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Research Article

On Riesz-Caputo Formulation for Sequential Fractional Variational Principles

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This paper deals with sequential Riesz-Caputo fractional variational problems with and without the presence of delay in the state variables and their derivatives. In both cases the necessary conditions for the optimal control are reported.

1. Introduction

Fractional calculus which is as old as the classical calculus has become a candidate in solving problems of complex systems which appear in various branches of science [1–10].

In the theory of time-delay systems the correct formulation of some initial value problems together with the representation of solutions as well as the asymptotic behavior of solutions are some of the general problems to be discussed. From the point of view of the fractional calculus we have almost the same problems to be discussed. The combination of these two main tools will lead us to a better description of the dynamics of the complex systems. In [10], fractional order modeling of delay dynamics was used to better characterize the delay behavior. Recently, it was shown that the delay parameter modifies the time window for the fractional order time derivative by allowing fading memory of an earlier time and introduces specific information about initial conditions (see [11] and the references therein).

Optimal control problems with time delay in calculus of variations were investigated in [12–14]. Variational optimal control problems within the framework of fractional calculus were considered in [15]. Very recently, the fractional variational problems in the presence of delay were discussed in [16, 17].

The aim of this paper is to find necessary conditions for sequential Riesz-Caputo fractional variational and optimal control problems with or without delay.

The structure of the paper is as follows: in Section 2 the basic definitions in fractional calculus are presented. Section 3 is devoted to the variational and optimal control problem within sequential Riesz-Caputo fractional derivatives in the absence of delay. In Section 4 the Riesz-Caputo variational principles with delay are discussed.

2. Basic Definitions

In this section, some basic definitions related to fractional derivatives are presented.

The left Riemann-Liouville fractional integral and the right Riemann-Liouville fractional integral are defined, respectively, by

$$
\begin{aligned}
{}_aI^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \\
I_b^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau,
\end{aligned}
\tag{2.1}
$$

where $f \in L_1(a,b)$. Here and in the following $\Gamma(\alpha)$ represents the gamma function.

The left Riemann-Liouville fractional derivative and the right Riemann-Liouville fractional derivatives are defined, respectively, by

$$
\begin{aligned}
{}_aD^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \\
D_b^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau,
\end{aligned}
\tag{2.2}
$$

where $\alpha > 0, n = [\alpha]+1$, and $f \in AC^n[a,b]$ (the space of complex-valued functions $f(t)$ having continuous derivatives up to $n-1$ on $[a,b]$) and $f^{(n-1)}(x) \in AC[a,b]$ (the space of absolutely continuous functions on $[a,b]$). In particular,

$$
\begin{gathered}
{}_aD^{0} f(t) = D_b^{0} f(t) = f(t), \\
{}_aD^{n} f(t) = f^{(n)}(t), \qquad D_b^{n} f(t) = (-1)^n f^{(n)}(t).
\end{gathered}
\tag{2.3}
$$

The fractional derivative of a constant takes the form

$$
{}_aD^{\alpha} C = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}
\tag{2.4}
$$

and the fractional derivative of a power of t has the following form:

$$
{}_aD^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\alpha+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)},
\tag{2.5}
$$

for $\beta > -1,\ \alpha \geq 0$.

The left Caputo fractional derivative is

$$ {}_a^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{2.6} $$

while the right Caputo fractional derivative is

$$ {}^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{2.7} $$

where $f \in AC^n[a,b]$ and α represents the order of the derivative such that $n = [\alpha] + 1$. By definition the Caputo fractional derivative of a constant is zero.

The Riemann-Liouville fractional derivatives and Caputo fractional derivatives are connected with each other by the following relations:

$$ \begin{aligned} {}_a^C D^\alpha f(t) &= {}_a D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \\ {}^C D_b^\alpha f(t) &= D_b^\alpha f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}. \end{aligned} \tag{2.8} $$

In [18, 19], the Riesz Riemann-Liouville fractional derivative or simply the Riesz fractional derivative and the Caputo Riesz fractional derivative are, respectively, represented as

$$ {}_a^R D_b^\alpha f(t) = \frac{1}{2}\left({}_a D^\alpha f(t) + (-1)^n D_b^\alpha f(t)\right), \tag{2.9} $$

$$ {}_a^{RC} D_b^\alpha f(t) = \frac{1}{2}\left({}_a^C D^\alpha f(t) + (-1)^n\, {}^C D_b^\alpha f(t)\right). \tag{2.10} $$

In [1], a formula for the fractional integration by parts on the whole interval $[a,b]$ was given by the following lemma.

Lemma 2.1. *Let $\alpha > 0$, $p, q \geq 1$, and $1/p + 1/q \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $1/p + 1/q = 1 + \alpha$).*

(a) *If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then*

$$ \int_a^b \varphi(t)\left({}_a I^\alpha \psi\right)(t)dt = \int_a^b \psi(t)\left(I_b^\alpha \varphi\right)(t)dt. \tag{2.11} $$

(b) *If $g \in I_b^\alpha(L_p)$ and $f \in {}_a I^\alpha(L_q)$, then*

$$ \int_a^b g(t)\left({}_a D^\alpha f\right)(t)dt = \int_a^b f(t)\left(D_b^\alpha g\right)(t)dt, \tag{2.12} $$

where ${}_aI^{\alpha}(L_P) = \{f : f = {}_aI^{\alpha}\varphi,\ \varphi \in L_p(a,b)\}$ *and* $I_b^{\alpha}(L_P) = \{f : f = I_b^{\alpha}\phi,\ \phi \in L_p(a,b)\}$.

Lemma 2.1 was generalized in [16, Lemma 2] and [17, Lemma 2], with the same conditions, to be applicable to subintervals $[a,r]$ and $[r,b]$ where $r \in (a,b)$.

3. The Sequential Riesz-Caputo Fractional Variational Problem

In this section we consider the following variational problem.

Minimize

$$J(y) = \int_a^b L\Big(t, y(t), {}_a^{RC}D_b^{\alpha}y(t), {}_a^{RC}D_b^{2\alpha}y(t), \dots, {}_a^{RC}D_b^{k\alpha}y(t)\Big)dt, \tag{3.1}$$

where $y \in AC^{2r}[a,b]$, $0 < \alpha \le 1$, $r-1 < k\alpha \le r$, $r \in \mathbb{N}$ $y^{(i)}(a) = c_i$, $y^{(i)}(b) = d_i$, $i = 0, 1, \dots, r-1$, c_i's and d_i's are constant, and $L : [a,b] \times \mathbb{R}^{k+1} \to \mathbb{R}$ has first and second partial derivatives with respect to all of its variables.

Theorem 3.1. *Let $J(y)$ be the functional of the form* (3.1) *defined on the set of functions $y(t)$ which have continuous Riesz-Caputo derivatives of order $i\alpha, i = 1, \dots, k$ and which satisfy the boundary conditions $y^{(i)}(a) = c_i$, $y^{(i)}(b) = d_i$. Then a necessary condition for $J(y)$ to have a minimum for a given $y(t)$ is that $y(t)$ satisfy the following Euler-Lagrange equation:*

$$\frac{\partial L}{\partial y} + \sum_{i=1}^{k} (-1)^i\, {}_a^{R}D_b^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha}y}\right) = 0. \tag{3.2}$$

Proof. Assume that $J(y)$ has a minimum for $y^*(t)$. Define a family of curves $y(t) = y^*(t) + \epsilon\eta(t)$ where $\epsilon \in \mathbb{R}$ and η is a function in $AC^{2r}[a,b]$ satisfying $\eta^{(i)}(a) = \eta^{(i)}(b) = 0$, $i = 0, 1, \dots, r-1$. The function

$$\gamma(\epsilon) = J\big(y^*(t) + \epsilon\eta(t)\big) \tag{3.3}$$

admits a minimum when $\gamma'(0) = 0$. That is,

$$\int_a^b \frac{\partial L}{\partial y*}\eta(t) + \sum_{i=1}^{k} \frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha}y*}\, {}_a^{RC}D_b^{i\alpha}\eta(t)dt = 0. \tag{3.4}$$

Using (2.10), (2.8), and [1, Lemma 2.7], (3.4) becomes

$$\int_a^b \left[\frac{\partial L}{\partial y*} + \sum_{i=1}^{k} (-1)^i\, {}_a^{R}D_b^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha}y*}\right)\right]\eta(t)dt = 0. \tag{3.5}$$

Since η is arbitrary, one has

$$\frac{\partial L}{\partial y^*} + \sum_{i=1}^{k} (-1)^i \, {}_a^R D_b^{i\alpha} \left(\frac{\partial L}{\partial \, {}_a^{RC} D_b^{i\alpha} y^*} \right) = 0. \tag{3.6}$$

□

When $\alpha = 1$, the results in Theorem 3.1 coincide with the results in [20, p. 59].

Example 3.2. Let us consider the following action:

$$\int_a^b \left(\frac{1}{2} y^2 + f(t) \, {}_a^{RC} D_b^{\alpha} y \right) dt, \tag{3.7}$$

where $f(t) \in AC[a,b]$, $0 < \alpha \leq 1$, $y(a) = c_1$, and $y(b) = d_1$. Then by Theorem 3.1, the necessary condition for optimality is

$$y = {}_a^R D_b^{\alpha} f(t). \tag{3.8}$$

One can generalize Theorem 3.1 as follows.

Corollary 3.3. *Consider the functional of the form*

$$\begin{aligned} J(y_1, y_2, \ldots, y_l) = \int_a^b L\Big(& t, y_1(t), {}_a^{RC} D_b^{\alpha} y_1(t), {}_a^{RC} D_b^{2\alpha} y_1(t), \ldots, {}_a^{RC} D_b^{k\alpha} y_1(t), \\ & y_2(t), {}_a^{RC} D_b^{\alpha} y_2(t), \; {}_a^{RC} D_b^{2\alpha} y_2(t), \ldots, {}_a^{RC} D_b^{k\alpha} y_2(t), \ldots, \\ & y_l(t), {}_a^{RC} D_b^{\alpha} y_l(t), {}_a^{RC} D_b^{2\alpha} y_l(t), \ldots, {}_a^{RC} D_b^{k\alpha} y_l(t) \Big) dt, \end{aligned} \tag{3.9}$$

defined on sets of continuous functions y_j, $j = 1, 2 \ldots, l$ that have Riesz Caputo fractional derivatives of order $i\alpha$, $i = 1, 2, \ldots, k$, $0 < \alpha \leq 1$, $(r - 1 < k\alpha \leq r)$ in the interval $[a, b]$ and satisfy the conditions

$$y_j^{(i)}(a) = c_{ij}, \quad y_j^{(i)}(b) = d_{ij}, \quad i = 0, 1, \ldots, r-1; \; j = 1, 2, ..l, \tag{3.10}$$

where a_{ij}, d_{ij} are constant and $L : [a, b] \times \mathbb{R}^{l(k+1)} \longrightarrow \mathbb{R}$ is a function with continuous first and second partial derivatives with respect to all of its arguments. For y_j, $j = 1, 2, \ldots, l$ to be a minimum of (3.9), *it is necessary that $y_j(x)$ satisfy the following Euler-Lagrange equations:*

$$\frac{\partial L}{\partial y_j} + \sum_{i=1}^{k} (-1)^i \, {}_a^R D_b^{i\alpha} \left(\frac{\partial L}{\partial \, {}_a^{RC} D_b^{i\alpha} y_j} \right) = 0, \quad j = 1, 2, \ldots, l. \tag{3.11}$$

3.1. The Optimal Control Case

The optimal control problem we consider is to find $u(t)$ that minimizes the performance index

$$J(y,u) = \int_a^b F\Big(t, u(t), y(t), {}^{RC}_{a}D^{\alpha}_{b}y(t), {}^{RC}_{a}D^{2\alpha}_{b}y(t), \dots, {}^{RC}_{a}D^{k\alpha}_{b}y(t)\Big)dt, \tag{3.12}$$

subject to the constraint

$$G\Big(t, u(t), y(t), {}^{RC}_{a}D^{\alpha}_{b}y(t), {}^{RC}_{a}D^{2\alpha}_{b}y(t), \dots, {}^{RC}_{a}D^{k\alpha}_{b}y(t)\Big) = 0, \tag{3.13}$$

such that

$$y^{(i)}(a) = c_i, \quad y^{(i)}(b) = d_i, \quad i = 0, 1, \dots, r-1, \tag{3.14}$$

where c_i, d_i are constant, $0 < \alpha \leq 1$, $r - 1 < k\alpha \leq r$, and F, $G : [a,b] \times \mathbb{R}^{k+2} \to \mathbb{R}$ are functions with continuous first and second partial derivatives with respect to their arguments.

To find the optimal control, one defines the modified performance index as

$$\widehat{J}(y,u) = \int_a^b [F(\cdot) + \lambda(t)G(\cdot)]dt, \tag{3.15}$$

where $\lambda(t)$ is called a Lagrange multiplier or adjoint variable. Using conditions (3.11), the following necessary conditions for optimal control are found:

$$\begin{gathered} \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} = 0, \\ \frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y} + \sum_{i=1}^{k}(-1)^i\, {}^{R}_{a}D^{i\alpha}_{b}\left(\frac{\partial F}{\partial\, {}^{RC}_{a}D^{i\alpha}_{b}y}\right) + {}^{R}_{a}D^{i\alpha}_{b}\left(\lambda\frac{\partial G}{\partial\, {}^{RC}_{a}D^{i\alpha}_{b}y}\right) = 0. \end{gathered} \tag{3.16}$$

When $\alpha = 1$, $k = 1$, (3.16) coincides with the results in [20, Theorem 6.2.2].

4. The Riesz-Caputo Sequential Fractional Variational Principles with Delay

In this section the following problem with delay is considered.

Minimize

$$\begin{aligned} J(y) = \int_a^b L\Big(&t, {}^{RC}_{a}D^{\alpha}_{b}y(t), {}^{RC}_{a}D^{2\alpha}_{b}y(t), \dots, {}^{RC}_{a}D^{k\alpha}_{b}y(t), \\ &y(t), y'(t), \dots, y^{(r)}(t), y(t-\tau), y'(t-\tau), \dots, y^{(r)}(t-\tau)\Big)dt \end{aligned} \tag{4.1}$$

such that

$$y(t) = f(t), \quad a-\tau \le t \le a, \qquad y^{(i)}(b) = d_i, \quad i = 0,1,2,\dots,r-1,\ r-1 < k\alpha \le r, \tag{4.2}$$

where d_i are constant, $y \in AC^{2r}[a,b]$, f is a smooth function, and $L : [a-\tau, b] \times \mathbb{R}^{k+2r+2} \to \mathbb{R}$ is a function with continuous first- and second-order partial derivatives with respect to its arguments.

Theorem 4.1. *If $y(t)$ satisfying* (4.2) *is a minimum of* (4.1), *then it is necessary that it satisfies the Euler-Lagrange equations*

$$\begin{aligned}
&\left(\frac{\partial L}{\partial y(t-\tau)}\right)(t+\tau) + \frac{\partial L}{\partial y(t)}(t) \\
&\quad + \sum_{i=1}^{r}\left[(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t-\tau)}\right)(t+\tau) + (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\right] \\
&\quad + \sum_{i=1}^{k}\left[(-1)^i\, {}_a^R D_{b-\tau}^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}\right)(t) - \frac{1}{2}\frac{1}{\Gamma(i\alpha)} D_{b-\tau}^{i\alpha}\right. \\
&\qquad \left.\times\left(\int_{b-\tau}^{b} D_b^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}\right)(s)(s-t)^{i\alpha-1} ds\right)\right] = 0,
\end{aligned} \tag{4.3}$$

for $a \le t \le b-\tau$

$$\begin{aligned}
&\frac{\partial L}{\partial y(t)}(t) + \sum_{i=1}^{r}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t) \\
&\quad + \sum_{i=1}^{k}(-1)^i\left[{}_{b-\tau}^{R} D_b^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}\right)(t) - \frac{1}{2}\frac{1}{\Gamma(i\alpha)}\, {}_{b-\tau}D^{i\alpha}\right. \\
&\qquad \left.\times\left(\int_{a}^{b-\tau} {}_aD^{i\alpha}\left(\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}\right)(s)(t-s)^{i\alpha-1} ds\right)\right] = 0,
\end{aligned} \tag{4.4}$$

for $b-\tau \le t \le b$, and the transversality condition

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_a^{b-\tau} \\
&\quad + \sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_{b-\tau}^{b} \\
&\quad + \sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t-\tau)}\right)(t+\tau)\eta^{(r-l-1)}(t)\Bigg|_a^{b-\tau} = 0,
\end{aligned} \tag{4.5}$$

where $\eta \in AC^{2r}[a,b]$ *is any arbitrary function such that* $\eta^{(i)}(b) = 0,\ i = 0,1,\ldots,r-1,\ \eta \equiv 0,\ a-\tau \leq t \leq a$.

Proof. Assume that $J(y)$ has a minimum for $y_0(t)$. Define a family of curves $y(t) = y_0(t)+\epsilon\eta(t)$ where $\epsilon \in \mathbb{R}$ and η is a function in $AC^{2r}[a,b]$ satisfying $\eta^{(i)}(b) = 0,\ i = 0,1,\ldots,r-1,\ \eta \equiv 0,\ a-\tau \leq t \leq a$. The function

$$\gamma(\epsilon) = J\big(y_0(t) + \epsilon\eta(t)\big) \tag{4.6}$$

admits a minimum when $\gamma'(0) = 0$. That is, at y_0, one has

$$\begin{aligned}\int_a^b \Bigg\{ &\frac{\partial L}{\partial y(t)}(t)\eta(t) + \frac{\partial L}{\partial y(t-\tau)}(t)\eta(t-\tau) + \sum_{i=1}^{k} \frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}(t)\, {}_a^{RC}D_b^{i\alpha}\eta(t) \\ &+ \sum_{i=1}^{r}\left[\frac{\partial L}{\partial y^{(i)}(t)}(t)\eta^{(i)}(t) + \frac{\partial L}{\partial y^{(i)}(t-\tau)}(t)\eta^{(i)}(t-\tau)\right]\Bigg\} dt = 0.\end{aligned} \tag{4.7}$$

Using (2.8), (2.10), and the fact that $\eta(t) \equiv 0,\ a-\tau \leq t \leq a$ and $\eta^{(i)}(b) = 0,\ i = 0,1,\ldots,r-1$, one gets

$$\begin{aligned}\int_a^b \Bigg\{ &\frac{\partial L}{\partial y(t)}(t)\eta(t) + \frac{\partial L}{\partial y(t-\tau)}(t)\eta(t-\tau) + \sum_{i=1}^{k} \frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}(t)\, {}_a^{R}D_b^{i\alpha}\eta(t) \\ &+ \sum_{i=1}^{r}\left[\frac{\partial L}{\partial y^{(i)}(t)}(t)\eta^{(i)}(t) + \frac{\partial L}{\partial y^{(i)}(t-\tau)}(t)\eta^{(i)}(t-\tau)\right]\Bigg\} dt = 0.\end{aligned} \tag{4.8}$$

Splitting the integrals, making the change of variables $t-\tau \leftrightarrow t$ and keeping in mind that $\eta(t) \equiv 0,\ a-\tau \leq t \leq a$, (4.8) becomes

$$\begin{aligned}\int_a^{b-\tau} \Bigg\{ &\frac{\partial L}{\partial y(t)}(t)\eta(t) + \frac{\partial L}{\partial y(t-\tau)}(t+\tau)\eta(t) + \sum_{i=1}^{k} \frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}(t)\, {}_a^{R}D_b^{i\alpha}\eta(t) \\ &+ \sum_{i=1}^{r}\left[\frac{\partial L}{\partial y^{(i)}(t)}(t)\eta^{(i)}(t) + \frac{\partial L}{\partial y^{(i)}(t-\tau)}(t+\tau)\eta^{(i)}(t)\right]\Bigg\} dt \\ &+ \int_{b-\tau}^{b}\left\{\frac{\partial L}{\partial y(t)}(t)\eta(t) + \sum_{i=1}^{k}\frac{\partial L}{\partial\, {}_a^{RC}D_b^{i\alpha} y(t)}(t)\, {}_a^{R}D_b^{i\alpha}\eta(t) + \sum_{i=1}^{r}\frac{\partial L}{\partial y^{(i)}(t)}(t)\eta^{(i)}(t)\right\} dt = 0.\end{aligned} \tag{4.9}$$

If one uses the usual integration by parts formula, the integration by parts formulas in [16, Lemma 2] and [17, Lemma 2], and the properties of the function η, one obtains

$$\begin{aligned}
&\int_a^{b-\tau}\Bigg\{\left(\frac{\partial L}{\partial y(t-\tau)}\right)(t+\tau)+\frac{\partial L}{\partial y(t)}(t)\\
&\quad+\sum_{i=1}^{r}\left[(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t-\tau)}\right)(t+\tau)+(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\right]\\
&\quad+\sum_{i=1}^{k}\Bigg[(-1)^i\,{}_a^{R}D_{b-\tau}^{i\alpha}\left(\frac{\partial L}{\partial{}_a^{RC}D_b^{i\alpha}y(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}D_{b-\tau}^{i\alpha}\\
&\qquad\times\left(\int_{b-\tau}^{b}D_b^{i\alpha}\left(\frac{\partial L}{\partial{}_a^{RC}D_b^{i\alpha}y(t)}\right)(s)(s-t)^{i\alpha-1}ds\right)\Bigg]\Bigg\}\eta(t)dt\\
&+\int_{b-\tau}^{b}\Bigg\{\frac{\partial L}{\partial y(t)}(t)+\sum_{i=1}^{r}(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\\
&\quad+\sum_{i=1}^{k}(-1)^i\Bigg[{}_{b-\tau}^{R}D_b^{i\alpha}\left(\frac{\partial L}{\partial{}_a^{RC}D_b^{i\alpha}y(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}\,{}_{b-\tau}D^{i\alpha}\\
&\qquad\times\left(\int_a^{b-\tau}{}_aD^{i\alpha}\left(\frac{\partial L}{\partial{}_a^{RC}D_b^{i\alpha}y(t)}\right)(s)(t-s)^{i\alpha-1}ds\right)\Bigg]\Bigg\}\eta(t)dt\\
&+\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_a^{b-\tau}\\
&+\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_{b-\tau}^{b}\\
&+\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y^{(i)}(t-\tau)}\right)(t+\tau)\eta^{(r-l-1)}(t)\Bigg|_a^{b-\tau}=0.
\end{aligned}\tag{4.10}$$

If one chooses η such that $\eta(a)=0$ and $\eta\equiv 0$ on $[b-\tau,b]$, (4.3) holds. If one chooses $\eta(b)=0$ and $\eta\equiv 0$ on $[a,b-\tau]$, (4.4) holds. Then since the first and second integrals in (4.10) become zero, (4.5) holds. □

The result in Theorem 4.1 coincides with the results in [14, Corollaries 2.1 and 2.2].

Example 4.2. Consider

$$\int_0^2\left[\frac{1}{2}y^2(t-1)+\frac{1}{2}y^2(t)+f(t)\,{}_0^{RC}D_2^{\alpha}y(t)\right]dt,\tag{4.11}$$

where $0 < \alpha \leq 1$, $y(2) =$ constant, and $y(t) = g(t)$ for $-1 \leq t \leq 0$. The necessary conditions for optimality according to Theorem 4.1 are

$$y(t) = \frac{1}{2}\left({}_0^R D_1^\alpha f\right)(t) + \frac{1}{4\Gamma(\alpha)} D_1^\alpha \left[\int_1^2 \left(D_2^\alpha f\right)(s)(s-t)^{\alpha-1}\right](t), \tag{4.12}$$

for $0 < t < 1$ and

$$y(t) = \left({}_1^R D_2^\alpha f\right)(t) - \frac{1}{2\Gamma(\alpha)} {}_1D^\alpha \left[\int_0^1 \left({}_0D^\alpha f\right)(s)(t-s)^{\alpha-1}\right](t), \tag{4.13}$$

for $1 < t < 2$.

Theorem 4.1 can be generalized as follows

Corollary 4.3. *Consider the functional of the form*

$$\begin{aligned} J(y_1, y_2, \ldots, y_m) = \int_a^b L\Big(t, & {}_a^{RC}D_b^\alpha y_1(t), {}_a^{RC}D_b^{2\alpha} y_1(t), \ldots, {}_a^{RC}D_b^{k\alpha} y_1(t), \\ & y_1(t), y_1'(t), \ldots, y_1^{(r)}(t), y_1(t-\tau), y_1'(t-\tau), \ldots, y_1^{(r)}(t-\tau), \\ & {}_a^{RC}D_b^\alpha y_2(t), {}_a^{RC}D_b^{2\alpha} y_2(t), \ldots, {}_a^{RC}D_b^{k\alpha} y_2(t), y_2(t), y_2'(t), \ldots, \\ & y_2^{(r)}(t), y_2(t-\tau), y_2'(t-\tau), \ldots, y_2^{(r)}(t-\tau), \ldots, \\ & {}_a^{RC}D_b^\alpha y_m(t), {}_a^{RC}D_b^{2\alpha} y_m(t), \ldots, {}_a^{RC}D_b^{k\alpha} y_m(t), y_m(t), \\ & y_m'(t), \ldots, y_m^{(r)}(t), y_m(t-\tau), y_m'(t-\tau), \ldots, y_m^{(r)}(t-\tau)\Big) dt, \end{aligned} \tag{4.14}$$

defined on sets of functions $y_j(t) \in AC^{2r}[a,b]$, $j = 1, 2, \ldots, m$ that have Riesz-Caputo fractional derivatives of order $i\alpha$, $i = 1, 2, \ldots, k$, $0 < \alpha \leq 1$, $r - 1 < k\alpha \leq r$, and satisfy the conditions

$$y_j^{(i)} = d_{ij}, \qquad y_j(t) = f_j(t), \quad t \in [a-\tau, a],\ \tau < b-a,\ a < b, \tag{4.15}$$

where d_{ij} are constant, $f_j(t)$ are smooth functions and $L : [a-\tau, b] \times \mathbb{R}^{m(k+2r+2)} \longrightarrow \mathbb{R}$ is a function having first-and second-order partial derivatives with respect to its arguments. For y_j, $j = 1, 2, \ldots, m$

satisfying (4.15) *to be a minimum of* (4.14), *it is necessary that* y_j *satisfy the Euler-Lagrange equations*

$$\begin{aligned}
&\left(\frac{\partial L}{\partial y_j(t-\tau)}\right)(t+\tau)+\frac{\partial L}{\partial y_j(t)}(t)\\
&\quad+\sum_{i=1}^{r}\left[(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y_j^{(i)}(t-\tau)}\right)(t+\tau)+(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y_j^{(i)}(t)}\right)(t)\right]\\
&\quad+\sum_{i=1}^{k}\left[(-1)^i\,{}_a^RD_{b-\tau}^{i\alpha}\left(\frac{\partial L}{\partial\,{}_a^{RC}D_b^{i\alpha}y_j(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}D_{b-\tau}^{i\alpha}\right.\\
&\qquad\left.\times\left(\int_{b-\tau}^{b}D_b^{i\alpha}\left(\frac{\partial L}{\partial\,{}_a^{RC}D_b^{i\alpha}y_j(t)}\right)(s)(s-t)^{i\alpha-1}ds\right)\right]=0,
\end{aligned}\tag{4.16}$$

for $a\le t\le b-\tau,\ j=1,2\ldots,m$

$$\begin{aligned}
&\frac{\partial L}{\partial y_j(t)}(t)+\sum_{i=1}^{r}(-1)^i\frac{d^i}{dt^i}\left(\frac{\partial L}{\partial y_j^{(i)}(t)}\right)(t)\\
&\quad+\sum_{i=1}^{k}(-1)^i\left[{}_{b-\tau}^{R}D_b^{i\alpha}\left(\frac{\partial L}{\partial\,{}_a^{RC}D_b^{i\alpha}y_j(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}\,{}_{b-\tau}D^{i\alpha}\right.\\
&\qquad\left.\times\left(\int_a^{b-\tau}{}_aD^{i\alpha}\left(\frac{\partial L}{\partial\,{}_a^{RC}D_b^{i\alpha}y_j(t)}\right)(s)(t-s)^{i\alpha-1}ds\right)\right]=0,
\end{aligned}\tag{4.17}$$

for $b-\tau\le t\le b,\ j=1,2\ldots,m$, *and the transversality condition*

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y_j^{(i)}(t)}\right)(t)\eta_j^{(r-l-1)}(t)\Bigg|_a^{b-\tau}\\
&\quad+\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y_j^{(i)}(t)}\right)(t)\eta_j^{(r-l-1)}(t)\Bigg|_{b-\tau}^{b}\\
&\quad+\sum_{i=1}^{r}\sum_{l=0}^{i-1}(-1)^l\frac{d^l}{dt^l}\left(\frac{\partial L}{\partial y_j^{(i)}(t-\tau)}\right)(t+\tau)\eta_j^{(r-l-1)}(t)\Bigg|_a^{b-\tau}=0,
\end{aligned}\tag{4.18}$$

where $\eta=(\eta_1,\eta_2,\ldots,\eta_m)$ *is an an arbitrary vector-valued function satisfying* $\eta(t)=(0,0,\ldots,0),\ t\in[a-\tau,a]$, *and* $\eta^{(i)}(b)=(0,0,\ldots,0),\ i=0,1,\ldots,r-1$.

4.1. The Optimal Control Case with Delay

Find the optimal control variable u such that u minimizes the performance index

$$J(y,u) = \int_a^b F\Big(t, u(t), {}_a^{RC}D_b^{\alpha}y(t), {}_a^{RC}D_b^{2\alpha}y(t), \dots, {}_a^{RC}D_b^{k\alpha}y(t), \\ y(t), y'(t), \dots, y^{(k)}(t), y(t-\tau), y'(t-\tau), \dots, y^{(k)}(t-\tau)\Big)dt \tag{4.19}$$

subject to the constraint

$$G\Big(t, u(t), {}_a^{RC}D_b^{\alpha}y(t), {}_a^{RC}D_b^{2\alpha}y(t), \dots, {}_a^{RC}D_b^{k\alpha}y(t), \\ y(t), y'(t), \dots, y^{(k)}(t), y(t-\tau), y'(t-\tau), \dots, y^{(k)}(t-\tau)\Big) = 0 \tag{4.20}$$

such that

$$y(t) = f(t), \quad a-\tau \le t \le a, \qquad y^{(i)} = d_i, \quad i = 0, 1, 2, \dots, r-1, \ 0 < \alpha \le 1, \tag{4.21}$$

where $r - 1 < k\alpha \le r$ d_i are constant, $y \in AC^{2r}[a,b]$, f is a smooth function and $F, G : [a-\tau, b] \times \mathbb{R}^{k+2r+3} \to \mathbb{R}$ are functions with continuous first and second order partial derivatives with respect to their arguments.

If one defines a modified performance index as

$$\hat{J}(y,u) = \int_a^b F(\cdot) + \lambda(t)G(\cdot)dt, \tag{4.22}$$

applying Corollary 4.3, it is necessary that

$$\left(\frac{\partial F}{\partial y(t-\tau)}\right)(t+\tau) + \frac{\partial F}{\partial y(t)}(t) \\ + \sum_{i=1}^{r}\left[(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial F}{\partial y^{(i)}(t-\tau)}\right)(t+\tau) + (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial F}{\partial y^{(i)}(t)}\right)(t)\right] \\ + \sum_{i=1}^{k}\left[(-1)^i\, {}_a^{R}D_{b-\tau}^{i\alpha}\left(\frac{\partial F}{\partial\, {}_a^{RC}D_b^{i\alpha}y(t)}\right)(t) - \frac{1}{2}\frac{1}{\Gamma(i\alpha)} D_{b-\tau}^{i\alpha} \right. \\ \left. \times\left(\int_{b-\tau}^{b} D_b^{i\alpha}\left(\frac{\partial F}{\partial\, {}_a^{RC}D_b^{i\alpha}y(t)}\right)(s)(s-t)^{i\alpha-1}ds\right)\right] + \left(\lambda\frac{\partial G}{\partial y(t-\tau)}\right)(t+\tau)$$

$$
\begin{aligned}
&+\left(\lambda\frac{\partial G}{\partial y(t)}\right)(t)+\sum_{i=1}^{r}\left[(-1)^{i}\frac{d^{i}}{dt^{i}}\left(\lambda\frac{\partial G}{\partial y^{(i)}(t-\tau)}\right)(t+\tau)+(-1)^{i}\frac{d^{i}}{dt^{i}}\left(\lambda\frac{\partial G}{\partial y^{(i)}(t)}\right)(t)\right]\\
&+\sum_{i=1}^{k}\left[(-1)^{i}\,{}_{a}^{R}D_{b-\tau}^{i\alpha}\left(\lambda\frac{\partial G}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}D_{b-\tau}^{i\alpha}\right.\\
&\qquad\left.\times\left(\int_{b-\tau}^{b}D_{b}^{i\alpha}\left(\lambda\frac{\partial G}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(s)(s-t)^{i\alpha-1}ds\right)\right]=0,\\
&\qquad\qquad\frac{\partial F}{\partial u(t)}(t)+\lambda(t)\frac{\partial G}{\partial u(t)}(t)=0,
\end{aligned}
\tag{4.23}
$$

for $a \le t \le b-\tau$,

$$
\begin{aligned}
&\frac{\partial F}{\partial y(t)}(t)+\sum_{i=1}^{r}(-1)^{i}\frac{d^{i}}{dt^{i}}\left(\frac{\partial F}{\partial y^{(i)}(t)}\right)(t)\\
&\quad+\sum_{i=1}^{k}(-1)^{i}\left[{}_{b-\tau}^{R}D_{b}^{i\alpha}\left(\frac{\partial F}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}\,{}_{b-\tau}D^{i\alpha}\right.\\
&\qquad\qquad\left.\times\left(\int_{a}^{b-\tau}{}_{a}D^{i\alpha}\left(\frac{\partial F}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(s)(t-s)^{i\alpha-1}ds\right)\right],\\
&\left(\lambda\frac{\partial G}{\partial y(t)}\right)(t)+\sum_{i=1}^{r}(-1)^{i}\frac{d^{i}}{dt^{i}}\left(\lambda\frac{\partial G}{\partial y^{(i)}(t)}\right)(t)\\
&\quad+\sum_{i=1}^{k}(-1)^{i}\left[{}_{b-\tau}^{R}D_{b}^{i\alpha}\left(\lambda\frac{\partial G}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(t)-\frac{1}{2}\frac{1}{\Gamma(i\alpha)}\,{}_{b-\tau}D^{i\alpha}\right.\\
&\qquad\qquad\left.\times\left(\int_{a}^{b-\tau}{}_{a}D^{i\alpha}\left(\lambda\frac{\partial F}{\partial\,{}_{a}^{RC}D_{b}^{i\alpha}y(t)}\right)(s)(t-s)^{i\alpha-1}ds\right)\right]=0,\\
&\qquad\qquad\frac{\partial F}{\partial u(t)}(t)+\lambda(t)\frac{\partial G}{\partial u(t)}(t)=0,
\end{aligned}
\tag{4.24}
$$

for $b-\tau < t < b$, and the transversality condition

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{l=0}^{i-1}(-1)^{l}\frac{d^{l}}{dt^{l}}\left(\frac{\partial F}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_{a}^{b-\tau}\\
&\quad+\sum_{i=1}^{k}\sum_{l=0}^{i-1}(-1)^{l}\frac{d^{l}}{dt^{l}}\left(\frac{\partial F}{\partial y^{(i)}(t)}\right)(t)\eta^{(r-l-1)}(t)\Bigg|_{b-\tau}^{b}\\
&\quad+\sum_{i=1}^{k}\sum_{l=0}^{i-1}(-1)^{l}\frac{d^{l}}{dt^{l}}\left(\frac{\partial F}{\partial y^{(i)}(t-\tau)}\right)(t+\tau)\eta^{(r-l-1)}(t)\Bigg|_{a}^{b-\tau}=0,
\end{aligned}
\tag{4.25}
$$

where η is any arbitrary function satisfying $\eta(t) = 0$, $t \in [a - \tau, t]$, and $\eta^{(i)}(b) = 0$, $i = 0, 1, 2, \ldots, r - 1$.

5. Conclusion

Fractional variational principles started to be used in several branches of science and engineering. However the delay is present in various phenomena having great impact in science and engineering. One of the main questions is to combine in an optimal way the properties of the fractional calculus and those of the delay having in mind to obtain a more general variational principles. In this paper we have used the fractional Riesz-Caputo derivative and the delay in the state variable. We mention that the definition of the Riesz-Caputo derivative contains both left and right fractional derivative. The necessary conditions for the optimal control were obtained. When $\alpha \to 1$ or the delay is absent the classical results are obtained.

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