PARTITIONED METHODS FOR COUPLED FLUID FLOW AND TURBULENCE FLOW PROBLEMS

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Numerical simulation of different physical processes in different regions is one of the wide variety of real world applications. Many important applications such as coupled surface water groundwater flows require the accurate solution of multi-domain, multi-physics coupling of unobstructed flows with filtration or porous media flows. There are large advantages in efficiency, storage, accuracy and programmer effort in using partitioned methods build from components optimized for the individual sub-processes. On the other hand, the multi-domain or multi-physical process describes different natures of the physical processes of each subproblem. They may require different meshes, time steps and methods. There is a natural desire to uncouple and solve such systems by solving each sub physics problem, to reduce the technical complexity and allow the use of optimized, legacy sub-problems' codes in fluid flow. Stabilization using filters is intended to model and extract the energy lost to resolved scales due to nonlinearity breaking down resolved scales to unresolved scales. This process is highly nonlinear. Including a particular form of the nonlinear filter allows for easy incorporation of more knowledge into the filter process and its computational complexity is comparable to many of the current models which use linear filters to select the eddies for damping.

The objective of the first part of this work is the development, analysis and validation of new partitioned algorithms for some coupled flow models, addressing some of the above problems. Particularly, this thesis studies: i) long time stability of four methods for splitting the evolutionary Stokes-Darcy problem into Stokes and Darcy sub problems, ii) analysis of a multi-rate splitting method for uncoupling evolutionary groundwater-surface water flows, and iii) a connection between filter stabilization and eddy viscosity models. For each problem, we give comprehensive analysis of stability and derive optimal error estimates of our proposed methods. Numerical experiments are performed to verify the theoretical results.

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PREFACE

I would like to express my deep appreciation and gratitude to my advisor Dr. William Layton. Throughout my years at University of Pittsburgh, he provided constant encouragement, sound advice, great patience and excellent teaching due to those this project was possible. With me, he is not only a great advisor but also a great friend, whose positive and dedicated attitude towards both math research and life deeply influenced me. His constant support and help in my job searching is always appreciated. I have thoroughly enjoyed the time we spent to discuss both math and non-math topics.

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1.0 INTRODUCTION

The flow of liquids occurs in many processes in nature and plays an important role in science and industry. Obtaining accurate, efficient and reliable prediction of quantities in fluid flows is crucial to understand and predict the related real-world phenomena. Many fluid flows in engineering and technology are solved by complex codes or coupled to other physical effects. The ability of fast refining these models when understanding is improved and using the legacy and best codes for subprocesses poses an important modeling problem. This thesis involves the development and testing of new numerical methods which help address the above difficulty in the modeling and simulation of some complex flows. In particular, we have studied

- partitioned methods for groundwater -surface water models.
- extension of the unified time step partitioned method to multi-rate time step method for groundwater -surface water models.
- nonlinear filtered projection method for higher reynolds number flows.

The following sections will describe each of the topics in details.

1.1 PARTITIONED TIME STEPPING METHODS FOR THE EVOLUTIONARY STOKES-DARCY PROBLEMS

Groundwater, forming two-thirds of the world's fresh water, is vital to human activities. One serious global problem nowadays is groundwater contamination, which occurs when man-made pollutants are dissolved in lakes and rivers and get into the groundwater, making it unsafe and unfit for human use. To predict and control the spread of such contamination requires the accurate solution of coupling of groundwater flows with surface water flows (the Stokes-Darcy problem). The essential problems of estimation of the propagation of pollutants into groundwater are that (i) the different physical processes suggest that codes optimized for each sub-process need to be used for solution of the coupled problem, (ii) the large domains plus the need to compute for several turn-over times for reliable statistics require calculations over long time intervals and (iii) values of some system parameters, e.g., hydraulic conductivity and specific storage, are frequently very small. To address these issues, we study the stability and errors over long time intervals of uncoupled methods for the fully time dependent Stokes-Darcy problem. We are particularly interested in analyzing and comparing the performance of the studied methods for small parameters.

In this work, we propose several implicit-explicit based and splitting based partitioned methods for uncoupling the evolutionary Stokes-Darcy problem. The Stokes-Darcy equation is as follows:

Let the two domains be Ω_f, Ω_p lie across an interface I from each other. The fluid velocity and porous media piezometric head (related to the Darcy pressure) satisfy

$$
\begin{gathered}
\rho u_t - \mu \triangle u = f_f, \text{ and } \nabla \cdot u = 0, \text{ in } \Omega_f, \\
S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p, \text{ in } \Omega_p, \\
\phi(x,0) = \phi_0, \text{ in } \Omega_p \text{ and } u(x,0) = u_0, \text{ in } \Omega_f, \\
\phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \text{ in } \partial\Omega_f \backslash I, \\
+ \text{ coupling conditions across } I.
\end{gathered} \tag{1.1}
$$

Let $\widehat{n}_{f/p}$ denote the indicated, outward pointing, unit normal vector on I. The coupling conditions are conservation of mass and balance of forces on I

$$
\begin{gathered}
u \cdot \widehat{n}_f - \mathcal{K} \nabla \phi \cdot \widehat{n}_p = 0, \text{ on I}, \\
p - \mu \, \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = \rho g \phi \text{ on I}.
\end{gathered}
$$

The last condition needed is the Beavers-Joseph-Saffman (-Jones) condition

$$
-\mu \, \nabla u \cdot \widehat{n}_f = \alpha \sqrt{\frac{\mu \rho g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i \equiv \chi u \cdot \widehat{\tau}_i, \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I,
$$

General experience with partitioned methods suggests that some price is inevitably paid. Our proposed methods with explicit coupling terms inherit restrictions on time step size Δt

$$\Delta t \leq C_p^* \min\{k, S_0\} \tag{1.2}$$

where S_0 is specific storage, k is hydraulic conductivity and C_p^* is a generic positive constant independent of mesh size, time step and final time. The values of S_0 and k are frequently very small, see [7], [33], and in those cases, the dependence indicated in (1.2) becomes too pessimistic. To overcome this problem, we propose and analyze four novel uncoupling methods for Stokes-Darcy equations, which have stronger stability properties, using ideas from splitting methods. These methods include ones stable uniformly in S_0 for moderate k and uniformly in k for moderate S_0. They are thus good options when *one* of the parameters is small.

The literature on numerical analysis of methods for the Stokes-Darcy coupled problem has grown extensively since [30], [67]. See [35] for a recent survey and [8], [18], [95], [97], [101], [114] and [67] for theory of the continuum model. There is less work on the fully evolutionary Stokes-Darcy problem. One approach is monolithic discretization by an implicit method followed by iterative solution of the non-symmetric system where subregion uncoupling is attained by using a domain decomposition preconditioner; see, e.g., [18], [19], [82], [87], [15], [28], [79], [81], [80], [60], [85], [112]. Partitioned methods allow parallel, non-iterative uncoupling into one (SPD) Stokes and one (SPD) Darcy system per time step. The first such partitioned method was studied in 2010 by Mu and Zhu [88]. This has been followed by an asynchronous (allow different time steps in the two subregions) partitioned method in [105] and higher order partitioned methods in [20], [69]. In most of these works, stability and convergence were studied over bounded time intervals $0 \leq t \leq T < \infty$ and the estimates included $e^{\alpha T}$ multipliers.

Understanding of the equilibrium Stokes-Darcy problem is now advanced, e.g., [57], [67], [29], [97], [95]. For the evolutionary problem, the monolithic approach (discretize the problem implicitly, assemble the fully coupled system at each time step, solve by an iterative method where uncoupling is attained by using a domain decomposition preconditioner) is an important complement to partitioned methods; it is developed in, e.g., [29], [20], [28], [32],

[31], [51], [14], [87], [60], [85], [87], and [112]. Partitioned methods require neither access to a fully coupled system nor iteration at each time step, e.g., [69], [68], [104], [88] (the first paper on partitioned methods for Stokes-Darcy), and [18], [19] (a interesting new approach and the first papers studying the Beavers-Joseph interface coupling). There is a very strong connection between application-specific partitioned methods and more general IMEX and splitting methods; see, e.g., [113], [110], [3], [27], [37], [53], [113], [75], [76], [116]. The idea used in CNsplit below to compute in parallel two approximations and then average occurs in the Dyakunov splitting method, e.g., [75], [76], [116], [50].

1.2 EXTENSION OF THE UNIFIED TIME STEP PARTITIONED METHOD TO MULTI-RATE TIME STEP METHOD FOR GROUNDWATER -SURFACE WATER MODELS

There are a rich number of studies on the mathematical analysis, numerical methods and applications for the Stokes-Darcy model, see, e.g., [2], [28], [30], [32], [44], [57], [58], [67], [85], [100]. The mathematical model consists of the evolutionary Stokes equations in the fluid region coupled with the evolutionary Darcy model in the porous medium, [18], [25], [88]. Important features of estimating transport of pollution between surface water and ground water include the different physical processes and models in the two regions, the availability of optimized codes for subdomain physics and the wide difference in the rates at which the flows progress in the unobstructed, free flowing region and in the porous media. With these issues in mind, we herein present, analyze and test an asynchronous or multi-rate(allowing different time steps in the sub regions), partitioned method for the fully evolutionary Stokes-Darcy problem. The essential features of the method we present in this work are that it allows different time steps in the fluid region and the porous region, requires only one, uncoupled Stokes solve per small time step and one Darcy solve per large time step without reference to the globally coupled problem and is stable over long time intervals.

Partitioned methods have great advantages for multi-physics, multi-domain problems, e.g., [68], [70], [88], [104]. Splitting methods, one approach for partitioning, have been widely used in applications [55], [50]. For first steps in partitioned method for Stokes-Darcy, see Mu and Zhu [88], extended to a multi-rate method in [105]. For the Stokes-Darcy problem, typical velocities are greater in the fluid region than in the porous media region. Therefore, there are significant advantages in accuracy and efficiency in using a small time step size in the fluid region and a large time step size in the porous media region. However, both partitioning and asynchronous time steps require interpolation of unknow values for the solves and this manufacturing of required value can introduce instabilities.

Our work herein is motivated by the search for more partitioned methods, which can accurately capture the features of the physical process while making it easy to calculate numerically. The interface coupling conditions are conservation of mass across the interface, balance of forces and the Beavers-Joseph-Saffman condition, [8], [57], [58], [97], [101]. More general application-oriented partitioned methods and more general IMEX and splitting methods have been widely studied, see, e.g., [113], [110], [3], [27], [37], [53], [113], [75], [116].

In comparison with the multirate method in [105], the mehod herein starts from a Darcy solve, from which an intermediate velocity in porous media is derived, and then has r Stokes solves in sequence and ends with a Darcy solve at the following time level, while the multistep method in [105] has a different sequence of Stokes and Darcy solves, resulting in different conditions of stability and convergence.

1.2.1 Algorithm

To streamline notations, choose a uniform time step $\triangle t$ in Ω_f,

$$\mathcal{P} = \left\{0 = t^0, t^1, t^2, \cdots, t^N = T\right\}, \ t^j = j\triangle t$$

The large time step in Ω_p is given by a separate notations hereafter, $\triangle s = r\triangle t$. Denote by

$$\mathcal{S} = \{0 = t^{m_0}, t^{m_1}, t^{m_2}, \cdots, t^{m_M} = T\} \subset \mathcal{P},$$

a subset satisfying $t^{m_k} = rt^k$ such that the positive constant r is fixed and $Mr = N$. To streamline our notation further, we shall suppress the subscript "h" and replace u_h^m, ϕ_h^m, p_h^m by u^m, ϕ^m, p^m, respectively. For t^m, $t^{m_k} \in [0, T]$, (u^m, ϕ^m, p^m) will denote the discrete approximation to $(u(t^m), \phi(t^m), p(t^m))$. In practice only the data at time t^0 would be provided. One important feature of the algorithm given bellow is that (u^m, p^m) can be calculated for $m = m_k, m_k + 1, \cdots m_{k+1} - 1$ in parallel with $\phi^{m_{k+1}}$.

• Given u^{m_k}, ϕ^{m_k}, do one step with the large time step $\triangle s$ to obtain $\phi^{m_k^*} \in H_{ph}$, such that $\forall\, \psi \in H_{ph}$

$$gS_0 \left(\frac{\phi^{m_k^*} - \phi^{m_k}}{\triangle s}, \psi \right) + \frac{1}{2} a_p \left(\phi^{m_k^*}, \psi \right) = \frac{1}{2} g \left(f_2^{m_k^*}, \psi \right) + \frac{1}{2} g \int_\Gamma \psi u^{m_k} \cdot n_f. \tag{1.3}$$

• Obtaining $\phi^{m_k^*}$ from the first step, do r step in fluid region with small time step $\triangle t = \triangle s / r$ to find (u^{m+1}, p^{m+1}) for $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$, such that $\forall\, (v, q) \in (H_{fh}, Q_h)$

$$\begin{aligned} \left(\frac{u^{m+1} - u^m}{\triangle t}, v \right) + a_f \left(u^{m+1}, v \right) + b \left(v, p^{m+1} \right) &= \left(f_1^{m+1}, v \right) - g \int_\Gamma \phi^{m_k^*} v \cdot n_f, \\ b \left(u^{m+1}, q \right) &= 0. \end{aligned} \tag{1.4}$$

• With $\phi^{m_k^*}, u^{m_{k+1}}$ obtained from Step 1 and Step 2, do one step in porous region with the large step $\triangle s$ to find $\phi^{m_{k+1}} \in H_{ph}$, such that $\forall\, \psi \in H_{ph}$

$$gS_0 \left(\frac{\phi^{m_{k+1}} - \phi^{m_k^*}}{\triangle s}, \psi \right) + \frac{1}{2} a_p \left(\phi^{m_{k+1}}, \psi \right) = \frac{1}{2} g \left(f_2^{m_{k+1}}, \psi \right) + \frac{1}{2} g \int_\Gamma \psi u^{m_{k+1}} \cdot n_f. \tag{1.5}$$

• Set $k = k + 1$ and repeat until $k = M - 1$.

The method treats the subphysics terms implicitly and the coupling terms on the fluid-porous interface explicitly thereby uncoupling the system at each time step into subdomain problems. It also allows smaller time steps in the unobstructed fluid region than in the porous region. One fundamental question of both partitioned and multirate methods is stability over long time intervals $0 \le t < \infty$. We resolve the stability issue here, give a complete error analysis and computational tests.

1.3 NONLINEAR FILTERED PROJECTION METHOD FOR HIGHER REYNOLDS NUMBER FLOWS

Recently, a new approach for the stabilization of the incompressible Navier-Stokes equations for higher Reynolds numbers was introduced based on the filtering of solution on every time step of a discrete scheme. In this work, the stabilization is shown to be equivalent to a certain closure model in LES. This allows a refined analysis, further understanding of desired filter properties and clearer interpretation of the results of numerical experiments. We also consider the application of the post-filtering in a projection (pressure correction) method, the standard splitting algorithm for time integration of the incompressible fluid equations.

A stabilization of a numerical time-integration algorithm for the incompressible Navier-Stokes equations

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p &= f \\ \operatorname{div} u &= 0 \end{aligned} \quad \text{in } \; \Omega\times(0,T]\ , \tag{1.6}$$

for large Reynolds numbers with the help of an additional filtering step was recently introduced in [65]. Denote by w^n or u^n approximations to the Navier-Stokes system velocity solution at time t_n, similarly p^n approximates pressure $p(t^n)$. Let $\triangle t = t^{n+1} - t^n$. The algorithm reads: For $n = 0, 1, \ldots$ and $u^0 = u(t^0)$ compute

1. intermediate velocity w^{n+1} from

$$\begin{cases} \dfrac{1}{\triangle t}(w^{n+1} - u^n) + (w^{n+1}\cdot\nabla)w^{n+1} + \nabla p^{n+1} - \nu\Delta w^{n+1} = f^{n+1}, \\ \operatorname{div} w^{n+1} = 0, \end{cases}$$

   subject to appropriate boundary conditions;

2. filter the intermediate velocity, $\overline{w^{n+1}} := F\, w^{n+1}$;
3. relax $u^{n+1} := (1-\chi)w^{n+1} + \chi\overline{w^{n+1}}$, with a relaxation parameter $\chi \in [0, 1]$.

Here F is a generic nonlinear filter acting from $L^2(\Omega)^3$ to $H^1(\Omega)^3$. The convergence of the finite element solutions of 1.–3. to the smooth Navier-Stokes solutions has been analyzed in [65], where the step 2. is called the post-filtering. One advantage of the approach is the implementation convenience within an existing CFD code for laminar flows and flexibility in the choice of a filter. Numerical results with the approach from [13,36,65,66] with composite nonlinear deferential (post)-filters, as defined in Section 4.2, consistently show more precise localization of model viscosity and its more precise correlation with the action of nonlinearity on the smallest resolved scales than plain Smagorinski type LES or VMS methods. Thus we deem the approach deserves further studies, should be put into perspective and related to developing LES models.

In this work, we show that introducing the post-filtering is closely related (and even equivalent in a sense which is made precise further in the paper) to adapting a certain closure model for LES. The connection to a LES model allows us to quantify the model dissipation introduced by the post-filtering, formulate a stability criteria, and have an insight into the choice of the filter and the relaxation parameter. In particular, it provides an explanation, why the stabilization by the post-filtering avoids adding excessive model viscosity to a regions of larger velocity gradients unlike some other eddy viscosity models. Since the entire approach is specifically designed for treating higher Reynolds number flows, it is natural to extend it to the Chorin-Temam-Yanenko type splitting algorithms, which are the prevailing method for the time-integration of the incompressible Navier-Stokes equations for fast unsteady flows. Such (rather natural) extension is presented in the paper together with the relevant error analysis. We note right away that the analysis demonstrates the convergence of numerical solutions to the Navier-Stokes smooth solution, while it would be also interesting to analyze the error of the numerical solutions to a (presumably smoother) solution of the corresponding LES model. However, the specific difficulty we faced in the latter case is the lacking of the monotone property by most of eddy viscosity indicator functionals, which were numerically proved to be useful if defining the filter F.

Though practically attractive, introducing such functionals makes the mathematical wellposhness of the LES model and accordingly the error analysis hard to accomplish and we are unaware of relevant results in this direction.

1.4 ANALYSIS TOOLS

In this section, we state some well-known results and assumptions which will be utilized in the analysis throughout this thesis. Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). We denote the $L^2(\Omega)$ norm and inner product by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. The space H^{-k} denotes the dual space of H_0^k.

Theorem 1.4.1. *(the trace theorem) Let $\partial\Omega$ be a graph of a Lipschitz continuous function. If $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega)$, then $u|_{\partial\Omega} \in L^2(\partial\Omega)$ and*

$$|u|_{L^2(\partial\Omega)} \leq C\|u\|^{1/2}\left(\|u\|^2 + \|\nabla u\|^2\right)^{1/4}.$$

Theorem 1.4.2. *(the Poincaré inequality) There is a constant $C = C(\Omega)$ such that*

$$\|u\| \leq C\|\nabla u\|$$

for every $u \in H_0^1(\Omega)$.

Theorem 1.4.3. *For any $u, v, w \in H_0^1(\Omega)$, there is $C = C(\Omega)$ such that*

$$\left|\int_\Omega u \cdot \nabla v \cdot w dx\right| \leq C\sqrt{\|u\|\|\nabla u\|}\|\nabla v\|\|\nabla w\|. \tag{1.7}$$

For the proof, see [83].

Lemma 1.4.4. *(discrete Grönwall inequality) Let $D \geq 0$ and κ_n, A_n, B_n, $C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_N + \Delta t\sum_{n=0}^{N} B_n \leq \Delta t\sum_{n=0}^{N}\kappa_n A_n + \Delta t\sum_{n=0}^{N} C_n + D \text{ for } N \geq 0.$$

Suppose that for all n, $\Delta t\kappa_n < 1$, and set $g_n = (1 - \Delta t\kappa_n)^{-1}$. Then,

$$A_N + \Delta t\sum_{n=0}^{N} B_n \leq exp\left(\Delta t\sum_{n=0}^{N} g_n\kappa_n\right)\left[\Delta t\sum_{n=0}^{N} C_n + D\right] \text{ for } N \geq 0.$$

For the details, see, e.g., [78].

1.5 THESIS OUTLINE

This thesis begins in Chapter 2 with a study of four splitting based partitioned algorithm for uncoupling groundwater - surface water coupling system. We show that they are more stable in motivating applications involving small physical parameters. A complete long time stability, the associated time step restrictions are given in Section 2.4. The convergence analysis of BEsplit2 and SDsplit methods are presented in Section 2.5. In Section 2.6 we give computational experiments to verify the accuracy and stability of our methods. .

In Chapter 3, we discuss the extension of the unified time step splitting methods to multi-rate splitting method which uncouple the Stokes-Darcy coupling system into two separate problems in the two subdomains. We show in Section 3.2 that these formulations have a stable solution for long time periods and its time step restriction for it to be stable. The main convergence results are presented in Theorem 3.3. The numerical experiments in Section 3.4 support these theoretical results.

Chapter 4 will be devoted for the analysis of a nonlinear filtered projection method for NSE. Section 4.1 presents the background of nonlinear differential filters and its connections to LES models. In Theorem 4.5.2, we present the numerical scheme and prove that the method is long time and uniformly in time stable. Section 4.6 gives a comprehensive error analysis and Section 4.7 follows with numerical tests which confirm the theory.

2.0 SPLITTING BASED PARTITIONED METHODS FOR THE EVOLUTIONARY STOKES-DARCY PROBLEMS

2.1 METHOD DESCRIPTIONS

Many important applications such as coupled surfacewater groundwater flows require the accurate solution of multi-domain, multi-physics coupling of unobstructed flows with filtration or porous media flows (the Stokes-Darcy problem). There are large advantages in efficiency, storage, accuracy and programmer effort in using partitioned methods built from components optimized for the individual sub-processes. Partitioned methods for the evolutionary Stokes-Darcy problem confront several intrinsic difficulties which include:

- Values of the hydraulic conductivity k can be small, for example 10^{-12} for sands to 10^{-15} for clay, [7].
- Values for the storativity coefficient S_0 range from 10^{-2} in unconfined aquifers to 10^{-5} in confined aquifers, [61].
- The scale of the problem varies from large $L = diam(\Omega)$ for geophysics and small L for biomedical applications.
- Turnover times in aquifers can be large due to small hydraulic conductivity values and large domains. Thus accurate calculations are needed over long time intervals.
- Differences in flow rates in the Stokes and the Darcy regions can require different time steps in the two domains for efficiency and accuracy.

These features mean that stability is a primary issue for partitioned methods for the Stokes-Darcy problem. Uncoupling / partitioning necessarily induces a time step restriction for long time stability. The severity of the restriction depends on the method chosen, the relaxation times of the individual subdomain problems and the strength of coupling of the underlying problem. We study herein stability vs the severity of the induced time step restriction for small $k_{\min}$, S_0 and long time intervals for uncoupling by splitting methods. Since the Stokes-Darcy problem and the methods we consider are linear, their error satisfies the same equations as the approximate solution with the body force replaced by a consistency error. Thus, for errors also, stability over long time intervals for small S_0, k is the key to a method with good error behavior.

The four methods we analyze methods uncouple each time step into a separate Stokes flow problem and Darcy flow problem. The strength of the coupling between the two subdomains varies with different ranges of physical parameters and is reflected in restrictions on time steps required for long time stability. Our estimates and tests suggest that these methods are stable for larger time steps that the IMEX based partitioned methods in [88], [69], [68], [104]. In particular, stability analysis and numerical tests herein indicate that splitting based partitioned methods are a very good option when either $k_{\min}$ or S_0 is small. Finding partitioned methods stable for large time step when both $k_{\min}$, S_0 are small is an open problem. Further, while the first order methods gave acceptable error levels, more accuracy is always desirable. Stable higher order partitioned methods for large time steps and small parameters are also not yet known.

Let the two domains be Ω_f, Ω_p lie across an interface I from each other. The fluid velocity and porous media piezometric head (related to the Darcy pressure) satisfy

$$\begin{gathered}
\rho u_t - \mu \triangle u = f_f, \text{ and } \nabla \cdot u = 0, \text{ in } \Omega_f, \qquad (2.1)\\
S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p, \text{ in } \Omega_p,\\
\phi(x,0) = \phi_0, \text{ in } \Omega_p \text{ and } u(x,0) = u_0, \text{ in } \Omega_f,\\
\phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \text{ in } \partial\Omega_f \backslash I,\\
+ \text{ coupling conditions across } I.
\end{gathered}$$

Let $\widehat{n}_{f/p}$ denote the indicated, outward pointing, unit normal vector on I. The coupling conditions are conservation of mass and balance of forces on I

$$u \cdot \widehat{n}_f - \mathcal{K}\nabla\phi \cdot \widehat{n}_p = 0, \text{ on I},$$
$$p - \mu \, \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = \rho g \phi \text{ on I}.$$

The last condition needed is the Beavers-Joseph-Saffman (-Jones) condition

$$-\mu \, \nabla u \cdot \widehat{n}_f = \alpha \sqrt{\frac{\mu \rho g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i \equiv \chi u \cdot \widehat{\tau}_i, \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I,$$

see [8], [101], [57]. This is a simplification of the original and more physically realistic Beavers-Joseph conditions, in which $u \cdot \widehat{\tau}_i$ is replaced by $(u - u_p) \cdot \widehat{\tau}_i$, e.g., [18], [19]. Here ρ, g are the fluid density and gravitational acceleration constant and

$$\begin{aligned}
\phi &= \text{Darcy pressure + elevation induced pressure = piezometric head},\\
u_p &= -\mathcal{K}\nabla\phi = \text{ velocity in porous media region, } \Omega_p,\\
u &= \text{ velocity in Stokes region, } \Omega_f,\\
f_f, \; f_p &= \text{body forces in fluid region and source in porous media region},\\
\mathcal{K} &= \text{ hydraulic conductivity tensor with } min_{\Omega_p}\lambda_{\min}(\mathcal{K}) =: k_{\min} > 0,\\
\mu &= \text{ viscosity of fluid},\\
S_0 &= \text{ specific mass storativity coefficient}.
\end{aligned}$$

We assume that all material and fluid parameters are positive and the boundary conditions are simple Dirichlet conditions on the exterior boundaries (not including the interface I). While this is only one of several important boundary conditions, [7], [98], *the algorithms herein and their numerical analysis can easily be extended to different combinations of exterior boundary conditions.*

2.2 NOTATIONS AND PRELIMINARIES

We denote the $L^2(I)$ norm by $||\cdot||_I$ and the $L^2(\Omega_{f/p})$ norms by $||\cdot||_{f/p}$, respectively; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Let

$$\begin{aligned} X_f \ &: \ = \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f \backslash I\}, \\ X_p \ &: \ = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p \backslash I\}, \\ Q_f \ &: \ = L_0^2(\Omega_f). \end{aligned}$$

To discretize the Stokes-Darcy problem in space by the finite element method, we select conforming finite element spaces

$$\begin{aligned} \text{Velocity:} \quad & X_f^h \subset X_f = \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f \backslash I\}, \\ \text{Darcy pressure:} \quad & X_p^h \subset X_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p \backslash I\}, \\ \text{Stokes pressure:} \quad & Q_f^h \subset Q_f = L_0^2(\Omega_f). \end{aligned}$$

based on a conforming FEM triangulations in Ω_f, Ω_p with maximum triangle diameter "h". No mesh compatibility at or continuity across the interface I between the FEM meshes in the two subdomains is assumed. It is known that provided a minimum angle condition holds functions in piecewise polynomial finite element spaces including X_f^h, X_p^h and even Q_f^h (for the elementwise gradient) satisfy an inverse inequality[1]:

$$||\nabla v_h|| \leq C_{INV} h^{-1} ||v_h||, \ \ h = \text{minimum meshwidth}. \tag{2.2}$$

The Stokes velocity-pressure FEM spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup / LBB^h condition for stability of the discrete pressure, e.g., [45], [43], [64]. We denote the discretely divergence free velocities by

$$V^h := X_f^h \cap \{v_h : (q_h, \nabla \cdot v_h)_f = 0, \text{ for all } q_h \in Q_f^h\}$$

[1] The constant C_{INV} depends upon the angles in the finite element mesh but not on the domain size. The analysis must either use $h_{\min}$ in stability restrictions and $h_{\max}$ in the interpolation inequalities or assume a quasi-uniform mesh. For notational simplicity we do the latter.

The $H_{DIV}(\Omega_f)$ norm is given by

$$||u||_{DIV} := \sqrt{||u||_f^2 + ||\nabla \cdot u||_f^2}.$$

Note that if $d = \dim(\Omega_f)$, $||\nabla \cdot u||_f \leq \sqrt{d}||\nabla u||_f$ and that the Poincaré - Friedrichs inequality holds in both domains:

$$||v||_{f/p} \leq C_{PF}(\Omega_{f/p})||\nabla v||_{f/p}, \forall v \in X_{f/p}. \tag{2.3}$$

We use versions of the trace theorem on the interface I:

$$||\phi||_I \leq C_p^* ||\phi||_p^{1/2} ||\nabla \phi||_p^{1/2} \text{ and } ||u||_I \leq C_f^* ||u||_f^{1/2} ||\nabla u||_f^{1/2} \tag{2.4}$$

We shall assume that the domains $\Omega_{f/p}$ are such that the second trace inequality holds:

$$\left| \int_I \phi u \cdot \widehat{n} ds \right| \leq C||u||_{DIV}||\phi||_{H^1(\Omega_p)}, \text{ for all } u \in X_f, \phi \in X_p. \tag{HDIV trace}$$

This inequality is standard if $\Omega_p = \Omega_f$ and $I = \partial\Omega_p$ and holds with $C = 1$ in that case, e.g., [43]. It also holds if Ω_p is contained in Ω_f and $I = \partial\Omega_p$ and visa versa. The most general domains and shared boundaries I which satisfy this inequality do not seem to be known. However, Moraiti [86] shows that it holds in many cases directly (without extra assumptions like $\phi \in H_{00}^{1/2}(I)$) such as when one domain is an image under a smooth map of the other. For example, we have the following special case of Moraiti [86].

Lemma 2.2.1. *Suppose $\Omega_{f/p}$ are open connected, regular sets in $\mathbb{R}^d$ sharing a boundary portion I which is an open connected set with $I \subset \{x = (x_1, \cdots, x_d) : x_d = 0\}$. Suppose Ω_p is the reflection of Ω_f across I, i.e., $(x_1, \cdots, x_d) \in \Omega_p$ if and only if $(x_1, \cdots, -x_d) \in \Omega_f$. Then (HDIV trace) holds with $C = 1$.*

Proof. We have that $\phi(x_1, \cdots, x_d) \in X_p$ means $\phi^* := \phi(x_1, \cdots, -x_d)$ is a well defined function on Ω_f with the same regularity, norms and boundary conditions. Since $\phi^* = \phi$ on I we have

$$\begin{aligned}\int_I \phi u \cdot \widehat{n} ds &= \int_I \phi^* u \cdot \widehat{n} ds = \int_{\Omega_f} \nabla \cdot (u\phi^*)\, dx = \\ &= \int_{\Omega_f} (\nabla \cdot u)\, \phi^* dx + \int_{\Omega_f} u \cdot \nabla \phi^* dx.\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality

$$\left| \int_I \phi u \cdot \widehat{n} ds \right| \leq ||u||_{DIV} ||\phi^*||_{H^1(\Omega_f)} = ||u||_{DIV} ||\phi||_{H^1(\Omega_p)}.$$

□

To present a convenient[2] variational formulation we first multiply the porous media equation through by ρg. Define the associated bilinear forms

$$\begin{aligned}a_f(u, v) &= (\mu \nabla u, \nabla v)_f + (\nabla \cdot u, \nabla \cdot v)_f + \sum_i \int_I \chi (u \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i) ds, \\ a_p(\phi, \psi) &= \rho g (\mathcal{K} \nabla \phi, \nabla \psi)_p, \quad \text{and} \\ c_I(u, \phi) &= \rho g \int_I \phi u \cdot \widehat{n}_f ds.\end{aligned}$$

A (monolithic) variational formulation of the coupled problem is to find $(u, p, \phi) : [0, \infty) \to X_f \times Q_f \times X_p$ satisfying the given initial conditions and, for all $v \in X_f, q \in Q_f, \psi \in X_p$

$$\begin{gathered}\rho(u_t, v)_f + a_f(u, v) - (p, \nabla \cdot v)_f + c_I(v, \phi) = (f_f, v)_f, \\ (q, \nabla \cdot u)_f = 0, \\ \rho g S_0 (\phi_t, \psi)_p + a_p(\phi, \psi) - c_I(u, \psi) = \rho g (f_p, \psi)_p.\end{gathered} \tag{2.5}$$

The bilinear forms $a_{f/p}(\cdot, \cdot)$ are symmetric, continuous and coercive. We include grad-div stabilization (the term $(\nabla \cdot u, \nabla \cdot v)_f$), an idea developed by [72], [91], [90], with coefficient (normally $O(1)$) chosen to be 1.

The key to the problem is the coupling term. The effect of the above pre-multiplications by ρg is to make the coupling exactly skew symmetric.

[2] Other variational formulations are possible. In (2.3) the volumetric porosity is implicit rather than explicit.

Lemma 2.2.2. *If (HDIV trace) holds we have for $u, \phi \in X_f, X_p$*

$$
\begin{aligned}
|c_I(u,\phi)| &\leq \frac{\mu}{2}||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{(C_f^* C_p^*)^2 (\rho g)^{3/2}}{4\sqrt{\mu k_{\min}}}||u||_f||\phi||_p, \\
|c_I(u,\phi)| &\leq \frac{\mu}{2}||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{\rho}{2}||u||_f^2 + \frac{(C_f^* C_p^*)^4 (\rho g)^{3}}{32\rho\mu k_{\min}}||\phi||_p^2, \\
& \textit{and} \\
|c_I(u,\phi)| &\leq \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}\left(||u||_j^2 + ||\nabla\cdot u||_f^2\right).
\end{aligned}
$$

In the discrete case, if the inverse estimate (2.2) holds we have for all $u^h, \phi^h \in X_f^h, X_p^h$

$$
|c_I(u^h,\phi^h)| \leq \rho g C_f^* C_p^* C_{INV} h^{-1}\left(\frac{1}{2}||u^h||_f^2 + \frac{1}{2}||\phi^h||_p^2\right).
$$

Proof. Using (2.2) and the arithmetic geometric mean inequality twice we obtain

$$
\begin{aligned}
c_I(u,\phi) = \rho g \int_I \phi u \cdot \widehat{n} ds \leq \rho g ||u||_I ||\phi||_I \\
\leq \rho g C_f^* C_p^* ||\phi||_p^{1/2} ||\nabla\phi||_p^{1/2} ||u||_f^{1/2} ||\nabla u||_f^{1/2} \\
\leq \frac{\mu}{2}||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{(C_f^* C_p^*)^2 (\rho g)^{3/2}}{4\sqrt{\mu k_{\min}}}||u||_f||\phi||_p.
\end{aligned}
$$

The second follows from the first by another application of the arithmetic-geometric mean inequality. For the third estimate we use (HDIV trace) and the Poincaré- Friedrichs inequality

$$
\begin{aligned}
|c_I(u,\phi)| \leq \rho g ||u||_{DIV} ||\phi||_{H^1(\Omega_p)} \leq \rho g ||u||_{DIV} \sqrt{1 + C_{PF}^2(\Omega_p)} ||\nabla\phi||_p \\
\leq \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}||u||_{DIV}^2.
\end{aligned}
$$

The fourth follows similarly using the inverse estimate:

$$
\begin{aligned}
|c_I(u^h,\phi^h)| \leq \rho g ||u^h||_I ||\phi^h||_I \leq \rho g C_f^* ||u||_f^{1/2} ||\nabla u||_f^{1/2} C_p^* ||\phi^h||_p^{1/2} ||\nabla\phi^h||_p^{1/2} \\
\leq \rho g C_f^* C_p^* C_{INV} h^{-1} ||u^h||_f ||\phi^h||_p \leq \rho g C_f^* C_p^* C_{INV} h^{-1}\left(\frac{1}{2}||u^h||_f^2 + \frac{1}{2}||\phi^h||_p^2\right).
\end{aligned}
$$

□

Let $\mathbf{W} = X_f \times X_p$, $\mathbf{W}_h = X_f^h \times X_p^h \subset \mathbf{W}$ and $Q^h \subset Q$ denote the conforming finite element subspaces.

Define the equilibrium projection operator:

$$P_h : (\mathbf{w}(t), p(t)) \in (\mathbf{W}, Q) \to (\mathbf{w_h}(t), p_h(t)) \in (\mathbf{W_h}, Q^h),\ \forall t \in [0, T]$$

by

$$a(\mathbf{w_h}(t), \mathbf{v}_h) + b(\mathbf{v_h}, p_h(t)) = a(\mathbf{w}(t), \mathbf{v_h}) + b(\mathbf{v_h}, p(t)) \ \ \forall \mathbf{v_h} \in \mathbf{W_h}$$
$$b(\mathbf{w_h}(t), q_h) = 0, \ \ \forall q_h \in Q^h$$

where

$$a(\mathbf{w}, \mathbf{v}) = a_f(u, v) + a_p(\phi, \psi) + c_I(u, \phi)$$

2.3 DISCRETE FORMULATION

We consider four uncoupling methods. BEsplit1 and 2 methods have superior stability properties in different cases of small physical parameters. The fourth method is second order accurate. The first method is a translation of the method from [113] to the Stokes-Darcy problem.

Method 1: SDsplit = a Stokes-Darcy time-split method. SDsplit is a first order accurate, three sub-step method adapted from [113]. The **SDsplit** approximations are: given (u_h^n, p_h^n, ϕ_h^n), find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1/2}) \in X_f^h \times Q_f^h \times X_p^h$ and $\phi_h^{n+1} \in X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$:

$$\rho g S_0(\frac{\phi_h^{n+1/2} - \phi_h^n}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_h^{n+1/2}, \psi_h) - \frac{1}{2} c_I(u_h^n, \psi_h) = \frac{1}{2} \rho g (f_p^{n+1/2}, \psi_h)_p.$$

$$\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)_f$$

$$+ c_I(v_h, \phi_h^{n+1/2}) = (f_f^{n+1}, v_h)_f, \quad \text{and} \quad (q_h, \nabla \cdot u_h^{n+1})_f = 0, \qquad \text{(SDsplit)}$$

$$\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^{n+1/2}}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_h^{n+1}, \psi_h) - \frac{1}{2} c_I(u_h^{n+1}, \psi_h) = \frac{1}{2} \rho g (f_f^{n+1}, \psi_h)_p.$$

SDsplit is uncoupled but sequential: $u_h^n \to \phi_h^{n+1/2} \to u_h^{n+1} \to \phi_h^{n+1}$.

Method 2: BEsplit1 = a Backward Euler time-split method. The BEsplit approximations are: given (u_h^n, p_h^n, ϕ_h^n) find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$,

$$
\begin{aligned}
\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)_f + c_I(v_h, \phi_h^n) &= (f_f^{n+1}, v_h)_f, \\
(q_h, \nabla \cdot u_h^{n+1})_f &= 0, \\
\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^n}{\triangle t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^{n+1}, \psi_h) &= \rho g(f_p^{n+1}, \psi_h)_p.
\end{aligned}
\tag{BEsplit1}
$$

The coupling term in the ϕ equation is evaluated at the newly computed value u_h^{n+1} so we compute $\phi_h^n \to u_h^{n+1} \to \phi_h^{n+1}$.

Method 3: BEsplit2. The order of cycling through the equations alters the computed results. **BEsplit2** is the previous method in the opposite order. It is given by: given (u_h^n, p_h^n, ϕ_h^n) find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$,

$$
\begin{aligned}
\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^n}{\triangle t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^n, \psi_h) &= \rho g(f_p^{n+1}, \psi_h)_p \\
\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + \rho(\nabla \cdot \frac{u_h^{n+1} - u_h^n}{\triangle t}, \nabla \cdot v_h)_f + a_f(u_h^{n+1}, v_h) & \\
-(p_h^{n+1}, \nabla \cdot v_h)_f + c_I(v_h, \phi_h^{n+1}) &= (f_f^{n+1}, v_h)_f, \\
(q_h, \nabla \cdot u_h^{n+1})_f &= 0.
\end{aligned}
\tag{BEsplit2}
$$

Our initial analysis revealed that control was needed for a term $||u_h^{n+1} - u_h^n||_{DIV}$. This led to the idea of inserting the grad-div stabilization term $(\nabla \cdot \left(u_h^{n+1} - u_h^n\right) / \triangle t, \nabla \cdot v_h)_f$ acting on the time discretization of u_t. This term is exactly zero for the continuous problem so it does not increase the method's consistency error.

Method 4: CNsplit= a Crank-Nicolson time-split method. CNsplit is second order accurate. It computes *in parallel*[3] two partitioned approximations $(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1})$

[3] Two processors can be working simultaneously with waiting only due to the different speeds of solving the subdomain problems.

and $(\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ whereupon *the new approximation to each variable is the average of the two computed approximations*:

$$(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) = \frac{1}{2}[(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1}) + (\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})]. \tag{CNsplit}$$

The two individual approximations satisfy, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$

$$\begin{aligned}
&\rho(\frac{\widehat{u}_h^{n+1} - \widehat{u}_h^n}{\triangle t}, v_h)_f + a_f(\frac{\widehat{u}_h^{n+1} + \widehat{u}_h^n}{2}, v_h) - (\frac{\widehat{p}_h^{n+1} + \widehat{p}_h^n}{2}, \nabla \cdot v_h)_f \\
&+ c_I(v_h, \widehat{\phi}_h^n) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla \cdot \widehat{u}_h^{n+1})_f = 0, \qquad \text{(CNsplit-a)}\\
&\rho g S_0(\frac{\widehat{\phi}_h^{n+1} - \widehat{\phi}_h^n}{\triangle t}, \psi_h)_p + a_p(\frac{\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n}{2}, \psi_h) - c_I(\widehat{u}_h^{n+1}, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p
\end{aligned}$$

and

$$\begin{aligned}
&\rho g S_0(\frac{\widetilde{\phi}_h^{n+1} - \widetilde{\phi}_h^n}{\triangle t}, \psi_h)_p + a_p(\frac{\widetilde{\phi}_h^{n+1} + \widetilde{\phi}_h^n}{2}, \psi_h) - c_I(\widetilde{u}_h^n, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p.\\
&\rho(\frac{\widetilde{u}_h^{n+1} - \widetilde{u}_h^n}{\triangle t}, v_h)_f + a_f(\frac{\widetilde{u}_h^{n+1} + \widetilde{u}_h^n}{2}, v_h) - (\frac{\widetilde{p}_h^{n+1} + \widetilde{p}_h^n}{2}, \nabla \cdot v_h)_f \qquad \text{(CNsplit-b)}\\
&+ c_I(v_h, \widetilde{\phi}_h^{n+1}) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla \cdot \widetilde{u}_h^{n+1})_f = 0.
\end{aligned}$$

The calculation can proceed as follows

Step 1: Pass previous values across the interface to the other domains

solve, in parallel for $\widehat{u}_h^{n+1}, \widetilde{\phi}_h^{n+1}$

Step 2: Pass each of $\widehat{u}_h^{n+1}, \widetilde{\phi}_h^{n+1}$ across the interface to the other domains

solve, in parallel, for $\widetilde{u}_h^{n+1}, \widehat{\phi}_h^{n+1}$.

Step 3: Average the two approximations on each domain

Averaging the equations of the two approximations shows that the averages u_h^n and ϕ_h^n satisfy

$$\begin{aligned}
&\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + a_f(\frac{u_h^{n+1} + u_h^n}{2}, v_h) - (\frac{p_h^{n+1} + p_h^n}{2}, \nabla \cdot v_h)_f + \qquad (2.6)\\
&+ c_I(v_h, \frac{\widetilde{\phi}_h^{n+1} + \widehat{\phi}_h^n}{2}) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla \cdot u_h^{n+1})_f = 0,\\
&\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^n}{\triangle t}, \psi_h)_p + a_p(\frac{\phi_h^{n+1} + \phi_h^n}{2}, \psi_h) - c_I(\frac{\widehat{u}_h^{n+1} + \widetilde{u}_h^n}{2}, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p
\end{aligned}$$

To assess consistency errors, the residual is estimated when the true solution $u(t), \phi(t)$ is inserted for all variables u, $\widetilde{u}$, $\widehat{u}$, ϕ, $\widetilde{\phi}$ and $\widehat{\phi}$ in (2.6). As this eliminates the differences between the "hat" and the "tilde" variables, it shows that CNsplit has the same consistency error as the (monolithic / fully coupled) Crank-Nicolson time discretization.

2.4 ANALYSIS OF STABILITY OF THE FOUR SPLITTING BASED PARTITIONED METHOD: SDSPLIT, BESPLIT1/2, CNSPLIT

Since the partitioned methods considered treat some variables in some steps explicitly, a time step restriction for stability in unavoidable. This section gives a stability proof by energy methods in the form that implies stability over long time intervals and elucidates the time step restriction required for the four methods.

2.4.1 SDsplit Stability

We prove conditional stability (with a time step restriction linked to the spacial meshwidth) of SDsplit in this subsection. The time step restriction is of the form

$$\triangle t < C \min \{S_0, k_{min}\} h.$$

To be precise, define

$$\triangle T_0 := \frac{2}{\rho g (C_f^* C_p^*)^2 C_{INV}} \min \left\{ \frac{S_0 \mu}{C_{PF}(\Omega_f)}, \frac{\rho k_{min}}{C_{PF}(\Omega_p)} \right\} h.$$

Theorem 2.4.1. *Suppose that for some α, $0 < \alpha < 1$,*

$$\triangle t \leq (1-\alpha) \triangle T_0. \tag{2.7}$$

Then SDsplit is stable:

$$
\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2+\rho g S_0||\phi_h^N||_p^2\right]+\triangle t\sum_{n=0}^{N-1}\triangle t\frac{\rho g S_0}{2}||\frac{\phi_h^{n+1/2}-\phi_h^n}{\triangle t}||_p^2\\
&+\frac{\alpha\rho g S_0}{2}\triangle t\sum_{n=0}^{N-1}\triangle t||\frac{\phi_h^{n+1/2}-\phi_h^{n+1}}{\triangle t}||_p^2+\frac{\alpha\rho}{2}\triangle t\sum_{n=0}^{N-1}\triangle t||\frac{u_h^{n+1}-u_h^n}{\triangle t}||_f^2\\
&\leq\frac{1}{2}\left[\rho||u_h^0||_f^2+\rho g S_0||\phi_h^0||_p^2\right]+\frac{\rho g C_{PF}^2(\Omega_p)}{2k_{min}}\triangle t\sum_{n=0}^{N-1}||f_p^{n+1/2}||_p^2\\
&+\frac{C_{PF}^2(\Omega_f)}{2\mu}\triangle t\sum_{n=0}^{N-1}||f_f^{n+1}||_f^2+\frac{\rho g C_{PF}^2(\Omega_p)}{4k_{min}}\triangle t\sum_{n=0}^{N-1}||f_p^{n+1}||_p^2.
\end{aligned} \tag{2.8}
$$

Proof. In the first 1/3 step of SDsplit, take $\psi=\triangle t\phi_h^{n+1/2}$. This gives

$$
\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1/2}||_p^2-||\phi_h^n||_p^2+||\phi_h^{n+1/2}-\phi_h^n||_p^2)+\frac{\triangle t}{2}a_p(\phi_h^{n+1/2},\phi_h^{n+1/2})\\
&=\frac{\triangle t}{2}\rho g(f_p^{n+1/2},\phi_h^{n+1/2})_p+\frac{\triangle t}{2}c_I(u_h^n,\phi_h^{n+1/2}).
\end{aligned}
$$

Take $v=\triangle t u_h^{n+1}, q=p_h^{n+1}$ in the 2/3 step and add. This gives

$$
\begin{aligned}
&\frac{1}{2}\rho(||u_h^{n+1}||_f^2-||u_h^n||_f^2+||u_h^{n+1}-u_h^n||_f^2)+\triangle t a_f(u_h^{n+1},u_h^{n+1})\\
&=\triangle t(f_f^{n+1},u_h^{n+1})_f-\triangle t c_I(u_h^{n+1},\phi_h^{n+1/2}).
\end{aligned}
$$

In the 3/3 step, take $\psi=\triangle t\phi_h^{n+1}$:

$$
\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2-||\phi_h^{n+1/2}||_p^2+||\phi_h^{n+1}-\phi_h^{n+1/2}||_p^2)+\frac{\triangle t}{2}a_p(\phi_h^{n+1},\phi_h^{n+1})\\
&=\frac{\triangle t}{2}\rho g(f_p^{n+1},\phi_h^{n+1})_p+\frac{\triangle t}{2}c_I(u_h^{n+1},\phi_h^{n+1}).
\end{aligned}
$$

Adding, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2 - ||\phi_h^n||_p^2) + \frac{1}{2}\rho(||u_h^{n+1}||_f^2 - ||u_h^n||_f^2) \\
&+\frac{1}{2}\rho g S_0(||\phi_h^{n+1/2} - \phi_h^n||_p^2 + ||\phi_h^{n+1} - \phi_h^{n+1/2}||_p^2) + \frac{1}{2}\rho||u_h^{n+1} - u_h^n||_f^2 \\
&+\frac{\triangle t}{2}a_p(\phi_h^{n+1/2}, \phi_h^{n+1/2}) + \frac{\triangle t}{2}a_p(\phi_h^{n+1}, \phi_h^{n+1}) + \triangle t a_f(u_h^{n+1}, u_h^{n+1}) \\
&= \frac{\triangle t}{2}\rho g(f_p^{n+1/2}, \phi_h^{n+1/2})_p + \triangle t(f_f^{n+1}, u_h^{n+1})_f + \frac{\triangle t}{2}\rho g(f_p^{n+1}, \phi_h^{n+1})_p \\
&+\frac{\triangle t}{2}c_I(u_h^n, \phi_h^{n+1/2}) - \triangle t c_I(u_h^{n+1}, \phi_h^{n+1/2}) + \frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1}).
\end{aligned}
$$

Consider the interface terms (the last line):

$$
\text{Interface Terms} = \frac{\triangle t}{2}c_I(u_h^n, \phi_h^{n+1/2}) - \triangle t c_I(u_h^{n+1}, \phi_h^{n+1/2}) + \frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1}).
$$

Rewrite the interface term as a difference by splitting the middle term. This gives

$$
\begin{aligned}
\text{Interface Terms} &= \frac{\triangle t}{2}c_I(u_h^n, \phi_h^{n+1/2}) - \frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1/2}) \\
&-\frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1/2}) + \frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1}) \\
&= \frac{\triangle t}{2}c_I(u_h^n - u_h^{n+1}, \phi_h^{n+1/2}) - \frac{\triangle t}{2}c_I(u_h^{n+1}, \phi_h^{n+1/2} - \phi_h^{n+1}).
\end{aligned}
$$

Lemma 2, the Poincaré-Friedrichs and inverse inequalities give the two bounds

$$
\begin{aligned}
&\frac{\triangle t}{2}|c_I(u^n - u^{n+1}, \phi^{n+1/2})| \leq \\
&\leq \frac{\rho g \triangle t}{4}||\mathcal{K}^{1/2}\nabla\phi_h^{n+1/2}||_p^2 + \frac{\rho g(C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p)h^{-1}\triangle t}{4k_{min}}||u_h^n - u_h^{n+1}||_f^2.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\triangle t}{2}|c_I(u_h^{n+1}, \phi_h^{n+1/2} - \phi_h^{n+1})| \leq \\
&\leq \frac{\mu\triangle t}{4}||\nabla u_h^{n+1}||_f^2 + \frac{\rho^2 g^2(C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f)h^{-1}\triangle t}{4\mu}||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2.
\end{aligned}
$$

Next, we bound the right-hand side in a standard way:

$$
\begin{aligned}
\frac{\triangle t}{2}\rho g(f_p^{n+1/2}, \phi_h^{n+1/2}) &\leq \frac{\rho g \triangle t}{8}||K^{1/2}\nabla\phi_h^{n+1/2}||_p^2 + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{2k_{min}}||f_p^{n+1/2}||_p^2, \\
\triangle t(f_f^{n+1}, u_h^{n+1}) &\leq \frac{C_{PF}^2(\Omega_f)\triangle t}{2\mu}||f_f^{n+1}||_f^2 + \frac{\mu\triangle t}{2}||\nabla u_h^{n+1}||_f^2, \\
\frac{\triangle t}{2}\rho g(f_p^{n+1}, \phi_h^{n+1}) &\leq \frac{\rho g\triangle t}{4}||K^{1/2}\nabla\phi_h^{n+1}||_p^2 + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{4k_{min}}||f_p^{n+1}||_p^2.
\end{aligned}
$$

For the left side. apply coercivity:

$$
\begin{aligned}
\frac{\triangle t}{2}a_p(\phi_h^{n+1/2}, \phi_h^{n+1/2}) &\geq \frac{\rho g\triangle t}{2}||K^{1/2}\nabla\phi_h^{n+1/2}||_p^2, \\
\triangle t a_f(u_h^{n+1}, u_h^{n+1}) &\geq \mu\triangle t||\nabla u_h^{n+1}||_f^2, \\
\frac{\triangle t}{2}a_p(\phi_h^{n+1}, \phi_h^{n+1}) &\geq \frac{\rho g\triangle t}{2}||K^{1/2}\nabla\phi_h^{n+1}||_p^2.
\end{aligned}
$$

Combine, we arrive at:

$$
\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2 - ||\phi_h^n||_p^2) + \frac{1}{2}\rho(||u_h^{n+1}||_f^2 - ||u_h^n||_f^2) + \frac{1}{2}\rho g S_0||\phi_h^{n+1/2} - \phi_h^n||_p^2 \\
&+(\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1}\triangle t}{4\mu})||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2 \\
&+(\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p) h^{-1}\triangle t}{4k_{min}})||u_h^{n+1} - u_h^n||_f^2 \\
&\leq \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{2k_{min}}||f_p^{n+1/2}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{2\mu}||f_f^{n+1}||_f^2 + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{4k_{min}}||f_p^{n+1}||_p^2.
\end{aligned}
$$

Sum this over $n = 0, 1, \cdots, N-1$. We have:

$$\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \frac{1}{2}\rho g S_0 \sum_{n=0}^{N-1} ||\phi_h^{n+1/2} - \phi_h^n||_p^2$$
$$+(\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu}) \sum_{n=0}^{N-1} ||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2$$
$$+(\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p) h^{-1} \triangle t}{4k_{min}}) \sum_{n=0}^{N-1} ||u_h^{n+1} - u_h^n||_f^2$$
$$\leq \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] + \frac{\rho g C_{PF}^2(\Omega_p) \triangle t}{2k_{min}} \sum_{n=0}^{N-1} ||f_p^{n+1/2}||_p^2 +$$
$$+\frac{C_{PF}^2(\Omega_f) \triangle t}{2\mu} \sum_{n=0}^{N-1} ||f_f^{n+1}||_f^2 + \frac{\rho g C_{PF}^2(\Omega_p) \triangle t}{4k_{min}} \sum_{n=0}^{N-1} ||f_p^{n+1}||_p^2.$$

Stability follows under the two conditions below, which are equivalent to the time step restriction $\triangle t \leq (1-\alpha)\triangle T_0$:

$$\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu} \geq \alpha \frac{\rho g S_0}{2},$$
$$\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_P) h^{-1} \triangle t}{4k_{min}} \leq \alpha \frac{\rho}{2}.$$

□

2.4.2 BEsplit1 Stability

Define

$$\triangle T_1 := 2\min\{\mu k_{\min} S_0 \frac{16\rho}{(C_f^* C_p^*)^4 (\rho g)^2}, 1\},$$
$$\triangle T_2 := \frac{2\min\{1, g S_0\}}{g C_f^* C_p^* C_{INV}} h,$$
$$\triangle T_3 = 2\rho g S_0 \mu h \left(\rho g C_f^* C_p^*\right)^{-2} (C_{INV} C_{PF}(\Omega_f))^{-1}$$
$$\triangle T_4 = \frac{2\min\{1, \rho\}}{\rho g (1 + C_{PF}^2(\Omega_p))} k_{\min},$$
$$Parameters := (1 + C_{PF}^2(\Omega_p))(C_{PF}^2(\Omega_f) + d) \frac{\rho g}{k_{\min}\mu}.$$

Note that $\triangle T_1$ and $\triangle T_4$ are independent of h but depend on $k_{\min}$ and S_0 as $\triangle T_1 \simeq S_0 k_{\min}$ and $\triangle T_4 \simeq k_{\min}$. $\triangle T_2$ and $\triangle T_3$ are independent of $k_{\min}$ but depend on h and S_0 as $\triangle T_{2/3} \simeq S_0 h$. The combination of physical parameters $Parameters$ is independent of h and S_0 but depends on all the other physical parameters. When $\mu = O(1)$, the meshwidth h in the porous medium is moderate and $k_{\min}$, S_0 are small the above restrictions mean

$$\text{either } \triangle t \leq C \max\{k_{\min}, S_0 k_{\min}, S_0 h\} \text{ or } C\sqrt{\mu k_{\min}} \geq 1.$$

Theorem 2.4.2 (Uniform in time stability of BEsplit1)**.** *Suppose either the problem parameters satisfy*

$$Parameters \leq 1,$$

or there is an $0 < \alpha < 1$ such that $\triangle t$ satisfies the time step restriction

$$\triangle t \leq (1-\alpha) \max\{\triangle T_1, \triangle T_2, \triangle T_3, \triangle T_4\}$$

Then, (BEsplit1) is stable uniformly in time. Specifically, if the time step restriction with $\triangle T_3$ is active then:

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho g S_0 ||\phi_h^N||_p^2\right] + \\
&+\triangle t \sum_{n=0}^{N-1} [\frac{\triangle t}{2}\rho||\frac{u_h^{n+1}-u_h^n}{\triangle t}||_f^2 \\
&+\alpha a_f(u_h^{n+1}, u_h^{n+1}) + a_p(\phi_h^{n+1}, \phi_h^{n+1})] \leq \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho g S_0 ||\phi_h^0||_p^2\right] \\
&+\triangle t \sum_{n=0}^{N-1} \left[(f_f^{n+1}, u_h^{n+1})_f + \rho g (f_p^{n+1}, \phi_h^{n+1})_p\right].
\end{aligned}$$

If any of the other time step restrictions are active then for any $N > 0$, there holds

$$\begin{aligned}
&\alpha\left[\rho||u_h^N||_f^2 + \rho g S_0 ||\phi_h^N||_p^2\right] + \\
&+\frac{\triangle t}{2} \sum_{n=0}^{N-1} [a_f(u_h^{n+1}+u_h^n, u_h^{n+1}+u_h^n) + a_p(\phi_h^{n+1}+\phi_h^n, \phi_h^{n+1}+\phi_h^n)] \\
&\leq \alpha\left[\rho||u_h^0||_f^2 + \rho g S_0 ||\phi_h^0||_p^2\right] + \\
&+\triangle t \sum_{n=0}^{N-1} \left[(f_f^{n+1}, u_h^{n+1}+u_h^n)_f + \rho g (f_p^{n+1}, \phi_h^{n+1}+\phi_h^n)_p\right].
\end{aligned}$$

Proof. In (BEsplit1) set $v_h = u_h^{n+1} + u_h^n, q_h = p_h^{n+1}$, average the incompressibility condition at successive time levels and add. We use

$$\begin{aligned} a_f(u_h^{n+1}, u_h^{n+1} + u_h^n) = \frac{1}{2} a_f(u_h^{n+1}, u_h^{n+1}) - \frac{1}{2} a_f(u_h^n, u_h^n) + \\ + \frac{1}{2} a_f(u_h^{n+1} + u_h^n, u_h^{n+1} + u_h^n). \end{aligned} \tag{2.9}$$

This gives:

$$\begin{aligned} \frac{1}{2}\left[2\rho||u_h^{n+1}||_f^2 + \triangle t a_f(u_h^{n+1}, u_h^{n+1})\right] - \frac{1}{2}\left[2\rho||u_h^n||_f^2 + \triangle t a_f(u_h^n, u_h^n)\right] + \\ + \frac{\triangle t}{2} a_f(u_h^{n+1} + u_h^n, u_h^{n+1} + u_h^n) + \triangle t c_I(\phi_h^n, u_h^{n+1} + u_h^n) = \triangle t (f_f^{n+1}, u_h^{n+1} + u_h^n)_f. \end{aligned} \tag{2.10}$$

Similarly, in the porous media equation, set $\psi_h = \phi_h^{n+1} + \phi_h^n$. We use here

$$\begin{aligned} a_p(\phi_h^{n+1}, \phi_h^{n+1} + \phi_h^n) = \frac{1}{2} a_p(\phi_h^{n+1}, \phi_h^{n+1}) - \frac{1}{2} a_p(\phi_h^n, \phi_h^n) + \\ + \frac{1}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n). \end{aligned}$$

This gives

$$\begin{aligned} \frac{1}{2}\left[2\rho g S_0||\phi_h^{n+1}||_p^2 + \triangle t a_p(\phi_h^{n+1}, \phi_h^{n+1})\right] - \frac{1}{2}\left[2\rho g S_0||\phi_h^n||_p^2 + \triangle t a_p(\phi_h^n, \phi_h^n)\right] \\ + \frac{\triangle t}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n) - \triangle t c_I(\phi_h^{n+1} + \phi_h^n, u_h^{n+1}) = \triangle t \rho g (f_p^{n+1}, \phi_h^{n+1} + \phi_h^n)_p. \end{aligned} \tag{2.11}$$

Add (2.10) and (2.11). Consider the sum of the two coupling terms that results

$$\begin{aligned} Coupling &= \triangle t\left[c_I(\phi_h^n, u_h^{n+1} + u_h^n) - c_I(\phi_h^{n+1} + \phi_h^n, u_h^{n+1})\right] = \\ &= \triangle t\left[c_I(\phi_h^n, u_h^n) - c_I(\phi_h^{n+1}, u_h^{n+1})\right]. \end{aligned}$$

Let us denote $C^n = c_I(\phi_h^n, u_h^n)$ and

$$\begin{aligned} E^n &= \frac{1}{2}\left[2\rho||u_h^n||_f^2 + 2\rho g S_0||\phi_h^n||_p^2 + \triangle t a_f(u_h^n, u_h^n) + \triangle t a_p(\phi_h^n, \phi_h^n)\right], \\ D^n &= \frac{1}{2} a_f(u_h^{n+1} + u_h^n, u_h^{n+1} + u_h^n) + \frac{1}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n). \end{aligned}$$

Adding the two energy estimates and using the above reduction of the coupling term reduces the total energy estimate to

$$\begin{aligned}\left[E^{n+1}-\triangle tC^{n+1}\right]-\left[E^{n}-\triangle tC^{n}\right]+\\ +\triangle tD^{n}=\ \triangle t\left(\ (f_f^{n+1},u_h^{n+1}+u_h^{n})_f+\rho g(f_p^{n+1},\phi_h^{n+1}+\phi_h^{n})_p\right).\end{aligned}$$

Summing this up from $n=0$ to $n=N-1$ results in

$$\begin{aligned}\left[E^{N}-\triangle tC^{N}\right]+\triangle t\sum_{n=0}^{N-1}D^{n}=\left[E^{0}-\triangle tC^{0}\right]+\\ +\triangle t\sum_{n=0}^{N-1}\left[(f_f^{n+1},u_h^{n+1}+u_h^{n})_f+\rho g(f_p^{n+1},\phi_h^{n+1}+\phi_h^{n})_p\right].\end{aligned}$$

Stability and the stated energy inequality thus follows provided

$$E^{N}-\triangle tC^{N}>0\ \text{ for every } N.$$

We have already shown that

$$\begin{aligned}D^{n} &\geq \frac{\mu}{2}||\nabla\left(u_h^{n+1}+u_h^{n}\right)||_f^2+\frac{\rho g k_{\min}}{2}||\nabla\left(\phi_h^{n+1}+\phi_h^{n}\right)||_p^2,\\ |C^{n}| &\leq \frac{\mu}{2}||\nabla u_h^{n}||_f^2+\frac{\rho g k_{\min}}{2}||\nabla\phi_h^{n}||_p^2+\frac{\rho}{2}||u_h^{n}||_f^2+\frac{(C_f^*C_p^*)^4\left(\rho g\right)^3}{32\rho\mu k_{\min}}||\phi_h^{n}||_p^2.\end{aligned}$$

Thus,

$$\begin{aligned}E^{n}-\triangle tC^{n}\geq\rho||u_h^{n}||_f^2+\rho gS_0||\phi_h^{n}||_p^2+\frac{\triangle t}{2}\left(\mu||\nabla u_h^{n}||_f^2+\rho g k_{\min}||\nabla\phi_h^{n}||_p^2\right)\\ -\triangle t[\frac{\mu}{2}||\nabla u_h^{n}||_f^2+\frac{\rho g k_{\min}}{2}||\nabla\phi_h^{n}||_p^2+\frac{\rho}{2}||u_h^{n}||_f^2+\frac{(C_f^*C_p^*)^4\left(\rho g\right)^3}{32\rho\mu k_{\min}}||\phi_h^{n}||_p^2].\end{aligned}\tag{2.12}$$

Thus stability follows provided

$$\begin{aligned}\triangle t\frac{(C_f^*C_p^*)^4\left(\rho g\right)^3}{32\rho\mu k_{\min}}\leq(1-\alpha)\rho gS_0,\ \text{ or}\\ \triangle t\leq(1-\alpha)\mu k_{\min}S_0\frac{32\rho}{(C_f^*C_p^*)^4\left(\rho g\right)^2}\equiv(1-\alpha)\triangle T_1.\end{aligned}$$

Alternate conditions are obtained using different estimates of the coupling / interface term. Indeed, using Lemma 2

$$|C^n| = |c_I(u_h^n, \phi_h^n)| \leq \rho g C_f^* C_p^* C_{INV} h^{-1} \left(\frac{1}{2} ||u_h^n||_f^2 + \frac{1}{2} ||\phi_h^n||_p^2 \right).$$

Thus stability follows provided

$$\begin{aligned}
\frac{\triangle t}{h} \rho g C_f^* C_p^* C_{INV} &\leq 2(1-\alpha) \min\{\rho, \rho g S_0\}, \text{ or} \\
\triangle t &\leq (1-\alpha) \frac{2 \min\{1, g S_0\}}{g C_f^* C_p^* C_{INV}} h \equiv (1-\alpha) \triangle T_2,
\end{aligned}$$

which is the second condition.

For the condition $Parameters \leq 1$, that by Lemma 2

$$\begin{aligned}
|C^n| &\leq \frac{\rho g k_{\min}}{2} ||\nabla \phi_h^n||_p^2 + \frac{\rho g (1 + C_{PF}^2(\Omega_p))}{2 k_{\min}} ||u_h^n||_{DIV}^2 \\
&\leq \frac{\rho g k_{\min}}{2} ||\nabla \phi_h^n||_p^2 + \frac{\rho g (1 + C_{PF}^2(\Omega_p))}{2 k_{\min}} (||u_h^n||_f^2 + d ||\nabla u_h^n||_f^2) \\
&\leq \frac{\rho g k_{\min}}{2} ||\nabla \phi_h^n||_p^2 + \frac{\rho g (1 + C_{PF}^2(\Omega_p))}{2 k_{\min}} (C_{PF}^2(\Omega_f) + d) ||\nabla u_h^n||_f^2
\end{aligned}$$

Thus the method is also stable if the problem data satisfies

$$\begin{gathered}
\frac{\rho g (1 + C_{PF}^2(\Omega_p))}{2 k_{\min}} (C_{PF}^2(\Omega_f) + d) \leq \frac{\mu}{2} \quad \text{or} \\
Parameters = (1 + C_{PF}^2(\Omega_p))(C_{PF}^2(\Omega_f) + d) \frac{\rho g}{k_{\min} \mu} \leq 1
\end{gathered}$$

The condition involving $\triangle T_3$ requires a separate stability proof. In (BEsplit1) set $v_h = u_h^{n+1}, q_h = p_h^{n+1}$ and add. We use

$$(u_h^{n+1} - u_h^n, u_h^{n+1})_f = \frac{1}{2} \left[||u_h^{n+1}||_f^2 - ||u_h^n||_f^2 \right] + \frac{1}{2} ||u_h^{n+1} - u_h^n||_f^2,$$

and similarly for ϕ. This gives:

$$\begin{gathered}
\frac{\rho}{2} \left[||u_h^{n+1}||_f^2 - ||u_h^n||_f^2 \right] + \frac{\rho}{2} ||u_h^{n+1} - u_h^n||_f^2 + \triangle t a_f(u_h^{n+1}, u_h^{n+1}) + \\
+ \triangle t c_I(\phi_h^n, u_h^{n+1}) = \triangle t (f_f^{n+1}, u_h^{n+1})_f.
\end{gathered}$$

Similarly, in the porous media equation, set $\psi_h = \phi_h^{n+1}$, we get

$$\frac{1}{2}\left[\rho g S_0||\phi_h^{n+1}||_p^2 - \rho g S_0||\phi_h^n||_p^2 + \rho g S_0||\phi_h^{n+1} - \phi_h^n||_p^2\right] + \triangle t a_p(\phi_h^{n+1}, \phi_h^{n+1})$$
$$-\triangle t c_I(\phi_h^{n+1}, u_h^{n+1}) = \triangle t \rho g (f_p^{n+1}, \phi_h^{n+1})_p.$$

Add these two equations and consider the sum of the two coupling terms that result:

$$|Coupling| = \triangle t|c_I(\phi_h^n, u_h^{n+1}) - c_I(\phi_h^{n+1}, u_h^{n+1})| = \triangle t|c_I(\phi_h^{n+1} - \phi_h^n, u_h^{n+1})|.$$

The following bound holds by an analogous proof as that of in Lemma 2:

$$\begin{aligned} |Coupling| &\leq \frac{\rho g S_0}{2}||\phi_h^{n+1} - \phi_h^n||_p^2 + \\ &+\triangle t\left[\frac{\triangle t}{2\rho g S_0}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1}||u_h^{n+1}||_f||\nabla u_h^{n+1}||_f\right] \\ &\leq \frac{\rho g S_0}{2}||\phi_h^{n+1} - \phi_h^n||_p^2 + \\ &+\triangle t\left[\frac{\triangle t}{2\rho g S_0 \mu}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1} C_{PF}(\Omega_f) a_f(u_h^{n+1}, u_h^{n+1})\right]. \end{aligned}$$

The remainder of the proof follows the above pattern and is complete, provided

$$\frac{\triangle t}{2\rho g S_0 \mu}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1} C_{PF}(\Omega_f) \leq 1 - \alpha, \text{ or}$$
$$\triangle t < (1-\alpha)\frac{2\rho g S_0 \mu}{\left(\rho g C_f^* C_p^*\right)^2 C_{INV} C_{PF}(\Omega_f)} h \equiv (1-\alpha)\triangle T_3.$$

For the $\triangle T_4$ condition, we exploit the added grad-div stabilization. By the third inequality of Lemma 2

$$|Coupling| \leq \triangle t\frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \triangle t\frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}||u||^2 + \triangle t\frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}||\nabla \cdot u||^2.$$

The last term can be subsumed into the grad-div stabilization term provided

$$\triangle t\frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}} \leq 1.$$

The other two terms are subsumed into the system energy. Stability thus follows provided

$$\rho||u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2 + \frac{\triangle t}{2}\left(\mu||\nabla u_h^n||_f^2 + \rho g k_{\min}||\nabla\phi_h^n||_p^2\right)$$
$$-\left[\triangle t\frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \triangle t\frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}||u||^2\right] > 0.$$

This requires

$$\triangle t \frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}} \leq \rho$$

Thus, stability follows under these two conditions, i.e., if

$$\triangle t \leq \min\{1,\rho\} \frac{2k_{\min}}{\rho g(1+C_{PF}^2(\Omega_p))} = \triangle T_4.$$

The rest of the proof follows by summing. □

2.4.3 BEsplit2 stability

Due to the similarity of the analysis for BEsplit2 to BEsplit1, we present the aspects of the proof that differ only. Define

$$\begin{aligned}
\triangle T_5 &: = \frac{2k_{\min} h}{g(C_f^* C_p^*)^2 C_{PF}(\Omega_p) C_{INV}} \\
\triangle T_6 &: = \frac{2}{g\,(1+C_{PF}^2(\Omega_p))} k_{\min}.
\end{aligned}$$

We prove uniform in time stability under a time step restriction of the form that occurred in BEsplit1 with $\triangle T_3$ replaced by $\triangle T_5$ and $\triangle T_4$ replaced by $\triangle T_6$. Thus, for small S_0 the active constraint is expected to be

$$\triangle t < \triangle T_6 \simeq C k_{\min}$$

which is independent of both h and S_0. Thus, BEsplit1/2 are promising for the quasi-static approximation and for problems with very small S_0 and moderate $k_{\min}$.

Theorem 2.4.3 (Uniform in time and S_0 stability)**.** *Consider the method (BEsplit2). Suppose that there is an $\alpha, 0 < \alpha < 1$, such that either the problem parameters satisfy*

$$Parameters \leq 1-\alpha,$$

or $\triangle t$ satisfies the time step restriction

$$\triangle t \leq (1-\alpha)\max\{\triangle T_1, \triangle T_2, \triangle T_5, \triangle T_6\}.$$

Then, BEsplit2 is stable uniformly in time and uniformly in S_0. Specifically, for any $N > 0$ we have the energy inequality (which also proves stability)

$$
\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho||\nabla\cdot u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \\
&+\triangle t\sum_{n=0}^{N-1}\left[\frac{\triangle t}{2}\rho g S_0||\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}||_p^2 + a_f(u_h^{n+1},u_h^{n+1}) + \alpha a_p(\phi_h^{n+1},\phi_h^{n+1})\right] \\
&\leq \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho||\nabla\cdot u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] + \triangle t\sum_{n=0}^{N-1}\left[(f_f^{n+1},u_h^{n+1})_f + \rho g(f_p^{n+1},\phi_h^{n+1})_p\right].
\end{aligned}
$$

Proof. The derivation of the stability conditions involving $Parameters$ and $\triangle T_1$, $\triangle T_2$ is very similar to the case of BEsplit1. We therefore move to the condition involving $\triangle T_5$ and T_6.

In (BEsplit2) set $\psi_h = \phi_h^{n+1}, v_h = u_h^{n+1}, q_h = p_h^{n+1}$, and add. We use

$$
-(u_h^n,u_h^{n+1})_f = -\frac{1}{2}(u_h^n,u_h^n)_f - \frac{1}{2}(u_h^{n+1},u_h^{n+1})_f + \frac{1}{2}(u_h^{n+1}-u_h^n,u_h^{n+1}-u_h^n)_f,
$$

and similarly for the $(\nabla\cdot u_h^n,\nabla\cdot u_h^{n+1})_f$ terms and the analogous terms in the ϕ equation. This gives:

$$
\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^{n+1}||_f^2 + \rho||\nabla\cdot u_h^{n+1}||_f^2 + \rho g S_0||\phi_h^{n+1}||_p^2\right] - \frac{1}{2}\left[\rho||u_h^n||_f^2 + \rho||\nabla\cdot u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2\right] + \\
&+\frac{1}{2}\left[\rho||u_h^{n+1}-u_h^n||_f^2 + \rho||\nabla\cdot(u_h^{n+1}-u_h^n)||_f^2 + \rho g S_0||\phi_h^{n+1}-\phi_h^n||_p^2\right] \\
&+\triangle t\left[a_f(u_h^{n+1},u_h^{n+1}) + a_p(\phi_h^{n+1},\phi_h^{n+1})\right] \\
&+\triangle t c_I(\phi_h^{n+1},u_h^{n+1}-u_h^n) = \triangle t(f_f^{n+1},u_h^{n+1})_f + \triangle t\rho g(f_p^{n+1},\phi_h^{n+1})_p.
\end{aligned}
$$

Consider the sum of the two coupling terms

$$
Coupling = \triangle t c_I(\phi_h^{n+1},u_h^{n+1}-u_h^n).
$$

For the condition involving $\triangle T_5$,

$$
\begin{aligned}
|Coupling| &\leq \triangle t\rho g C_f^* C_p^* C_{PF}^{\frac{1}{2}}(\Omega_p)(C_{INV}h^{-1})^{\frac{1}{2}}||\nabla\phi_h^{n+1}||_p||u_h^{n+1}-u_h^n||_f \\
&\leq \frac{1}{2}\rho||u_h^{n+1}-u_h^n||_f^2 + \frac{g(C_f^*C_p^*)^2 C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle t^2}{2k_{min}}a_p(\phi_h^{n+1},\phi_h^{n+1})
\end{aligned}
$$

Subsuming the above two terms in the obvious places, the method is stable if

$$\triangle t \leq \frac{2k_{min}h}{g(C_f^* C_p^*)^2 C_{PF}(\Omega_p) C_{INV}} = \triangle T_5.$$

For the stability condition involving $\triangle T_6$, we have, using Lemma 2 and $a_p(\phi_h^{n+1}, \phi_h^{n+1}) \geq \rho g k_{\min} ||\nabla \phi_h^{n+1}||_p$,

$$\begin{aligned}
|Coupling| &\leq \triangle t\,(\rho g)\,||\phi_h^{n+1}||_{H^1(\Omega_p)}||u_h^{n+1} - u_h^n||_{DIV} \\
&\leq \triangle t\,(\rho g)\,\sqrt{1 + C_{PF}^2(\Omega_p)}||\nabla \phi_h^{n+1}||_p||u_h^{n+1} - u_h^n||_{DIV} \\
&\leq \frac{1}{2}\left[\rho||u_h^{n+1} - u_h^n||_f^2 + \rho||\nabla \cdot (u_h^{n+1} - u_h^n)||_f^2\right] \\
&+ \frac{1}{2}\triangle t^2 \frac{g}{k_{\min}}\left(1 + C_{PF}^2(\Omega_p)\right) a_p(\phi_h^{n+1}, \phi_h^{n+1}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^{n+1}||_f^2 + \rho||\nabla \cdot u_h^{n+1}||_f^2 + \rho g S_0||\phi_h^{n+1}||_p^2\right] - \frac{1}{2}\left[\rho||u_h^n||_f^2 + \rho||\nabla \cdot u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2\right] + \\
&\quad + \frac{1}{2}\rho g S_0||\phi_h^{n+1} - \phi_h^n||_p^2 + \triangle t[a_f(u_h^{n+1}, u_h^{n+1}) + \\
&\quad + (1 - \frac{1}{2}\triangle t g\left(1 + C_{PF}^2(\Omega_p)\right) k_{\min}^{-1}) a_p(\phi_h^{n+1}, \phi_h^{n+1})] \\
&\quad \leq \triangle t(f_f^{n+1}, u_h^{n+1})_f + \triangle t \rho g(f_p^{n+1}, \phi_h^{n+1})_p.
\end{aligned}$$

Stability then follows under the time step restriction

$$(1 - \frac{1}{2}\triangle t g\left(1 + C_{PF}^2(\Omega_p)\right) k_{\min}^{-1}) \geq \alpha > 0$$

which is equivalent to

$$\triangle t \leq (1 - \alpha)\,\frac{2}{g\,(1 + C_{PF}^2(\Omega_p))} k_{\min} \equiv (1 - \alpha)\triangle T_6.$$

□

2.4.4 Stability of CNsplit

CNsplit computes two partitioned approximations $(\widehat{u}_h^n, \widehat{p}_h^n, \widehat{\phi}_h^n)$ and $(\widetilde{u}_h^n, \widetilde{p}_h^n, \widetilde{\phi}_h^n) \in X_f^h \times Q_f^h \times X_p^h$ for $n \geq 1$ whereupon

$$(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) = \frac{1}{2}[(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1}) + (\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})], \qquad \text{(CNsplit)}$$

that is, *the new approximation to each variable is the average of the two computed approximations*. Since the unit ball in a Hilbert space is convex, stability of $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1})$ follows from stability of $(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1})$ and $(\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})$. We thus prove stability of the two individual sub-problems. Define

$$\triangle T_6 := \frac{\sqrt{2S_0}}{\sqrt{g}C_p^* C_f^* C_{INV}} h$$

We prove long time stability under a time step condition of the form

$$\triangle t < C\sqrt{S_0}h.$$

Theorem 2.4.4 (Stability of one step of CNsplit)**.** *Consider (CNsplit-a) one step of the **CNsplit** method. Suppose there is an $0 < \alpha < 1/2$ such that $\triangle t$ satisfies the time step restriction*

$$\triangle t \leq (1-\alpha)\triangle T_6$$

Then, CNsplit-a is stable uniformly in time over possibly long time intervals. Specifically, for every $N \geq 1$

$$\begin{aligned}
&\alpha\left[\rho||\widehat{u}_h^N||_f^2 + \rho g S_0||\widehat{\phi}_h^N||_p^2\right] \\
&+\triangle t \sum_{n=0}^{N-1} \frac{1}{2}\left[a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) + a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)\right] \\
&\leq \rho||\widehat{u}_h^0||_f^2 + \rho g S_0||\widehat{\phi}_h^0||_p^2 - \triangle t c_I(\widehat{\phi}_h^0, \widehat{u}_h^0) \\
&+\triangle t \sum_{n=0}^{N-1}\left[(f_f^{n+1/2}, \widehat{u}_h^{n+1} + \widehat{u}_h^n)_f + \rho g(f_p^{n+1/2}, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)_p\right].
\end{aligned}$$

Proof. In (CNsplit-a) set $v_h = \widehat{u}_h^{n+1} + \widehat{u}_h^n, q_h = \widehat{p}_h^{n+1}$, average the incompressibility condition at successive time levels and add. This gives:

$$\rho||\widehat{u}_h^{n+1}||_f^2 - \rho||\widehat{u}_h^n||_f^2 + \frac{\triangle t}{2} a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) +$$
$$+\triangle t c_I(\widehat{\phi}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) = \triangle t(f_f^{n+1/2}, \widehat{u}_h^{n+1} + \widehat{u}_h^n)_f.$$

Similarly, in the porous media equation, set $\psi_h = \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n$. This gives

$$\rho g S_0||\widehat{\phi}_h^{n+1}||_p^2 - \rho g S_0||\widehat{\phi}_h^n||_p^2 + \frac{\triangle t}{2} a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)$$
$$-\triangle t c_I(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{u}_h^{n+1}) = \triangle t \rho g(f_p^{n+1/2}, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n).$$

Add and consider the sum of the two coupling terms

$$\begin{aligned} Coupling &= \triangle t \left[c_I(\widehat{\phi}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) - c_I(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{u}_h^{n+1}) \right] \\ &= \triangle t \left[c_I(\widehat{\phi}_h^n, \widehat{u}_h^n) - c_I(\widehat{\phi}_h^{n+1}, \widehat{u}_h^{n+1}) \right] \end{aligned}$$

Let us denote $C^n = c_I(\widehat{\phi}_h^n, \widehat{u}_h^n)$ and

$$\begin{aligned} E^n &= \rho||\widehat{u}_h^n||_f^2 + \rho g S_0||\widehat{\phi}_h^n||_p^2, \\ D^n &= \frac{1}{2} a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) + \frac{1}{2} a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n). \end{aligned}$$

Adding the two energy estimates and using the above reduction of the coupling term reduces the total energy estimate to

$$\left[E^{n+1} - \triangle t C^{n+1}\right] - \left[E^n - \triangle t C^n\right] +$$
$$+\triangle t D^n = \triangle t \Big((f_f^{n+1/2}, \widehat{u}_h^{n+1} + \widehat{u}_h^n)_f + \rho g(f_p^{n+1/2}, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)_p \Big)$$

Sum this inequality from $n = 0$ to $N - 1$. The energy inequality thus follows provided

$$E^N - \triangle t C^N \geq \alpha E^N \text{ for every } N.$$

Consider $\triangle t C^N$. Dropping super and subscripts and applying Lemma 2 gives

$$\begin{aligned} \triangle t|C| &\leq \triangle t \rho g C_p^* C_f^* C_{INV} h^{-1} ||u||_f ||\phi||_p \\ &\leq \frac{\rho g S_0}{2} ||\phi||_p^2 + \frac{\triangle t^2}{2\rho g S_0} \left[\rho g C_p^* C_f^* C_{INV} h^{-1}\right]^2 ||u||_f^2. \end{aligned}$$

We thus have stability provided

$$\frac{\triangle t^2}{2\rho g S_0} \left[\rho g C_p^* C_f^* C_{INV} h^{-1}\right]^2 < \rho \text{ or } \triangle t < \triangle T_6.$$

Under the time step restriction $\triangle t \leq \sqrt{1-\alpha}\triangle T_6$ which is implied by $\triangle t \leq (1-\alpha)\triangle T_6$ we have

$$\rho||\widehat{u}_h^{n+1}||_f^2 + \rho g S_0 ||\widehat{\phi}_h^{n+1}||_p^2 - \triangle t c_I(\widehat{\phi}_h^{n+1}, \widehat{u}_h^{n+1}) \geq \alpha \left[\rho||\widehat{u}_h^{n+1}||_f^2 + \rho g S_0 ||\widehat{\phi}_h^{n+1}||_p^2\right].$$

This proves stability of the first half step. □

Now we consider the second half step.

Theorem 2.4.5 (Stability of one step of CNsplit)**.** *Consider (CNsplit-b). Suppose there is an $\alpha, 0 < \alpha < 1$, such that $\triangle t$ satisfies the time step restriction*

$$\triangle t \leq (1-\alpha)\triangle T_6$$

Then, it is stable over long time intervals. Specifically, for every $N \geq 1$

$$\begin{aligned} &\alpha \left[\rho||\widetilde{u}_h^N||_f^2 + \rho g S_0 ||\widetilde{\phi}_h^N||_p^2\right] \\ &+\triangle t \sum_{n=0}^{N-1} \frac{1}{2}\left[a_f(\widetilde{u}_h^{n+1} + \widetilde{u}_h^n, \widetilde{u}_h^{n+1} + \widetilde{u}_h^n) + a_p(\widetilde{\phi}_h^{n+1} + \widetilde{\phi}_h^n, \widetilde{\phi}_h^{n+1} + \widetilde{\phi}_h^n)\right] \\ &\leq \left[\rho||\widetilde{u}_h^0||_f^2 + \rho g S_0 ||\widetilde{\phi}_h^0||_p^2 + \triangle t c_I(\widetilde{\phi}_h^0, \widetilde{u}_h^0)\right] \\ &+\triangle t \sum_{n=0}^{N-1} \left[(f_f^{n+1/2}, \widetilde{u}_h^{n+1} + \widetilde{u}_h^n)_f + \rho g(f_p^{n+1/2}, \widetilde{\phi}_h^{n+1} + \widetilde{\phi}_h^n)_p\right]. \end{aligned}$$

The proof is essentially the same as for the first half-step and is thus omitted.

2.5 ERROR ANALYSIS OF BESPLIT2 AND SDSPLIT

2.5.1 ERROR ANALYSIS OF BEsplit2

$$\text{BESplit 2}\rho g S_0(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t},\psi_h)_p + a_p(\phi_h^{n+1},\psi_h) - C_I(u_h^n,\psi_h) = \rho g(f_2^{n+1},\psi_h)_p \tag{2.13}$$

$$\rho(\frac{u_h^{n+1}-u_h^n}{\triangle t},v_h)_f + a_f(u_h^{n+1},v_h) - (p_h^{n+1},\nabla\cdot v_h) + C_I(v_h,\phi_h^{n+1}) = (f_1^{n+1},v_h)_f \tag{2.14}$$

$$(q_h,\nabla\cdot u_h^{n+1}) = 0 \ \text{ for } \forall q_h\in Q_h$$

Define $u_m = P_h u(t^m), \phi_m = P_h\phi(t^m), p_m = P_h p(t^m)$ The true solution satisfy:

$$\rho g S_0(\phi_t(t^{n+1}),\psi_h)_p + a_p(\phi(t^{n+1})\psi_h) - C_I(u(t^{n+1}),\psi_h) = \rho g(f_2^{n+1},\psi_h)_p$$

$$\rho(u_t(t^{n+1}),v_h) + a_f(u(t^{n+1}),v_h)_f - (p(t^{n+1}),\nabla\cdot v_h) + C_I(v_h,\phi(t^{n+1})) = (f_1^{n+1},v_h)_f$$

Rewrite the equations of the true solution and using the property of the projection:

$$\begin{aligned}\rho g S_0(\frac{\phi_{n+1}-\phi_n}{\triangle t},\psi_h)_p + a_p(\phi_{n+1},\psi_h) - C_I(u_{n+1},\psi_h)\\ = \rho g S_0(\frac{\phi_{n+1}-\phi_n}{\triangle t} - \phi_t(t^{n+1}),\psi_h)_p + \rho g(f_2^{n+1},\psi_h)_p\end{aligned} \tag{2.15}$$

$$\begin{aligned}\rho(\frac{u_{n+1}-u_n}{\triangle t},v_h)_f + a_f(u_{n+1},v_h) - (p_{n+1},\nabla\cdot v_h) + C_I(v_h,\phi_{n+1})\\ = \rho(\frac{u_{n+1}-u_n}{\triangle t} - u_t(t^{n+1}),v_h) + (f_1^{n+1},v_h)_f\end{aligned} \tag{2.16}$$

Define the error $e_\phi^n = P_h\phi(t^{n+1}) - \phi_h^{n+1} = \phi_{n+1} - \phi_h^{n+1}, e_u^n = P_h u(t^n) - u_h^n = u_n - u_h^n$, we have the error equations (2.15)-(2.13) and (2.16)-(2.14) :

$$\begin{aligned}\rho g S_0(\frac{e_\phi^{n+1}-e_\phi^n}{\triangle t},\psi_h)_p + a_p(e_\phi^{n+1},\psi_h) - C_I(u_{n+1}-u_h^n,\psi_h)\\ = \rho g S_0(\frac{\phi_{n+1}-\phi_n}{\triangle t} - \phi_t(t^{n+1}),\psi_h)_p\end{aligned} \tag{2.17}$$

$$\begin{aligned}\rho(\frac{e_u^{n+1}-e_u^n}{\triangle t},v_h)_f + a_f(e_u^{n+1},v_h) - (p_{n+1}-p_h^{n+1},\nabla\cdot v_h) + C_I(v_h,e_\phi^{n+1})\\ = \rho(\frac{u_{n+1}-u_n}{\triangle t} - u_t(t^{n+1}),v_h)_f\end{aligned} \tag{2.18}$$

In (2.17), take $\psi_h = 2\triangle t e_\phi^{n+1}$ and in (2.18), take $v_h = 2\triangle t e_u^{n+1}$ and add up

$$\rho g S_0(||e_\phi^{n+1}||_p^2 - ||e_\phi^n||_p^2) + \rho(||e_u^{n+1}||^2 - ||e_u^n||_f^2) + \rho g S_0||e_\phi^{n+1} - e_\phi^n||_p^2 + \rho||e_u^{n+1} - e_u^n||_f^2$$
$$+2\triangle t a_p(e_\phi^{n+1}, e_\phi^{n+1}) + 2\triangle t a_f(e_u^{n+1}, e_u^{n+1})$$
$$= 2\rho g S_0(\phi_{n+1} - \phi_n - \triangle t\phi_t(t^{n+1}), e_\phi^{n+1})_p + 2\rho(u_{n+1} - u_n - \triangle t u_t(t^{n+1}), v_h)_f$$
$$+2\triangle t C_I(u_{n+1} - u_h^n, e_\phi^{n+1}) - 2\triangle t C_I(e_u^{n+1}, e_\phi^{n+1}) \tag{2.19}$$

Rewrite the interface term on the RHS of (2.19)

$$2\triangle t C_I(u_{n+1} - u_h^n, e_\phi^{n+1}) - 2\triangle t C_I(e_u^{n+1}, e_\phi^{n+1})$$
$$= 2\triangle t\rho g\int_\Gamma (u_{n+1}e_\phi^{n+1} - u_h^n e_\phi^{n+1} - e_\phi^{n+1}e_u^{n+1})\cdot n_f$$
$$= 2\triangle t\rho g\int_\Gamma (u_n e_\phi^{n+1} - u_h^n e_\phi^{n+1} + (u_{n+1} - u_n)e_\phi^{n+1} - e_\phi^{n+1}e_u^{n+1})\cdot n_f$$
$$= 2\triangle t\rho g\int_\Gamma (e_u^n e_\phi^{n+1} + (u_{n+1} - u_n)e_\phi^{n+1} - e_\phi^{n+1}e_u^{n+1})\cdot n_f$$
$$= 2\triangle t\rho g\int_\Gamma ((e_u^n - e_u^{n+1})e_\phi^{n+1} + (u_{n+1} - u_n)e_\phi^{n+1})\cdot n_f$$

Bounding the interface term using typical inequalities

$$||2\triangle t C_I(u_{n+1} - u_h^n, e_\phi^{n+1}) - 2\triangle t C_I(e_u^{n+1}, e_\phi^{n+1})||^2$$
$$\leq 2\triangle t\rho g C_f C_g||e_u^n - e_u^{n+1}||_f^{1/2}||\nabla(e_u^n - e_u^{n+1})||_f^{1/2}||e_\phi^{n+1}||_p^{1/2}||\nabla e_\phi^{n+1}||_p^{1/2}$$
$$+2\triangle t\rho g C_f C_g||u_{n+1} - u_n||_f^{1/2}||\nabla(u_{n+1} - u_n)||_f^{1/2}||e_\phi^{n+1}||_p^{1/2}||\nabla e_\phi^{n+1}||_p^{1/2}$$
$$\leq \frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p)C_{inv}h^{-1}}{k_{min}}||e_u^n - e_u^{n+1}||_f^2 + \frac{1}{4}\rho g\triangle t||K^{1/2}\nabla e_\phi^{n+1}||_p^2$$
$$+\frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p)C_{inv}h^{-1}}{k_{min}}||u_{n+1} - u_n||_f^2 + \frac{1}{4}\rho g\triangle t||K^{1/2}\nabla e_\phi^{n+1}||_p^2 \tag{2.20}$$

In the RHS of (2.19)

$$\phi_{n+1} - \phi_n - \triangle t\phi_t(t^{n+1}) = \phi_{n+1} - \phi_n - (\phi(t^{n+1}) - \phi(t^n)) + (\phi(t^{n+1}) - \phi(t^n)) - \triangle t\phi_t(t^{n+1})$$
$$= w_{p,1}^{n+1} + w_{p,2}^{n+1}$$

$$||w_{p,1}^{n+1}||_p^2 = \int_\Omega (\int_{t^n}^{t^{n+1}} (P_h - I)\phi_t(t)dt)^2 dx \leq \triangle t\int_{t^n}^{t^{n+1}} ||(P_h - I)\phi_t(t)||_p^2 dt$$

And

$$||w_{p,2}^{n+1}||_p^2 = \int_\Omega (\int_{t^n}^{t^{n+1}} (t - t^n)\phi_{tt}(t)dt)^2 dx \leq \triangle t^3 \int_{t^n}^{t^{n+1}} ||\phi_{tt}(t)||_p^2 dt$$

Similarly in (2.19)

$$\begin{aligned} u_{n+1} - u_n - \triangle t u_t(t^{n+1}) &= u_{n+1} - u_n - (u(t^{n+1}) - u(t^n)) + (u(t^{n+1}) - u(t^n)) - \triangle t u_t(t^{n+1}) \\ &= w_{u,1}^{n+1} + w_{u,2}^{n+1} \end{aligned}$$

And we can show

$$\begin{aligned} ||w_{f,1}^{n+1}||_f^2 \leq \triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)u_t(t)||_f^2 dt \\ ||w_{f,2}^{n+1}||_f^2 \leq \triangle t^3 \int_{t^n}^{t^{n+1}} ||u_{tt}(t)||_f^2 dt \end{aligned}$$

Thus in (2.19), we can bound the term

$$\begin{aligned} &2\rho g S_0(\phi_{n+1} - \phi_n - \triangle t \phi_t(t^{n+1}), e_\phi^{n+1})_p + 2\rho(u_{n+1} - u_n - \triangle t u_t(t^{n+1}), v_h)_f \\ &= 2\rho g S_0(w_{p,1}^{n+1} + w_{p,2}^{n+1}, e_\phi^{n+1})_p + 2\rho(w_{f,1}^{n+1} + w_{f,2}^{n+1}, v_h)_f \\ &\leq \frac{4\rho g S_0^2}{\triangle t}||w_{p,1}^{n+1} + w_{p,2}^{n+1}||_p^2 + \frac{1}{4}\rho g \triangle t ||K^{1/2}\nabla e_\phi^{n+1}||_p^2 + \frac{4\rho^2}{\triangle t \mu}||w_{f,1}^{n+1} + w_{f,2}^{n+1}||_f^2 + \frac{\mu \triangle t}{4}||\nabla e_u^{n+1}||_f^2 \end{aligned} \tag{2.21}$$

Combining (2.19), (3.9), (2.21), we have

$$\begin{aligned} &\rho g S_0(||e_\phi^{n+1}||_p^2 - ||e_\phi^n||_p^2) + \rho(||e_u^{n+1}||_f^2 - ||e_u^n||_f^2) + \rho g S_0||e_\phi^{n+1} - e_\phi^n||_p^2 + \rho||e_u^{n+1} - e_u^n||_f^2 \\ &+ 2\triangle t a_p(e_\phi^{n+1}, e_\phi^{n+1}) + 2\triangle t a_f(e_u^{n+1}, e_u^{n+1}) \\ &\leq \frac{4\rho g S_0^2}{\triangle t}||w_{p,1}^{n+1} + w_{p,2}^{n+1}||_p^2 + \frac{1}{4}\rho g \triangle t ||K^{1/2}\nabla e_\phi^{n+1}||_p^2 + \frac{4\rho^2}{\triangle t \mu}||w_{f,1}^{n+1} + w_{f,2}^{n+1}||_f^2 + \frac{\mu \triangle t}{4}||\nabla e_u^{n+1}||_f^2 \\ &+ \frac{4\triangle t \rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}}||e_u^n - e_u^{n+1}||_f^2 + \frac{1}{4}\rho g \triangle t ||K^{1/2}\nabla e_\phi^{n+1}||_p^2 \\ &+ \frac{4\triangle t \rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}}||u_{n+1} - u_n||_f^2 + \frac{1}{4}\rho g \triangle t ||K^{1/2}\nabla e_\phi^{n+1}||_p^2 \end{aligned} \tag{2.22}$$

Adding up the inequality from 1 to $N-1$

$$\rho g S_0(||e_\phi^N||_p^2 - ||e_\phi^0||_p^2) + \rho(||e_u^N||_f^2 - ||e_u^0||_f^2) + \rho g S_0 \sum_{i=0}^{N-1}(||e_\phi^{i+1} - e_\phi^i||_p^2) + \rho \sum_{i=0}^{N-1}(||e_u^{i+1} - e_u^i||_f^2)$$
$$+2\triangle t \sum_{i=0}^{N-1} a_p(e_\phi^{i+1}, e_\phi^{i+1}) + 2\triangle t \sum_{i=0}^{N-1} a_f(e_u^{i+1}, e_u^{i+1})$$
$$\leq \frac{4\rho g S_0^2}{\triangle t}\sum_{i=0}^{N-1}||w_{p,1}^{i+1} + w_{p,2}^{i+1}||_p^2 + \frac{3}{4}\rho g \sum_{i=0}^{N-1}\triangle t||K^{1/2}\nabla e_\phi^{i+1}||_p^2$$
$$+\frac{4\rho^2}{\triangle t\mu}\sum_{i=0}^{N-1}||w_{f,1}^{i+1} + w_{f,2}^{i+1}||_f^2 + \frac{\mu\triangle t}{4}\sum_{i=0}^{N-1}||\nabla e_u^{i+1}||_f^2$$
$$+\frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}}\sum_{i=0}^{N-1}(||e_u^i - e_u^{i+1}||_f^2 + ||u_{i+1} - u_i||_f^2) \tag{2.23}$$

From the coercivity of $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$

$$2\triangle t \sum_{i=0}^{N-1} a_p(e_\phi^{i+1}, e_\phi^{i+1}) \geq 2\rho g\triangle t \sum_{i=0}^{N-1}||K^{1/2}\nabla e_\phi^{i+1}||_p^2$$
$$2\triangle t \sum_{i=0}^{N-1} a_f(e_u^{i+1}, e_u^{i+1}) \geq 2\mu\triangle t \sum_{i=0}^{N-1}||\nabla e_u^{i+1}||_f^2$$

Simplifying (2.23),

$$\rho g S_0||e_\phi^N||_p^2 + \rho||e_u^N||_f^2 + \rho g S_0\sum_{i=0}^{N-1}(||e_\phi^{i+1} - e_\phi^i||_p^2)$$
$$+(\rho - \frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}})\sum_{i=0}^{N-1}((||e_u^i - e_u^{i+1}||_f^2$$
$$\leq \frac{4\rho g S_0^2}{\triangle t}\sum_{i=0}^{N-1}||w_{p,1}^{i+1} + w_{p,2}^{i+1}||_p^2 + \frac{4\rho^2}{\triangle t\mu}\sum_{i=0}^{N-1}\triangle t\mu||w_{f,1}^{i+1} + w_{f,2}^{i+1}||_f^2$$
$$+\frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}}\sum_{i=0}^{N-1}||u_{i+1} - u_i||_f^2$$
$$\leq \frac{4\rho g S_0^2}{\triangle t}(\triangle t\int_{t^0}^{t^N}||(P_h - I)\phi_t(t)||_p^2 dt + \triangle t^3\int_{t^0}^{t^N}||\phi_{tt}(t)||_p^2 dt)$$
$$+\frac{4\rho^2}{\triangle t\mu}(\triangle t\int_{t^0}^{t^N}||(P_h - I)u_t(t)||_f^2 dt + \triangle t^3\int_{t^0}^{t^N}||u_{tt}(t)||_f^2 dt)$$
$$+\frac{4\triangle t\rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}}\int_{t^0}^{t^N}||u_t(t)||_f^2 dt \leq C(\triangle t^2 + h^4) \tag{2.24}$$

provided that

$$\rho - \frac{4\triangle t \rho g C_f^2 C_g^2 C_{PF}(\Omega_p) C_{inv} h^{-1}}{k_{min}} \geq 0$$
$$\triangle t \leq \frac{k_{min} h}{4 g C_f^2 C_p^2 C_{PF}(\Omega_p) C_{inv}}$$

2.5.2 ERROR ANALYSIS OF SDsplit

$$\text{SDSplit}\rho g S_0(\frac{\phi_h^{n+1/2} - \phi_h^n}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_h^{n+1/2}, \psi_h) - \frac{1}{2} C_I(u_h^n, \psi_h) = \frac{1}{2}\rho g(f_2^{n+1/2}, \psi_h)_p \quad (2.25)$$
$$\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h) + C_I(v_h, \phi_h^{n+1}) = (f_1^{n+1}, v_h)_f \quad (2.26)$$
$$(q_h, \nabla \cdot u_h^{n+1}) = 0 \text{ for } \forall q_h \in Q_h$$
$$\frac{1}{2}\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^{n+1/2}}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_h^{n+1}, \psi_h) - \frac{1}{2} C_I(u_h^{n+1}, \psi_h) = \frac{1}{2}\rho g(f_2^{n+1}, \psi_h)_p \quad (2.27)$$

Define $u_n = P_h u(t^n), \phi_n = P_h \phi(t^n), \phi_{n+1/2} = P_h \phi(t^{n+1/2}), p_n = P_h p(t^n)$ Rewrite the equations of the true solution and using the property of the projection:

$$\rho g S_0(\frac{\phi_{n+1/2} - \phi_n}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_{n+1/2}, \psi_h) - \frac{1}{2} C_I(u_{n+1/2}, \psi_h) =$$
$$\rho g S_0(\frac{\phi_{n+1/2} - \phi_n}{\triangle t} - \phi_t(t^{n+1/2}), \psi_h)_p + \frac{1}{2}\rho g(f_2^{n+1/2}, \psi_h)_p \quad (2.28)$$
$$\rho(\frac{u_{n+1} - u_n}{\triangle t}, v_h)_f + a_f(u_{n+1}, v_h) - (p_{n+1}, \nabla \cdot v_h) + C_I(v_h, \phi_{n+1})$$
$$= \rho(\frac{u_{n+1} - u_n}{\triangle t} - u_t(t^{n+1}), v_h) + (f_1^{n+1}, v_h)_f \quad (2.29)$$
$$\rho g S_0(\frac{\phi_{n+1} - \phi_{n+1/2}}{\triangle t}, \psi_h)_p + \frac{1}{2} a_p(\phi_{n+1}, \psi_h) - \frac{1}{2} C_I(u_{n+1}, \psi_h) =$$
$$\rho g S_0(\frac{\phi_{n+1} - \phi_{n+1/2}}{\triangle t} - \phi_t(t^{n+1}), \psi_h)_p + \frac{1}{2}\rho g(f_2^{n+1}, \psi_h)_p \quad (2.30)$$

Define the error $e_\phi^n = P_h\phi(t^{n+1}) - \phi_h^{n+1} = \phi_{n+1} - \phi_h^{n+1}, e_u^n = P_h u(t^n) - u_h^n = u_n - u_h^n$, we have the error equations (2.28)-(2.25), (2.29)-(2.26) and (2.30)-(2.27) :

$$\rho g S_0(\frac{e_\phi^{n+1/2} - e_\phi^n}{\triangle t}, \psi_h)_p + \frac{1}{2}a_p(e_\phi^{n+1/2}, \psi_h) - \frac{1}{2}C_I(u_{n+1/2} - u_h^n, \psi_h)$$
$$= \rho g S_0(\frac{\phi_{n+1} - \phi_n}{\triangle t} - \phi_t(t^{n+1}), \psi_h)_p \quad (2.31)$$
$$\rho(\frac{e_u^{n+1} - e_u^n}{\triangle t}, v_h)_f + a_f(e_u^{n+1}, v_h) - (p_{n+1} - p_h^{n+1}, \nabla \cdot v_h) + C_I(v_h, \phi_{n+1} - \phi_h^{n+1/2})$$
$$= \rho(\frac{u_{n+1} - u_n}{\triangle t} - u_t(t^{n+1}), v_h)_f \quad (2.32)$$
$$\rho g S_0(\frac{e_\phi^{n+1} - e_\phi^{n+1/2}}{\triangle t}, \psi_h)_p + \frac{1}{2}a_p(e_\phi^{n+1}, \psi_h) - \frac{1}{2}C_I(e_\phi^{n+1}, \psi_h)$$
$$= \rho g S_0(\frac{\phi_{n+1} - \phi_{n+1/2}}{\triangle t} - \phi_t(t^{n+1}), \psi_h)_p \quad (2.33)$$

In (2.31), take $\psi_h = 2\triangle t e_\phi^{n+1/2}$ and in (2.32), take $v_h = 2\triangle t e_u^{n+1}$, $\psi_h = 2\triangle t e_\phi^{n+1}$ in (2.33) and add up

$$\rho g S_0(||e_\phi^{n+1/2}||_p^2 - ||e_\phi^n||_p^2) + \rho(||e_u^{n+1}||_f^2 - ||e_u^n||_f^2) + \rho g S_0(||e_\phi^{n+1}||_p^2 - ||e_\phi^{n+1/2}||_p^2)$$
$$+\rho g S_0||e_\phi^{n+1/2} - e_\phi^n||_p^2 + \rho||e_u^{n+1} - e_u^{n+1/2}||_f^2 + \rho g S_0||e_\phi^{n+1} - e_\phi^{n+1/2}||_p^2$$
$$+2\triangle t a_p(e_\phi^{n+1/2}, e_\phi^{n+1/2}) + 2\triangle t a_f(e_u^{n+1}, e_u^{n+1}) + +2\triangle t a_p(e_\phi^{n+1}, e_\phi^{n+1})$$
$$= 2\rho g S_0(\phi_{n+1/2} - \phi_n - \triangle t\phi_t(t^{n+1/2}), e_\phi^{n+1/2})_p + 2\rho(u_{n+1} - u_n - \triangle t u_t(t^{n+1}), e_u^{n+1})_f$$
$$+2\rho g S_0(\phi_{n+1} - \phi_{n+1/2} - \triangle t\phi_t(t^{n+1}), e_\phi^{n+1})_p$$
$$+\text{Interface Term} \quad (2.34)$$

Where

$$\text{Interface Term} = \triangle t C_I(u_{n+1/2} - u_h^n, e_\phi^{n+1/2}) - 2\triangle t C_I(e_u^{n+1}, \phi_{n+1} - \phi_h^{n+1/2})$$
$$+\triangle t C_I(e_u^{n+1}, e_\phi^{n+1}) = \triangle t C_I(u_{n+1/2} - u_n, e_\phi^{n+1/2}) + \triangle t C_I(e_u^n, e_\phi^{n+1/2})$$
$$-2\triangle t C_I(e_u^{n+1}, e_\phi^{n+1/2}) - 2\triangle t C_I(e_u^{n+1}, \phi_{n+1} - \phi_{n+1/2}) + \triangle t C_I(e_u^{n+1}, e_\phi^{n+1})$$
$$= \triangle t C_I(u_{n+1/2} - u_n, e_\phi^{n+1/2}) - 2\triangle t C_I(e_u^{n+1}, \phi_{n+1} - \phi_{n+1/2}) + \triangle t C_I(e_u^n, e_\phi^{n+1/2})$$
$$-2\triangle t C_I(e_u^{n+1}, e_\phi^{n+1/2}) + \triangle t C_I(e_u^{n+1}, e_\phi^{n+1})$$
$$= \triangle t C_I(u_{n+1/2} - u_n, e_\phi^{n+1/2}) - 2\triangle t C_I(e_u^{n+1}, \phi_{n+1} - \phi_{n+1/2}) + \triangle t C_I(e_u^n - e_u^{n+1}, e_\phi^{n+1/2})$$
$$+\triangle t C_I(e_u^{n+1}, e_\phi^{n+1} - e_\phi^{n+1/2})$$

We can bound the interface term by using the standard inequalities

$$\text{Interface Term} \leq \frac{2\rho g \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}} ||u_{n+1/2} - u_n||_f^2 + \frac{\rho g \triangle t}{8} ||K^{1/2} \nabla e_\phi^{n+1/2}||_p^2$$
$$+ \frac{2\rho^2 g^2 \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu} ||\phi_{n+1} - \phi_{n+1/2}||_p^2 + \frac{\mu \triangle t}{2} ||\nabla e_u^{n+1}||_f^2$$
$$+ \frac{2\rho g \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}} ||e_u^n - e_u^{n+1}||_f^2 + \frac{\rho g \triangle t}{8} ||K^{1/2} \nabla e_\phi^{n+1/2}||_p^2$$
$$+ \frac{2\rho^2 g^2 \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu} ||e_\phi^{n+1} - e_\phi^{n+1/2}||_p^2 + \frac{\mu \triangle t}{4} ||\nabla e_u^{n+1}||_f^2$$

In (2.34),

$$\phi_{n+1/2} - \phi_n - \triangle t \phi_t(t^{n+1/2}) = \phi_{n+1/2} - \phi_n - (\phi(t^{n+1/2}) - \phi(t^n))$$
$$+(\phi(t^{n+1/2}) - \phi(t^n)) - \triangle t \phi_t(t^{n+1/2})$$
$$= w_{p,1}^{n+1/2} + w_{p,2}^{n+1/2}$$

where

$$||w_{p,1}^{n+1/2}||^2 = \int_\Omega (\int_{t^n}^{t^{n+1/2}} (P_h - I)\phi_t(t) dt)^2 dx$$
$$\leq \int_\Omega \int_{t^n}^{t^{n+1/2}} ((P_h - I)\phi_t(t))^2 dt \triangle t dx$$
$$\leq \triangle t \int_{t^n}^{t^{n+1/2}} ||(P_h - I)\phi_t(t)||_p^2 dt$$
$$\leq \triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)\phi_t(t)||_p^2 dt$$

And

$$||w_{p,2}^{n+1/2}||_p^2 = \int_\Omega (\int_{t^n}^{t^{n+1/2}} (t - t^n)\phi_{tt}(t) dt)^2 dx$$
$$\leq \int_\Omega \int_{t^n}^{t^{n+1/2}} (t - t^n)^2 dt \int_{t^n}^{t^{n+1/2}} (\phi_{tt}(t) dt)^2 dt dx$$
$$\leq \triangle t^3 \int_{t^n}^{t^{n+1}} ||\phi_{tt}(t)||_p^2 dt$$

Sililarly

$$u_{n+1} - u_n - \triangle t u_t(t^{n+1}) = u_{n+1} - u_n - (u(t^{n+1} - u(t^n)) + (u(t^{n+1} - u(t^n)) \triangle t u_t(t^{n+1})$$
$$= w_{f,1}^{n+1} + w_{f,2}^{n+1}$$

Where

$$||w_{f,1}^{n+1}||_f^2 \leq \triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)u_t(t)||_f^2 dt$$
$$||w_{f,2}^{n+1}||_f^2 \leq \triangle t^3 \int_{t^n}^{t^{n+1}} ||u_{tt}(t)||_f^2 dt$$

And

$$\begin{aligned}\phi_{n+1} - \phi_{n+1/2} - \triangle t \phi_t(t^{n+1}) &= \phi_{n+1} - \phi_{n+1/2} - (\phi(t^{n+1}) - \phi(t^{n+1/2})) \\ +(\phi(t^{n+1}) - \phi(t^{n+1/2})) - \triangle t \phi_t(t^{n+1}) & \\ &= w_{p,1}^{n+1} + w_{p,2}^{n+1}\end{aligned}$$

Where

$$||w_{p,1}^{n+1}||_p^2 \leq \triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)\phi_t(t)||_p^2 dt$$
$$||w_{p,2}^{n+1}||_p^2 \leq \triangle t^3 \int_{t^n}^{t^{n+1}} ||\phi_{tt}(t)||_p^2 dt$$

In (2.34), the consistent errors can be bounded by

$$\begin{aligned}&2\rho g S_0(\phi_{n+1/2} - \phi_n - \triangle t \phi_t(t^{n+1/2}), e_\phi^{n+1/2})_p + 2\rho(u_{n+1} - u_n - \triangle t u_t(t^{n+1}), e_u^{n+1})_f \\ &+2\rho g S_0(\phi_{n+1} - \phi_{n+1/2} - \triangle t \phi_t(t^{n+1}), e_\phi^{n+1})_p \\ &\leq \frac{16\rho g S_0^2}{k_{min}\triangle t}(\triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)\phi_t(t)||_p^2 dt + \triangle t^3 \int_{t^n}^{t^{n+1}} ||\phi_{tt}(t)||_p^2 dt) \\ &+\frac{8\rho g \triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1/2}||_p^2 + \frac{8\rho g \triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1}||_p^2 \\ &+\frac{4\rho^2}{\mu \triangle t}(\triangle t \int_{t^n}^{t^{n+1}} ||(P_h - I)u_t(t)||_f^2 dt + \triangle t^3 \int_{t^n}^{t^{n+1}} ||u_{tt}(t)||_f^2 dt) + \frac{\mu \triangle t}{4}||\nabla e_u^{n+1}||_f^2\end{aligned}$$

(2.34) becomes

$$
\begin{aligned}
&\rho g S_0(||e_\phi^{n+1}||_p^2 - ||e_\phi^n||_p^2) + \rho(||e_u^{n+1}||_f^2 - ||e_u^n||_f^2) \\
&+\rho g S_0||e_\phi^{n+1/2} - e_\phi^n||_p^2 + \rho||e_u^{n+1} - e_u^{n+1/2}||_f^2 + \rho g S_0||e_\phi^{n+1} - e_\phi^{n+1/2}||_p^2 \\
&+2\triangle t a_p(e_\phi^{n+1/2}, e_\phi^{n+1/2}) + 2\triangle t a_f(e_u^{n+1}, e_u^{n+1}) + 2\triangle t a_p(e_\phi^{n+1}, e_\phi^{n+1}) \\
&\leq \frac{2\rho g\triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}}||u_{n+1/2} - u_n||_f^2 + \frac{\rho g\triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1/2}||_p^2 \\
&+\frac{2\rho^2 g^2\triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu}||\phi_{n+1} - \phi_{n+1/2}||_p^2 + \frac{\mu\triangle t}{2}||\nabla e_u^{n+1}||_f^2 \\
&+\frac{2\rho g\triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}}||e_u^n - e_u^{n+1}||_f^2 + \frac{\rho g\triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1/2}||_p^2 \\
&+\frac{2\rho^2 g^2\triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu}||e_\phi^{n+1} - e_\phi^{n+1/2}||_p^2 + \frac{\mu\triangle t}{4}||\nabla e^{n+1}||_f^2 \\
&+\frac{16\rho g S_0^2}{k_{min}\triangle t}(\triangle t\int_{t^n}^{t^{n+1}}||(P_h - I)\phi_t(t)||_p^2 dt + \triangle t^3\int_{t^n}^{t^{n+1}}||\phi_{tt}(t)||_p^2 dt) \\
&+\frac{8\rho g\triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1/2}||_p^2 + \frac{8\rho g\triangle t}{8}||K^{1/2}\nabla e_\phi^{n+1}||_p^2 + \frac{4\rho^2}{\mu\triangle t}(\triangle t\int_{t^n}^{t^{n+1}}||(P_h - I)u_t(t)||_f^2 dt \\
&+\triangle t^3\int_{t^n}^{t^{n+1}}||u_{tt}(t)||_f^2 dt) + \frac{\mu\triangle t}{4}||\nabla e_u^{n+1}||_f^2 \qquad (2.35)
\end{aligned}
$$

With the coercivity of $a_p(\cdot,\cdot)$ and $a_f(\cdot,\cdot)$, we have

$$
\begin{aligned}
\triangle t a_p(e_\phi^{n+1/2}, e_\phi^{n+1/2}) &\geq \rho g\triangle t||K^{1/2}\nabla e_\phi^{n+1/2}||_p^2 \\
\triangle t a_p(e_\phi^{n+1}, e_\phi^{n+1}) &\geq \rho g\triangle t||K^{1/2}\nabla e_\phi^{n+1}||_p^2 \\
\triangle t a_f(e_u^{n+1}, e_u^{n+1}) &\geq \mu\triangle t||\nabla e_u^{n+1}||_f^2
\end{aligned}
$$

Combining the same terms and adding up the inequality from 0 to $N-1$

$$
\begin{aligned}
&\rho g S_0 ||e_\phi^{N+1}||_p^2 + \rho ||e_u^{N+1}||_f^2 + \sum_{i=0}^{N-1} (\rho - \frac{2\rho g \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}}) ||e_u^{i+1} - e_u^{i+1/2}||_f^2 \\
&+ \sum_{i=0}^{N-1} (\rho g S_0 - \frac{2\rho^2 g^2 \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu}) ||e_\phi^{i+1} - e_\phi^{i+1/2}||_p^2 \\
&\leq \rho g S_0 ||e_\phi^0||_p^2 + \rho ||e_u^0||^2 + \sum_{i=0}^{N-1} \frac{2\rho g \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}} ||u_{i+1/2} - u_i||_f^2 \\
&+ \sum_{i=0}^{N-1} \frac{2\rho^2 g^2 \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu} ||\phi_{i+1} - \phi_{i+1/2}||_p^2 \\
&+ \frac{16 \rho g S_0^2}{k_{min} \triangle t} (\triangle t \int_{t^0}^{t^{N+1}} ||(P_h - I)\phi_t(t)||_p^2 dt + \triangle t^3 \int_{t^0}^{t^{N+1}} ||\phi_{tt}(t)||_p^2 dt) \\
&+ \frac{8\rho g \triangle t}{8} ||K^{1/2} \nabla e_\phi^{n+1/2}||_p^2 + \frac{8\rho g \triangle t}{8} ||K^{1/2} \nabla e_\phi^{n+1}||_p^2 + \frac{4\rho^2}{\mu \triangle t} (\triangle t \int_{t^0}^{t^{N+1}} ||(P_h - I) u_t(t)||_f^2 dt \\
&+ \triangle t^3 \int_{t^0}^{t^{N+1}} ||u_{tt}(t)||_f^2 dt) \qquad (2.36)
\end{aligned}
$$

$$
\rho g S_0 ||e_\phi^{N+1}||_p^2 + \rho ||e_u^{N+1}||_f^2 \leq C(\triangle t^2 + h^4)
$$

provided that

$$
\begin{aligned}
\rho - \frac{2\rho g \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_p)}{k_{min}} &\geq 0 \text{ and} \\
\rho g S_0 - \frac{2\rho^2 g^2 \triangle t C_f^2 C_g^2 C_{INV} h^{-1} C_{PF}(\Omega_f)}{\mu} &\geq 0
\end{aligned}
$$

that is

$$
\triangle t \leq \min \left\{ \frac{\mu S_0 h}{\rho g C_f^2 C_p^2 C_{INV} C_{PF}(\Omega_f)}, \frac{k_{min} h}{2 g C_f^2 C_p^2 C_{INV} C_{PF}(\Omega_p)} \right\}
$$

2.6 NUMERICAL EXPERIMENTS

We present numerical experiments to test the algorithms proposed in this chapter. First, using the exact solution introduced in [88], we test accuracy. One new aspect is that we also test mass conservation errors across the interface I, the last columns of Tables 2.1 through 2.4. While mixed methods are expected to have better conservation properties than the non-mixed formulation we use, we find the mass conservation errors are quite acceptable in this limited test. Second, we test stability over longer time intervals and small values of $k_{\min}$ and S_0. In these tests the splitting based partitioned methods appear to be stable for larger time step sizes than the IMEX based partitioned methods and that good partitioned methods are available when one parameter is small. When both are small, a very small time step is required for stability for the four methods. The code was implemented using the software package *FreeFEM++*.

2.6.1 Test 1: Convergence rates.

For the first test we select the velocity and pressure field given in [88]. Let the domain Ω be composed of $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$ with the interface $\Gamma = (0,1) \times \{1\}$. The exact velocity field is given by

$$
\begin{aligned}
u_1(x,y,t) &= (x^2(y-1)^2 + y)\cos t\,, \\
u_2(x,y,t) &= \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right)\cos t\,, \\
p(x,y,t) &= (2 - \pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\cos t\,, \\
\phi(x,y,t) &= (2 - \pi\sin(\pi x))(1 - y - \cos(\pi y))\cos t.
\end{aligned}
$$

To check the rates of convergence, take the time interval $0 \le t \le 1$ and in this first test the physical parameters ρ, g, μ, K, S_0 and α are simply set to 1. We utilize Taylor-Hood $P2-P1$ finite elements for the Stokes subdomain and continuous piecewise quadratic finite element for the Darcy subdomain. The boundary conditions on the exterior boundaries (not including the interface I) are inhomogeneous Dirichlet: $u_h = u_{exact}, \phi_h = \phi_{exact}$ on the exterior boundaries. The initial data and source terms are chosen to correspond the exact solution.

For convenience, we denote $\|\cdot\|_I = \|\cdot\|_{L^2(0,T;L^2(I))}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T;L^2(\Omega_{f|p}))}$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(0,T;L^2(\Omega_{f|p}))}$. We show below in Table 2.1–2.4 the errors of approximated velocity and Darcy pressure in several different norms. In the last columns of the tables are the errors in mass conservation on I.

From the tables, we see that SDsplit, BEsplit1 and BEsplit2 are first order methods while CNsplit is second order accuracy, as predicted. Further, the error levels of the first order methods seem quite acceptable as are the mass conservation errors across I.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f - u_h^p)\cdot\mathbf{n}\|_I$
1/5	2.921e-3	7.194e-2	4.030e-3	4.626e-3	2.280e-1
1/10	8.954e-4	2.181e-2	1.183e-2	1.661e-3	4.070e-2
1/20	4.198e-4	5.751e-3	6.367e-4	9.080e-4	9.566e-3
1/40	2.105e-4	1.959e-3	3.399e-4	4.977e-4	2.376e-3
1/80	1.057e-4	8.328e-4	1.771e-4	2.668e-4	5.047e-4

Table 2.1: The convergence performance for SDsplit method. The time step Δt is set to be equal to mesh size h.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f - u_h^p)\cdot\mathbf{n}\|_I$
1/5	3.448e-3	7.371e-2	4.289e-3	4.766e-3	2.278e-1
1/10	1.657e-3	2.343e-2	1.163e-3	1.665e-3	4.694e-2
1/20	8.405e-4	7.200e-3	5.409e-4	8.126e-3	9.531e-3
1/40	4.239e-4	2.923e-3	2.705e-4	4.081e-4	2.369e-3
1/80	2.128e-4	1.367e-3	1.356e-4	2.046e-4	5.035e-4

Table 2.2: The convergence performance for BEsplit1 method. The time step Δt is set to be equal to mesh size h.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f-u_h^p)\cdot\mathbf{n}\|_I$
1/5	2.768e-3	7.130e-2	9.738e-3	1.649e-2	2.547e-1
1/10	9.282e-4	2.164e-2	4.833e-3	8.441e-3	7.087e-2
1/20	4.390e-4	5.610e-3	2.447e-3	4.231e-3	2.722e-2
1/40	2.196e-4	1.860e-3	1.233e-3	2.119e-3	1.212e-2
1/80	1.100e-4	7.739e-4	6.188e-4	1.060e-3	6.258e-3

Table 2.3: The convergence performance for BEsplit2 method. The time step Δt is set to be equal to mesh size h.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f-u_h^p)\cdot\mathbf{n}\|_I$
1/5	3.044e-3	7.789e-2	7.647e-3	1.112e-2	2.284e-1
1/10	4.323e-4	2.259e-2	1.520e-3	2.085e-3	4.795e-2
1/20	5.466e-5	5.193e-3	3.654e-4	4.961e-4	9.849e-3
1/40	7.829e-6	1.270e-3	9.081e-5	1.227e-4	2.487e-3
1/80	1.573e-6	3.187e-4	2.265e-5	3.056e-5	5.273e-4

Table 2.4: The convergence performance for CNsplit method. The time step Δt is set to be equal to mesh size h.

2.6.2 Test 2: Stability in case of small parameters.

In this test, we compare the performance of our proposed methods for uncoupling Stokes-Darcy flows for three cases: small $k_{\min}$ and $O(1)$ S_0, $O(1)$ $k_{\min}$ and small S_0, and small $k_{\min}$ and small S_0. The last case is separated into several sub-cases to distinguish 'extremely small' and 'moderately small' S_0 and $k_{\min}$. Our test here is to check the largest time step for which the four methods are stable over long time intervals. Since the problem is linear we can take the body force terms to be zero. The true solution decays as $t \to \infty$, so any growth in the approximate solution is an instability. We take the initial condition

$$
\begin{aligned}
u_1(x,y,0) &= (x^2(y-1)^2 + y)\,, \\
u_2(x,y,0) &= \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right)\,, \\
p(x,y,0) &= (2 - \pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\,, \\
\phi(x,y,0) &= (2 - \pi\sin(\pi x))(1 - y - \cos(\pi y)).
\end{aligned}
$$

Define the kinetic energy $E^n = \|u_h^n\|_f^2 + \|\phi_h^n\|_p^2$. The final time T_f in our experiment is 10.0 and the system parameters are simply set to be 1.0, except hydraulic conductivity $k_{\min}$ and storativity coefficient S_0. We take the mesh size $h = 1/10$ and run the experiment with different time-step sizes. With each value of Δt, we compute the kinetic energy at final time, i.e., E^N where $N = T_f/\Delta t$. However, we use 10^{250} as a 'cut-off' value for E^n. If E^n exceeds 10^{250} at some n, we stop and output E^n, the kinetic energy at that point. By looking at these figures, we can estimate the largest Δt for which numerical methods is stable.

Since Stokes flows and porous media flows are not typically high velocity flows, and since the domains are large with associated significant costs for subdomain solves, the ability to take large time steps is desirable. In the stability tests for small parameter $k_{\min}$ or S_0 the three first order methods are superior. They are stable for larger time steps, as predicted by the theory. The CNsplit method generally requires a much smaller time step to attain stability. Thus, in some of the figures, the largest time steps needed for the stability of CNsplit are not shown in some cases. To present the CNsplit case, Figure 2.7 gives a graph showing stability of CNsplit alone with numerous small values of S_0 and $k_{\min}$.

2.7 CONCLUSIONS

In both our analysis and tests on problems k_{min} and S_0 are small it seems that stability over long time intervals (and the associated time step restriction) is a key issue in uncoupling the Stokes-Darcy problem. With one small parameter, the first order splitting methods had significant advantages in stability and are a good option when $k_{\min}$ or S_0 is small.

Many other open problems remain. Finding partitioned methods stable for large time steps when both $k_{\min}$, S_0 are small is an open problem. Further, while the first order methods gave acceptable error levels, more accuracy is always desirable. The stability of higher order partitioned methods for large time steps and small parameters also is also largely an open problem. We have not tried to optimize the dependence of the time step barriers upon the domain size. This is an important and open problem, especially for domains with large aspect ratios. At this point we do not know if a partitioned method exists with time step restriction independent of $S_0, k_{\min}, \mu$ and h. If $k_{\min}, \mu \to 0$ the problem reduces to $u_t + C\phi = 0$ and $\phi_t - Cu = 0$ and any such algorithm would be an explicit method for an abstract wave-like equation written as a first order system. The behavior of numerical methods (both partitioned time stepping methods and iterative decoupling methods for use with monolithic time discretizations) in the quasi-static limit (as $S_0 \to 0$) is an open question critical in applications to aquifers since quasi static models are common, e.g., [23] for an example and [86] for a first step to its resolution. In many problems k_{min} and S_0 are both small and the double asymptotics of both parameters is important and open. Since fluid flow acts on different time scales in free flow and in porous media, developing algorithms with good properties that allow different time step sizes in the two domains (multi-rate or asynchronous methods) is an important and largely open challenge.

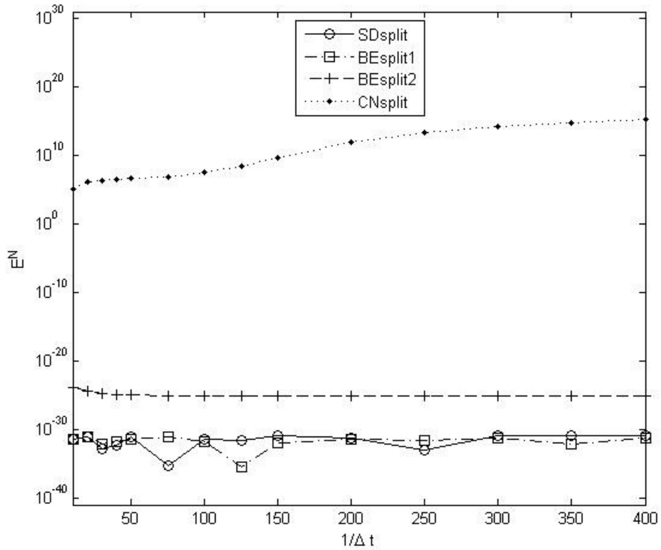


Figure 2.1: E^N using different time step sizes and splitting methods with $k_{\min} = 1$ and $S_0 = 10^{-12}$.

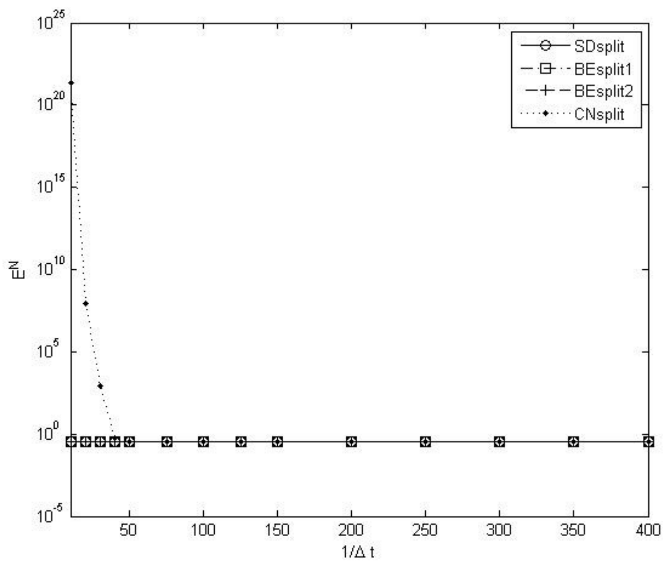


Figure 2.2: E^N using different time step sizes and splitting methods with $k_{\min} = 10^{-12}$ and $S_0 = 1$.

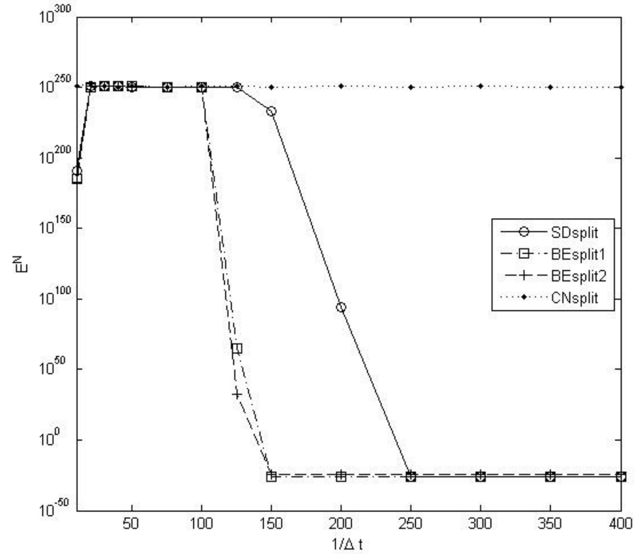


Figure 2.3: E^N using different time step sizes and splitting methods with $k_{\min} = 10^{-3}$ and $S_0 = 10^{-3}$.

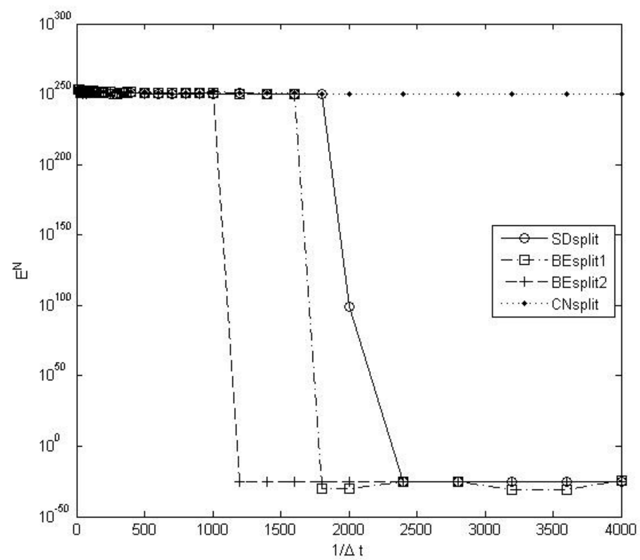


Figure 2.4: E^N using different time step sizes and splitting methods with $k_{\min} = 10^{-4}$ and $S_0 = 10^{-4}$.

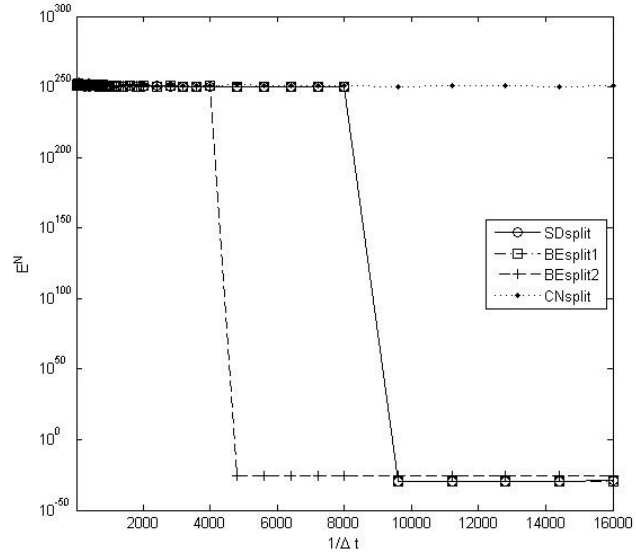


Figure 2.5: E^N using different time step sizes and splitting methods with $k_{\min} = 10^{-4}$ and $S_0 = 10^{-12}$.

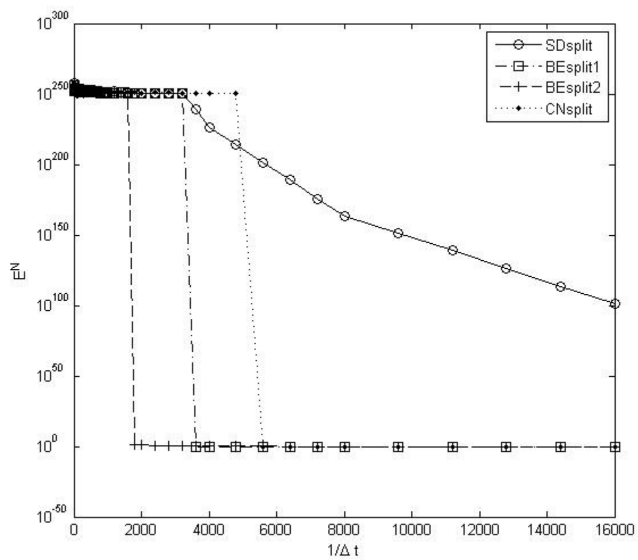


Figure 2.6: E^N using different time step sizes and splitting methods with $k_{\min} = 10^{-12}$ and $S_0 = 10^{-4}$.

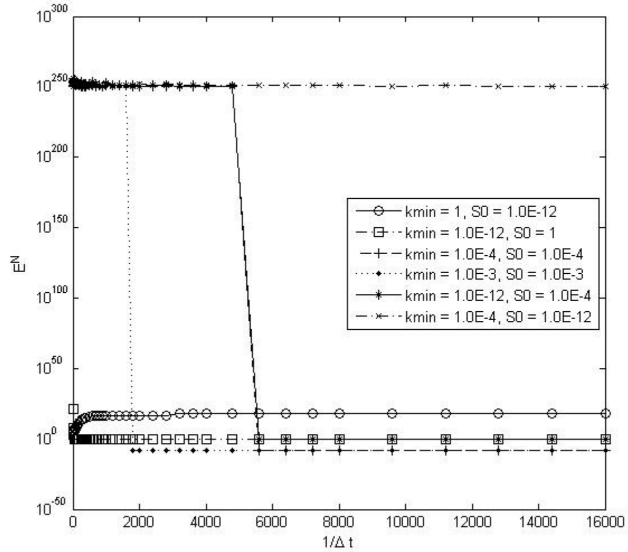


Figure 2.7: Stability of CNsplit at different small values of $k_{\min}$ and S_0.

3.0 ANALYSIS OF A MULTI-RATE SPLITTING METHOD FOR UNCOUPLING EVOLUTIONARY GROUNDWATER-SURFACE WATER FLOWS

3.1 NOTATIONS AND NUMERIC ALGORITHM

Partitioned methods have great advantages for multi-physics, multi-domain problems, e.g., [68], [70], [88], [104]. Splitting methods, one approach for partitioning, have been widely used in applications [55], [50]. For first steps in partitioned method for Stokes-Darcy, see Mu and Zhu [88], extended to a multi-rate method in [105]. For the Stokes-Darcy problem, typical velocities are greater in the fluid region than in the porous media region. Therefore, there are significant advantages in accuracy and efficiency in using a small time step size in the fluid region and a large time step size in the porous media region. However, both partitioning and asynchronous time steps require interpolation of unknown values for the solves and this manufacturing of required value can introduce instabilities. Our work herein is motivated by the search for more partitioned methods, which can accurately capture the features of the physical process while making it easy to calculate numerically. The interface coupling conditions are conservation of mass across the interface, balance of forces and the Beavers-Joseph-Saffman condition, [8], [57], [58], [97], [101]. More general application-oriented partitioned methods and more general IMEX and splitting methods have been widely studied, see, e.g., [113], [110], [3], [27], [37], [53], [113], [75], [116].

In comparison with the multi-rate method in [105], the method herein starts from a Darcy solve, from which an intermediate velocity in porous media is derived, and then has r(a constant) Stokes solves in sequence and ends with a Darcy solve at the following time level, while the multistep method in [105] has a different sequence of Stokes and Darcy solves, resulting in different conditions of stability and convergence.

Let $W = X_f \times X_p$, we consider a triangulation $\mathcal{T}_h$ of the domain $\bar{\Omega}_f \cup \bar{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra if $d = 3$. Here we make the same assumptions for the triangulation as in [88] that:

(1) each triangle or tetrahedra, say T, is such that $int(T) \neq \emptyset$;

(2) $int(T_1) \cap int(T_2) = \emptyset$ for each pair of different $T_1, T_2 \in \mathcal{T}_h$, and if $T_1 \cap T_2 = F \neq \emptyset$, then F is a common face or edge or vetex to T_1 and T_2;

(3) $diam(T) \leq h$ for all $T \in \mathcal{T}_h$;

(4) $\mathcal{T}_h$ is regular; that is, there exists a constant $C_r \geq 1$ such that

$$\max_{T\in\mathcal{T}_h} \frac{diam(T)}{\rho_T} \leq C_r \;\; \forall\, h > 0$$

with $\rho_T = \sup diam(B)$: B is a ball contained in T;

(5) the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains Ω_f and Ω_p are compatible on the interface Γ; that is, they share the same edges (if $d = 2$) or faces (if $d = 3$) therein;

(6) the triangulation $\mathcal{T}_{\Gamma h}$ induced on Γ is quasi-uniform; that is, it is regular and there exists a constant $C_\Gamma > 0$ such that

$$\min_{T\in\mathcal{T}_{\Gamma h}} h_T \geq C_\Gamma h \;\; for\ all\ h > 0$$

And the equilibrium projection operator is defined as in section 2.2

$$P_h : (\mathbf{w}(t), p(t)) \in (W, Q) \to (\mathbf{w_h}(t), p_h(t)) \in (W_h, Q_h),\ \forall\, t \in [0, T]$$

such that

$$a(\mathbf{w_h}(t), \mathbf{v}_h) + b(\mathbf{v}_h, p_h(t)) = a(\mathbf{w}(t), \mathbf{v}_h) + b(\mathbf{v}_h, p(t)),\ \forall\, v_h \in W_h$$
$$b(\mathbf{w_h}(t), q_h) = 0,\ \forall\, q_h \in Q_h$$

P_h is a linear operator. Furthermore, from [88] and [77], if $\mathbf{w} \in W \cap \left\{(H^2(\Omega_f))^d \times H^2(\Omega_p)\right\}$, the following approximation properties hold:

$$||P_h\mathbf{w}(t) - \mathbf{w}(t)||_0 \le Ch^2$$
$$||P_h\mathbf{w}(t) - \mathbf{w}(t)||_1 \le Ch$$
$$||P_h p(t) - p(t)|| \le Ch$$

3.1.1 Algorithm

To streamline notations, choose a uniform timestep $\triangle t$ in Ω_f,

$$\mathcal{P} = \left\{0 = t^0, t^1, t^2, \cdots, t^N = T\right\}, \ t^j = j\triangle t$$

The large time step in Ω_p is given by a separate notations hereafter, $\triangle s = r\triangle t$. Denote by

$$\mathcal{S} = \{0 = t^{m_0}, t^{m_1}, t^{m_2}, \cdots, t^{m_M} = T\} \subset \mathcal{P},$$

a subset satisfying $t^{m_k} = rt^k$ such that the positive constant r is fixed and $Mr = N$. To streamline our notation further, we shall suppress the subscript "h" and replace u_h^m, ϕ_h^m, p_h^m by u^m, ϕ^m, p^m, respectively. For t^m, $t^{m_k} \in [0,T]$, (u^m, ϕ^m, p^m) will denote the discrete approximation to $(u(t^m), \phi(t^m), p(t^m))$. In practice only the data at time t^0 would be provided. One important feature of the algorithm given bellow is that (u^m, p^m) can be calculated for $m = m_k, m_k+1, \cdots m_{k+1}-1$ in parallel with $\phi^{m_{k+1}}$.

• Given u^{m_k}, ϕ^{m_k}, do one step with the large time step $\triangle s$ to obtain $\phi^{m_k^*} \in H_{ph}$, such that $\forall \, \psi \in H_{ph}$

$$gS_0\left(\frac{\phi^{m_k^*} - \phi^{m_k}}{\triangle s}, \psi\right) + \frac{1}{2}a_p\left(\phi^{m_k^*}, \psi\right) = \frac{1}{2}g\left(f_2^{m_k^*}, \psi\right) + \frac{1}{2}g\int_\Gamma \psi u^{m_k} \cdot n_f. \tag{3.1}$$

• Obtaining $\phi^{m_k^*}$ from the first step, do r step in fluid region with small time step $\triangle t = \triangle s/r$ to find (u^{m+1}, p^{m+1}) for $m = m_k, m_k+1, \cdots, m_{k+1}-1$, such that $\forall \, (v,q) \in (H_{fh}, Q_h)$

$$\begin{aligned} \left(\frac{u^{m+1} - u^m}{\triangle t}, v\right) + a_f\left(u^{m+1}, v\right) + b\left(v, p^{m+1}\right) &= \left(f_1^{m+1}, v\right) - g\int_\Gamma \phi^{m_k^*} v \cdot n_f, \\ b\left(u^{m+1}, q\right) &= 0. \end{aligned} \tag{3.2}$$

• With $\phi^{m_k^*}, u^{m_{k+1}}$ obtained from Step 1 and Step 2, do one step in porous region with the large step $\triangle s$ to find $\phi^{m_{k+1}} \in H_{ph}$, such that $\forall\ \psi \in H_{ph}$

$$gS_0\left(\frac{\phi^{m_{k+1}} - \phi^{m_k^*}}{\triangle s}, \psi\right) + \frac{1}{2}a_p\left(\phi^{m_{k+1}}, \psi\right) = \frac{1}{2}g\left(f_2^{m_{k+1}}, \psi\right) + \frac{1}{2}g\int_\Gamma \psi u^{m_{k+1}} \cdot n_f. \tag{3.3}$$

• Set $k = k + 1$ and repeat until $k = M - 1$.

3.2 STABILITY OF THE MULTI-RATE SPLITTING METHOD ON STOKES-DARCY EQUATION

In this section, we prove conditional stability when the smaller time step in fluid region $\triangle t$ is within some restriction C, the restriction of larger time step can be also derived with the ratio of $r = \triangle s/\triangle t$ fixed.

Theorem 3.2.1. *Under the time step restriction*

$$\triangle t \leq \triangle t \leq \frac{2h}{gr(C_f^*)^2(C_p^*)^2 C_{INV}} \min\left\{\frac{S_0\nu}{rC_{PF}(\Omega_f)}, \frac{k_{min}}{3C_{PF}(\Omega_g)}\right\}, \tag{3.4}$$

where the constant $C_{PF}(\Omega_{f/p})$ and C_f^*, C_p^*, C_{INV} are from 2.3, 2.4 and 2.2, the asynchronous algorithm is stable over $0 \leq t < \infty$. We have the stability inequality:

$$\begin{aligned}\frac{1}{2}||u^{m_{l+1}}||_f^2 + \frac{1}{2}gS_0||\phi^{m_{l+1}}||_p^2 + \frac{1}{2}gS_0\sum_{k=0}^{l}||\phi^{m_k^*} - \phi^{m_k}||_p^2 \\ \leq \frac{gC_{PF}^2(\Omega_p)r\triangle t}{2k_{min}}\sum_{k=0}^{l}||f_2^{m_k^*}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{2\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||f_1^{i+1}||_f^2 \\ + \frac{gC_{PF}^2(\Omega_p)r\triangle t}{4k_{min}}\sum_{k=0}^{l}||f_2^{m_{k+1}}||_p^2.\end{aligned}$$

Proof. In (3.1), take $\psi = \triangle s \phi^{m_k^*}$, this gives

$$\frac{1}{2}gS_0(||\phi^{m_k^*}||_p^2 - ||\phi^{m_k}||_p^2 + ||\phi^{m_k^*} - \phi^{m_k}||_p^2) + \frac{1}{2}\triangle s a_p(\phi^{m_k^*}, \phi^{m_k^*})$$
$$= \frac{1}{2}\triangle s g(f_2^{m_k^*}, \phi^{m_k^*}) + \frac{1}{2}\triangle s g\int_\Gamma \phi^{m_k^*} u^{m_k} \cdot n_f. \tag{3.5}$$

Taking $v = \triangle t u^{m_{k+1}}$ in the second step (3.2), using divergence-free property, and summing over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$ give

$$\frac{1}{2}(||u^{m_{k+1}}||_f^2 + \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1} - u^i||_f^2 - ||u^{m_k}||_f^2) + \triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(u^{i+1}, u^{i+1})$$
$$= \triangle t \sum_{i=m_k}^{m_{k+1}-1} (f_1^{i+1}, u^{i+1}) - g\triangle t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma \phi^{m_k^*} u^{i+1} \cdot n_f. \tag{3.6}$$

In the third step (3.3), taking $\psi = \triangle s \phi^{m_{k+1}}$, we obtain:

$$\frac{1}{2}gS_0(||\phi^{m_{k+1}}||_p^2 - ||\phi^{m_k^*}||_p^2 + ||\phi^{m_{k+1}} - \phi^{m_k^*}||_p^2) + \frac{1}{2}\triangle s a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}})$$
$$= \frac{1}{2}\triangle s g(f_2^{m_{k+1}}, \phi^{m_{k+1}}) + \frac{1}{2}\triangle s g\int_\Gamma \phi^{m_{k+1}} u^{m_{k+1}} \cdot n_f. \tag{3.7}$$

Combining (3.5), (3.6), (3.7), we obtain:

$$\frac{1}{2}gS_0(||\phi^{m_{k+1}}||_p^2 - ||\phi^{m_k}||_p^2) + \frac{1}{2}(||u^{m_{k+1}}||_f^2 - ||u^{m_k}||_f^2)$$
$$+\frac{1}{2}gS_0(||\phi^{m_k^*} - \phi^{m_k}||_p^2 + ||\phi^{m_{k+1}} - \phi^{m_k^*}||_p^2) + \frac{1}{2}\sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1} - u^i||_f^2$$
$$+\frac{1}{2}\triangle s a_p(\phi^{m_k^*}, \phi^{m_k^*}) + \frac{1}{2}\triangle s a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) + \triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(u^{i+1}, u^{i+1})$$
$$= \frac{1}{2}\triangle s g(f_2^{m_k^*}, \phi^{m_k^*}) + \triangle t \sum_{i=m_k}^{m_{k+1}-1} (f_1^{i+1}, u^{i+1}) + \frac{1}{2}\triangle s g(f_2^{m_{k+1}}, \phi^{m_{k+1}})$$
$$+\frac{1}{2}\triangle s g\int_\Gamma \phi^{m_k^*} u^{m_k} \cdot n_f - g\triangle t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma \phi^{m_k^*} u^{i+1} \cdot n_f + \frac{1}{2}\triangle s g\int_\Gamma \phi^{m_{k+1}} u^{m_{k+1}} \cdot n_f. \tag{3.8}$$

Now, for the interface terms in the above energy equations (3.8):

$$\begin{aligned}\text{Interface terms} &= \frac{1}{2}\triangle sg\int_\Gamma \phi^{m_k^*}u^{m_k}\cdot n_f - g\triangle t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{i+1}\cdot n_f \\ &\quad+\frac{1}{2}\triangle sg\int_\Gamma \phi^{m_{k+1}}u^{m_{k+1}}\cdot n_f \\ &= \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{m_k}\cdot n_f - \triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{i+1}\cdot n_f \\ &\quad+\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_{k+1}}u^{m_{k+1}}\cdot n_f.\end{aligned}$$

Rewriting the interface terms as differences by splitting the middle term, this gives:

$$\begin{aligned}\text{Interface terms} &= \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{m_k}\cdot n_f - \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{i+1}\cdot n_f \\ &\quad-(\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_k^*}u^{i+1}\cdot n_f - \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \phi^{m_{k+1}}u^{m_{k+1}}\cdot n_f).\end{aligned} \tag{3.9}$$

The Poincaré and inverse inequalities (2.3) and (2.4) now give the bounds:

$$\begin{aligned}&\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}|\int_\Gamma \phi^{m_k^*}(u^{m_k}-u^{i+1})\cdot n_f| \\ &\le \frac{1}{2}\triangle tg(C_f^*)(C_p^*)\sum_{i=m_k}^{m_{k+1}-1}||\phi^{m_k^*}||_p^{1/2}||\nabla\phi^{m_k^*}||_p^{1/2}||(u^{m_k}-u^{i+1})||_f^{1/2}||\nabla(u^{m_k}-u^{i+1})||_f^{1/2} \\ &\le \frac{1}{2}\triangle tg(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_p)C_{INV}^{1/2}h^{-1/2}\sum_{i=m_k}^{m_{k+1}-1}||\nabla\phi^{m_k^*}||_p||(u^{m_k}-u^{i+1})||_f \\ &\le \frac{\triangle t}{2\sqrt{k_{min}}}g(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_p)C_{INV}^{1/2}h^{-1/2}\sum_{i=m_k}^{m_{k+1}-1}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p||(u^{m_k}-u^{i+1})||_f \\ &\le \sum_{i=m_k}^{m_{k+1}-1}(\frac{g\triangle t}{4}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2 + \frac{g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}\triangle t}{4k_{min}}||(u^{m_k}-u^{i+1})||_f^2) \\ &\le \frac{rg\triangle t}{4}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2 + \frac{g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}\triangle t}{4k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||(u^{m_k}-u^{i+1})||_f^2 \\ &\le \frac{rg\triangle t}{4}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2 + \frac{g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}r\triangle t}{4k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||(u^{i+1}-u^{i})||_f^2.\end{aligned} \tag{3.10}$$

For the last two terms in the equation (3.9), we use the identity:

$$
\begin{aligned}
&-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_{\Gamma}(\phi^{m_k^*}u^{i+1}-\phi^{m_{k+1}}u^{m_{k+1}})\cdot n_f\\
&=-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_{\Gamma}\phi^{m_k^*}(u^{i+1}-u^{m_{k+1}})\cdot n_f-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_{\Gamma}u^{m_{k+1}}(\phi^{m_k^*}-\phi^{m_{k+1}})\cdot n_f.
\end{aligned}
\tag{3.11}
$$

The Trace, Poincar$\acute{e}$, Young and Holder inequalities give the bound :

$$
\begin{aligned}
&\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}|\int_{\Gamma}\phi^{m_k^*}(u^{i+1}-u^{m_{k+1}})\cdot n_f|\\
&\leq\frac{rg\triangle t}{8}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2+\frac{g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}r\triangle t}{2k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||(u^{i+1}-u^i)||_f^2.
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}|\int_{\Gamma}u^{m_{k+1}}(\phi^{m_k^*}-\phi^{m_{k+1}})\cdot n_f|\\
&\leq\frac{1}{2}\triangle trg(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_f)C_{INV}^{1/2}h^{-1/2}||\nabla u^{m_{k+1}}||_f||\phi^{m_k^*}-\phi^{m_{k+1}}||_p\\
&\leq\frac{\nu\triangle t}{4}||\nabla u^{m_{k+1}}||_f^2+\frac{g^2r^2(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_f)h^{-1}\triangle t}{4\nu}||\phi^{m_k^*}-\phi^{m_{k+1}}||_p^2.
\end{aligned}
\tag{3.13}
$$

Next, using the Young and Holder's inequality, we bound the other three terms on the right-hand side of the equation (3.8) in a standard way:

$$
\frac{1}{2}\triangle sg(f_2^{m_k^*},\phi^{m_k^*})=\frac{1}{2}r\triangle tg(f^{m_k^*},\phi^{m_k^*})\leq\frac{rg\triangle t}{8}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2+\frac{gC_{PF}^2(\Omega_p)r\triangle t}{2k_{min}}||f_2^{m_k^*}||_p^2,\tag{3.14}
$$

$$
\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f_1^{i+1},u^{i+1})\leq\sum_{i=m_k}^{m_{k+1}-1}(\frac{C_{PF}^2(\Omega_f)\triangle t}{2\nu}||f_1^{i+1}||_f^2+\frac{\nu\triangle t}{2}||\nabla u^{i+1}||_f^2),\tag{3.15}
$$

$$
\frac{1}{2}\triangle sg(f_2^{m_{k+1}},\phi^{m_{k+1}})\leq\frac{rg\triangle t}{4}||\kappa^{1/2}\nabla\phi^{m_{k+1}}||_p^2+\frac{gC_{PF}^2(\Omega_p)r\triangle t}{4k_{min}}||f_2^{m_{k+1}}||_p^2.\tag{3.16}
$$

For the left side of the energy equation (3.8), we apply coercivity:

$$\frac{1}{2}\triangle s a_p(\phi^{m_k^*}, \phi^{m_k^*}) \geq \frac{rg\triangle t}{2}||\kappa^{1/2}\nabla\phi^{m_k^*}||_p^2, \tag{3.17}$$

$$\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(u^{i+1}, u^{i+1}) \geq \nu\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||\nabla u^{i+1}||_f^2, \tag{3.18}$$

$$\frac{1}{2}\triangle s a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) \geq \frac{rg\triangle t}{2}||\kappa^{1/2}\nabla\phi^{m_{k+1}}||_p^2. \tag{3.19}$$

Combining all the inequalities from (3.10)-(3.19), we arrive at:

$$\begin{aligned}
&\frac{1}{2}gS_0(||\phi^{m_{k+1}}||_p^2 - ||\phi^{m_k}||_p^2) + \frac{1}{2}(||u^{m_{k+1}}||_f^2 - ||u^{m_k}||_f^2) + \frac{1}{2}gS_0||\phi^{m_k^*} - \phi^{m_k}||_p^2 \\
&+(\frac{1}{2}gS_0 - \frac{g^2r^2(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_f)h^{-1}\triangle t}{4\nu})||\phi^{m_k^*} - \phi^{m_{k+1}}||_p^2 \\
&+(\frac{1}{2} - \frac{3g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}r\triangle t}{4k_{min}})\sum_{i=m_k}^{m_{k+1}-1}||(u^{i+1} - u^i)||_f^2 \\
&\leq \frac{gC_{PF}^2(\Omega_p)r\triangle t}{2k_{min}}||f_2^{m_k^*}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{2\nu}\sum_{i=m_k}^{m_{k+1}-1}||f_1^{i+1}||_f^2 + \frac{gC_{PF}^2(\Omega_p)r\triangle t}{4k_{min}}||f_2^{m_{k+1}}||_p^2.
\end{aligned}$$

Summing this over $k = 0, 1, \cdots, l$ with $0 \leq l \leq M-1$, we have:

$$\begin{aligned}
&\frac{1}{2}||u^{m_{l+1}}||_f^2 + \frac{1}{2}gS_0||\phi^{m_{l+1}}||_p^2 + \frac{1}{2}gS_0\sum_{k=0}^{l}||\phi^{m_k^*} - \phi^{m_k}||^2 \\
&+(\frac{1}{2}gS_0 - \frac{g^2r^2(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_f)h^{-1}\triangle t}{4\nu})\sum_{k=0}^{l}||\phi^{m_k^*} - \phi^{m_{k+1}}||_p^2 \\
&+(\frac{1}{2} - \frac{3g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}r\triangle t}{4k_{min}})\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||(u^{i+1} - u^i)||_f^2 \\
&\leq \frac{gC_{PF}^2(\Omega_p)r\triangle t}{2k_{min}}\sum_{k=0}^{l}||f_2^{m_k^*}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{2\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||f_1^{i+1}||_f^2 \\
&+\frac{gC_{PF}^2(\Omega_p)r\triangle t}{4k_{min}}\sum_{k=0}^{l}||f_2^{m_{k+1}}||_p^2.
\end{aligned}$$

Stability follows under the two conditions below:

$$\frac{1}{2}gS_0 - \frac{g^2r^2(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_f)h^{-1}\triangle t}{4\nu} \geq 0$$

and

$$\frac{1}{2} - \frac{3g(C_f^*)^2(C_p^*)^2C_{INV}C_{PF}(\Omega_p)h^{-1}r\triangle t}{4k_{min}} \geq 0.$$

These two are equivalent to the time step restriction:

$$\triangle t \leq \frac{2h}{gr(C_f^*)^2(C_p^*)^2C_{INV}} \min\left\{\frac{S_0\nu}{rC_{PF}(\Omega_f)}, \frac{k_{min}}{3C_{PF}(\Omega_p)}\right\}.$$

□

Remark 3.2.2. *Recall that* $\triangle s = r\triangle t$ *with a fixed* r*. The time step restriction in Theorem* 1 *can be rephrased as:*

$$\triangle s \leq \frac{2h}{g(C_f^*)^2(C_p^*)^2C_{INV}} \min\left\{\frac{S_0\nu}{rC_{PF}(\Omega_f)}, \frac{k_{min}}{3C_{PF}(\Omega_p)}\right\}.$$

3.3 ERROR ANALYSIS

In this section, we estimate the error for the algorithm. Here we are using the following notations. Define $u_c^m = u(t^m)$, $\phi_c^m = \phi(t^m)$, $p_c^m = p(t^m)$, and define $u_m = P_h u(t^m)$, $\phi_m = P_h\phi(t^m)$, $p_m = P_h p(t^m)$ to be the projection of the true solutions on to the finite element spaces, then we set $e_c^m = u_c^m - u_m$, $\epsilon_c^m = \phi_c^m - \phi_m$, $\eta_c^m = p_c^m - p_m$, and $e^m = u_m - u^m$, $\epsilon^m = \phi_m - \phi^m$, $\eta^m = p_m - p^m$. Obviously, we observe that $u(t^m) - u^m = e_c^m + e^m$ and $\phi(t^m) - \phi^m = \epsilon_c^m + \epsilon^m$, from approximation properties, we have$||e_c^m||_f + ||\epsilon_c^m||_p \leq Ch^2$, $||\nabla e_c^m||_f + ||\nabla\epsilon_c^m||_p \leq Ch$. Moreover, we assume that $u^0 = u_0 = P_h u(t^0)$, $\phi^0 = \phi_0 = P_h\phi(t^0)$, which imply $e^0 = \epsilon^0 = 0$. Rewriting the true solutions of the Stokes-Darcy equations, for$(\mathbf{v}, q) \in (\mathbf{W}_h, Q_h)$, we have

$$
\begin{aligned}
gS_0\left(\frac{\phi_{m_k^*} - \phi_{m_k}}{\triangle s}, \psi\right) + \frac{1}{2}a_p(\phi_{m_k^*}, \psi) - \frac{1}{2}g\int_\Gamma \psi u_{m_k^*}\cdot n_f \\
= \frac{1}{2}gS_0(w_{p,s}^{m_k^*}, \psi) + \frac{1}{2}g(f_2^{m_k^*}, \psi) \qquad (3.20)\\
\left(\frac{u_{m+1} - u_m}{\triangle t}, v\right) + a_f(u_{m+1}, v) + g\int_\Gamma v\phi_{m+1}\cdot n_f + b(v, p_{m+1}) \\
= (w_{f,t}^{m+1}, v) + (f_1^{m+1}, v) \qquad (3.21)\\
gS_0\left(\frac{\phi_{m_{k+1}} - \phi_{m_k^*}}{\triangle s}, \psi\right) + \frac{1}{2}a_p(\phi_{m_{k+1}}, \psi) - \frac{1}{2}g\int_\Gamma \psi u_{m_{k+1}}\cdot n_f \\
= \frac{1}{2}gS_0(w_{p,s}^{m_{k+1}}, \psi) + \frac{1}{2}g(f_2^{m_{k+1}}, \psi), \qquad (3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
w_{p,s}^{m_k^*} &= \frac{\phi_{m_k^*} - \phi_{m_k}}{\triangle s} - \phi_s(t^{m_k^*}) \\
&= \left[\frac{\phi_{m_k^*} - \phi_{m_k}}{\triangle s} - \frac{\phi(t^{m_k^*}) - \phi(t^{m_k})}{\triangle s}\right] + \left[\frac{\phi(t^{m_k^*}) - \phi(t^{m_k})}{\triangle s} - \phi_s(t^{m_k^*})\right] \\
&= w_{p,s,1}^{m_k^*} + w_{p,s,2}^{m_k^*}. \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
w_{f,t}^{m+1} &= \frac{u_{m+1}-u_m}{\triangle t} - u_t(t^{m+1}) \\
&= \left[\frac{u_{m+1}-u_m}{\triangle t} - \frac{u_(t^{m+1})-u(t^m)}{\triangle t}\right] + \left[\frac{u(t^{m+1})-u(t^m)}{\triangle t} - u_t(t^{m+1})\right] \\
&= w_{f,t,1}^{m+1} + w_{f,t,2}^{m+1}, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
w_{p,s}^{m_{k+1}} &= \frac{\phi_{m_{k+1}}-\phi_{m_k^*}}{\triangle s} - \phi_s(t^{m_{k+1}}) \\
&= \left[\frac{\phi_{m_{k+1}}-\phi_{m_k^*}}{\triangle s} - \frac{\phi(t^{m_{k+1}})-\phi(t^{m_k^*})}{\triangle s}\right] + \left[\frac{\phi(t^{m_{k+1}})-\phi(t^{m_k^*})}{\triangle s} - \phi_s(t^{m_{k+1}})\right] \\
&= w_{p,s,1}^{m_{k+1}} + w_{p,s,2}^{m_{k+1}}. \qquad (3.25)
\end{aligned}
$$

It is easy to verify the following properties of $w_{f,t}$, $w_{p,s}$, from the definition

$$
w_{f,t,1}^{m+1} = (P_h - I)\frac{u(t^{m+1})-u(t^m)}{\triangle t} = \frac{1}{\triangle t}\int_{t^m}^{t^{m+1}} (P_h - I)u_t(t)dt. \qquad (3.26)
$$

then from Cauthy-Schwarz inequality, we have

$$
\begin{aligned}
||w_{f,t,1}^{m+1}||_f^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}} (P_h - I)u_t(t)dt)^2 dx \\
&\leq \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}} ((P_h - I)u_t(t))^2 dt \int_{t^m}^{t^{m+1}} 1^2 dtdx \\
&\leq \frac{1}{\triangle t}\int_{t^m}^{t^{m+1}} ||(P_h - I)u_t(t)||_f^2 dt. \qquad (3.27)
\end{aligned}
$$

and

$$
w_{f,t,2}^{m+1} = \frac{u(t^{m+1})-u(t^m)-\triangle t u_t(t^{m+1})}{\triangle t} = -\frac{\int_{t^m}^{t^{m+1}} (t-t^m)u_{tt}(t)dt}{\triangle t}, \qquad (3.28)
$$

which means

$$
\begin{aligned}
||w_{f,t,2}^{m+1}||_f^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}} (t-t^m)u_{tt}(t)dt)^2 dx \\
&\leq \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}} (u_{tt}(t))^2 dt \int_{t^m}^{t^{m+1}} (t-t^m)dtdx \\
&\leq \triangle t \int_{t^m}^{t^{m+1}} ||u_{tt}(t)||_f^2 dt. \qquad (3.29)
\end{aligned}
$$

The similar property hold for $w_{p,s,1}^{m_k^*}, w_{p,s,2}^{m_k^*}, w_{p,s,1}^{m_{k+1}}, w_{p,s,2}^{m_{k+1}}$ while considering the large time step size $\triangle s$,

$$\begin{aligned}
||w_{p,s,1}^{m_k^*}||_p^2 &= \frac{1}{\triangle s^2}\int_\Omega (\int_{t^{m_k}}^{t^{m_k^*}} (P_h - I)\phi_s(s)ds)^2 dx \\
&\leq \frac{1}{\triangle s^2}\int_\Omega \int_{t^{m_k}}^{t^{m_k^*}} ((P_h - I)\phi_s(s))^2 ds \int_{t^{m_k^*}}^{t^{m_{k+1}}} 1^2 ds dx \\
&\leq \frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_k^*}} ||(P_h - I)\phi_s(s)||_p^2 ds \\
&\leq \frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}} ||(P_h - I)\phi_s(s)||_p^2 ds, \qquad (3.30)
\end{aligned}$$

$$\begin{aligned}
||w_{p,s,2}^{m_k^*}||_p^2 &= \frac{1}{\triangle s^2}\int_\Omega (\int_{t^{m_k}}^{t^{m_k^*}} (t - t^{m_k^*})\phi_{ss}(s)ds)^2 dx \\
&\leq \frac{1}{\triangle s^2}\int_\Omega \int_{t^{m_k}}^{t^{m_k^*}} (\phi_{ss}(s))^2 ds \int_{t^{m_k^*}}^{t^{m_{k+1}}} (t - t^m)^2 ds dx \\
&\leq \triangle s \int_{t^{m_k}}^{t^{m_k^*}} ||\phi_{ss}(s)||_p^2 ds \\
&\leq \triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_{ss}(s)||_p^2 ds. \qquad (3.31)
\end{aligned}$$

Similarly,

$$||w_{p,s,1}^{m_{k+1}}||_p^2 \leq \frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}} ||(P_h - I)\phi_s(s)||_p^2 ds. \qquad (3.32)$$

$$||w_{p,s,2}^{m_{k+1}}||_p^2 \leq \triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_{ss}(s)||_p^2 ds. \qquad (3.33)$$

Also we can show

$$\begin{aligned}
\sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_{m_k^*}||^2 &= \sum_{i=m_k}^{m_{k+1}-1} ||P_h(\phi(t^{i+1}) - \phi(t^{m_k^*}))||^2 \\
&\leq \sum_{i=m_k}^{m_{k+1}-1} C||\phi(t^{i+1}) - \phi(t^{m_k^*})||^2.
\end{aligned}$$

Then bound this term using the Cauthy-Schwartz inequality

$$\begin{aligned}
\sum_{i=m_k}^{m_{k+1}-1} C||\phi(t^{i+1}) - \phi(t^{m_k^*})||^2 &\le \sum_{i=m_k}^{m_{k+1}-1} C \int_{\Omega_p} (\phi(t^{i+1}) - \phi(t^{m_k^*}))^2 dx \\
&\le \sum_{i=m_k}^{m_{k+1}-1} C \int_{\Omega_p} (\int_{t^{m_k^*}}^{t^{i+1}} \phi_s(s) ds)^2 dx \\
&\le \sum_{i=m_k}^{m_{k+1}-1} C \int_{\Omega_p} \int_{t^{m_k^*}}^{t^{i+1}} (\phi_s(s))^2 ds \int_{t^{m_k^*}}^{t^{i+1}} 1^2 ds dx \\
&\le \sum_{i=m_k}^{m_{k+1}-1} C \triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_s(s)||_p^2 ds \\
&= Cr\triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_s(s)||_p^2 ds. \qquad (3.34)
\end{aligned}$$

Similarly, we have

$$||u_{m_k^*} - u_{m_k}||_f^2 \le C\triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||u_t(t)||_f^2 dt. \qquad (3.35)$$

Subtracting (3.1) from (3.20) gives

$$gS_0 \left(\frac{\epsilon^{m_k^*} - \epsilon^{m_k}}{\triangle s}, \psi \right) + \frac{1}{2} a_p(\epsilon^{m_k^*}, \psi) = \frac{1}{2} g S_0 (w_{p,s}^{m_k^*}, \psi) + \frac{1}{2} g \int_\Gamma \psi (u_{m_k^*} - u^{m_k}) \cdot n_f. \qquad (3.36)$$

Considering the smaller time step size $\triangle t$ and subtracting (3.2) from (3.21), we get

$$\left(\frac{e^{m+1} - e^m}{\triangle t}, v \right) + a_f(e^{m+1}, v) + b(v, \eta^{m+1}) = (w_{f,t}^{m+1}, v) - g \int_\Gamma (\phi_{m+1} - \phi^{m_k^*}) v \cdot n_f. \qquad (3.37)$$

Subtracting (3.3) from (3.22), we have

$$\begin{aligned}
& gS_0 \left(\frac{\epsilon^{m_{k+1}} - \epsilon^{m_k^*}}{\triangle s}, \psi \right) + \frac{1}{2} a_p(\epsilon^{m_{k+1}}, \psi) \\
&= \frac{1}{2} g S_0 (w_{p,s}^{m_{k+1}}, \psi) + \frac{1}{2} g \int_\Gamma \psi (u_{m_{k+1}} - u^{m_{k+1}}) \cdot n_f. \qquad (3.38)
\end{aligned}$$

Next we will show that under a certain time step restriction for the small time step $\triangle t$, the multi-step splittling method has first order convergence with respect to the time step size $\triangle t$ and is of second order accurate with respect to the spacing size h to the true solution at each time level for large time step $\triangle s$.

Theorem 3.3.1. *Suppose the true solution is smooth, the initial approximation are sufficiently accurate and that the time step and mesh width $\triangle t$, h satisfy*

$$\triangle t \leq \frac{h}{rg(C_f^*)^2(C_p^*)^2 C_{INV}} \min\left\{\frac{k_{min}}{2C_{PF}(\Omega_p)}, \frac{2S_0\nu}{rC_{PF}(\Omega_f)}\right\}$$

then the following error estimate at the large time steps holds

$$\frac{1}{2}gS_0||\epsilon^{m_{l+1}}||_p^2 + \frac{1}{2}||e^{m_{l+1}}||_f^2 \leq C(\triangle t^2 + h^4)$$

where $C = C(u, \phi, \Omega_{f/p}, material\ parameters)$

Proof. In equation (3.36), we take $\psi = \triangle s \epsilon^{m_k^*}$

$$\begin{aligned}&\frac{1}{2}gS_0(||\epsilon^{m_k^*}||_p^2 - ||\epsilon^{m_k}||_p^2 + ||\epsilon^{m_k^*} - \epsilon^{m_k}||_p^2) + \frac{1}{2}\triangle s a_p(\epsilon^{m_k^*}, \epsilon^{m_k^*})\\ &\quad = \frac{1}{2}\triangle s g S_0(w_{p,s}^{m_k^*}, \epsilon^{m_k^*}) + \frac{1}{2}\triangle s g\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u^{m_k})\cdot n_f. \end{aligned} \tag{3.39}$$

Taking $v = \triangle t e^{m+1}$ in equation (3.37), using the divergence-free property, summing it over $m = m_k, m_k+1, \cdots, m_{k+1}-1$, yield

$$\begin{aligned}&\frac{1}{2}(||e^{m_{k+1}}||_f^2 - ||e^{m_k}||_f^2) + \frac{1}{2}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1} - e^i||_f^2 + \triangle t\sum_{i=m_k}^{m_{k+1}-1} a_f(e^{i+1}, e^{i+1})\\ &\quad = \triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, e^{i+1}) - \triangle t g\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\phi_{i+1} - \phi^{m_k^*})e^{i+1}\cdot n_f. \end{aligned} \tag{3.40}$$

Take $\psi = \triangle s \epsilon^{m_{k+1}}$ in equation (3.38)

$$\begin{aligned}&\frac{1}{2}gS_0(||\epsilon^{m_k}||_p^2 - ||\epsilon^{m_k^*}||_p^2 + ||\epsilon^{m_{k+1}} - \epsilon^{m_k^*}||_p^2) + \frac{1}{2}\triangle s a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}})\\ &\quad = \frac{1}{2}\triangle s g S_0(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) + \frac{1}{2}\triangle s g\int_\Gamma \epsilon^{m_{k+1}}(u_{m_{k+1}} - u^{m_{k+1}})\cdot n_f. \end{aligned} \tag{3.41}$$

Adding up the above inequalities (3.39), (3.40) and (3.41), we obtain

$$
\begin{aligned}
&\frac{1}{2}gS_0(||\epsilon^{m_{k+1}}||_p^2 - ||\epsilon^{m_k}||_p^2) + \frac{1}{2}\sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1} - e^i||_f^2 \\
&+\frac{1}{2}gS_0||\epsilon^{m_k^*} - \epsilon^{m_k}||_p^2 + \frac{1}{2}gS_0||\epsilon^{m_{k+1}} - \epsilon^{m_k^*}||_p^2 + \frac{1}{2}(||e^{m_{k+1}}||_f^2 - ||e^{m_k}||_f^2) \\
&+\frac{1}{2}\triangle s a_p(\epsilon^{m_k^*}, \epsilon^{m_k^*}) + \triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(e^{i+1}, e^{i+1}) + \frac{1}{2}\triangle s a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&= \frac{1}{2}\triangle sgS_0(w_{p,s}^{m_k^*}, \epsilon^{m_k^*}) + \triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{f,t}^{i+1}, e^{i+1}) + \frac{1}{2}\triangle sgS_0(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&+\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u^{m_k})\cdot n_f - \triangle tg \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\phi_{i+1} - \phi^{m_k^*})e^{i+1}\cdot n_f \\
&+\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_{k+1}}(u_{m_{k+1}} - u^{m_{k+1}})\cdot n_f. \qquad (3.42)
\end{aligned}
$$

First, look at the last three interface terms on the righthand side of the equation (3.42)

$$
\begin{aligned}
\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u^{m_k})\cdot n_f &= \frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u_{m_k})\cdot n_f + \frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}e^{m_k}\cdot n_f \\
&= \frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u_{m_k})\cdot n_f + \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{m_k}\cdot n_f,
\end{aligned}
\qquad (3.43)
$$

and

$$
\begin{aligned}
&-\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma (\phi_{i+1} - \phi^{m_k^*})e^{i+1}\cdot n_f \\
&= -\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma (\phi_{i+1} - \phi_{m_k^*})e^{i+1}\cdot n_f - \triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{i+1}\cdot n_f. \qquad (3.44)
\end{aligned}
$$

For the last term in (3.42), we have

$$\begin{aligned}\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_{k+1}}(u_{m_{k+1}} - u^{m_{k+1}})\cdot n_f &= \frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_{k+1}} e^{m_{k+1}}\cdot n_f \\ &= \frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_{k+1}} e^{m_{k+1}}\cdot n_f. \end{aligned} \tag{3.45}$$

Summing up the above three equations (3.43)-(3.45), then the interface terms on the RHS of (4.23) can be rewritten as

$$\begin{aligned}&\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u^{m_k})\cdot n_f - \triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\phi_{i+1} - \phi^{m_k^*})e^{i+1}\cdot n_f \\ &\quad+\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_{k+1}}(u_{m_{k+1}} - u^{m_{k+1}})\cdot n_f \\ &= \frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u_{m_k})\cdot n_f - \triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\phi_{i+1} - \phi_{m_k^*})e^{i+1}\cdot n_f \\ &\quad+\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{m_k}\cdot n_f - \triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{i+1}\cdot n_f \\ &\quad+\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_{k+1}}e^{m_{k+1}}\cdot n_f. \end{aligned} \tag{3.46}$$

We bound the terms in the equation (3.46) using trace and Young and Holder's inequalities (??)

$$\begin{aligned}&|\frac{1}{2}\triangle sg\int_\Gamma \epsilon^{m_k^*}(u_{m_k^*} - u_{m_k})\cdot n_f| \\ &\le \frac{1}{2}\triangle sg(C_f^*)(C_p^*)||\epsilon^{m_k^*}||_p^{1/2}||\nabla\epsilon^{m_k^*}||_p^{1/2}||u_{m_k^*} - u_{m_k}||_f^{1/2}||\nabla(u_{m_k^*} - u_{m_k})||_f^{1/2} \\ &\le \frac{1}{2}\triangle sg(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_p)C_{INV}^{1/2}h^{-1/2}||\nabla\epsilon^{m_k^*}||_p||u_{m_k^*} - u_{m_k}||_f \\ &\le \frac{g\triangle s}{8}||\nabla K^{1/2}\epsilon^{m_k^*}||_p^2 + \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle s}{2k_{min}}||u_{m_k^*} - u_{m_k}||_f^2. \end{aligned} \tag{3.47}$$

and

$$|-\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\phi_{i+1}-\phi_{m_k^*})e^{i+1}\cdot n_f|$$

$$\leq \triangle tg(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_f)C_{INV}^{1/2}h^{-1/2}\sum_{i=m_k}^{m_{k+1}-1}||\nabla e^{i+1}||_f||\phi_{i+1}-\phi_{m_k^*}||_p$$

$$\leq \sum_{i=m_k}^{m_{k+1}-1}\frac{\nu\triangle t}{4}||\nabla e^{i+1}||_f^2+\sum_{i=m_k}^{m_{k+1}-1}\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}\triangle t}{\nu}||\phi_{i+1}-\phi_{m_k^*}||_p^2. \quad (3.48)$$

For the last three terms on the righthand side of the equation (3.46), split the middle one into halves and associate the first two and the last two:

$$\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{m_k}\cdot n_f-\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}e^{i+1}\cdot n_f+\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_{k+1}}e^{m_{k+1}}\cdot n_f$$

$$=\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}(e^{m_k}-e^{i+1})\cdot n_f-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\epsilon^{m_k^*}e^{i+1}-\epsilon^{m_{k+1}}e^{m_{k+1}})\cdot n_f$$

$$=\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}(e^{m_k}-e^{i+1})\cdot n_f$$

$$-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\epsilon^{m_k^*}(e^{i+1}-e^{m_{k+1}})+(\epsilon^{m_k^*}-\epsilon^{m_{k+1}}))e^{m_{k+1}}\cdot n_f. \quad (3.49)$$

Bounding the terms in the equation (3.49) using the Trace, Young and Holder's inequalities again, we obtain

$$|\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma \epsilon^{m_k^*}(e^{m_k}-e^{i+1})\cdot n_f|$$

$$\leq \frac{1}{2}\triangle tg(C_f^*)(C_p^*)C_{PF}^{1/2}(\Omega_f)C_{INV}^{1/2}h^{-1/2}\sum_{i=m_k}^{m_{k+1}-1}||\nabla\epsilon^{m_k^*}||_p||e^{m_k}-e^{i+1}||_f$$

$$\leq \frac{gr\triangle t}{8}||\nabla\kappa^{1/2}\epsilon^{m_k^*}||_p^2+\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle t}{2k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||e^{m_k}-e^{i+1}||_f^2$$

$$\leq \frac{gr\triangle t}{8}||\nabla\kappa^{1/2}\epsilon^{m_k^*}||_p^2+\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle tr}{2k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}-e^{i}||_f^2. \quad (3.50)$$

Similarly

$$
\begin{aligned}
&|-\frac{1}{2}\triangle tg\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\epsilon^{m_k^*}(e^{i+1}-e^{m_{k+1}})+(\epsilon^{m_k^*}-\epsilon^{m_{k+1}}))e^{m_{k+1}}\cdot n_f| \\
\leq& \frac{gr\triangle t}{8}||\nabla\kappa^{1/2}\epsilon^{m_k^*}||_p^2+\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle t}{2k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||e^{m_k}-e^{i+1}||_f^2 \\
&+\frac{\nu\triangle t}{4}||\nabla e^{m_{k+1}}||_f^2+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}r^2\triangle t}{4\nu}||\epsilon^{m_k^*}-\epsilon^{m_{k+1}}||_p^2 \\
\leq& \frac{gr\triangle t}{8}||\nabla\kappa^{1/2}\epsilon^{m_k^*}||_p^2+\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle tr}{2k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}-e^{i}||_f^2 \\
&+\frac{\nu\triangle t}{4}||\nabla e^{m_{k+1}}||_f^2+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}r^2\triangle t}{4\nu}||\epsilon^{m_k^*}-\epsilon^{m_{k+1}}||_p^2. \qquad (3.51)
\end{aligned}
$$

Next, look at the first three terms on the RHS of the equation (3.42)

$$
\begin{aligned}
&\frac{1}{2}\triangle sgS_0(w_{p,s}^{m_k^*},\epsilon^{m_k^*})+\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1},e^{i+1})+\frac{1}{2}\triangle sgS_0(w_{p,s}^{m_{k+1}},\epsilon^{m_{k+1}}) \\
&\leq \frac{g\triangle s}{8}||\kappa^{1/2}\nabla\epsilon^{m_k^*}||_p^2+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}||w_{p,s}^{m_k^*}||_p^2+\frac{\nu\triangle t}{4}\sum_{i=m_k}^{m_{k+1}-1}||\nabla e^{i+1}||_f^2 \\
&+\frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}\sum_{i=m_k}^{m_{k+1}-1}||w_{f,t}^{i+1}||_f^2+\frac{g\triangle s}{4}||\kappa^{1/2}\nabla\epsilon^{m_{k+1}}||_p^2+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}||w_{p,s}^{m_{k+1}}||_p^2.
\end{aligned}
$$

For the terms on the lefthand side of the equation (3.42), we have:

$$
\frac{1}{2}\triangle sa_p(\epsilon^{m_k^*},\epsilon^{m_k^*})\geq\frac{rg\triangle t}{2}||\kappa^{1/2}\nabla\epsilon^{m_k^*}||_p^2, \qquad (3.52)
$$

$$
\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_f(e^{i+1},e^{i+1})\geq\nu\triangle t\sum_{i=m_k}^{m_{k+1}-1}||\nabla e^{i+1}||_f^2, \qquad (3.53)
$$

$$
\frac{1}{2}\triangle sa_p(\epsilon^{m_{k+1}},\epsilon^{m_{k+1}})\geq\frac{rg\triangle t}{2}||\kappa^{1/2}\nabla\epsilon^{m_{k+1}}||_p^2. \qquad (3.54)
$$

Combining all the above inequalities (3.42)-(3.54), we arrive at

$$
\begin{aligned}
&\frac{1}{2}gS_0(||\epsilon^{m_{k+1}}||_p^2 - ||\epsilon^{m_k}||_p^2) + \frac{1}{2}(||e^{m_{k+1}}||_f^2 - ||e^{m_k}||_f^2) \\
&+(\frac{1}{2}gS_0 - \frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}r^2\triangle t}{4\nu})||\epsilon^{m_{k+1}} - \epsilon^{m_k^*}||_p^2 \\
&+(\frac{1}{2} - \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}r\triangle t}{k_{min}})\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1} - e^i||_f^2 \\
&\leq \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle s}{2k_{min}}||u_{m_k^*} - u_{m_k}||_f^2 \\
&+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}\triangle t}{\nu}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1} - \phi_{m_k^*}||_p^2 \\
&+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}||w_{p,s}^{m_k^*}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}\sum_{i=m_k}^{m_{k+1}-1}||w_{f,t}^{i+1}||_f^2 + \frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}||w_{p,s}^{m_{k+1}}||_p^2.
\end{aligned}
$$

Recall that the initial conditions for the algorithm are chosen so that $e^0 = 0$, $\epsilon^0 = 0$. Summing the inequality over $k = 0, 1, \cdots, l$, combinning with the inequalities (3.23)-(3.35) we arrive at

$$
\begin{aligned}
\frac{1}{2}gS_0||\epsilon^{m_{l+1}}||_p^2 + \frac{1}{2}||e^{m_{l+1}}||_f^2 \le \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}\triangle s}{2k_{min}h}\sum_{k=0}^{l}||u_{m_k^*} - u_{m_k}||_f^2 \\
+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}\triangle t}{\nu h}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1} - \phi_{m_k^*}||_p^2 \\
+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}\sum_{k=0}^{l}||w_{p,s}^{m_k^*}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||w_{f,t}^{i+1}||_f^2 \\
+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}\sum_{k=0}^{l}||w_{p,s}^{m_{k+1}}||_p^2 \\
\le \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}\triangle s^2C}{2k_{min}h}\sum_{k=0}^{l}\int_{t^{m_k}}^{t^{m_{k+1}}}||u_t||_f^2dt \\
+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}r\triangle t^2}{\nu h}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\int_{t^i}^{t^{i+1}}||\phi_s||_p^2ds \\
+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h - I)\phi_s(s)||_p^2ds + \triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||_p^2ds) \\
+\frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(\frac{1}{\triangle t}\int_{t^i}^{t^{i+1}}||(P_h - I)u_t(t)||_f^2dt + \triangle t\int_{t^i}^{t^{i+1}}||u_{tt}||_f^2dt) \\
+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h - I)\phi_s(s)||_p^2ds + \triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||_p^2ds). \qquad (3.55)
\end{aligned}
$$

provided that we have

$$\frac{1}{2}gS_0 - \frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}h^{-1}r^2\triangle t}{4\nu} \ge 0$$

and

$$\frac{1}{2} - \frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}h^{-1}r\triangle t}{k_{min}} \ge 0$$

This is equivalent to the restriction of $\triangle t$ in Theorem 2.

From (3.34) and (3.35), we can estimate the terms on the RHS of (3.55)

$$\begin{aligned}
&\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}\triangle s^2C}{2k_{min}h}\sum_{k=0}^{l}\int_{t^{m_k}}^{t^{m_{k+1}}}||u_t||_f^2dt \\
&+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}r\triangle t^2}{\nu h}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\int_{t^i}^{t^{i+1}}||\phi_s||_p^2ds \\
&=\frac{g(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_p)C_{INV}\triangle s^2C}{2k_{min}h}\int_0^T||u_t||_f^2dt \\
&+\frac{g^2(C_f^*)^2(C_p^*)^2C_{PF}(\Omega_f)C_{INV}r\triangle t^2}{\nu h}\int_0^T||\phi_s||_p^2ds \\
&\leq C\triangle t^2. \qquad (3.56)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h-I)\phi_s(s)||_p^2ds+\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||_p^2ds) \\
&+\frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(\frac{1}{\triangle t}\int_{t^i}^{t^{i+1}}||(P_h-I)u_t(t)||_f^2dt+\triangle t\int_{t^i}^{t^{i+1}}||u_{tt}||_f^2dt) \\
&+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h-I)\phi_s(s)||_p^2ds+\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||_p^2ds) \\
&=\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{2k_{min}}(\frac{1}{\triangle s}\int_0^T||(P_h-I)\phi_s(s)||_p^2ds+\triangle s\int_0^T||\phi_{ss}||_p^2ds) \\
&+\frac{C_{PF}^2(\Omega_f)\triangle t}{\nu}(\frac{1}{\triangle t}\int_0^T||(P_h-I)u_t(t)||_f^2dt+\triangle t\int_0^T||u_{tt}||_f^2dt) \\
&+\frac{gS_0^2C_{PF}^2(\Omega_p)\triangle s}{4k_{min}}(\frac{1}{\triangle s}\int_0^T||(P_h-I)\phi_s(s)||_p^2ds+\triangle s\int_0^T||\phi_{ss}||_p^2ds)\leq C(\triangle t^2+h^4). \qquad (3.57)
\end{aligned}$$

From (3.56) and (3.57), we have

$$\frac{1}{2}gS_0||\epsilon^{m_{l+1}}||^2+\frac{1}{2}n||e^{m_{l+1}}||^2\leq C(\triangle t^2+h^4).$$

□

Remark 3.3.2. *For fixed ratio of $\triangle s/\triangle t$, we impose a time step restriction for the small time step $\triangle t$ of the form $\triangle t\leq Ch$ to estimate the error. Since convergence implies stability, Theorem 2 also gives a stability condition depending on the physical parameters.*

3.4 NUMERICAL TESTS OF STABILITY AND CONVERGENCE RATE

This section consists of two testing parts. The first one is a test of stability. It reveals that the methods are stable for beyond the range of $\triangle t$ given by (3.4) in our analysis. The second one confirms the predicted convergence rate and the efficiency of using different time steps and spacing size.

3.4.1 Stability

In this test, we take $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = [0,1] \times [0,1]$ with interface $\Gamma = (0,1) \times 1$. The exact solution is given by

$$(u1, u2) = ([x^2(y-1)^2 + y]cos(wt), [-\frac{2}{3}x(y-1)^3]cos(wt) + [2 - \pi sin(\pi x)]cos(t)),$$

$$p = [2 - \pi sin(\pi x)]sin(0.5\pi y)cos(t),$$

$$\phi = [2 - \pi sin(\pi x)][1 - y - cos(\pi y)]cos(t).$$

Here we take $r = \triangle s/\triangle t = 5$, and the initial conditions, boundary conditions, and the forcing terms follows the solution.

We constructed the finite element spaces are by using the well-known MINI elements for the Stokes problem and the linear Lagrangian elements for the Darcy flow. The code was implemented using the software package FreeFEM++. For the uncoupled scheme, a multi-frontal Gauss LU factorization implemented to solve the SPD sub-systems.

First, we want to look at the numerical test for stability for $k_{min} = 1, 1.0e-4, \text{and} 1.0e-8$. Define the kinetic energy $E^n = \|u_h^n\|_f^2 + \|\phi_h^n\|_p^2$. The final time T_f in our experiment is 1.0 and all the system parameters are simply set to be 1.0 except k_{min}. We fix the mesh size at $h = 1/8$. With $\Delta t = 0.005$ we take the corresponding $\triangle s = 5\triangle t$. We simply choose the initial condition and the exterior boundary condition to be the exact solution. The external force terms are solved when plug the true solution in the equations. We generally compute the kinetic energy on large time step size and figure 3.4.1 shows the quantity. The horizontal axe represents time step while the vertical axe is the corresponding kinetic energy.

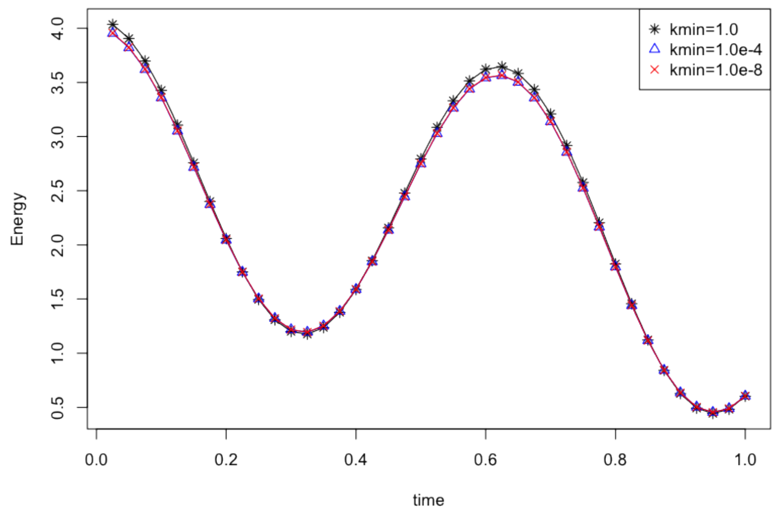


Figure 3.1: Energy versus time on large time step points with $\Delta t = 0.005$ and $r = 5$ for different k_{min}

3.4.2 Error Estimate

Firstly we will compare the performance of the original SDsplit in section 2 with uniform time step in two subregions with the performance of the multi-rate splitting method. Define the error kinetic energy Error Energy $= ||u(t_n) - u_h^n||_f^2 + ||\phi(t_n) - \phi_h^n||_p^2$. We simply set all the system parameters to be 1.0, fix the mesh size at $h = 1/8$ and final time at $T = 1.0$. Again with $\Delta t = 0.005$, the corresponding $\Delta s = 5\Delta t$. The initial condition and the boundary condition are set to be the exact solution. We generally compute the kinetic error energy on large time step size and figure 3.4.2 reveals that the multi-rate splitting scheme is more accurate than the uniform time step splitting scheme.

Next, we will focus on examining the order of convergence with respect to the spacing h or the time step $\triangle t$ with the fixed ratio of $\triangle s/\triangle t = 5$. We here use the method developed in Mu and Zhu [88] to examine the order of convergence with respect to the time step $\triangle t$ and

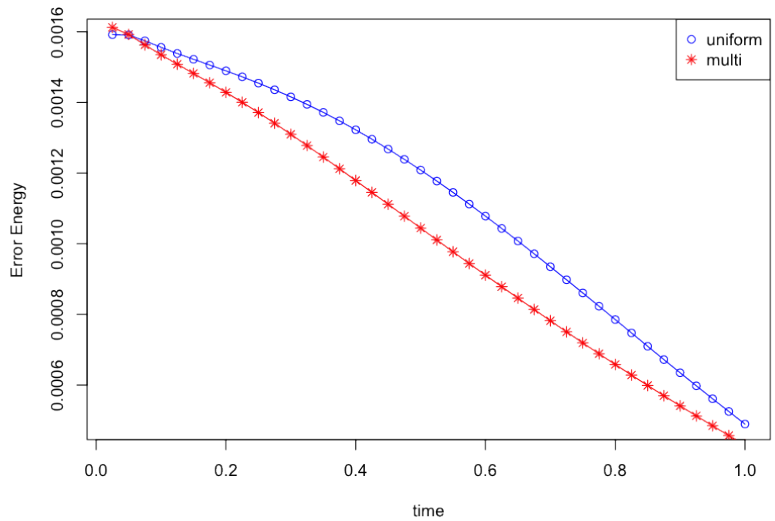


Figure 3.2: Error of SDsplit method versus multi-rate splitting method on large time step with $\Delta t = 0.005$ and $r = 5$

the mesh size h due to the approximation errors $O(\triangle t^{\gamma}) + O(h^{\mu})$. For example, assuming

$$v_h^{\triangle t}(x, t^m) \approx v(x, t^m) + C_1(x, t^m)\triangle t^{\gamma} + \tilde{C}_1(x, t^m)h^{\mu} \tag{3.58}$$

Thus,

$$\rho_{v,h,i} = \frac{||v_h^{\triangle t(x,t^m)} - v_{\frac{h}{2}}^{\triangle t(x,t^m)}||_i}{||v_{\frac{h}{2}}^{\triangle t(x,t^m)} - v_{\frac{h}{4}}^{\triangle t(x,t^m)}||_i} \approx \frac{4^{\mu} - 2^{\mu}}{2^{\mu} - 1}.$$

$$\rho_{v,\triangle t,i} = \frac{||v_h^{\triangle t(x,t^m)} - v_h^{\frac{\triangle t}{2}(x,t^m)}||_i}{||v_h^{\frac{\triangle t}{2}(x,t^m)} - v_h^{\frac{\triangle t}{4}(x,t^m)}||_i} \approx \frac{4^{\gamma} - 2^{\gamma}}{2^{\gamma} - 1}.$$

Here, v can be u, p, ϕ and i can be $0, 1$. While $\rho_{v,h,i}, \rho_{v,\triangle t,i}$ approach 4.0 or 2.0, the convergence order will be 2.0 and 1.0, respectively.

In Table 1, we study the convergence order with a fixed time step $\triangle t = 0.01$ and $\triangle s = 5\triangle t$

h	$\|\|u_h^m - u_{\frac{h}{2}}^m\|\|_0$	$\rho_{u,h,0}$	$\|\|\phi_h^m - \phi_{\frac{h}{2}}^m\|\|_0$	$\rho_{\phi,h,0}$	$\|\|p_h^m - p_{\frac{h}{2}}^m\|\|_0$	$\rho_{p,h,0}$
$\frac{1}{2}$	0.209968	3.79667	0.136084	3.28272	0.695738	1.25399
$\frac{1}{4}$	0.0553032	3.86264	0.0414546	4.06725	0.695738	1.25399
$\frac{1}{8}$	0.0143175	4.05795	0.0101923	4.19066	0.0860786	2.79538
$\frac{1}{16}$	0.00352825		0.00243215		0.0860786	
h	$\|\|\nabla(u_h^m - u_{\frac{h}{2}}^m)\|\|_0$	$\rho_{u,h,1}$	$\|\|\nabla(\phi_h^m - \phi_{\frac{h}{2}}^m)\|\|_0$	$\rho_{\phi,h,1}$		
$\frac{1}{2}$	1.60657	1.91156	1.30526	1.68152		
$\frac{1}{4}$	0.840451	1.91004	0.77624	1.90587		
$\frac{1}{16}$	0.440018	2.13987	0.40729	1.98435		
$\frac{1}{32}$	0.205628		0.205251			

Table 3.1: Examining the second order convergence for spacing h with fixed time step $\triangle t = 0.01$ and at time $t_m = 1.0$

$\triangle t$	$\|\|u_{\triangle t}^m - u_{\frac{\triangle t}{2}}^m\|\|_0$	$\rho_{u,\triangle t,0}$	$\|\|\phi_{\triangle t}^m - \phi_{\frac{1}{2}\triangle t}^m\|\|_0$	$\rho_{\phi,\triangle t,0}$	$\|\|p_{\triangle t}^m - p_{\frac{\triangle t}{2}}^m\|\|_0$	$\rho_{p,\triangle t,0}$
0.1	0.0195322	1.77476	0.0547219	1.75764	0.0230683	1.51918
0.05	0.0111668	1.82594	0.0360206	1.89842	0.0132468	1.71832
0.0025	0.00593724	1.98954	0.0209627	1.9492	0.00727495	1.88277
0.00125	0.00309429		0.011134		0.00397785	
$\triangle t$	$\|\|\nabla(u_{\triangle t}^m - u_{\frac{1}{2}\triangle t}^m)\|\|_0$	$\rho_{u,\triangle t,1}$	$\|\|\nabla(\phi_{\triangle t}^m - \phi_{\frac{\triangle t}{2}}^m)\|\|_0$	$\rho_{\phi,\triangle t,1}$		
0.1	0.00225926	1.74142	0.0547219	1.75764		
0.05	0.00127299	1.82088	0.0360206	1.89842		
0.025	0.000697171	1.82886	0.0209627	1.9492		
0.00125	0.000350418		0.011134			

Table 3.2: Examining the first order convergence for time step $\triangle t$ with fixed spacing $h = \frac{1}{8}$, and at time $t_m = 1.0$

and varying spacing $h = 1/2, 1/4, 1/8, 1/16, 1/32$. Observe that $\rho_{u,h,0}$ and $\rho_{\phi,h,0}$ are over 4.0 when mesh size is smaller and $\rho_{u,h,1}$, $\rho_{\phi,h,1}$ and $\rho_{p,h,0}$ approach 2.0, which suggest that the order of convergence in space for u and ϕ are 2 and for p is 1. However, in table 2, we study the convergence order with a fixed spacing $h = 1/8$ and varying time step size $\triangle t = 0.1, 0.05, 0.025, 0.0125$ and $\triangle s = 5\triangle t$. The numerical results strongly show that all $\rho_{u,\triangle t,0}, \rho_{\phi,\triangle t,0}, \rho_{p,\triangle t,0}$ are less than 2 but increasing, which suggested that the order of convergence is approaching O($\triangle t$) from below.

3.5 CONCLUSION

A multi-rate decoupled method with different time steps in each sub-domain for the coupled Stokes-Darcy problem is proposed and analyzed in this work. The method required only subdomain/sub physics solves and exchanged interface data without reference to the globally coupled problem. Under a time step restriction we prove stability over bounded time intervals of the method. An error estimate is presented with respect to both time step sizes $\triangle t$ and spacing h. Numerical experiments confirmed the analytical results of the decoupled approach.

Interesting open problems include modifying the boundary condition on $\partial\Omega_{f/p}$, seeing how the extreme case of the parameters, large T, small k_{min}, small S_0, affect the stability, efficiency and accuracy of the partitioned method. Other multi-rate decoupled methods with higher order of convergence or less restrictive time step conditions are also to be found.

4.0 A CONNECTION BETWEEN FILTER STABILIZATION AND EDDY VISCOSITY MODELS

4.1 FILTER STABILIZATION AND LES MODEL

It is well known, see, e.g., [12] or [73], that explicit filtering is related to adding eddy or artificial viscosity. The connection of the filter stabilization as defined above to LES modeling is easily recovered by noting that shifting the index $n+1 \to n$ on steps 2 and 3 and using step 1 gives the implicit discretization of the NS, with explicitly treated nonlinear dissipation term:

$$\begin{cases} \frac{1}{\triangle t}(w^{n+1} - w^n) + (w^{n+1} \cdot \nabla) w^{n+1} + \nabla p^{n+1} - \nu \Delta w^{n+1} + \frac{\chi}{\triangle t} G w^n = f^{n+1}, \\ \operatorname{div} w^{n+1} = 0, \end{cases} \tag{4.1}$$

with

$$G := I - F, \qquad I \text{ is the identity operator.}$$

Assume $\chi = \chi_0 \triangle t$, where χ_0 is a time- and mesh-independent constant, then (4.1) can be treated as the time-stepping scheme for

$$\begin{cases} w_t + (w \cdot \nabla) w + \nabla p - \nu \Delta w + \chi_0 G\, w = f, \\ \operatorname{div} w = 0. \end{cases} \tag{4.2}$$

We note that numerical experiments in [36,38] suggested that $\chi = O(\triangle t)$ is indeed the right scaling of the relaxation parameter with respect to the time step. These arguments show that the numerical integrator (A1) with filter stabilization is the splitting scheme for solving

(4.2). Furthermore, (4.2) can be observed as a LES model, with $\chi_0 G\, w$ corresponding to the the Reynolds stress tensor closure:

$$\nabla \cdot (\overline{w \otimes w} - \overline{w} \otimes \overline{w}) \approx \chi_0 G\, w.$$

This simple observation leads to a refined analysis and better interpretation of the numerical results and the method properties.

We start by showing several numerical properties of the approach. Throughout the paper we use $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the L^2 scalar product and the norm, respectively. For the sake of analysis, assume the homogeneous Dirichlet boundary conditions for velocity. Taking the L^2 scalar product of (4.1) with $2\triangle t w^{n+1}$ and integrating by parts gives

$$\|w^{n+1}\|^2 - \|w^n\|^2 + \frac{1}{2}\|w^{n+1} - w^n\|^2 + \nu\triangle t\|\nabla w^{n+1}\|^2 + \chi(Gw^n, w^{n+1}) = \triangle t(f^{n+1}, w^{n+1}). \quad (4.3)$$

For a self-adjoint filtering operator, i.e. $(Gu, v) = (Gv, u)$ for any $u, v \in H_0^1(\Omega)^3$, the equality (4.3) can be alternatively written as

$$\begin{aligned} \|w^{n+1}\|^2 - \|w^n\|^2 + \nu\triangle t\|\nabla w^{n+1}\|^2 + \frac{\chi}{2}\left((Gw^{n+1}, w^{n+1}) + (Gw^n, w^n)\right) \\ = \triangle t(f, w^{n+1}) + \frac{1}{2}\left(\chi(G(w^{n+1} - w^n), w^{n+1} - w^n) - \|w^{n+1} - w^n\|^2\right). \quad (4.4) \end{aligned}$$

Considering the last two terms on the right-hand side, we immediately get the sufficient condition of the energy stability of (4.1) for the case of self-adjoint filters:

$$\chi(Gu, u) \le \|u\|^2 \quad \forall\ u \in H_0^1(\Omega)^3. \quad (4.5)$$

If G is not necessarily self-adjoint, one may rewrite (4.3) as

$$\begin{aligned} \|w^{n+1}\|^2 - \|w^n\|^2 + \frac{1}{2}\|w^{n+1} - w^n\|^2 + \nu\triangle t\|\nabla w^{n+1}\|^2 + \chi(Gw^n, w^n) \\ = \triangle t(f, w^{n+1}) + \chi(Gw^n, w^n - w^{n+1}). \end{aligned}$$

Thanks to the Cauchy inequality one gets for any $\theta \in \mathbb{R}$:

$$\begin{aligned} \|w^{n+1}\|^2 - \|w^n\|^2 + \nu\triangle t\|\nabla w^{n+1}\|^2 + (1-\theta)\chi(Gw^n, w^n) \\ \le \triangle t(f, w^{n+1}) - \chi\Big(\theta(Gw^n, w^n) - \frac{\chi}{2}(Gw^n, Gw^n)\Big). \quad (4.6) \end{aligned}$$

In this more general case, one may consider the following sufficient condition for the energy stability. Fixing, for example, $\theta = \frac{1}{2}$, assures the sum of the last two terms in (4.6) is positive if

$$\chi(Gu, Gu) \leq (Gu, u) \quad \forall\, u \in H_0^1(\Omega)^3. \tag{4.7}$$

Assume G is self-adjoint and w^n approximates a smooth in time Navier-Stokes solution, then (4.4) leads to the following energy balance relation of the numerical method:

$$\|w^N\|^2 + \nu \sum_{n=1}^{N} \triangle t \|\nabla w^n\|^2 + \chi_0 \sum_{n=1}^{N} \triangle t (Gw^n, w^n) = \|w^0\|^2 + \sum_{n=1}^{N} \triangle t (f^n, w^n) + O(\triangle t).$$

In particular, we may conclude that the filter stabilization introduces the *model dissipation* of

$$\chi_0 \sum_{n=1}^{N} \triangle t (Gw^n, w^n). \tag{4.8}$$

Finally, we notice that the filtering and relaxation steps in (A1) can be rearranged as

$$\frac{u^{n+1} - w^{n+1}}{\triangle t} = -\chi_0 G\, w^{n+1},$$

which is the explicit Euler method for integrating

$$u_t = -\chi_0 G\, u \quad \text{on } [t_n, t_{n+1}], \quad \text{with } u(t_n) = w(t_{n+1}). \tag{4.9}$$

The coupling of a DNS method with the evolution equation (4.9) is known as another way of introducing explicit filtering in modelling of dynamical systems, e.g. [12]. This suggests that an improvement leading to higher order methods for integrating (4.9) might be possible.

In the next section, we shall study properties of the operator G for a class of nonlinear differential filters.

4.2 NONLINEAR DIFFERENTIAL FILTERS

Linear differential filters have a long history in LES, see [39]. We also point to [49] and references therein for applications of linear differential filters in the Lagrange-averaging turbulence models. In this section, we consider a family of *nonlinear* differential filters for the filtering procedure. Some conclusions will be drawn concerning the stability conditions (4.5), (4.7) and equivalence to other approachers in the LES modelling. We use the following notation:

$$V := \left\{ v \in H_0^1(\Omega)^3 \,:\, \operatorname{div} v = 0 \right\}, \quad H = \left\{ v \in \; L^2(\Omega)^3 : \operatorname{div} v = 0, v \cdot n|_{dO} = 0 \right\}.$$

By $\mathbb{P}$ we denote the L^2 orthogonal projector from $L^2(\Omega)^3$ onto H.

For a given sufficiently smooth vector function u we define $F\,w$ as the solution to

$$(\delta^2 a(u) \nabla (F\,w), \nabla v) + (F\,w, v) = (w, v) \quad \forall v \in X, \tag{4.10}$$

with an indicator functional $0 \le a(u) \le 1$ and filtering radius δ^2, which generally may depend on x and t, $\delta_{\max} = \max_{x,t} |\delta|$. Here $X = H_0^1(\Omega)^3$ or $X = V$, if the filter is div-free preserving. We note that it is not immediately clear if the problem (4.10) is well-posed. In practice, this is not an issue, since in a finite dimension setting, e.g. for a finite element method, the bilinear form from the left-hand side of (4.10) is elliptic and thus (4.10) is well-posed. Otherwise, we may assume $0 < \varepsilon \le a(u) \le 1$ for some sufficiently small positive ε. If we assume this, none of our results further in the paper depend on the parameter ε. It is standard to base the indicator functional on the input function w itself, that is $u = w$ and we will denote $\overline{w} := F\,w$ in this case. However, in the course of analysis we need to consider (auxiliary) filtering with $u \neq w$. If we need to show explicitly the function used for the indicator, we shall write $F(u)w$ instead of $F\,w$ or $F(w)w$ instead of $\overline{w}$.

The action of $G = I - F$, $w_g := G\,w$, is defined formally as the solution to

$$(\delta^2 a(u) \nabla w_g, \nabla v) + (w_g, v) = (\delta^2 a(u) \nabla w, \nabla v) \quad \forall v \in X. \tag{4.11}$$

The operator G is self-adjoint on X and in the operator notation it can be written as

$$G = -\left[I - \Delta_a\right]^{-1} \Delta_a, \tag{4.12}$$

with

$$\Delta_a := \begin{cases} \operatorname{div}(\delta^2 a(u)\nabla) & \text{if } X = H_0^1(\Omega)^3, \\ \mathbb{P}\operatorname{div}(\delta^2 a(u)\nabla) & \text{if } X = V. \end{cases}$$

Since operator Δ_a is self-adjoint and positive definite, one see from (4.12) that $G \le I$ and thus the sufficient stability condition (4.5) holds for any $\chi \in [0,1]$. This can be easily verified in a formal way by substituting $v = F\,w$ in (4.10) to get $(w, F\,w) \ge 0$ and thus $(w, Gw) = (w, w - F\,w) \le \|w\|^2$ for any $w \in H_0^1(\Omega)^3$. Moreover, varying θ in (4.6) and using (4.7), one shows the energy stability estimate for any $\chi \in [0,2]$. However, such refinement is not important for our further analysis.

With the help of (4.8) and (4.12), we now quantify the model dissipation introduced by the differential filters. To make notation shorter and without loss of generality, let $\chi = \chi_0 \triangle t$.

First, representation (4.12) immediately implies $G \le -\Delta_a$. Thus the additional dissipation introduced by the differential filtering does not exceed those introduced by the LES closure model:

$$\operatorname{div}(\overline{w \otimes w} - \overline{w} \otimes \overline{w}) \approx -\chi_0 \Delta_a w. \tag{4.13}$$

It is easy to show that for a discrete case and if the condition

$$\delta \lesssim \text{ spacial mesh width}$$

holds and $0 \le a(u) \le 1$, then the dissipation introduced by the differential filtering (4.10) is *equivalent* to the dissipation of the closure model (4.13).

We make the above statement more precise for a finite element discretization. To this end, assume a consistent triangulation $\mathcal{T}$ of Ω, satisfying the minimal angle condition

$$\inf_{K \in \mathcal{T}} \rho(K)/r(K) =: \alpha_0 > 0$$

where $\rho(K)$ and $r(K)$ are the diameters of inscribed and superscribed circles (spheres in 3D) for a triangle (tetrahedron) K. We have the following result.

Theorem 4.2.1. *Assume X is the finite element space of continuous functions which are polynomials of degree $p \geq 1$ on every element K and $\max_{x\in K} |\delta(x)| \leq C_\delta\, r(K)$ for any $K \in \mathcal{T}$, with a constant C_δ independent of K. Then for any $w \in X$ the equivalence*

$$\widetilde{c}\,(\delta^2 a(u)\nabla w, \nabla w) \leq (G\, w, w) \leq (\delta^2 a(u)\nabla w, \nabla w) \tag{4.14}$$

holds with a constant $\widetilde{c} > 0$ independent of w, the indicator $a(\cdot)$, and the filtering radius δ. The constant $\widetilde{c} > 0$ may depend on p, C_δ, and α_0.

Proof. Consider the finite element inverse inequality

$$\|\nabla w\|_{L^2(K)} \leq c_0 \rho(K)^{-1} \|w\|_{L^2(K)}, \qquad \forall\, w \in X, \tag{4.15}$$

where the constant c_0 depends only on the polynomial degree p and α_0. The inequality (4.15), the assumption on δ and the minimal angle condition imply

$$\|\delta \nabla w\|_{L^2(K)} \leq \widetilde{C} \|w\|_{L^2(K)}, \tag{4.16}$$

where the constant $\widetilde{C}$ depends only on p, C_δ, and α_0. Squaring (4.16), summing over all $K \in \mathcal{T}$, and recalling that $a(\cdot) \leq 1$, implies

$$(\delta^2 a(u)\nabla w, \nabla w) \leq \widetilde{C}^2 \|w\|^2. \tag{4.17}$$

Denote $w_g = G\, w$ for some $w \in X$. We set $v = w_g$ and $v = -w$ in (4.11) and sum up the equalities to get

$$\begin{aligned} 0 &= (\delta^2 a(u)\nabla w_g, \nabla w_g) + (w_g, w_g) - 2(\delta^2 a(u)\nabla w, \nabla w_g) - (w_g, w) + (\delta^2 a(u)\nabla w, \nabla w) \\ &= \|w_g\|^2 - (w_g, w) + (\delta^2 a(u)\nabla(w - w_g), \nabla(w - w_g)). \end{aligned}$$

Thus, it holds $\|w_g\|^2 \leq (w_g, w)$, i.e. the condition (4.7). Now we set $v = w$ in (4.11) and use (4.7) and (4.17) to estimate

$$\begin{aligned} (\delta^2 a(u)\nabla w, \nabla w) &= (\delta^2 a(u)\nabla w_g, \nabla w) + (w_g, w) \\ &\leq \frac{1}{2}(\delta^2 a(u)\nabla w_g, \nabla w_g) + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w) + (w_g, w) \\ &\leq \frac{1}{2}\widetilde{C}^2 \|w_g\|^2 + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w) + (w_g, w) \\ &\leq (\frac{1}{2}\widetilde{C}^2 + 1)(w_g, w) + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w). \end{aligned}$$

We proved the lower bound in (4.14).

To show the upper bound we set $v = w_g$ and $v = w$ in (4.11) and sum up the equalities to get

$$0 = (\delta^2 a(u)\nabla w_g, \nabla w_g) + (w_g, w_g) + (w_g, w) - (\delta^2 a(u)\nabla w, \nabla w).$$

This yields the upper bound in (4.14): $(w_g, w) \leq (\delta^2 a(u)\nabla w, \nabla w)$.

□

Few conclusions can be drawn from the equivalence result (4.14) concerning the relation of the filter stabilization to some other eddy-viscosity models.

The use of the linear differential filter ($a \equiv 1$), as considered in [36], is equivalent to the method of artificial viscosity. This means that the model dissipation is equivalent to the isotropic diffusion scaled with $\chi_0\delta^2$. Given what is known about the method of artificial viscosity, it is not surprising that the method is not very accurate in this case. Thus, more elaborated indicators functionals should be used. Generally, we may think of $a(u)$ as a real valued functional, depending on $u, \nabla u$, and selected with the intent that

$$a(u(x)) \approx 0 \quad \text{for laminar regions or persistent flow structures,}$$
$$a(u(x)) \approx 1 \quad \text{for flow structures which decay rapidly.}$$

The choice of the Smagorinsky type indicator function, $a(u) = |\nabla u|$, does not necessarily satisfy the condition $a(u) \leq 1$. In this case, we do not have the equivalence result of the filter stabilization to the Smagorinsky LES model. Only the upper bound in (4.14) is guaranteed to hold. Thus the dissipation introduced by the filtering with $a(u) = |\nabla u|$ is likely *less* than that of the Smagorinsky model. This can be a desirable property, since the Smagorinsky LES model is known to be severely over-diffusive for certain flows, e.g. [102], and several ad hoc corrections were introduced such as van Driest damping, dynamic models, and others, see [34, 40, 93].

Several reasonable indicator functions $a(u)$ are known to satisfy the boundedness condition: $0 \leq a(u) \leq 1$. These are the re-normalized Smagorinsky type indicator [11], the indicator based on the Q-criteria [115] and the Vreman indicators [111]; also an indicator based on the normalized helical density distribution was considered in [13]. Given several indicators $a_i(\cdot)$, $i = 1, \ldots, N$, the combined indicator can be defined as the geometric mean: $a(\cdot) := \left(\prod_{i=1}^{N} a_i(\cdot)\right)^{\frac{1}{N}}$.

We remark, that the convergence results proved further in this paper do *not* rely on any smoothness properties or particular form of $a(\cdot)$.

The last remark in this section is that Theorem 4.2.1 does not give much insight if enforcing the divergence constraint in the filter is important or not. However, if we assume $X = V$ in (4.10), i.e., the filtered velocity satisfies the divergence free condition, then this slightly simplifies the error analysis in Section 4.6.

4.3 THREE EXAMPLES OF INDICATOR FUNCTIONS AND NONLINEAR FILTERS

The most mathematically convenient indicator function, recovering variants of the Smagorinsky model, is $a(u) = |\nabla u|$ (suitably normalized) due to its strong monotonicity property. However, it is well known that the Smagorinsky model is not sufficiently selective. Indeed, this choice incorrectly selects laminar shear flow (where$|\nabla u|$is constant but large) as sites of large turbulent fluctuations. Insights into construction of indicator functions of increased accuracy can be obtained from theories of intermittence and eduction. In some respects, theories of intermittence are complementary to theories of eduction of coherent and persistent flow structures. Both therefore have insights that can be used to sharpen the indicator function used in nonlinear filtering. In this section we show how several can be adapted to give indicator functions. Since the geometric average of indicator functions is a more selective indicate function, examples are not isolated but give a path for successive improvements.

4.3.1 The Q criterion

Let the deformation and spin tensors be denoted, respectively

$$\nabla^s u := \frac{1}{2}(\nabla u + \nabla u^{tr}) \;\; \nabla^{ss} u := \frac{1}{2}(\nabla u - \nabla u^{tr})$$

The most popular method for eduction of coherent votices is the Q criterion of Hunt, Wray and Moin [54] which marks as persistent and coherent vortex those regions where

$$Q(u,u) := \frac{1}{2}(\nabla^{ss}u : \nabla^{ss}u - \nabla^{s}u : \nabla^{s}u) > 0$$

Thus $Q > 0$ occurs in those regions where spin (local rigid body rotation) dominates. It is known to be a necessary condition (in 3d) and both necessary and sufficient (in 2d) for slower than exponential local separation of trajectories.

An indicator function is obtained by rescaling $Q(u,u)$ so the the condition $Q(u,u) > 0$ implies $a(u) \approx 0$ so that $u \approx \bar{u}$. There are many plausible ways to do this. We shall test the following.

Definition 4.3.1. *The Q-criterion based indicator function is given by*

$$a_Q(u) := \frac{1}{2} - \frac{1}{\pi} arctan(\delta^{-1}\frac{Q(u,u)}{Q(u,u)+\delta^2}) \tag{4.18}$$

4.3.2 Vreman's eddy viscosity

Perhaps the most advanced and elegant eddy viscosity model has recently been proposed by Vreman [111]. In a very deep construction, using only the gradient tensor he constructs an eddy viscosity coefficient formula that vanishes identically for 320 types of flow structures that are known to be coherent (non turbulent). Define

$$|\nabla w|_F^2 = \sum_{i,j=1,2,3} (\frac{\partial u_j}{\partial x_i})^2, \ \beta_{i,j} := \sum_{m=1,2,3} \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m}$$
$$B(u) := \beta_{11}\beta_{22} - \beta_{12}^2 + \beta_{11}\beta_{33} - -\beta_{13}^2 - \beta_{22}\beta_{33} - \beta_{23}^2$$

In $2d$, $B(u)$ simplifies to

$$B(u) = [(\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_1}{\partial x_2})^2][(\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2] - [\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_2}]^2$$

With C a positive tuning constant, it is given as follows

$$\text{Vreman's eddy viscosity coefficient} = C\delta^2 \begin{cases} \sqrt{\dfrac{B(u)}{|\nabla u|_F^4}}, \ \text{if}|\nabla u|_F \neq 0 \\ 0, \ \text{if}|\nabla u|_F = 0 \end{cases}$$

Since $0 \leq B(u)/|\nabla u|_F^4 \leq 1$ we take as indicator function the following.

Definition 4.3.2. *The Vreman based indicator function is*

$$a_V(u) = \sqrt{B(u)/|\nabla u|_F^4}$$

4.4 PROJECTION SCHEME WITH FILTER STABILIZATION

One idea behind introducing the filter stabilization or explicit filtering was to provide CFD software users and developers with a simple way to enhance existing codes for laminar incompressible flows to compute high Reynolds number flows. This goal is accomplished by making the filtering procedure algorithmically independent of a time integration method. Driven by this intention, we consider the Chorin [26] splitting (projection) scheme with the additional separate filtering step. Projection methods are the common numerical approach to the incompressible Navier-Stokes equations and form a family of splitting algorithms, cf. [42,94]. We perform the numerical analysis for the simplest first order method given below. From the algorithmic standpoint, the generalization to higher order projection methods is straightforward, although analysis may become considerably more involved.

Projection methods split the time evolution of the velocity vector field according to the momentum equation and the projection of the velocity to satisfy the divergence-free condition. The filtering step can be introduced before or after the projection step. If the filter is div-free preserving, then it is reasonable to put it after the projection. We shall study the following algorithm:

Step 1: Solve the convection-diffusion type problem: Given u^n, w^*, find $\widetilde{w^{n+1}}$:

$$\begin{cases} \dfrac{1}{\triangle t}(\widetilde{w^{n+1}} - u^n) + (w^* \cdot \nabla)\widetilde{w^{n+1}} - \nu\Delta\widetilde{w^{n+1}} = f^{n+1}, \\ \widetilde{w^{n+1}}|_{\partial\Omega} = 0. \end{cases} \tag{4.19}$$

The velocity w^* is typically an interpolation from previous times, e.g. $w^* := w^n$ or higher order interpolation. For the sake of analysis we consider $w^* = w^n$.

Step 2: Project $\widetilde{w^{n+1}}$ on the div-free subspace: Find p^{n+1} and w^{n+1} solving the Neumann pressure Poisson problem:

$$\begin{cases} \frac{1}{\triangle t}(w^{n+1} - \widetilde{w^{n+1}}) + \nabla p^{n+1} = 0, \\ \operatorname{div} w^{n+1} = 0, \\ n \cdot w^{n+1}|_{\partial\Omega} = 0. \end{cases} \tag{4.20}$$

Step 3: Filter: $\overline{w^{n+1}} := F\, w^{n+1}$;

Step 4: Relax:

$$u^{n+1} := (1-\chi)w^{n+1} + \chi\overline{w^{n+1}}, \tag{4.21}$$

with some $\chi \in [0,1]$.

Similar to what was shown in section 4.1, shifting the index $n+1 \to n$ on steps 2–4 and substituting into (4.19) gives for $\chi = \chi_0 \triangle t$

$$\begin{cases} \frac{1}{\triangle t}(\widetilde{w^{n+1}} - \widetilde{w^n}) + (w^* \cdot \nabla)\widetilde{w^{n+1}} + \nabla p^{n+1} - \nu\Delta\widetilde{w^{n+1}} + \chi_0 G\widetilde{w^n} - \triangle t\chi_0 G\nabla p^{n+1} = f^{n+1}, \\ \operatorname{div} \widetilde{w^{n+1}} - \triangle t\Delta p^{n+1} = 0, \end{cases} \tag{4.22}$$

From (4.22) we see that the splitting scheme (4.19)–(4.21) is formally the first order accurate time-discretization of the LES model (4.2).

Further, we show that the splitting scheme (4.19)–(4.21) is stable. There are two well-known approaches to accomplish the error analysis of projection methods. The one of Rannacher and Prohl [94], [99] uses the relation between projection and quasi-compressibility methods as it is seen from (4.22). However, this analysis needs considerable effort to get extended to equations different from the plain Navier-Stokes equations. Another framework is mainly due to Shen (see [59, 103]), where convergence results were shown based on energy type estimates. In our error analysis we follow (to a certain extend) arguments from these two papers.

4.5 STABILITY

To show the stability of the splitting scheme, we need the following simple auxiliary result:

Lemma 4.5.1. *For w^{n+1} and u^{n+1} from the algorithm (4.19)–(4.21) and the filter F defined in (4.10), it holds*

$$\|w^{n+1}\| \geq \|u^{n+1}\|.$$

Proof. From the definition (4.10) we obtain:

$$(\delta^2 a(w^{n+1})\nabla\overline{w^{n+1}}, \nabla\overline{w^{n+1}}) + \|\overline{w^{n+1}}\|^2 = (w^{n+1}, \overline{w^{n+1}})$$
$$= \frac{1}{2}(\|w^{n+1}\|^2 + \|\overline{w^{n+1}}\|^2 - \|w^{n+1} - \overline{w^{n+1}}\|^2).$$

This yields

$$\|w^{n+1}\|^2 = 2(\delta^2 a(w^{n+1})\nabla\overline{w^{n+1}}, \nabla\overline{w^{n+1}}) + \|\overline{w^{n+1}}\|^2 + \|\overline{w^{n+1}} - w^{n+1}\|^2.$$

Hence, $\|w^{n+1}\| \geq \|\overline{w^{n+1}}\|$. From (4.21), we get

$$\|u^{n+1}\| \leq (1-\chi)\|w^{n+1}\| + \chi\|\overline{w^{n+1}}\| \leq \|w^{n+1}\| \quad \text{for } \chi \in [0,1].$$

□

Now we are ready to prove the following stability result.

Theorem 4.5.2. *The algorithm (4.19)–(4.21) is stable in the sense of the following a priori estimate:*

$$\|w^l\|^2 + \sum_{n=0}^{l-1}\|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \sum_{n=0}^{l-1}\|\widetilde{w^{n+1}} - u^n\|^2 + \sum_{n=0}^{l-1}\nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2$$
$$\leq \|w^0\|^2 + \sum_{n=0}^{l-1}\nu^{-1}\triangle t\|f(t_{n+1})\|_{-1}^2 \quad (4.23)$$

for any $l = 1, 2, \ldots$.

Proof. Take the L^2 scalar product of (4.19) with $2\triangle t\widetilde{w^{n+1}}$:

$$2(\widetilde{w^{n+1}}-u^n,\widetilde{w^{n+1}})+2\nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2 = 2\triangle t(f^{n+1},\widetilde{w^{n+1}}) \le \nu^{-1}\triangle t\|f^{n+1}\|_{-1}^2+\nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2.$$

Rewriting and simplifying this leads to:

$$\|\widetilde{w^{n+1}}\|^2 - \|u^n\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2 \le \nu^{-1}\triangle t\|f^{n+1}\|_{-1}^2. \qquad (4.24)$$

The L^2 scalar of (4.20) with $2\triangle t\, w^{n+1}$ and div $w^{n+1} = 0$ gives

$$2(w^{n+1} - \widetilde{w^{n+1}}, w^{n+1}) = 0 \quad \Longrightarrow \quad \|w^{n+1}\|^2 - \|\widetilde{w^{n+1}}\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 = 0.$$

Substituting $\|\widetilde{w^{n+1}}\|^2$ with $\|w^{n+1}\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2$ in (4.24) yields

$$\|w^{n+1}\|^2 - \|u^n\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2 \le \nu^{-1}\triangle t\|f^{n+1}\|_{-1}^2.$$

The application of Lemma 4.5.1 gives

$$(\|w^{n+1}\|^2 - \|w^n\|^2) + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\triangle t\|\nabla\widetilde{w^{n+1}}\|^2 \le \nu^{-1}\triangle t\|f^{n+1}\|_{-1}^2.$$

Summing up the inequality from $n = 0, \ldots, l-1$, we arrive at (4.23). □

4.6 ERROR ESTIMATES

We shall use $\langle \cdot, \cdot \rangle$ to denote the duality product between H^{-s} and $H_0^s(\Omega)$ for all $s \geq 0$. In the following, we assume that the given data and solution to the equations (1.6) subject to the homogeneous Dirichlet velocity boundary conditions satisfy

$$\begin{cases} u_0 \in (H^2(\Omega))^d \cap V, \\ f \in L^\infty(0, T; (L^2(\Omega))^d) \cap L^2(0, T; (H^1(\Omega))^d), \\ f_t \in L^2(0, T; H^{-1}), \\ \sup_{t \in [0,T]} \|\nabla u(t)\| \leq \tilde{C}. \end{cases} \tag{4.25}$$

We will use c and C as a generic positive constant which may depend on Ω, ν, T, constants from various Sobolev inequalities, u_0, f, and the solution u through the constant $\tilde{C}$ in (4.25).

Under the assumption (4.25) one can prove the following inequalities, cf. [52]:

$$\sup_{t \in [0,T]} \{\|u(t)\|_2 + \|u_t(t)\| + \|\nabla p(t)\|\} \leq C, \tag{4.26}$$

$$\int_0^T \|\nabla u_t(t)\|^2 + t\|u_{tt}\|^2 dt \leq C, \tag{4.27}$$

which will be used in the sequel. Further we often use the following well-known [107] estimates for the bilinear form $b(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w \, \mathrm{d}x$:

$$b(u, v, w) \leq \begin{cases} c\|\nabla u\| \|\nabla v\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} \|\nabla w\|, \\ c\|u\|_2 \|v\| \|\nabla w\|, \\ c\|\nabla u\| \|v\|_2 \|w\|. \end{cases}$$

and $b(u, v, w) = -b(u, w, v)$ for $u \in H$.

Define the Stokes operator $Au = -\mathbb{P}\Delta u, \ \forall\, u \in D(A) = V \cap H^2(\Omega)^3$. We will use the following properties: A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$, and its inverse A^{-1} is compact in H and satisfies the following relations [59, 103]:

$$\exists\, c, C > 0, \text{ such that } \forall u \in H : \begin{cases} \|A^{-1}u\|_2 \le c\|u\| \text{ and } \|A^{-1}u\| \le c\|u\|_{V'}, \\ c\|u\|_{V'}^2 \le (A^{-1}u, u) \le C\|u\|_{V'}^2. \end{cases}$$

Before we proceed with the error analysis, we prove several auxiliary results given below in Lemma 4.6.1. The lemma gives estimates on the difference between a velocity w and the filtered velocity $F(u)w$.

Lemma 4.6.1. *Consider the differential filter F defined in (4.10) with some admissible u. For any $w \in V$ and $Fw \in V$ it holds*

$$\|w - Fw\| \quad \le \delta_{\max}\|\nabla w\|, \tag{4.28}$$

$$\|w - Fw\|_{V'} \le \delta_{\max}^2\|\nabla w\|. \tag{4.29}$$

Proof. Denote $e = w - Fw$. The equation (4.10) gives

$$(\delta^2 a(u)\nabla e, \nabla v) + (e, v) = -(\operatorname{div}(\delta^2 a(u)\nabla w), v), \ \forall\, v \in \ V$$

Letting $v = e$ yields

$$\begin{aligned} \|\delta\sqrt{a(u)}\nabla e\|^2 + \|e\|^2 &= -(\operatorname{div}(\delta^2 a(u)\nabla w), e) \le \|\delta\sqrt{a(u)}\nabla w\|\|\delta\sqrt{a(u)}\nabla e\| \\ &\le \|\delta\sqrt{a(u)}\nabla e\|^2 + \frac{1}{4}\|\delta\sqrt{a(u)}\nabla w\|^2 \le \|\delta\sqrt{a(u)}\nabla e\|^2 + \frac{1}{4}\delta_{\max}^2\|\nabla w\|^2. \end{aligned}$$

This proves (4.28). To show (4.29), we note that setting $v = F\,w - w$ in (4.10) gives with $F\,w - w$ gives

$$(\delta^2 a(u)\nabla F\,w, \nabla(Fw - w)) = -\|Fw - w\|^2 \le 0$$

Hence, we obtain:

$$\|\delta\sqrt{a(u)}\nabla Fw\|^2 \le \|\delta\sqrt{a(u)}\nabla w\|^2. \tag{4.30}$$

Allowing $v = A^{-1}(w - Fw)$ in (4.10) leads to the following relations:

$$\begin{aligned}\|w - Fw\|_{V'}^2 &= (w - Fw, A^{-1}(w - Fw)) = (\delta^2 a(u)\nabla F\, w, A^{-1}(w - Fw)) \\ &\leq \|\delta^2 a(u)\nabla Fw\|_{V'}\|\nabla A^{-1}(w - Fw)\| \leq \frac{1}{2}(\|\delta^2 a(u)\nabla Fw\|^2 + \|w - Fw\|_{V'}^2) \\ &\leq \frac{1}{2}\delta_{\max}^2\|\delta\sqrt{a(u)}\nabla Fw\|^2 + \frac{1}{2}\|w - Fw\|_{V'}^2.\end{aligned}$$

The last estimate and (4.30) implies (4.29). □

Further in this section, we first show that $\overline{w^{n+1}}$, w^{n+1} and u^{n+1} are all strongly $O((\triangle t)^{\frac{1}{2}} + \delta)$ approximations to $u(t_{n+1})$ in $L^2(\Omega)^3$ provided $\chi = \chi_0\triangle t$. Then we use this result to improve the error estimates to weakly $O(\triangle t + \delta^2)$ approximations. This analysis largely follows the framework from [59] and [103] for the pure (non-filtered) Navier-Stokes equations, so we shall refer to these papers and [92] for some arguments which do not depend on the filtering procedure.

Lemma 4.6.2. *Let u be the solution to the Navier-Stokes system, satisfying (4.25). Denote*

$$\widetilde{\epsilon^{n+1}} = u(t_{n+1}) - \widetilde{w^{n+1}}; \quad \epsilon^{n+1} = u(t_{n+1}) - w^{n+1} \ \textit{and}\ e^{n+1} = u(t_{n+1}) - u^{n+1}.$$

The following estimate holds

$$\|\widetilde{\epsilon^l}\|^2 + \sum_{n=0}^{l-1}(\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \|\widetilde{\epsilon^{n+1}} - e^n\|^2) + \sum_{n=0}^{l-1} 2\nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \leq C(\triangle t + \delta_{\max}^2). \quad (4.31)$$

Proof. Let R^n denote the truncation error defined by

$$\frac{1}{\triangle t}(u(t_{n+1}) - u(t_n)) - \nu\triangle u(t_{n+1}) + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + \nabla p(t_{n+1}) = f^{n+1} + R^n, \quad (4.32)$$

where R^n is the integral residual of the Taylor series, i.e,

$$R^n = \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}}(t - t_n)u_{tt}(t)dt.$$

By subtracting (4.19) from (4.32), we obtain

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}} - e^n) - \nu\triangle\widetilde{\epsilon^{n+1}} = (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) - \nabla p(t_{n+1}) + R^n. \quad (4.33)$$

Taking the L^2 scalar product of (4.33) with $2\triangle t\widetilde{\epsilon^{n+1}}$, we get

$$\|\widetilde{\epsilon^{n+1}}\|^2 - \|e^n\|^2 + \|\widetilde{\epsilon^{n+1}} - e^n\|^2 + 2\nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 = 2\triangle t(R^n, \widetilde{\epsilon^{n+1}}) - 2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}}) + 2\triangle t b^*(w^n, \widetilde{w^{n+1}}, \widetilde{\epsilon^{n+1}}) - 2\triangle t b^*(u(t_{n+1}), u(t_{n+1}), \widetilde{\epsilon^{n+1}}). \quad (4.34)$$

The terms on the right-hand side are bounded exactly the same way as in [59] p.64 and [103] p.512, leading to the estimates:

$$\triangle t|b^*(w^n, \widetilde{w^{n+1}}, \widetilde{\epsilon^{n+1}}) - b^*(u(t_{n+1}), u(t_{n+1}), \widetilde{\epsilon^{n+1}})| \leq \frac{\nu\triangle t}{2}\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + C\triangle t\|e^n\|^2 + C(\triangle t)^2\int_{t_n}^{t_{n+1}}\|u_t\|^2 dt, \quad (4.35)$$

$$2\triangle t(R^n, \widetilde{\epsilon^{n+1}}) \leq \frac{\nu\triangle t}{4}\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + C(\triangle t)^2\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt, \quad (4.36)$$

$$2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}}) = 2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - e^n) \leq \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2. \quad (4.37)$$

Combining the inequalities (4.34) ,(4.35), (4.36), (4.37), and rearranging terms, we obtain

$$\|\widetilde{\epsilon^{n+1}}\|^2 - \|e^n\|^2 + \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \leq 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2 + C\triangle t\|e^n\|^2 + C(\triangle t)^2(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt + \int_{t_n}^{t_{n+1}}\|u_t\|^2 dt) \quad (4.38)$$

The step 4 of the algorithm (4.19)–(4.21) yields

$$e^n = (1-\chi)\epsilon^n + \chi F(w^{n+1})\epsilon^n + \chi(u(t_n) - F(w^{n+1})u(t_n)). \quad (4.39)$$

The definition of the filter and recalling that ϵ^n is the L^2 projection of $\widetilde{\epsilon^n}$ give $\|F(w^{n+1})\epsilon^n\| \leq \|\epsilon^n\| \leq \|\widetilde{\epsilon^n}\|$. We use this to deduce from (4.39) the following estimate:

$$\|e^n\| = (1-\chi)\|\epsilon^n\| + \chi\|F(w^{n+1})\epsilon^n\| + \chi\|u(t_n) - F(w^{n+1})u(t_n)\| \leq \|\widetilde{\epsilon^n}\| + \chi\|u(t_n) - F(w^{n+1})u(t_n)\|.$$

Now we apply (4.28) and square the resulting inequality to get (for the sake of convenience we assume $\triangle t \le C$ and recall $\chi = \chi_0 \triangle t$):

$$\|e^n\|^2 \le (1+\triangle t)\|\widetilde{\epsilon^n}\|^2 + C\triangle t \delta_{\max}^2. \tag{4.40}$$

We substitute (4.40) to the left-hand side of (4.38) for $\|e^n\|$, use $\|\epsilon^n\| \le \|\widetilde{\epsilon^n}\|$ and arrive at

$$\begin{aligned}\|\widetilde{\epsilon^{n+1}}\|^2 - \|\widetilde{\epsilon^n}\|^2 + \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \\ \le 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2 + C\triangle t\|\widetilde{\epsilon^n}\|^2 \\ +C(\triangle t)^2\left(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt + \int_{t_n}^{t_{n+1}} \|u_t\|^2 dt\right) + C\triangle t\delta_{\max}^2.\end{aligned} \tag{4.41}$$

Summing up (4.41) from $n = 0$ to $n = l-1$, assuming that $\widetilde{w^0} = w^0 = u_0$ (this implies $\|e^0\| = \|\epsilon^0\| = 0$), we obtain

$$\begin{aligned}\|\widetilde{\epsilon^l}\|^2 + \sum_{n=0}^{l-1}\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \frac{1}{2}\sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1}\nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \\ \le \sum_{n=0}^{l-l} C\triangle t\|\widetilde{\epsilon^n}\|^2 + 2(\triangle t)^2\sum_{n=0}^{l-1}\|\nabla p(t_{n+1})\|^2 + C(\triangle t)^2(\int_{t_0}^{t_l} t\|u_{tt}\|_{-1}^2 dt + \int_{t_0}^{t_l}\|u_t\|^2 dt) + C\delta_{\max}^2 \\ \le \sum_{n=0}^{l-1} C\triangle t\|\widetilde{\epsilon^n}\|^2 + C\triangle t + C\delta_{\max}^2\end{aligned}$$

Applying the discrete Gronwall inequality yields (4.31). □

Now, we will use the result of the lemma and improve the predicted order of convergence for the velocity. The main result in this section is the following theorem, stating that all $\widetilde{w^{n+1}}$, w^{n+1} and u^{n+1} are first-order approximations to the Navier-Stokes solution.

Theorem 4.6.3. *Assume the solution to the Navier-Stokes system satisfies (4.25) and $\chi = \chi_0 \triangle t$. Suppose $\partial\Omega \in C^{1,1}$ or Ω is convex. It holds*

$$\triangle t \sum_{n=1}^{l}(\|\widetilde{\epsilon^n}\|^2 + \|\epsilon^n\|^2 + \|e^n\|^2) \le C((\triangle t)^2 + \delta_{\max}^4). \tag{4.42}$$

Additionally assume $\int_0^T \|\nabla p_t\|^2 \le C$ and the filtering radius is bounded as $\delta_{\max}^4 \le C\,\triangle t$, then p^n is an approximation to $p(t_n)$ in $L^2(\Omega)/R$ in the following sense:

$$\triangle t \sum_{n=1}^{l}\|p^n - p(t_n)\|^2 \le C(\triangle t + \delta_{\max}^2). \tag{4.43}$$

Proof. Literally repeating the arguments from [59], pp. 66-69, one shows the estimate

$$\|\epsilon^{n+1}\|_{V'}^2 - \|e^n\|_{V'}^2 + \|\epsilon^{n+1} - e^n\|_{V'}^2 + \nu\triangle t\|\epsilon^{n+1}\|^2 \le C\Big(\triangle t\|\epsilon^{n+1}\|_{V'}^2$$
$$+ (\triangle t)^2 \int_{t_n}^{t_{n+1}} (t\|u_{tt}\|_{-1}^2 + \|u_t\|^2)dt + (\triangle t)^2\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + \triangle t\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \triangle t\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2\Big) \tag{4.44}$$

The estimate (4.29) gives $\|F\epsilon^n\|_{V'} \le \|\epsilon^n\|_{V'} + \delta_{\max}^2\|\nabla\epsilon^n\|$. Here and in the rest of the proof the filtering is based on the w^{n+1} velocity, that is $F\cdot := F(w^{n+1})\cdot$. Due to the assumption $\partial\Omega \in C^{1,1}$ or Ω is convex, the L^2 projection on H is H^1 stable, i.e. $\|\nabla\epsilon^n\| \le C\|\nabla\widetilde{\epsilon^n}\|$ and therefore we conclude

$$\|F\epsilon^n\|_{V'} \le \|\epsilon^n\|_{V'} + C\delta_{\max}^2\|\nabla\widetilde{\epsilon^n}\|.$$

Using this and (4.29), we get from (4.39) for $\chi = \chi_0\triangle t$

$$\begin{aligned}
\|e^n\|_{V'} &= (1-\chi)\|\epsilon^n\|_{V'} + \chi\|F\epsilon^n\|_{V'} + \chi\|u(t_n) - Fu(t_n)\|_{V'} \\
&\le \|\epsilon^n\|_{V'} + C\triangle t\left(\delta_{\max}^2\|\nabla\widetilde{\epsilon^n}\| + \|u(t_n) - Fu(t_n)\|_{V'}\right) \\
&\le \|\epsilon^n\|_{V'} + C\triangle t\delta_{\max}^2\left(\|\nabla\widetilde{\epsilon^n}\| + 1\right).
\end{aligned}$$

Squaring the inequality we get after elementary calculations

$$\|e^n\|_{V'}^2 \le (1+\triangle t)\|\epsilon^n\|_{V'}^2 + C\triangle t\delta_{\max}^4\left(\|\nabla\widetilde{\epsilon^n}\|^2 + 1\right).$$

We substitute the above estimate to the left-hand side of (4.44) and arrive at

$$\begin{aligned}
&\|\epsilon^{n+1}\|_{V'}^2 - \|\epsilon^n\|_{V'}^2 + \|\epsilon^{n+1} - e^n\|_{V'}^2 + \nu\triangle t\|\epsilon^{n+1}\|^2 \\
&\quad\le C\Big(\triangle t(\|\epsilon^{n+1}\|_{V'}^2 + \|\epsilon^n\|_{V'}^2) + (\triangle t)^2\int_{t_n}^{t_{n+1}} (t\|u_{tt}\|_{-1}^2 + \|u_t\|^2)dt + (\triangle t)^2\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \\
&\qquad + \triangle t(\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2) + \triangle t\delta_{\max}^4(1 + \|\nabla\widetilde{\epsilon^n}\|^2)\Big).
\end{aligned}$$

Assume for the sake of convenience $\delta_{\max} \le C$. Summing up the inequalities for $n = 0, \ldots, l-1$, we get

$$\begin{aligned}
&\|\epsilon^l\|_{V'}^2 + \sum_{n=0}^{l-1} \|\epsilon^{n+1} - e^n\|_{V'}^2 + \sum_{n=0}^{l-1} \nu \triangle t \|\epsilon^{n+1}\|^2 \\
&\quad \le C \left(\sum_{n=0}^{l-1} \triangle t \|\epsilon^{n+1}\|_{V'}^2 + (\triangle t)^2 \int_{t_0}^{t_l} (\|u_{tt}\|_{V'}^2 + \|u_t\|^2) dt + \delta_{\max}^4 \sum_{n=0}^{l-1} \triangle t \|\nabla \widetilde{\epsilon^n}\|^2 \right. \\
&\qquad \left. + \sum_{n=0}^{l-1} \triangle t \|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1} \triangle t \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \triangle t \delta_{\max}^4 \right). \qquad (4.45)
\end{aligned}$$

Now we use the result of the Lemma 4.6.2 to bound

$$\begin{aligned}
&\triangle t \|\epsilon^l\|_{V'}^2 + \delta_{\max}^4 \sum_{n=0}^{l-1} \triangle t \|\nabla \widetilde{\epsilon^{n+1}}\|^2 + \sum_{n=0}^{l-1} \triangle t \|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1} \triangle t \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 \\
&\qquad \le C((\triangle t)^2 + \triangle t \delta_{\max}^2 + \delta_{\max}^4).
\end{aligned}$$

Thus, applying the Gronwall inequality to (4.45) yields

$$\|\epsilon^l\|_{V'}^2 + \sum_{n=0}^{l-1} \|\epsilon^{n+1} - e^n\|_{V'}^2 + \sum_{n=0}^{l-1} \nu \triangle t \|\epsilon^{n+1}\|^2 \le C((\triangle t)^2 + \delta_{\max}^4). \qquad (4.46)$$

Here we also used $\triangle t \delta_{\max}^2 \le (\triangle t)^2 + \delta_{\max}^4$. Finally, the Lemma 4.6.2 helps us to estimate

$$\begin{aligned}
\triangle t \sum_{n=0}^{l-1} \|\widetilde{\epsilon^{n+1}}\|^2 &\le \triangle t \sum_{n=0}^{l-1} \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \triangle t \sum_{n=0}^{l-1} \|\epsilon^{n+1}\|^2 \le C((\triangle t)^2 + \delta_{\max}^4). \\
\triangle t \sum_{n=0}^{l} \|e^n\|^2 &\le \triangle t \sum_{n=0}^{l-1} \|\epsilon^{n+1} - e^n\|^2 + \triangle t \sum_{n=0}^{l-1} \|\epsilon^{n+1}\|^2 \le C((\triangle t)^2 + \delta_{\max}^4).
\end{aligned}$$

These estimates together with (4.46) proves the velocity error estimate of the theorem.

Further we show that the pressure is weakly $\frac{1}{2}$ order convergent to the true solution. Denote the pressure error as $q^n = p^n - p(t_n)$. We may assume $(q^n, 1) = 0$. It holds

$$-\nabla q^{n+1} = -\frac{1}{\triangle t}(\epsilon^{n+1} - e^n) + \nu \triangle \widetilde{\epsilon^{n+1}} + (w^n \cdot \nabla)\widetilde{w^{n+1}} - (u(t_{n+1}) \cdot \nabla) u(t_{n+1}). + R^n \qquad (4.47)$$

Repeating the arguments from [59] and using the Nečas inequality, see [89], one deduces from (4.47)

$$\|q^{n+1}\| \le c \sup_{v\in H_0^1(\Omega)^3} \frac{(\nabla q^{n+1}, v)}{\|\nabla v\|}$$
$$\le \frac{1}{\triangle t}\|\epsilon^{n+1} - e^n\|_{-1} + C(\|R^n\|_{-1} + \|\nabla\widetilde{\epsilon^{n+1}}\| + \|\nabla \epsilon^{n+1}\| + \|u(t_{n+1}) - u(t_n)\|).$$

Therefore, by using (4.31), we get

$$\triangle t \sum_{n=0}^{l-1} \|q^{n+1}\|^2 \le \frac{1}{\triangle t}\sum_{n=0}^{l-1} \|\nabla(\epsilon^{n+1} - e^n)\|_{-1}^2 + C(\triangle t + \delta_{max}^2) \tag{4.48}$$

To bound the first term on the right-hand side of (4.48) one estimates:

$$\|\epsilon^{n+1} - e^n\|_{-1} \le c\|\epsilon^{n+1} - e^n\| \le c(\|\epsilon^{n+1} - \epsilon^n\| + \|\epsilon^n - e^n\|) \le c(\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\| + \|\epsilon^n - e^n\|). \tag{4.49}$$

The estimate for the second term on the right-hand side of (4.49) follows from (4.39):

$$\|\epsilon^n - e^n\| \le \chi_0\triangle t(\|\epsilon^n - F\epsilon^n\| + \|u(t_n) - Fu(t_n)\|) \le \chi_0\triangle t(\|\epsilon^n\| + \|F\epsilon^n\| + \|u(t_n) - Fu(t_n)\|).$$

Thanks to (4.28), (4.31), and $\|F\epsilon^n\| \le \|\epsilon^n\|$ we continue the above estimate as

$$\|\epsilon^n - e^n\| \le C((\triangle t)^{\frac{3}{2}} + \triangle t\delta_{\max}). \tag{4.50}$$

Below we shall prove the bound

$$\sum_{n=0}^{l-1} \|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 \le C((\triangle t)^2 + \triangle t\delta_{\max}^2).$$

From (4.19) and (4.21) we get

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}} - e^n) - \nu\Delta\widetilde{\epsilon^{n+1}} + \nabla p(t_{n+1}) + (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) = R^n. \tag{4.51}$$

The projection step (4.20) gives $\epsilon^n = \widetilde{\epsilon^n} + \triangle t\nabla p^n$, so (4.39) yields

$$e^n = (1-\chi)(\widetilde{\epsilon^n} + \triangle t\nabla p^n) + \chi F\epsilon^n + \chi(u(t_n) - Fu(t_n)).$$

Substituting this in (4.51) implies

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})-\nu\Delta\widetilde{\epsilon^{n+1}}+(1-\chi)\nabla(p(t_{n+1})-p^n)+\chi\nabla p(t_{n+1})-\frac{\chi}{\triangle t}(F\epsilon^n-\widetilde{\epsilon^n})$$
$$-\frac{\chi}{\triangle t}(u(t_n)-Fu(t_n))+(w^n\cdot\nabla)\widetilde{w^{n+1}}-(u(t_{n+1})\cdot\nabla)u(t_{n+1})=R^n. \quad (4.52)$$

The inner product of (4.52) with $\triangle t(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})$ gives

$$\begin{aligned}
&\|\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}\|^2+\frac{\nu\triangle t}{2}(\|\nabla\widetilde{\epsilon^{n+1}}\|^2-\|\nabla\widetilde{\epsilon^n}\|^2+\|\nabla(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\|^2)\\
&\quad=\triangle t(R^n,\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})+(1-\chi)\triangle t(p(t_{n+1})-p^n,\operatorname{div}(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}))\\
&\qquad+\triangle t((w^n\cdot\nabla)\widetilde{w^{n+1}}-(u(t_{n+1})\cdot\nabla)u(t_{n+1}),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\\
&\quad-\chi\triangle t(\nabla p(t_{n+1}),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})+\chi(F\epsilon^n-\widetilde{\epsilon^n},\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})+\chi(u(t_n)-Fu(t_n),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\\
&=\triangle t(R^n,\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})+(1-\chi)\triangle t\Big[(q^n,\operatorname{div}(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}))+(p(t_{n+1})-p(t_n),\operatorname{div}(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}))\Big]\\
&\quad-\chi\Big[\triangle t(\nabla p(t_{n+1}),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})-(F\epsilon^n-\widetilde{\epsilon^n},\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})-(u(t_n)-Fu(t_n),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\Big]\\
&\qquad+\triangle t((w^n\cdot\nabla)\widetilde{w^{n+1}}-(u(t_{n+1})\cdot\nabla)u(t_{n+1}),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\\
&\qquad=I_1+I_2+I_3+I_4+I_5+I_6+I_7. \quad (4.53)
\end{aligned}$$

The last term I_7 is estimated in [92]:

$$\begin{aligned}
&\triangle t|((w^n\cdot\nabla)\widetilde{w^{n+1}}-(u(t_{n+1})\cdot\nabla)u(t_{n+1}),\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})|\\
&\quad\le\sigma\|\widetilde{\epsilon}^{n+1}-\widetilde{\epsilon}^n\|^2+C((\triangle t)^2\|\widetilde{\epsilon}^{n+1}\|^2+(\triangle t)^2\|\epsilon^{n+1}\|^2+\triangle t^{\frac{3}{2}}\|\nabla\epsilon^n\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2\\
&\qquad+\frac{\nu\triangle t}{2}\|\nabla(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\|^2+(\triangle t)^3)
\end{aligned}$$

for some $\sigma>0$, which can be taken sufficiently small. Applying (4.31) and $\|\nabla\epsilon^n\|\le C\|\widetilde{\nabla\epsilon^n}\|$ leads to

$$I_7\le\sigma\|\widetilde{\epsilon}^{n+1}-\widetilde{\epsilon}^n\|^2+C((\triangle t)^3+(\triangle t)^2\delta_{\max}^2)+\triangle t^{\frac{3}{2}}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2+\frac{\nu\triangle t}{2}\|\nabla(\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n})\|^2. \quad (4.54)$$

For I_4, I_5, and I_6 one has

$$I_4 = -\chi\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C\chi^2(\triangle t)^2\|\nabla p(t_{n+1})\|^2 + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2, \tag{4.55}$$

$$\begin{aligned} I_5 &= \chi(F\epsilon^n - \widetilde{\epsilon^n}, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C\chi^2(\|F\epsilon^n\|^2 + \|\widetilde{\epsilon^n}\|^2) + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 \\ &\le C((\triangle t)^3 + (\triangle t)^2\delta_{\max}^2) + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2, \end{aligned} \tag{4.56}$$

$$I_6 = \chi(u(t_n) - Fu(t_n), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C(\triangle t)^2\delta_{\max}^4 + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2. \tag{4.57}$$

The terms I_1, I_2 and I_3 are estimated in [59]. Using those estimates and (4.54)–(4.57) in (4.53) yields for sufficiently small $\sigma > 0$:

$$\begin{aligned} &\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\triangle t}{2}(\|\nabla\widetilde{\epsilon^{n+1}}\|^2 - \|\nabla\widetilde{\epsilon^n}\|^2) + (1-\chi)(\triangle t)^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) \\ &\quad \le C\left\{(\triangle t)^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|^2dt + (\triangle t)^2\int_{t_n}^{t_{n+1}}\|\nabla p_t\|^2dt + (\triangle t)^4\|\nabla p(t_{n+1})\|^2\right. \\ &\qquad \left. + (\triangle t)^3 + (\triangle t)^2\delta_{\max}^2 + \triangle t^{\frac{3}{2}}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2\right\}. \end{aligned} \tag{4.58}$$

We sum up the estimate for $n = 0, \ldots, l-1$ and apply our assumptions for the Navier-Stokes solution. This leads to the bound

$$\sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\triangle t}{2}\|\nabla\widetilde{\epsilon^l}\|^2 \le C((\triangle t)^2 + \triangle t\delta_{\max}^4 + \triangle t^{\frac{3}{2}}\sum_{n=0}^{l-1}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2).$$

The application of the discrete Gronwall inequality, (4.31) and the assumption $\delta_{\max}^4 \le C\triangle t$ yields

$$\begin{aligned} \sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\triangle t}{2}\|\nabla\widetilde{\epsilon^l}\|^2 &\le C\left((\triangle t)^2 + \triangle t\delta_{\max}^2\right)\exp\left\{(\triangle t)^{\frac{1}{2}}\sum_{n=0}^{l-1}\|\nabla\widetilde{\epsilon^{n+1}}\|^2\right\} \\ &\le C\left((\triangle t)^2 + \triangle t\delta_{\max}^2\right)\exp\left\{C((\triangle t)^{\frac{1}{2}} + (\triangle t)^{-\frac{1}{2}}\delta_{\max}^4)\right\} \\ &\le C((\triangle t)^2 + \triangle t\delta_{\max}^4). \end{aligned}$$

Therefore, (4.48)–(4.50) yield the desired bound:

$$\triangle t\sum_{n=0}^{l-1}\|q^{n+1}\|^2 \le C(\triangle t + \delta_{\max}^2).$$

□

4.7 NUMERICAL TESTINGS

In this section, we present numerical experiments to test the algorithms presented in Chapter 4. We used $FREEFEM++$ using Taylor-Hood elements(X_h =continuous piecewise quadratics and Q_h =continuous piecewise linears).

4.7.1 TEST OF STABILITY

We begin by testing the stability of Theorem 4.5.2 in linear and nonlinear filter in some $2d$ flows with known exact solution. In $2d$ helicity is exactly zero. We consider

$$\begin{aligned} \text{Linear Filtering} &\Leftrightarrow a(\cdot) \equiv 1 \\ \text{Nonlinear Filtering by Q-criterion} &\Leftrightarrow a = a_Q(\cdot) \\ \text{Nonlinear Filtering by Vreman's formula} &\Leftrightarrow a = a_V(\cdot) \end{aligned}$$

For the test we select the velocity field given by the Green-Taylor vortex, frozen at time t=1. The Green-Taylor vortex is used as a numerical test in many papers, e.g., Chorin [22], Tafti [106], John and Layton [56], Berselli and Grisanti [4] and Berselli [5]. The exact velocity field is given by

$$\begin{aligned} u_1(x,y,t) &= -cos(w\pi x)sin(w\pi y)e^{-2w^2\pi^2 t/\tau}, \\ u_2(x,y,t) &= sin(w\pi x)cos(w\pi y)e^{-2w^2\pi^2 t/\tau}, \\ p(x,y,t) &= -\frac{1}{4}(cos(2w\pi x) + cos(2w\pi y))e^{-2w^2\pi^2 t/\tau}. \end{aligned}$$

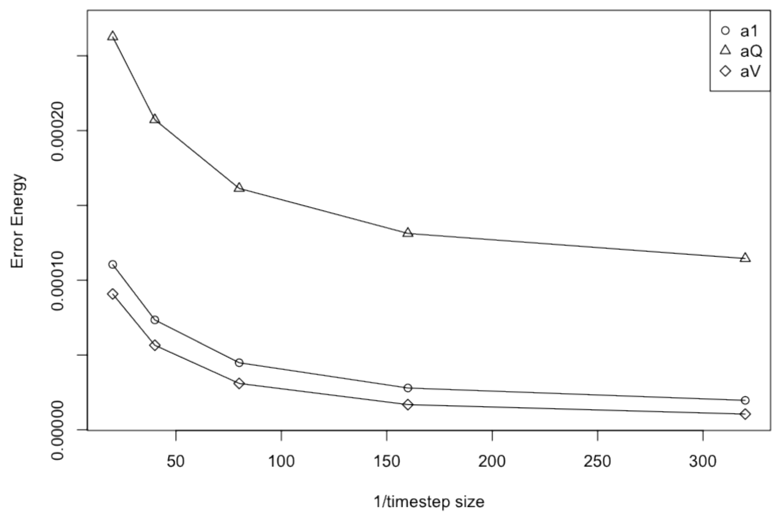
We take

$$w = 1,\ T = 1(\text{fixed}),\ \tau = Re = 100,\ \Omega = (0,1)^2,\ \delta = h = 1/m$$

Where m is the number of subdivisions of the interval $(0,1)$. When the relaxation time $\tau = Re$, the predicted solution is a solution of the NSE with $f = 0$. Convergence rates are calculated from the error at two successive values of h and $\triangle t$ at final time $T = 1$. The boundary conditions could be taken to be periodic (the easiest case), but instead we take the boundary condition on the filtering problem to be inhomogeneous Dirichlet

$$u^h = u_{exact}, \quad \text{on } \partial\Omega$$

The plot 4.7.1 shows that with linear or nonlinear filter, as we decrease the size of the time step, the velocity energy term does not blow up and this support our proof in theorem 4.5.2.



4.7.2 TEST OF THE RATE OF CONVERGENCE

We begin testing the errors and the rates of convergence with linear and nonlinear filters which are presented for the 3 methods in Table 4.1. All discrete filters achieved their predicted rates of convergence. Lemma 4.6.2 indicates the strongly first order convergence for spacing h and half order convergence for time step Δt, when verifying the result, we started with $\Delta t = 0.05$ and $h = 1/4$, in the following step we decrease the time step size to one fourth, and the spacing size to half of the value in the previous step, the ratio of L2 norm of the error term on the LHS of 4.31 at two consecutive step should be approaching to 2. Numerical results shows the theoretical results in Lemma 4.6.2 is not optimal and can be further improved. Theorem 4.6.3 indicates that the squared error energy term on the LHS is of second order with respect to time step size and is of fourth order with respect to the spacing size. We start the numerical test with $\Delta t = 0.05$ and $h = 1/2$, in the following step, we decrease the time step to one fourth of the previous step and the spacing size by half, the ratio of the squared error energy term at two consecutive step is assumed to be approaching to 16. This result is confirmed in table 4.7.2.

	$a(u)$	1	1	Q-based	Q-based	Vreman-based	Vreman-based
$\triangle t$	h	$\|\|u_{NSE} - u_h\|\|_2$	rate	$\|\|u_{NSE} - u_h\|\|_2$	rate	$\|\|u_{NSE} - u_h\|\|_2$	rate
0.05	$\frac{1}{4}$	0.020236		0.024074		0.0193774	
$\frac{0.05}{4}$	$\frac{1}{8}$	0.00571722	3.54	0.0081305	2.96	0.00523308	3.70
$\frac{0.05}{16}$	$\frac{1}{16}$	0.00194931	2.93	0.0026404	3.08	0.00163201	3.21
$\frac{0.05}{64}$	$\frac{1}{32}$	0.000620057	3.14	0.000743168	3.55	0.000488043	3.34

Table 4.1: Examining the strongly first order convergence for spacing h and half order convergence for time step $\triangle t$

	$a(u)$	1	1	Q-based	Q-based	Vreman-based	Vreman-based
$\triangle t$	h	$Error$	rate	$Error$	rate	$Error$	rate
0.05	$\frac{1}{2}$	0.00337251		0.00340044		0.00340229	
$\frac{0.05}{4}$	$\frac{1}{4}$	0.000100713	17.32	0.000124105	14.51	0.000104528	19.68
$\frac{0.05}{16}$	$\frac{1}{8}$	$5.81492e-06$	13.15	8.55445e-06	11.60	$5.31219e-06$	14.61
$\frac{0.05}{64}$	$\frac{1}{16}$	$4.42276e-07$	12.60	7.3727e-07	14.12	$3.63708e-07$	14.94
$\frac{0.05}{256}$	$\frac{1}{32}$	$3.50889e-08$	14.00	5.22619e-08	15.25	$2.43474e-08$	15.31

Table 4.2: Examining the weakly second order convergence for spacing h and first order convergence for time step $\triangle t$, Error=$\triangle t \sum_{n=1}^{l}(\|\widetilde{\epsilon^n}\|^2 + \|\epsilon^n\|^2 + \|e^n\|^2)$

5.0 CONCLUDING REMARKS AND FUTURE RESEARCH

The major contribution of this work is the development and analysis of partitioned methods for coupled fluid flow problems and filter stabilization for high Reynolds number turbulence flows.

1. partitioned time stepping methods for Stokes-Darcy problems,
2. extension of the unified time step partitioned method to multi-rate partitioned methods for Stokes-Darcy problems,
3. pressure corrected and nonlinear filtered method for Navier-Stokes equation.

Our methods have substantial algorithmic advantage, since they effectively break the complex coupled system into the subproblems and allow the use of optimized and legacy codes. By this way, they help to reduce the technical costs and programming effort. It has been shown that the proposed algorithms are stable and convergent at the optimal rates. In particular, long time and uniform in time stability are obtained for Stokes-Darcy flows, which surpasses previous results. The goal of any uncoupled method is to give results not appreciably worse than the associated fully coupled approach (which is expected to be more accurate). In the numerical experiments presented herein, our methods well meet this goal.

For uncoupling a coupled problem, our methods face some limitations as a trade-off for algorithmic advantage. The time stepping methods normally require time step restrictions for stability. These conditions are particularly troublesome in applications involving small or large physical parameters. We partially address the issue in this thesis; however, schemes with stronger stability properties are still in need in many cases. Also, deriving the exact dependence of the stability and/or error behavior on model parameters remains largely an open question. Certainly, studying higher order accurate uncoupling strategies, or strategies which allow the use of different time step and mesh size for subproblem solvers is another important and promising direction for future works.

In this work, we also show that introducing the filter stabilization is closely related to adapting a certain eddy-viscosity model for LES. It would be natural to extend the nonlinear filtered method to other splitting method for the time-integration of the incompressible Navier-Stokes equations for fast unsteady flows. The following research projects would help develop further computational capabilities for complex flow systems:

1. Developing algorithms with higher order time accuracy for Stokes-Darcy flows, allowing large time steps when both $k_{\min}$ and S_0 are small.
2. Studying the mass conservation errors across the interface I for the Stokes-Darcy flow. One interesting direction is developing and analyzing the combination of uncoupling schemes and mixed formulations, which are expected to have better conservation properties.
3. Extending the nonlinear filtered approach to other splitting algorithms for the time integration of the NSE.
4. Developing pressure corrected and nonlinear filtered algorithms with higher order time accuracy for NSE.

BIBLIOGRAPHY

[1] M. Anitescu, W. Layton and F. Pahlevani, Implicit for local effects, explicit for nonlocal is unconditionally stable, ETNA 18 (2004), 174-187.

[2] T. Arbogast and D.S. Brunson, A computational method for approximate a Darcy-Stokes system governing a vuggy porous medium, Comput. Geosci., 11 (2007), 201-218.

[3] U. Asher, S. Ruuth and B. Wetton, Implicit-Explicit methods for time dependent partial differential equations, SINUM 32 (1995), 797-823.

[4] D. Barbato, L. C. Berselli, C. R. Grisanti, Analytical and numerical results for the rational large eddy simulation model, J. Math. Fluid Mech., 9 (2007), 44-74.

[5] L. C. Berselli, On the large eddy simulation of the Taylor-Green vortex, J. Math. Fluid Mech., 7 (2005), S164-S191.

[6] S. Badia and R. Codina, Unified stabilized finite element formulations for the Stokes and the Darcy problems, SIAM J. Numer. Anal 47 (2009), 1971-2000.

[7] J. Bear, *Hydraulics of Groundwater*. McGraw-Hill, New York, 1979.

[8] G. S. Beavers, D. Joseph, Boundary conditions at a naturally permeable wall, JFM 30 (1967), 197-207.

[9] F. Brezzi, M. Fortin, Mixed and hybrid finite element methods, Springer-Verlag, (1991). MR1115205(92d:65187).

[10] E. Burman and P. Hansbo, A unified stabilized method for Stokes and Darcy's equations, JCAM 198 (2007), 35-51.

[11] J. Borggaard, T. Iliescu and J.P. Roop, A Bounded Artificial Viscosity Large Eddy Simulation Model, SIAM J Numer Anal 47 (2009), 622-645.

[12] J. P. Boyd, Two comments on filtering for Chebyshev and Legendre spectral and spectral element methods: Preserving the boundary conditions and interpretation of the filter as a diffusion, J Comp Phys 143 (1998), 283-288.

[13] A. L. Bowers, L. G. Rebholz, A. Takhirov and C. Trenchea, Improved accuracy in regularization models of incompressible flow via adaptive nonlinear filtering, International Journal for Numerical Methods in Fluids, 70 (2012), 805-828.

[14] M. Cai, M. Mu and J. Xu, Decoupled preconditioners for Stokes-Darcy coupling problems with applications in porous media, submitted, 2007.

[15] M. Cai, M. Mu and J. Xu, Preconditioning techniques for a mixed Stokes/Darcy model in porous media applications, J. Comput. Appl. Math. 233 (2009), 346-355.

[16] A. Çeşmelioğlu, B. Rivière, Analysis of time-dependent Navier–Stokes flow coupled with Darcy flow, J. Numerical Math., 16 (2008), 249–280.

[17] J. C. Chrispell, V. J. Ervin, E. W. Jenkins, Coupled generalized non-linear Stokes flow with flow through a porous media, SIAM J. Numer. Anal, 47 (2009), 929-952.

[18] Y. Cao, M. Gunzburger and F. Hua, X. Wang, Coupled Stokes-Darcy Model with Beavers-Joseph Interface Boundary Condition, Comm. Math. Sci., 8 (2010), 1-25.

[19] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang and W. Zhao, Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions, SINUM 47 (2010), 4239-4256.

[20] Y. Cao, M. Gunzburger, X.-M. He, and X. Wang, Robin-Robin domain decomposition methods for the steady Stokes-Darcy model with Beaver-Joseph interface condition, submitted, 2011.

[21] Y. Cao, M. Gunzburger, X.-M. He, and X Wang, Parallel, non-iterative multi-physics domain decomposition methods for the time-dependent Stokes-Darcy problem, submitted, 2011.

[22] A. J. Chorin, Numerical solution for the Navier-Stokes equations, Math. Co,p., 22 (1968), 745-762.

[23] A. Cesmelioglu, B. Riviere, Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow, Journal of Numerical Mathematics, 16 (2008), 249-280.

[24] J. M. Connors and J. S. Howell. A fluid-fluid interaction method using decoupled subproblems and differing time steps, Numerical Methods for Partial Differential Equations, 28 (2012), 1283-1308.

[25] J. M. Connors, J. S. Howell and W. Layton, Decoupled time stepping methods for fluid-fluid interaction, SIAM J. Numer. Anal., 50 (3), 1297-1319.

[26] A. Chorin, Numerical solution of the Navier-Stokes equations, Math Comp 22 (1968), 745-762.

[27] M. Crouzeix, Une méthode multipas implicite-explicite pour l'approximation des équationes d'évolution paraboliques, Numer. Math. 35 (1980), 257-276.

[28] M. Discacciati, Domain decomposition methods for the coupling of surface and groundwater flows, Ph.D. dissertation, École Polytechnique Fédérale de Lausanne, 2004.

[29] M. Discacciati, E. Miglio and A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math. 43 (2001), 57-74.

[30] M. Discacciati, E. Miglio, and A.Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math, 43 (2002), 57-74.

[31] M. Discacciati and A. Quarteroni. Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations. In Brezzi et al, editor, Numerical Analysis and Advanced Applications -ENUMATH 2001, pp. 3-20. Springer, Milan, 2003.

[32] M. Discacciati and A. Quarteroni, Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations, Comput. Visual Sci., 6 (2004), 93-103. MR2061270. (2005e:65142).

[33] M. Discacciati, A. Quarteroni, and A. Valli, Robin-Robin domain decomposition methods for the Stokes-Darcy coupling, SIAM J. Numer. Anal., 45 (2007), 1246-1268.

[34] E. R. Van Driest, On turbulent flow near a wall, J Aerospace Sci 23, (1956), 1007–1011.

[35] M. Discacciati, A. Quarteroni, Navier-Stokes/Darcy coupling: modeling, analysis, and numerical approximation, Rev. Mat. Complut., 22 (2009), 315-426.

[36] V. Ervin, W. Layton and M. Neda, Numerical Analysis of Filter Based Stabilization for Evolution Equations, SIAM J. Numer. Anal., 50(2012), 2307-2335 .

[37] J. Frank, W. Hundsdorfer and J. Verwer, Stability of Implicit-Explicit linear multistep methods, Appl. Numer. Math., 25 (1997), 193-205.

[38] P. Fischer, J. Mullen, Filter-based stabilization of spectral element methods, C R Acad Sci Paris 332 (2001), 265.

[39] J. Mullen, P. Fischer, Filtering techniques for complex geometry fluid flows, Commun Numer Meth Engng, 15 (1999), 9-18.

[40] M. Germano, U. Piomelli, P. Moin, and W.H. Cabot, A dynamic subgrid-scale eddy viscosity model, Phys Fluids A 3 (1991), 1760.

[41] G. N. Gatica, S. Meddahi and R. Oyarz´ua, A conforming mixed finite-element method for the coupling of fluid flow with porous media flow, IMA J. Numer. Anal., 29 (2009), 86–108.

[42] J. L. Guermond, P. Minev and J. Shen. An overview of projection methods for incompressible flows, Comput Meth Appl Mech Eng 195, (2006), 6011–6045.

[43] V. Girault and P. A. Raviart, *Finite Element Methods for Navier Stokes Equations*, Springer-Verlag, 1986.

[44] J. Galvis and M. Sarkis, Balancing domain decomposition methods for mortar coupling Stokes-Darcy Systems, Proceedings of the 16th International Conference on Domain Decomposition Methods, 2005.

[45] M. D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms*, Academic Press, 1989.

[46] H. Holden, K. N. Karlsen, K.-A. Lie and N. H. Risebro, *Splitting methods for Partial Differential Equations with rough solutions*, European Math., Soc. Zurich, 2010.

[47] F. Hua, Modeling, analysis and simulation of Stokes-Darcy system with Beavers-Joseph interface condition, Ph.D. dissertation, The Florida State University (2009).

[48] F. Hecht and O. Pironneau, FreeFem++ webpage: http://www.freefem.org.

[49] D. D. Holm and E. S. Titi, Computational Models of Turbulence: The LANS Model and the Role of Global Analysis, SIAM News 38 (2005), 1-5.

[50] H. Holden, K. N. Karlsen, K. A. Lie and N. H. Risebro, Splitting methods fdor Partial Differential Equations with rough solutions, European Math., Soc. Zurich, 2010.

[51] R. H. W. Hoppe, P. Porta and Y. Vassilevski, Computational issues related to iterative coupling of subsurface and channel flows, Calcolo 44 (2007), 1-20.

[52] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, SIAM J Numer Anal 19 (1982), 275-311 .

[53] W. Hundsdorfer and J. Verwer, *Numerical solution of time dependent advection diffusion reaction equations*, Springer, Berlin, 2003.

[54] J. C. Hunt, A. A. Wray and P. Moin, Eddies stream and convergence zones in turbulence flows, CTR report, CTR-S88, 1988.

[55] W. Hundsdorfer, Accuracy and stability of splitting with Stabilizing Corrections, Applied Numerical Mathematics, Volume 42, Issues 1-3, Numerical Solution of Differential and Differential-Algebraic Equations, 4-9 September 2000, Halle, Germany, August 2002, Pages 213-233, ISSN 0168-9274, DOI: 10.1016/S0168-9274(01)00152-0.

[56] V. John and W. Layton, Analysis of Numerical Errors in Large Eddy Simulations, SIAM J. Numer. Anal., 40 (2002), 995-1020.

[57] W. Jäeger and A. Mikelić, On the interface boundary conditions of Beavers, Joseph and Saffman, SIAM J. Applied Math. 60 (2000) 1111-1127.

[58] W. Jäeger, A. Mikelićand N. Neuss, Asymptotic analysis of the laminar viscous flow over a porous bed, SIAM J. Sci. Comput., 22 (2001), 2006-2028. MR1856299(2002f:76065).

[59] J. Shen, On error estimates of projection methods for Navier-Stokes equations: first-order schemes, SIAM J Numer Anal 29 (1992), 57-77.

[60] B. Jiang, A parallel domain decomposition method for coupling of surface and groundwater flows, Comput. Methods Appl. Mech. Engrg. 198 (2009), 947 - 957.

[61] A. Johnson, Compilation of specific yields for various materials, US Geological Survey Water Supply Paper 1667-D , 1967.

[62] T. Karper, K. A. Mardal, and R. Winther, Unified finite element discretizations of coupled Darcy-Stokes flow, Numer. Methods Partial Differential Equations 25 (2009), 311-326.

[63] W. Layton, Fluid-Porous Interface Conditions with the "Inertia Term" $1/2\ |U_{FLUID}|^2$ are not Galilean Invariant, tech report, 2009.

[64] W. Layton, *Introduction to the Numerical Analysis of Incompressible, Viscous Flows*, SIAM, Philadelphia 2007.

[65] W. Layton, L. G. Rebholz, and C. Trenchea, Modular nonlinear filter stabilization of methods for higher Reynolds numbers flow, Journal of Mathematical Fluid Mechanics, 14 (2012), 325-354.

[66] W. Layton, L. Roehe and H. Tran, Explicitly uncoupled VMS stabilization of fluid flow, Comp Meth Appl Mech Eng 200 (2011), 3183-3199.

[67] W. Layton, F. Schieweck and I. Yotov, Coupling fluid flow with porous media flow, SIAM J. Numer. Anal., 40 (2003), 2195-2218. MR1974181 (2004c:76048).

[68] W. Layton, H. Tran and C. Trenchea , Analysis of Long Time Stability and Errors of Two Partitioned Methods for Uncoupling Evolutionary Groundwater - Surface Water Flows, SIAM J. Numer. Anal., 51 (2013), 248-272.

[69] W. Layton and C. Trenchea, Stability of two IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations, Appl. Numer. Math., 62 (2012), 112-120.

[70] W. Layton, H. Tran and X. Xiong, Long Tme Stability of Four Methods for Splitting the Evolutionary Stokes-Darcy Problem into Stokes and Darcy Subproblems, J. Comput. Appl. Math., 236 (2012), 3198-3217.

[71] P Lesaint, FEM for Symmetric Hyperbolic Equations, Numer Math 21 (1973), 244-255.

[72] G. Lube and M. Olshanskii, Stable finite element calculations of incompressible flows using the rotation form of convection, *IMA J. Num. Anal.*, 22 (2002), 437-461.

[73] J. Mathew, R. Lechner, H. Foysi, J. Sesterhenn and R. Friedrich, An explicit filtering method for large eddy simulation of compressible flows, Physics of Fluids 15 (2003), 2279-2289.

[74] K. A. Mardal, X. C. Tai, and R. Winther, A robust finite element method for Darcy-Stokes flow, SIAM J. Numer. Anal. 40 (2002), 1605-1631.

[75] G. I. Marchuk, *Splitting Methods* , Nauka, Moscow, 1988.

[76] G. I. Marchuk, Splitting and alternate direction methods, *Handbook of Numerical Analysis*, PG Ciarlet and JL Lions editors, North Holland, Amsterdam, (1990), 197-464.

[77] M. Mu and J. Xu, A two-grid method of a method of a mixed Stokes-Darcy model for coupling fluid flow iwth porous media flow, SIAM J. Numer. Anal., 45 (2007), 1801-1813.

[78] J. G. Heywood, R. Rannacher, Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization, SIAM J. Numer. Anal., 27 (1990), 353-384.

[79] M. Discacciati, A. Quarteroni, Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations, 6 (2004), 93-103.

[80] R. H. W Hoppe, P. Porta, Y. Vassilevski, Computational issues related to iterative coupling of subsurface and channel flows, 44 (2007), 1-20.

[81] M. Discacciati, A. Quarteroni, Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations, 2003, Numerical mathematics and advanced applications, Springer Italia, Milan, 3-20.

[82] Y. Cao, M. Gunzburger, X. He, X. Wang, Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition, Numer. Math., 117 (2010), 601-629.

[83] W. Layton, Introduction to the numerical analysis of incompressible viscous flows, Computational Science & Engineering, Vol. 6, Society for Industrial and Applied Mathematics (SIAM), 2008.

[84] K. A. Mardal, X.-C. Tai and R. Winther, A robust finite element method for Darcy-Stokes flow, SIAM J. Numer. Anal., 40 (2002), 1605-1631.

[85] E. Miglio, A. Quarteroni and F. Saleri, Coupling of free surface flow and groundwater flows, Computers and Fluids 23 (2003), 73-83.

[86] M. Moraiti, On the quasi-static approximation in the Stokes-Darcy model of groundwater-surfacewater flows, J. of Math. Anal. and App., 394 (2012), 796-808.

[87] M. Mu and J. Xu, A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow, SINUM 45 (2007) 1801-1813.

[88] M. Mu and X. Zhu, Decoupled schemes for a non-stationary mixed Stokes -Darcy model, Math Comp 79 (2010), 707-731.

[89] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Masson, 1967.

[90] M. A. Olshanskii and A. Reusken, Grad-Div stabilization for the Stokes equations, *Mathematics of Computation*, 73 (2004), 1699-1718.

[91] M. A. Olshanskii and A. Reusken, Navier-Stokes equations in rotation form: a robust multigrid solver for the velocity problem, *SIAM J. Scientific Computing*, 23 (2002), 1682–1706

[92] M. A. Olshanskii, A. Sokolov, S. Turek, Error analysis of a projection method for the Navier-Stokes equations with Coriolis force, Journal of Mathematical Fluid Mechanics 12, (2010), 485-502.

[93] U. Piomelli, Large-eddy simulation: achievements and challenges, Progress in Aerospace Sciences 35, (1999), 335-362.

[94] A. Prohl, Projection and quasi-compressibility methods for solving the incompressible Navier-Stokes equations, BG Teubner, 1997.

[95] L. E. Payne, J. C. Song and B. Straughan, Continuous dependence and convergence results for Brinkman and Forcheimer models with variable viscosity, Proc. Royal Soc. London A 455(1999), 2173-2190.

[96] L. E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions, J Math. Pure Appl. 77(1998), 1959-1977.

[97] L. E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions, J. Math. Pure Appl., 77 (1998), 317-354. MR1623387(99c:35200).

[98] G. F. Pinder and M. A. Celia, *Subsurface hydrology*, John Wiley and Sons, Hoboken, New Jersey, 2006.

[99] R. Rannacher, On Chorin's projection method for the incompressible Navier-Stokes equations, The Navier-Stokes equations II—theory and numerical methods, 167-183, Lecture Notes in Math., 1530, Springer, Berlin, 1992.

[100] B. Rivière and I. Yotov, Locally conservative coupling of Stokes and Darcy flows, SIAM J. Numer. Anal., 40 (2005), 1959-1977. MR2139232(2006a:76035).

[101] P. G. Saffman, On the boundary condition at the interface of a porous medium, Studies in Appl. Math. 1(1971), 93-101.

[102] P. Sagaut, Large eddy simulation for Incompressible flows, Springer, Berlin, 2001.

[103] J. Shen, Remarks on the pressure error estimate for the projection methods, Numer Math 67 (1994), 513-520.

[104] L. Shan and H. Zheng , Partitioned Time Stepping Method for Fully Evolutionary Stokes-Darcy Flow with Beavers-Joseph Interface, SIAM J. Numer. Anal., 51 (2013), 813-839.

[105] L. Shan, H. Zheng and W. Layton, A Non-Iterative, Domain Decomposition Method with Different Time Step Sizes for the Evolutionary Stokes-Darcy Model, tech report, www.mathematics.pitt.edu/research/technical-reports.php, 2011.

[106] D. Tafti, Comparison of some upwind-biased high-order formulations with a second order central-difference scheme for time integration of the incompressible Navier-Stokes equations, Comput. and Fluid, 25 (1996), 647-665.

[107] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, SIAM Philadelphia, PA, 1983.

[108] J. J. M. Urquiza, D. N'Dri, A. Garon, M. C. Delfour, Coupling Stokes and Darcy equations, Applied Numerical Mathematics, 58 (2008), 525-538.

[109] J. Verwer, Convergence and component splitting for the Crank-Nicolson Leap-Frog integration method, CWI report MAS-E0902, 2009.

[110] J. M. Varah, Stability restrictions on a second order, three level finite difference schemes for parabolic equations, SINUM 17 (1980), 300-309.

[111] A. W. Vreman, An eddy-viscosity subgrid-scale model for turbulent shear flow: algebraic theory and applications, Phys Fluids, 16 (2004), 3670-3681.

[112] D. Vassilev, I. Yotov, Domain decomposition for coupled Stokes and Darcy flows, accepted by Computer Methods in Applied Mechanics and Engineering.

[113] J. Verwer, Convergence and component splitting for the Crank-Nicolson Leap-Frog integration method, CWI report MAS-E0902, 2009.

[114] X. Wang, On the coupled continuum pipe flow model (CCPF) for flows in karst aquifer, DCDS-B 13 (2010), 489-501.

[115] A. A. Wray J.C. Hunt and P. Moin, Eddies stream and convergence zones in turbulent flows, In Studying Turbulence Using Numerical Simulation Databases, Proceedings of the 1988 Summer Program, 193-208.

[116] N. N. Yanenko. *The Method of Fractional Steps*. Springer, Berlin, 1971.

[117] S. Zhang, X. Xie, and Y. Chen, Low order nonconforming rectangular finite element methods for Darcy-Stokes problems, J. Comput. Math. 27 (2009), 400-424.