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# Quantum-Classical Correspondence in Nonrelativistic Electrodynamics

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# Quantum-classical correspondence in nonrelativistic electrodynamics

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A form of classical electrodynamic field exists which gives exact agreement with the operator field of quantum electrodynamics (QED) for the Lamb shift of a harmonically bound point electron. Here it is pointed out that this form of classical theory with its physically acceptable interpretation is the result of an unconventional resolution of a mathematically ambiguous term in classical field theory. Finally a quantum-classical correspondence principle is shown to exist in the sense that the classical field and expectation value of the QED operator field are identical if retardation is neglected in the latter.

There is confusion in the literature regarding the limitations of classical electrodynamics to describe the Lamb shift of a bound electron. Usually it is assumed that the Abraham-Lorentz model [1], although physically flawed, is the natural outcome of a consistent application of classical electrodynamics to the fundamental problem of the self interaction of an electron with its own radiation field. On the other hand at least two other applications exist in the literature [2-3] in which exact agreement is obtained with quantum electrodynamics (QED) for this quantity.

In this paper it is pointed out that the confusion arises from writing down the classical field in two different mathematical forms, one of which,

$$\vec{A}(\vec{r},t) = \frac{e}{c}\int_0^\infty dt'\theta(t-t')[G^{(+)} - G^{(-)}]\,\vec{v}'(t') \quad , \qquad (1)$$

where $\theta(t-t')$ is the theta function, $G^{(+)}$ ($G^{(-)}$) are the retarded (advanced) Green's functions,

$$G^{(\pm)} = \frac{\delta(t' \pm R/c - t)}{R} \quad , \qquad (2)$$

$\vec{v}'(t') = \frac{d\vec{r}'(t')}{dt'}$ is the velocity for an electron whose trajectory is $\vec{r}'(t')$, and $R = |\vec{r} - \vec{r}'(t')|$, contains mathematical ambiguities at R = 0 in the second term which are resolved in favor of the vanishing of the term. The second form [2-3], however, arises from writing the field in terms of inverse Fourier integrals over the radial variable in $\vec{k}$ space and evaluating the field at R = 0 *before* performing the integration over k = ω/c. In other words if the k integration is performed first, then Eq. (1) is the result; however if the k integration is performed after finding the field at R = 0, then the result is a field whose application to the Lamb shift gives exact agreement with the QED result, where the k integral is recognizable from QED as the unbounded frequency integral which diverges linearly as ω. The familiar high-frequency divergence simply reflects the capacity of a point charged particle to radiate at infinitely high frequencies either quantum mechanically or classically. The latter problem, which appears to arise from an incompleteness of the physical model, is dealt with of course using

physical arguments to renormalize the result such that the infinite contributions are cancelled.

While this would seem to be an unsatisfactory situation from a mathematical point of view, yet mathematical physics abounds with examples in which ambiguities are resolved in favor of obtaining a result with a satisfactory physical interpretation. Here we point out that the usual choice of dropping the second term of Eq. (1), in the sense that it seems inevitably to lead to physically unsuccessful concepts such as the finite extent and charge distribution of the electron, is not as compelling physically as the prescription just outlined [2-3].

For completeness we outline below the derivation of the two forms of the classical field. The Maxwell equation for the vector field arising from the motion of a point electron is,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = -\frac{4\pi}{c}\, e\vec{v}'(t)\,\delta[\vec{r} - \vec{r}'(t)] \quad , \qquad (3)$$

Eq. (3) can be expeditiously solved by first Fourier transforming in space and then solving the resulting second-order differential equation in the time, namely,

$$(\frac{\partial^2}{\partial t^2} + \omega^2)\vec{a}(\vec{k},t) = 4\pi ec\upsilon'(t)e^{-i\vec{k}\cdot\vec{r}'(t)} = \vec{\rho}(\vec{k},t) \tag{4a}$$

$$\omega = kc \ , \tag{4b}$$

whose solution is

$$\vec{a}(\vec{k},t) = \frac{i}{2\omega}\int_0^{\infty} dt' e^{-i\omega(t_> - t_<)}\vec{\rho}(\vec{k},t') \ , \tag{5}$$

where $t_>$ ($t_<$) is the greater (lesser) of t and t'. Eq. (5) can be written out explicitly,

$$\vec{a}(\vec{k},t) = \frac{i}{2\omega}\left(\int_0^{t} dt' e^{-i\omega(t-t')} + \int_t^{\infty} dt' e^{i\omega(t-t')}\right)\vec{\rho}(\vec{k},t') \ . \tag{6}$$

In fact the integral from t to infinity can be recast as the integral from zero to infinity minus the integral from zero to t. Then on dropping the term containing the constant integral, the result is still a solution to Eq. (4a), namely,

$$\vec{a}_1(\vec{k},t) = \frac{i}{2\omega}\left(\int_0^{t} dt' e^{-i\omega(t-t')} - \int_0^{t} dt' e^{i\omega(t-t')}\right)\vec{\rho}(\vec{k},t') \ , \tag{7}$$

because the term we dropped is a solution of the homogeneous equation. One may easily verify that Eq. (7) is indeed a particular

solution of Eq. (4a) by direct substitution.

Boundary conditions have been used corresponding to the propagation of positive-frequency solutions forward in time. In the temporal inverse Fourier transformation satisfying this condition, the contour is closed in the lower half-plane with detour including (excluding) the positive (negative) real axis. These boundary conditions are discussed in detail for the solution of the Dirac equation in Bjorken and Drell [4].

The final expression for the field follows on taking the spatial inverse Fourier transformation of Eq. (7),

$$\vec{A}(\vec{r},t) = \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{r}}\, a_1(\vec{k},t)$$

$$= \frac{ie}{\pi c}\int_0^t dt' \int_0^\infty d\omega \frac{[e^{-i\omega(t-t')} - e^{i\omega(t-t')}]}{R} \sin(\frac{\omega R}{c})\, v'(t)$$

$$= \frac{e}{c}\int_0^t dt' \frac{[\delta(t-t'-R/c) - \delta(t-t'+R/c)]}{R}\, v'(t) \quad , \qquad (8)$$

where the identity,

$$\frac{1}{\pi}\int_0^\infty d\omega \cos\omega(t - t' \pm R/c) = \delta(t - t' \pm R/c) \qquad (9)$$

has been used. On recognizing that the integral from zero to t can be replaced by the intergral from zero to infinity if the term in brackets on the right side of Eq. (8) is weighted by the theta function $\theta(t-t')$, we can immediately write Eq. (8) in the compact form given by Eq. (1). On the other hand the other prescription for writing the field [2-3] is simply to defer the integration over $\omega$ until the field, as given by the second line on the right side of Eq. (8), is evaluated at R = 0 and used in the Lamb shift application, whereupon the $\omega$ integral is the familiar divergent integral over emitted frequencies.

In summary the result depends critically on the order in which the limits are taken: $\omega \to \infty$ followed by $R \to 0$ or $R \to 0$ followed by $\omega \to \infty$. The second choice leads to agreement with QED, while the first choice leads to the well known models of radiation reaction which are plagued by unphysical behavior (run-away and acausality).

Finally we demonstrate the classical-QED correspondence, which was first stated in [3]. First however we require the projection of the classical field (as given by the second line of Eq. (8) evaluated at R = 0) onto the unit vector $\hat{\varepsilon}$ in the direction of polarization of the

emitted radiation, namely,

$$\langle\vec{A}\rangle = \frac{1}{4\pi}\int d\Omega 2\hat{\varepsilon}(\hat{\varepsilon}\cdot\vec{A}) = -\frac{ie}{4\pi^2 c^2}\int d\Omega \int_{-\infty}^{+\infty} d\omega\omega \int_0^t dt'(e^{i\omega(t-t')} - e^{i\omega(t'-t)})\hat{\varepsilon}[\hat{\varepsilon}\cdot\vec{v}'(t')] \quad , \quad (10)$$

where the factor of 2 accounts for the two possible polarization states. On the other hand the QED operator field is written for a two-level model of an atom [5],

$$\vec{A}_Q(0,t) = -\frac{ie}{4\pi^2 c^2}\int d\Omega \int_{-\infty}^{+\infty} d\omega\omega \int_0^t dt'(e^{i\omega(t-t')} - e^{i\omega(t'-t)})\hat{\varepsilon}[\hat{\varepsilon}\cdot\vec{p}_H/m] \quad (11)$$

where Eq. (11) is evaluated at the center of the atom such that retardation is neglected and where $\vec{p}_H$ is the Heisenberg momentum operator, where, in order to compare with the form given in [5], we have used the operator identity,

$$i\frac{e\vec{p}_H}{m} = [R_+ - R_-]\omega_0\vec{d} \quad , \quad (12)$$

where $\omega_0$ and d are the transition frequency and dipole moment of the two-level atom, respectively, and $R_+$, $R_-$ are the raising and lowering operators. Note that Eq. (10) is indeed the expectation

value of Eq. (11); thus within the principal approximation of Eq. (11), neglect of retardation, the correspondence principle is established. A result identical to Eq. (11) has also been presented [3] when the model is that for the linear oscillator rather than a two-level atom.

It is not surprising that the correspondence exists only when retardation is neglected in the QED field because here the distinction between a classical point particle and a quantum point particle described by a spatially distributed wave function is made. What may be surprising however to a community which believes that quantum field theory is necessary to describe effects such as the Lamb shift is the mathematical similarity of the classical electrodynamic and QED descriptions once the decision is made in the former to represent the field as a mode sum over continuously distributed frequencies of the emitted radiation. This decision effectively lifts the mathematical ambiguity at R = 0 of Eq. (1) in favor of an unbounded frequency model, in agreement with QED but in disagreement with the model which diverges Coulombically as $R^{-1}$ as R goes to zero.

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