^\infty \alpha_n \beta_n < \infty$, then* $\liminf_{n\to\infty} \beta_n = 0$.

Lemma 6 (see [15]). *For $\lambda > 0$, $\mu > 0$, and $x \in E$, the following identity holds*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda x \right). \tag{9}$$

3. Convergence to a Common Fixed Point of Nonexpansive Mappings

In this part, we prove our main theorem for finding a common fixed point of nonexpansive mappings in Banach space.

Theorem 7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and let T, S, and R be three nonexpansive commuting mappings of C satisfying Condition (I) and $F \neq \emptyset$. Given that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\sum \alpha_n < \infty$, $\sum \gamma_n \beta_n = \infty$, $\sum (1 - \gamma_n) < \infty$ for all $n \geq 1$.*

Define a sequence $\{x_n\}_{n=0}^\infty$ in C by algorithm (2); then $\{x_n\}_{n=0}^\infty$ strongly converges to a common fixed point of T, S, and R.

Proof. First, we observe that $\{x_n\}$ is bounded; if we take an arbitrary fixed point q of F, noting that $\|y_n - q\| \leq \|x_n - q\|$ and $\|z_n - q\| \leq \|x_n - q\|$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n y_n + (1-\alpha_n) R z_n - q\| \\ &\leq \alpha_n \|y_n - R z_n\| + \|R z_n - q\| \\ &\leq (1 + 2\alpha_n) \|x_n - q\|. \end{aligned} \tag{10}$$

By Lemma 1 and $\sum \alpha_n < \infty$, $\lim_{n\to\infty} \|x_n - q\|$ exists. Denote

$$\lim_{n\to\infty} \|x_n - q\| = c. \tag{11}$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Now

$$\begin{aligned}\|x_{n+1}-q\| &= \|\alpha_n y_n + (1-\alpha_n) R z_n - q\| \\ &\le \alpha_n \|y_n - R z_n\| + \|z_n - q\|.\end{aligned} \tag{12}$$

Since $\|z_n - q\| \le \|x_n - q\|$, this implies that

$$\lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|x_n - q\| = c. \tag{13}$$

Moreover, $\|Sx_n - q\| \le \|x_n - q\|$ implies that

$$\limsup_{n\to\infty} \|Sx_n - q\| \le c. \tag{14}$$

Thus,

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|\gamma_n x_n + (1-\gamma_n) S x_n - q\| \\ &= \lim_{n\to\infty} \|\gamma_n (x_n - q) + (1-\gamma_n)(Sx_n - q)\|, \end{aligned} \tag{15}$$

given by Lemma 2 that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{16}$$

By (10) and $\sum \alpha_n < \infty$, then we have

$$\begin{aligned}\|x_{n+m} - q\| &\le (1 + 2\alpha_{n+m-1}) \|x_{n+m-1} - q\| \\ &\le e^{2\alpha_{n+m-1}} \|x_{n+m-1} - q\| \\ &\le \cdots \\ &\le e^{2\sum_{i=n}^{n+m-1} \alpha_i} \|x_n - q\|. \end{aligned} \tag{17}$$

That is,

$$\|x_{n+m} - q\| \le M \|x_n - q\|, \tag{18}$$

where $M = e^{2\sum_{i=n}^{n+m-1} \alpha_i}$ for all $m, n \in \mathbb{N}$, for all $q \in F$ and for $M > 0$.

Next, we prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since $q \in F$ arbitrarily and $\lim_{n\to\infty}\|x_n - q\|$ exists, consequently, $d(x_n, F)$ exists by Lemma 3. From Lemma 3 and (16), we get

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{19}$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Let $\varepsilon > 0$; since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore, there exists a constant n_0 such that, for all $n \ge n_0$, we have $d(x_n, F) \le \varepsilon/2M$. There must exist $p_1 \in F$, such that

$$d(x_n, p_1) \le \frac{\varepsilon}{2M}. \tag{20}$$

From (18), it can be obtained that, when $n \ge n_0$ and $m \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\le 2M \|x_n - p_1\| \\ &\le 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned} \tag{21}$$

This implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a closed convex subset C of a Banach space E. Thus, it must converge to a point in C; let $\lim_{n\to\infty} x_n = p$.

For all $\epsilon > 0$, as $\lim_{n\to\infty} x_n = p$, thus, there exists a number n_1 such that, when $n_2 \ge n_1$,

$$\|x_{n_2} - p\| \le \frac{\epsilon}{4}. \tag{22}$$

In fact, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that using number n_2 above, when $n \ge n_2$, we have $d(x_n, F) \le \epsilon/8$. In particular, $d(x_{n_2}, F) \le \epsilon/8$. Thus, there must exist $\overline{p} \in F$, such that

$$\|x_{n_2} - \overline{p}\| = d(x_{n_2}, \overline{p}) \le \frac{\epsilon}{8}. \tag{23}$$

From (22) and (23), we get

$$\begin{aligned} &\|Sp - p\| \\ &\quad = \|Sp - \overline{p} + Sx_{n_2} - \overline{p} + \overline{p} - x_{n_2} + x_{n_2} - p + \overline{p} - Sx_{n_2}\| \\ &\quad \le \|Sp - \overline{p}\| + \|x_{n_2} - \overline{p}\| + \|x_{n_2} - p\| + 2\|Sx_{n_2} - \overline{p}\| \\ &\quad \le \|p - \overline{p}\| + 3\|x_{n_2} - \overline{p}\| + \|x_{n_2} - p\| \\ &\quad \le 4\|x_{n_2} - \overline{p}\| + 2\|x_{n_2} - p\| \\ &\quad \le \frac{4\epsilon}{8} + \frac{2\epsilon}{4} = \epsilon. \end{aligned} \tag{24}$$

As ϵ is an arbitrary positive number, thus, $Sp = p$.

Let

$$u_{n+1} = \gamma_n x_n + \beta_n T x_n + (1 - \beta_n - \gamma_n) T x_n. \tag{25}$$

Then we have

$$\begin{aligned} &\|u_{n+1} - q\|^2 \\ &\quad = \|\gamma_n x_n + \beta_n T x_n + (1-\beta_n-\gamma_n) T x_n - q\|^2 \\ &\quad \le \gamma_n \|x_n - q\|^2 + \beta_n \|Tx_n - q\|^2 \\ &\qquad + (1-\beta_n-\gamma_n)\|Tx_n - q\|^2 - \gamma_n\beta_n\varphi\|x_n - Tx_n\| \\ &\quad \le \|x_n - q\|^2 - \gamma_n\beta_n\varphi\|x_n - Tx_n\|; \end{aligned} \tag{26}$$

hence,

$$\gamma_n\beta_n\varphi\|x_n - Tx_n\| \le \|x_n - q\|^2 - \|u_{n+1} - q\|^2, \tag{27}$$

for $q \in F$. Summing from $n = 1$ to ∞, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \gamma_n\beta_n\varphi\|x_n - Tx_n\| \\ &\qquad \le \sum_{n=1}^{\infty} \left(\|x_n - q\|^2 - \|u_{n+1} - q\|^2\right) \\ &\qquad \le 2K \sum_{n=1}^{\infty} \|u_{n+1} - x_n\| \\ &\qquad \le 4K^2 \sum_{n=1}^{\infty} (1 - \gamma_n), \end{aligned} \tag{28}$$

where $K = \sup_{n\in\mathbb{N}}\{\| x_n - q \|\}$; since $\sum_{n=1}^{\infty}(1-\gamma_n) < \infty$, we get

$$\sum_{n=1}^{\infty}\gamma_n\beta_n\varphi\|x_n - Tx_n\| \leq 4K^2\sum_{n=1}^{\infty}(1-\gamma_n) < \infty. \quad (29)$$

Since $\sum_{n=1}^{\infty}\gamma_n\beta_n = \infty$, from Lemma 5, we get $\liminf_{n\to\infty}\varphi\|Tx_n - x_n\| = 0$. Hence,

$$\liminf_{n\to\infty}\|Tx_n - x_n\| = 0. \quad (30)$$

Since T and R are nonexpansive mappings, we have

$$\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&\quad = \|Tx_{n+1} - RTx_n + RTx_n + \alpha_n RTx_n \\
&\qquad -\alpha_n RTx_n - \alpha_n y_n - (1-\alpha_n) Rz_n\| \\
&\quad \leq \|Tx_{n+1} - RTx_n\| + (1-\alpha_n)\|RTx_n - Rz_n\| \\
&\qquad + \alpha_n\|RTx_n - y_n\| \\
&\quad \leq \|x_{n+1} - Rx_n\| + (1-\alpha_n)\|Tx_n - z_n\| \\
&\qquad + \alpha_n\|RTx_n - y_n\| \\
&\quad \leq \alpha_n\|y_n - Rx_n\| + 2(1-\alpha_n)\|z_n - x_n\| \\
&\qquad + (1-\alpha_n)\|Tx_n - x_n\| \\
&\qquad + \alpha_n\|Tx_n - y_n\| + \alpha_n\|Rx_n - x_n\| \\
&\quad \leq \alpha_n\|y_n - x_n\| + 2\alpha_n\|Rx_n - x_n\| \\
&\qquad + 2(1-\alpha_n)(1-\gamma_n)\|Sx_n - x_n\| \\
&\qquad + (1-\alpha_n)\|Tx_n - x_n\| + \alpha_n\beta_n\|Tx_n - x_n\| \\
&\quad \leq \|Tx_n - x_n\| + 2\alpha_n\|Rx_n - x_n\| \\
&\qquad + 2(1-\alpha_n)(1-\gamma_n)\|Sx_n - x_n\|.
\end{aligned} \quad (31)$$

Since $\sum\alpha_n < \infty$, $\sum_{n=1}^{\infty}(1-\gamma_n) < \infty$, it follows from Lemma 1 that $\lim_{n\to\infty}\|Tx_n - x_n\|$ exists. Therefore, from (30), we get

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \quad (32)$$

Let

$$v_{n+1} = \gamma_n x_n + \beta_n Rx_n + (1-\beta_n-\gamma_n) Rx_n. \quad (33)$$

Using the same argument we can get

$$\liminf_{n\to\infty}\|Rx_n - x_n\| = 0. \quad (34)$$

Since R is a nonexpansive mapping, we have

$$\begin{aligned}
&\|Rx_{n+1} - x_{n+1}\| \\
&\quad = \|Rx_{n+1} - Rx_n + Rx_n + \alpha_n Rx_n \\
&\qquad -\alpha_n Rx_n - \alpha_n y_n - (1-\alpha_n) Rz_n\| \\
&\quad \leq \|x_{n+1} - x_n\| + (1-\alpha_n)\|x_n - z_n\| \\
&\qquad + \alpha_n\|Rx_n - x_n\| + \alpha_n\|y_n - x_n\| \\
&\quad \leq \alpha_n\|Rx_n - x_n\| + 2\alpha_n\|y_n - x_n\| \\
&\qquad + (1-\alpha_n)\|Rz_n - x_n\| + (1-\alpha_n)\|z_n - x_n\| \\
&\quad \leq \|Rx_n - x_n\| + 2\alpha_n\|y_n - x_n\| + 2(1-\alpha_n)\|z_n - x_n\| \\
&\quad = \|Rx_n - x_n\| + 2\alpha_n(1-\beta_n)\|Tx_n - x_n\| \\
&\qquad + 2(1-\alpha_n)(1-\gamma_n)\|Sx_n - x_n\|.
\end{aligned} \quad (35)$$

Since $\sum\alpha_n < \infty$, $\sum(1-\gamma_n) < \infty$, it follows from Lemma 1 that $\lim_{n\to\infty}\|Rx_n - x_n\|$ exists. Therefore, from (34), we get

$$\lim_{n\to\infty}\|Rx_n - x_n\| = 0. \quad (36)$$

Then using the same argument, we can show that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of T, S, and R. □

4. Convergence to a Common Fixed Point of Asymptotically Nonexpansive Mappings

4.1. There Exists One Sequence $\{x_n\}$. In this part, we prove our main theorem for finding a common fixed point of asymptotically nonexpansive mappings in Banach space in the case of one sequence.

Theorem 8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and let T and S be two asymptotically nonexpansive mappings of C satisfying Condition (I) and $\overline{F} \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{k_n\}$ with $k_n \geq 1$ such that $\sum\alpha_n < \infty$, $\sum(1-\beta_n) < \infty$ and $\sum(k_n - 1) < \infty$ for all $n \geq 1$.*

Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by algorithm (3); then $\{x_n\}_{n=0}^{\infty}$ strongly converges to a common fixed point of T and S.

Proof. First, we observe that $\{x_n\}$ is bounded; if we take an arbitrary fixed point q of $\overline{F}$, noting that $\|y_n - q\| \leq k_n\|x_n - q\|$ and $\|z_n - q\| \leq k_n\|x_n - q\|$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n y_n + (1-\alpha_n) z_n - q\| \\
&\leq \alpha_n\|y_n - q\| + (1-\alpha_n)\|Z_n - q\| \\
&\leq (1 + k_n - 1)\|x_n - q\|.
\end{aligned} \quad (37)$$

By Lemma 1 and $\sum(k_n-1)<\infty$, thus, $\lim_{n\to\infty}\|x_n-q\|$ exists. Denote $\lim_{n\to\infty}\|x_n-q\|=c$, and put $k_\infty=\sup\{k_n:n\geq 1\}<\infty$. Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Now

$$\begin{aligned}\|x_{n+1}-q\|&=\|\alpha_n y_n+(1-\alpha_n)z_n-q\|\\&\leq\alpha_n\|y_n-z_n\|+\|z_n-q\|.\end{aligned}\tag{38}$$

Since $\|z_n-q\|\leq k_n\|x_n-q\|$, this implies that

$$\lim_{n\to\infty}\|z_n-q\|=\lim_{n\to\infty}\|x_n-q\|=c.\tag{39}$$

Moreover, $\|S^n x_n-q\|\leq k_n\|x_n-q\|$ implies that

$$\limsup_{n\to\infty}\|S^n x_n-q\|\leq c.\tag{40}$$

Thus,

$$\begin{aligned}c&=\lim_{n\to\infty}\|z_n-q\|=\lim_{n\to\infty}\|\gamma_n x_n+(1-\gamma_n)S^n x_n-q\|\\&=\lim_{n\to\infty}\|\gamma_n(x_n-q)+(1-\gamma_n)(S^n x_n-q)\|,\end{aligned}\tag{41}$$

given by Lemma 2 that

$$\lim_{n\to\infty}\|S^n x_n-x_n\|=0.\tag{42}$$

Now,

$$\begin{aligned}\|z_n-x_n\|&=\|\gamma_n x_n+(1-\gamma_n)S^n x_n-x_n\|\\&\leq(1-\gamma_n)\|(S^n x_n-x_n)\|.\end{aligned}\tag{43}$$

Hence, by (42),

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{44}$$

Also note that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|\alpha_n y_n+(1-\alpha_n)z_n-x_n\|\\&\leq\alpha_n\|y_n-x_n\|+(1-\alpha_n)\|z_n-x_n\|,\end{aligned}\tag{45}$$

so that condition $\sum\alpha_n<\infty$ and (44) give

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{46}$$

Next, we show $\lim_{n\to\infty}\|x_{n+1}-Sx_n\|=0$.

We have

$$\begin{aligned}&\|x_{n+1}-Sx_{n+1}\|\\&\quad\leq\left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+\left\|S^{n+1}x_{n+1}-S^{n+1}x_n\right\|\\&\qquad+\left\|S^{n+1}x_n-Sx_{n+1}\right\|\\&\quad\leq\left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+k_\infty\|x_{n+1}-x_n\|\\&\qquad+k_\infty\|S^n x_n-x_{n+1}\|\\&\quad\leq\left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+2k_\infty\|x_{n+1}-x_n\|\\&\qquad+k_\infty\|S^n x_n-x_n\|.\end{aligned}\tag{47}$$

Hence, by (42) and (46), we get

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.\tag{48}$$

By (37) we assume that $k_n=1+r_n$, so $\sum r_n<\infty$ for $\sum(k_n-1)<\infty$, and now,

$$\begin{aligned}\|x_{n+m}-q\|&\leq(1+r_{n+m-1})\|x_{n+m-1}-q\|\\&\leq e^{r_{n+m-1}}\|x_{n+m-1}-q\|\\&\leq\cdots\\&\leq e^{\sum_{i=n}^{n+m-1}r_i}\|x_n-q\|.\end{aligned}\tag{49}$$

That is,

$$\|x_{n+m}-q\|\leq M\|x_n-q\|,\tag{50}$$

where $M=e^{\sum_{i=n}^{n+m-1}r_i}$, for all $m,n\in\mathbb{N}$, for all $q\in\overline{F}$ and for $M>0$.

Next, we prove that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence.

As proved in Theorem 7, it is easy to see that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in a closed convex subset C of a Banach space E. Thus, it must converge to a point in C; let $\lim_{n\to\infty}x_n=p$.

For all $\epsilon>0$, as $\lim_{n\to\infty}x_n=p$, thus, there exists a number n_1 such that, when $n_2\geq n_1$,

$$\left\|x_{n_2}-p\right\|\leq\frac{\epsilon}{2+2k_\infty}.\tag{51}$$

In fact, $\lim_{n\to\infty}d(x_n,\overline{F})=0$ implies that using number n_2 above, when $n\geq n_2$, we get $d(x_n,\overline{F})\leq\epsilon/(2+6k_\infty)$. In particular, $d(x_{n_2},\overline{F})\leq\epsilon/(2+6k_\infty)$. Thus, there must exist $\overline{p}\in\overline{F}$, such that

$$\left\|x_{n_2}-\overline{p}\right\|=d\left(x_{n_2},\overline{p}\right)=\frac{\epsilon}{2+6k_\infty}.\tag{52}$$

From (51) and (52), using the same argument in Theorem 7, we get $Sp=p$.

Now, we return to prove $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

It is easy to see that $\lim_{n\to\infty}\|y_n-x_n\|=0$, and then combined with (44), we have $\lim_{n\to\infty}\|z_n-y_n\|=0$; thus, $\lim_{n\to\infty}\|z_n-q\|=\lim_{n\to\infty}\|y_n-q\|$, and then we get

$$\lim_{n\to\infty}\|y_n-q\|=\lim_{n\to\infty}\|x_n-q\|=c.\tag{53}$$

Moreover, $\|T^n x_n-q\|\leq k_n\|x_n-q\|$ implies that

$$\limsup_{n\to\infty}\|T^n x_n-q\|\leq c;\tag{54}$$

thus,

$$\lim_{n\to\infty}\|x_n-T^n x_n\|=0.\tag{55}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.\tag{56}$$

Thus, we can show that $\{x_n\}_{n=0}^\infty$ converges strongly to a common fixed point of T and S immediately. □

4.2. There Exist Two Sequences $\{x'_n\}$ *and* $\{x''_n\}$. In this part, we prove our main theorem for finding a common fixed point of asymptotically nonexpansive mappings in Banach space in the case of two sequences.

Theorem 9. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E, and let T and S be two asymptotically nonexpansive mappings of C satisfying Condition (I) and $\overline{F} \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in (0,1) and $\{k_n\}$ with $1 \leq k_n < \infty$ such that $\sum \alpha_n < \infty$, $\sum \beta_n < \infty$, $\beta_n > \gamma_n$ and $\sum(k_n - 1) < \infty$ for all $n \geq 1$.*

Define two sequences $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ in C by the algorithm (5); then $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ strongly converge to a common fixed point of T and S.

Proof. By the boundedness of C, we obverse that both $\{x'_n\}$ and $\{x''_n\}$ are bounded; if we take an arbitrary fixed point q of $\overline{F}$, noting that $\|y_n - q\| \leq k_n\|x'_n - q\|$ and $\|z_n - q\| \leq k_n\|x''_n - q\|$, we have

$$\begin{aligned} \left\|x'_{n+1} - q\right\| &= \|\alpha_n z_n + (1 - \alpha_n) y_n - q\| \\ &\leq \alpha_n \|z_n - y_n\| + \|y_n - q\| \\ &= (1 + \alpha_n k_n + k_n - 1) \left\|x'_n - q\right\| \\ &\quad + \alpha_n k_n \left\|x''_n - q\right\|. \end{aligned} \tag{57}$$

By Lemma 1 and $\sum \alpha_n < \infty$, $\sum(k_n - 1) < \infty$, thus, $\lim_{n\to\infty}\|x'_n - q\|$ exists. Denote

$$\lim_{n\to\infty} \left\|x'_n - q\right\| = c'. \tag{58}$$

Similarly, we have

$$\lim_{n\to\infty} \left\|x''_n - q\right\| = c''. \tag{59}$$

Since both $\{x'_n\}$ and $\{x''_n\}$ are bounded, put $k_\infty = \sup\{k_n : n \geq 1\} < \infty$. We get that $\{y_n\}$ and $\{z_n\}$ are bounded. Now

$$\begin{aligned} \left\|x'_{n+1} - q\right\| &= \|\alpha_n z_n + (1 - \alpha_n) y_n - q\| \\ &\leq \alpha_n \|z_n - y_n\| + \|y_n - q\|. \end{aligned} \tag{60}$$

Since $\|y_n - q\| \leq k_n\|x'_n - q\|$, this implies that

$$\lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \left\|x'_n - q\right\| = c'. \tag{61}$$

Moreover, $\|T^n x'_n - q\| \leq k_n\|x'_n - q\|$ implies that

$$\limsup_{n\to\infty} \left\|T^n x'_n - q\right\| \leq c'. \tag{62}$$

Thus,

$$\begin{aligned} c' &= \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{n\to\infty} \left\|\beta_n x'_n + (1 - \beta_n) T^n x'_n - q\right\| \\ &= \lim_{n\to\infty} \left\|\beta_n \left(x'_n - q\right) + (1 - \beta_n) \left(T^n x'_n - q\right)\right\|, \end{aligned} \tag{63}$$

given by Lemma 2 that

$$\lim_{n\to\infty} \left\|T^n x'_n - x'_n\right\| = 0. \tag{64}$$

Also note that

$$\begin{aligned} &\left\|x'_{n+1} - x'_n\right\| \\ &\quad = \left\|\alpha_n z_n + (1 - \alpha_n) y_n - x'_n\right\| \\ &\quad \leq \alpha_n \|z_n - y_n\| + (1 - \beta_n) \left\|\left(T^n x'_n - x'_n\right)\right\|, \end{aligned} \tag{65}$$

so that condition $\sum \alpha_n < \infty$ and (64) give

$$\lim_{n\to\infty} \left\|x'_{n+1} - x'_n\right\| = 0. \tag{66}$$

We have

$$\begin{aligned} &\left\|x'_{n+1} - Tx'_{n+1}\right\| \\ &\quad \leq \left\|x'_{n+1} - T^{n+1}x'_{n+1}\right\| + \left\|T^{n+1}x'_{n+1} - T^{n+1}x'_n\right\| \\ &\qquad + \left\|T^{n+1}x'_n - Tx'_{n+1}\right\| \\ &\quad \leq \left\|x'_{n+1} - T^{n+1}x'_{n+1}\right\| \\ &\qquad + k_\infty \left\|x'_{n+1} - x'_n\right\| + k_\infty \left\|T^n x'_n - x'_{n+1}\right\| \\ &\quad \leq \left\|x'_{n+1} - T^{n+1}x'_{n+1}\right\| + 2k_\infty \left\|x'_{n+1} - x'_n\right\| \\ &\qquad + k_\infty \left\|T^n x'_n - x'_n\right\|. \end{aligned} \tag{67}$$

Hence, by (64) and (66), we get

$$\lim_{n\to\infty} \left\|x'_n - Tx'_n\right\| = 0, \tag{68}$$

and then we assume that $k_n = 1 + r_n$, so $\sum r_n < \infty$ for $\sum(k_n - 1) < \infty$; now by (57), we obtain that

$$\begin{aligned} &\left\|x'_{n+m} - q\right\| \\ &\quad \leq (1 + \alpha_{n+m-1})(1 + r_{n+m-1}) \left\|x'_{n+m-1} - q\right\| + s_{n+m-1} \\ &\quad \leq e^{\alpha_{n+m-1}} e^{r_{n+m-1}} \left\|x'_{n+m-1} - q\right\| + s_{n+m-1} \\ &\quad \leq \ldots \\ &\quad \leq e^{\sum_{i=n}^{n+m-1} \alpha_i} e^{\sum_{i=n}^{n+m-1} r_i} \left\|x'_n - q\right\| \\ &\qquad + e^{\sum_{i=n}^{n+m-1} \alpha_i} e^{\sum_{i=n}^{n+m-1} r_i} \sum_{i=n}^{n+m-1} s_i, \end{aligned} \tag{69}$$

where $s_i = \alpha_i k_i \|x''_n - q\|$.

By $\sum \alpha_n < \infty$ and the convergence of $\{r_n\}$, that is,

$$\left\|x'_{n+m} - q\right\| \leq M \left(\left\|x'_n - q\right\| + \sum_{i=n}^{\infty} s_i \right), \tag{70}$$

where $M = e^{\sum_{i=n}^{n+m-1} \alpha_i} e^{\sum_{i=n}^{n+m-1} r_i}$, $s_i = \alpha_i k_i \|x''_n - q\|$, for all $m, n \in \mathbb{N}$, $q \in \overline{F}$ and for $M > 0$.

Next, we prove that $\{x'_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since $q \in \overline{F}$ arbitrarily and $\lim_{n\to\infty}\|x_n' - q\|$ exists, consequently, $d(x_n', \overline{F})$ exists by Lemma 3. From Lemma 3 and (68), we get

$$\lim_{n\to\infty} f\left(d\left(x_n', \overline{F}\right)\right) \le \lim_{n\to\infty}\left\|x_n' - Tx_n'\right\| = 0. \quad (71)$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore, we have

$$\lim_{n\to\infty} d\left(x_n', \overline{F}\right) = 0. \quad (72)$$

Let $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n', \overline{F}) = 0$ and $\sum_{i=1}^{\infty} s_i < \infty$, therefore, there exists a constant n_0 such that, for all $n \ge n_0$, we have

$$d\left(x_n', \overline{F}\right) \le \frac{\varepsilon}{3M}, \qquad \sum_{j=n_0}^{\infty} s_j \le \frac{\varepsilon}{6M}, \quad (73)$$

in particular,

$$d\left(x_{n_0}', \overline{F}\right) \le \frac{\varepsilon}{3M}. \quad (74)$$

There must exist $p_1 \in \overline{F}$ such that

$$d\left(x_{n_0}', p_1\right) \le \frac{\varepsilon}{3M}. \quad (75)$$

From (70), it can be obtained that, when $n \ge n_0$,

$$\begin{aligned}\left\|x_{n+m}' - x_n'\right\| &\le \left\|x_{n+m}' - p_1\right\| + \left\|x_n' - p_1\right\| \\ &\le 2M\left(\left\|x_{n_0}' - p_1\right\| + \sum_{j=n_0}^{n_0+m-1} s_j\right) \\ &\le 2M\left(\frac{\varepsilon}{3M} + \frac{\varepsilon}{6M}\right) = \varepsilon.\end{aligned} \quad (76)$$

This implies that $\{x_n'\}_{n=0}^{\infty}$ is a Cauchy sequence in a closed convex subset C of a Banach space E. Thus, it must converge to a point in C; let $\lim_{n\to\infty} x_n' = x'$.

For all $\epsilon > 0$, on lines similar to Theorem 8, from $\lim_{n\to\infty} x_n' = x'$, it can be proved that $\|Tx' - x'\| \le \epsilon$. As ϵ is an arbitrary positive number, thus, $Tx' = x'$. Similarly, we have $\lim_{n\to\infty}\|Sx_n'' - x_n''\| = 0$, and then $Sx'' = x''$ ($x_n'' \to x''$ as $n \to \infty$).

Finally, we prove $x' = x''$.

Let $w_{n+1} = \alpha_n T^n x_n' + (1-\alpha_n) S^n x_n'$, and put $\|x_n'' - q\| \ge \|x_n' - q\|$, for all $n \ge 1$. Then,

$$\begin{aligned}\left\|w_{n+1} - q\right\| &\le \alpha_n k_n \left\|x_n' - q\right\| \\ &\quad + (1-\alpha_n) k_n \left\|x_n'' - q\right\| \\ &\le k_n \max\left\{\left\|x_n' - q\right\|, \left\|x_n'' - q\right\|\right\} \\ &\le k_n \left\|x_n'' - q\right\|.\end{aligned} \quad (77)$$

We have $\lim_{n\to\infty}\|w_{n+1} - q\| \le \lim_{n\to\infty}\|x_n'' - q\|$. Now,

$$\begin{aligned}\left\|x_{n+1}'' - q\right\| &= \left\|\alpha_n y_n + (1-\alpha_n) z_n - q\right\| \\ &\le \beta_n \left\|\alpha_n x_n' + (1-\alpha_n) x_n''\right\| \\ &\quad + \left\|\alpha_n T^n x_n' + (1-\alpha_n) S^n x_n'' - q\right\| \\ &= \beta_n \left\|\alpha_n x_n' + (1-\alpha_n) x_n''\right\| + \left\|w_{n+1} - q\right\|.\end{aligned} \quad (78)$$

Since $\sum \beta_n < \infty$ and the boundedness of $\{x_n'\}$ and $\{x_n''\}$, we get

$$\lim_{n\to\infty}\left\|w_{n+1} - q\right\| = c''. \quad (79)$$

Moreover, $\|T^n x_n' - q\| \le k_n\|x_n' - q\|$ and $\|S^n x_n'' - q\| \le k_n\|x_n'' - q\|$ imply that

$$\limsup_{n\to\infty}\left\|T^n x_n' - q\right\| \le c'', \qquad \limsup_{n\to\infty}\left\|S^n x_n'' - q\right\| \le c''. \quad (80)$$

Thus,

$$\begin{aligned}c'' &= \lim_{n\to\infty}\left\|w_{n+1} - q\right\| \\ &= \left\|\alpha_n T^n x_n' + (1-\alpha_n) S^n x_n'' - q\right\| \\ &= \left\|\alpha_n\left(T^n x_n' - q\right) + (1-\alpha_n)\left(S^n x_n'' - q\right)\right\|,\end{aligned} \quad (81)$$

given by Lemma 2 that

$$\lim_{n\to\infty}\left\|S^n x_n'' - T^n x_n'\right\| = 0. \quad (82)$$

So

$$\begin{aligned}&\lim_{n\to\infty}\left\|x_{n+1}'' - x_{n+1}''\right\| \\ &\quad = \lim_{n\to\infty}\left|2\alpha_n - 1\right|\left\|z_n - y_n\right\| \\ &\quad = \lim_{n\to\infty}\left|2\alpha_n - 1\right|\left\|\gamma_n\left(x_n'' - S^n x_n''\right)\right. \\ &\qquad\qquad \left. - \beta_n\left(x_n' - T^n x_n'\right) + \left(S^n x_n'' - T^n x_n'\right)\right\|,\end{aligned} \quad (83)$$

so we obtain that $\lim_{n\to\infty}\|x_{n+1}' - x_{n+1}''\| = 0$ for (64) and (82), and it means $x' = x''$. This completes the proof. □

5. Application: Convergence to a Zero of Accretive Operators

Let E be a real Banach space. Recall that an operator (possibly multivalued) A with domain $D(A)$ and range $R(A)$ in E is said to be accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\left\langle y_1 - y_2, j\left(x_1 - x_2\right)\right\rangle \ge 0, \quad (84)$$

where J is the normalized duality map from E to the dual space E^* given by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad x \in E. \quad (85)$$

An accretive operator A is m-accretive if $R(I + rA) = E$ for all $r > 0$. Denote the zero set of A by

$$F := A^{-1}(0) = \{z \in D(A) : 0 \in Az\}. \quad (86)$$

For an m-accretive operator A with $F \neq \emptyset$ and $C = \overline{D(A)}$ convex, the problem of finding a zero of A, that is,

$$\text{find } z \in C \quad \text{such that } 0 \in Az, \quad (87)$$

has extensively been investigated due to its applications in related problems such as minimization problems, variational inequality problems, and nonlinear evolution equations.

It is known that the *resolvent* of A, defined by

$$J_r = (I + rA)^{-1}, \quad (88)$$

for $r > 0$, is a nonexpansive mapping from E to C and it is straightforward to see that F coincides with the fixed point set of J_r for any $r > 0$. Therefore, (87) is equivalent to the fixed point problem $z = J_r z$. Then an interesting approach to solving this problem is via iterative methods for nonexpansive mappings. We need the resolvent identity [15].

Theorem 10. *Let E be a uniformly convex Banach space, and let A be an m-accretive operator in E such that* $\mathrm{Fix}(J_{r_1}) \bigcap \mathrm{Fix}(J_{r_2}) \bigcap \mathrm{Fix}(J_{r_3}) = A^{-1}(0) \neq \emptyset$, $J_{r_i} : E \to E$ *is nonexpansive commuting mappings for all $r_i > 0$ ($i = 1, 2, 3$) satisfying Condition (I). Given $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in (0,1), such that $\sum \alpha_n < \infty$, $\sum \gamma_n \beta_n = \infty$, $\sum (1 - \gamma_n) < \infty$ and $r_{i,n} \geq \varepsilon$ for some $\varepsilon > 0$ for all $n \geq 1$.*

Define a sequence $\{x_n\}_{n=0}^{\infty}$ by algorithm (4)*; then $\{x_n\}_{n=0}^{\infty}$ strongly converges to a zero of A.*

Proof. Take any arbitrary $q \in A^{-1}(0)$; it follows from Lemma 1 that $\lim_{n\to\infty} \|x_n - q\|$ exists. From Lemma 2, it can be shown that $\lim_{n\to\infty} \|J_{r_{i,n}} x_n - x_n\| = 0$ $(i = 1, 2, 3)$. Since $J_{r_i} : E \to E$ is nonexpansive for all $r_i > 0$, it follows from Lemma 6 that $\lim_{n\to\infty} \|J_{r_i} x_n - x_n\| = 0$. Therefore, all the conditions in Theorem 7 are satisfied. The conclusion of Theorem 10 can be obtained from Theorem 7 immediately. □

Theorem 11. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E, and let A be an m-accretive operator in C such that* $\mathrm{Fix}(J_{r_1}) \bigcap \mathrm{Fix}(J_{r_2}) = A^{-1}(0) \neq \emptyset$, $J_{r_i} : C \to C$ *is nonexpansive for all $r_i > 0$ ($i = 1, 2$) satisfying Condition (I). Given $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in (0,1) such that $\sum \alpha_n < \infty$, $\sum \beta_n < \infty$, $\beta_n > \gamma_n$ and $r_{i,n} \geq \varepsilon$ for some $\varepsilon > 0$ for all $n \geq 1$.*

Define two sequences $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ in C by algorithm (6)*; then $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ strongly converge to a zero of A.*

Proof. Only a sketch of the proof is given here.

Take any arbitrary $q \in \mathrm{Fix}(J_{r_1}) \bigcap \mathrm{Fix}(J_{r_2})$; it follows from Lemma 1 that $\lim_{n\to\infty} \|x'_n - q\|$ and $\lim_{n\to\infty} \|x''_n - q\|$ exist. From Lemma 2, it can be shown that $\lim_{n\to\infty} \|J_{r_{1,n}} x'_n - x'_n\| = 0$ and $\lim_{n\to\infty} \|J_{r_{2,n}} x''_n - x''_n\| = 0$. Since $J_{r_i} : E \to E$ is nonexpansive for all $r_i > 0$ $(i = 1, 2)$, it follows from lemma 2.6 that $\lim_{n\to\infty} \|J_{r_1} x'_n - x'_n\| = 0$ and $\lim_{n\to\infty} \|J_{r_2} x''_n - x''_n\| = 0$. Therefore, all the conditions in Theorems 7, 8, and 9, are satisfied and using the same argument in those theorems, the conclusion of Theorem 11 can be obtained immediately. □

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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