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Uniform convergence of discretization error for a singular perturbation problem

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Abstract

Cell-centered discretization of the convection-diffusion equation with large Péclet number Pe is analyzed, in the presence of a parabolic boundary layer. It is shown theoretically how by suitable mesh refinement in the boundary layer the accuracy can be made to be uniform in Pe, at the cost of a $\ln Pe$ increase of the number of grid cells, in the case of upwind discretization. Numerical experiments are presented indicating that this can be achieved with a Pe-independent number of grid cells, both with upwind and central discretization, and with vertex-centered discretization.

1 Introduction

It is sometimes thought that it is impossible to accurately compute flows at high Reynolds numbers, because numerical discretization errors ("artificial diffusion") will dominate the physical viscous forces as $Re \to \infty$. We will show that this argument is invalid by a theoretical and practical study of a singular perturbation problem that shares some essential properties with the equations of fluid dynamics, but is much simpler. Our aim is to show that a straightforward discretization method has global error and computing work that are uniform in the small parameter ε.

2 Problem statement

Consider the following special case of the convection-diffusion equation:

$$-\frac{\partial\varphi}{\partial x} - \varepsilon\left(\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2}\right) = q(x,y)\ , \quad (x,y) \in \Omega = (0,1)\times(0,1) \tag{2.1}$$

with boundary conditions

$$\begin{array}{rclcrcl} \varphi(1,y) & = & f_R(y)\ , & -\frac{\partial\varphi(0,y)}{\partial x} & = & f_L(y) \\ \varphi(x,0) & = & f_S(x)\ , & \frac{\partial\varphi(x,1)}{\partial y} & = & 0 \end{array} \tag{2.2}$$

We want to show that a straightforward cell-centered (block-centered) discretization has work and accuracy uniform in ε as $\varepsilon \downarrow 0$, or $Pe \to \infty$, with $Pe = 1/\varepsilon$ the Péclet number.

According to singular perturbation theory (cf. [1], [2], [3], [4] for $\varepsilon \ll 1$ there is a parabolic boundary layer of thickness $O(\sqrt{\varepsilon})$ at $y = 0$. There may also be an ordinary boundary layer of thickness $O(\varepsilon)$ at $x = 0$, but our data are such that this does not occur. It is assumed that δ is chosen such that the boundary layer is inside the strip

$$\Omega_f = \{(x,y) \in \Omega :\ 0 < y < \delta\} \tag{2.3}$$

How to choose δ is one of our main topics. In order to achieve our aim of ε-uniform accuracy and work, the grid is refined in Ω_f. A sketch of the grid is given in figure 2.1. The refined part of the grid is called G_f, the remainder G_c and the interface is called Γ. G_f and G_c are uniform. A cell-centered finite volume discretization is obtained in the usual way. The grid points are the cell centers. Cells and their centers are labeled by two-tuples (i,j): Ω_{ij} is the cell with center at (x_i, y_j). Furthermore, for example, $(i+1/2, j)$ refers to the center of the right vertical edge of cell (i,j). Integration of (2.1) over Ω_{ij} gives

$$F^x|_{i-1/2,j}^{i+1/2,j} + F^y|_{i,j-1/2}^{i,j+1/2} = q_{ij}|\Omega_{ij}| \tag{2.4}$$

with the "fluxes" F^x and F^y defined by

$$\begin{array}{rcl} F^x_{i+1/2,j} & = & -K_j\{\varphi_{i+1,j} + \varepsilon(\varphi_{i+1,j} - \varphi_{ij})/H_1\} \\ F^y_{i,j+1/2} & = & -2H_1\varepsilon(\varphi_{i,j+1} - \varphi_{ij})/(K_j + K_{j+1}) \end{array} \tag{2.5}$$

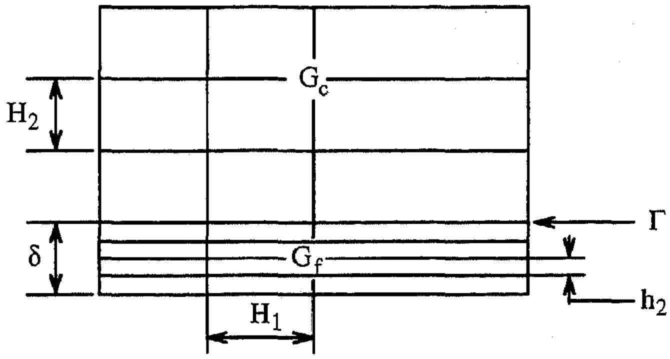

Figure 2.1: Computational grid

where K_j is the vertical dimension of Ω_{ij} : $K_j = h_2$ in G_f and $K_j = H_2$ in G_c. In (2.5), upwind discretization has been used for the first derivative. The cells are numbered $i = 1, ..., I$ and $j = 1, ..., J$ in the x- and y-directions, respectively. In order to facilitate the error analysis that is to follow additional unknowns $\varphi_{i,1/2}$ and $\varphi_{I+1/2,j}$ are placed on the Dirichlet boundaries. The system of equations is extended with the boundary conditions:

$$\varphi_{i,1/2} = f_S(x_i) \ , \quad \varphi_{I+1/2,j} = f_R(y_j) \tag{2.6}$$

At the boundaries,

$$\begin{aligned} F^x_{1/2,j} &= -K_j\{\varphi_{1,j} - \varepsilon f_L(y_j)\} \\ F^x_{I+1/2,j} &= -K_j\{\varphi_{I+1/2,j} + 2\varepsilon(\varphi_{I+1/2,j} - \varphi_{Ij})/H_1\} \\ F^y_{i,1/2} &= -2H_1\varepsilon(\varphi_{i1} - \varphi_{i,1/2})/K_1 \ , \\ F^y_{i,J-1/2} &= 0 \end{aligned} \tag{2.7}$$

The resulting discrete system is denoted by

$$L_h\varphi_h = \boldsymbol{q}_h \tag{2.8}$$

3 Expansion for local and global truncation error

The local truncation error $\boldsymbol{\tau}_h$ is defined as

$$\boldsymbol{\tau}_h = L_h(\varphi - \varphi_h) \tag{3.1}$$

where φ is the algebraic vector containing the values in the grid points of the exact solution of (2.1). The global truncation error $\boldsymbol{e}_h$ is defined as

$$\boldsymbol{e}_h = \varphi - \varphi_h \tag{3.2}$$

so that

$$L_h\boldsymbol{e}_h = \boldsymbol{\tau}_h \tag{3.3}$$

Our aim is to estimate $\boldsymbol{e}_h$, uniformly in ε. To this end we first estimate $\boldsymbol{\tau_h}$, carefully keeping track of its dependence on ε. We write $\tau_h = \tau_h^x + \tau_h^y$ with τ_h^x, τ_h^y the local truncation error generated by F^x, F^y. Taylor expansion gives

$$
\begin{aligned}
|\tau_{1j}^x| &\leq H_1^2 K_j C_1, \quad j = 1, \dots J && (3.4)\\
|\tau_{ij}^x| &\leq H_1^2 K_j C_2, \quad i = 2, \dots I-1, \quad j = 1, \dots J && (3.5)\\
|\tau_{Ij}^x| &\leq H_1 K_j C_3, \quad j = 1, \dots J && (3.6)\\
|\tau_{i1}^y| &\leq H_1 h_2 C_4, \quad i = 1, \dots I && (3.7)\\
|\tau_{ij}^y| &\leq H_1 h_2^3 C_5, \quad i = 1, \dots I, \quad j = 2, \dots, j_\Gamma - 1 && (3.8)\\
|\tau_{ij}^y| &\leq H_1 H_2^3 C_6, \quad i = 1, \dots I, \quad j = j_\Gamma + 2, \dots J-1 && (3.9)\\
|\tau_{iJ}^y| &\leq H_1 H_2^2 C_7, \quad i = 1, \dots I && (3.10)
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &= sup\{|\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}| : (x,y) \in \Omega\} + sup\{|\frac{\varepsilon}{24}\frac{\partial^3\varphi}{\partial x^3}| : (x,y) \in \Omega\} && (3.11)\\
C_2 &= sup\{|\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}| : (x,y) \in \Omega\} + sup\{|\frac{\varepsilon}{12}\frac{\partial^4\varphi}{\partial x^4}| : (x,y) \in \Omega\} && (3.12)\\
C_3 &= sup\{|\frac{1}{2}\frac{\partial\varphi}{\partial x}| : (x,y) \in \Omega\} + sup\{|\frac{\varepsilon}{4}\frac{\partial^2\varphi}{\partial x^2}| : (x,y) \in \Omega\} && (3.13)\\
C_4 &= sup\{|\frac{\varepsilon}{4}\frac{\partial^2\varphi}{\partial y^2}| : (x,y) \in \Omega_f\} && (3.14)\\
C_5 &= sup\{|\frac{\varepsilon}{12}\frac{\partial^4\varphi}{\partial y^4}| : (x,y) \in \Omega_f\} && (3.15)\\
C_6 &= sup\{|\frac{\varepsilon}{12}\frac{\partial^4\varphi}{\partial y^4}| : (x,y) \in \Omega_c\} && (3.16)\\
C_7 &= sup\{|\frac{\varepsilon}{24}\frac{\partial^3\varphi}{\partial y^3}| : 0 < x < 1, \ 1 - H_2 < y < 1\} && (3.17)
\end{aligned}
$$

Here $\Omega_c = \Omega \setminus \Omega_f$. Because of finite volume integration a factor of $H_1 K_j$ is gained. Hence, a second order scheme has $\tau_h = O(\Delta^4)$, $\Delta = \max(H_1, K_j)$. We see that (due to upwind discretization) the scheme is first order in the interior and at the Neumann boundaries, but $O(1)$, i.e. inconsistent (in the maximum norm) at Dirichlet boundaries.

Near the interface Γ between G_f and G_c (i.e. for $j = j_\Gamma, j_\Gamma + 1$) we need an expansion for τ_{ij}^y. We find

$$
\begin{aligned}
\tau_{ij_\Gamma}^y = & -\tfrac{1}{4}\varepsilon\tfrac{\partial^2\varphi}{\partial y^2} H_1(H_2 - h_2) - \tfrac{1}{24}\varepsilon\tfrac{\partial^3\varphi}{\partial y^3} H_1(H_2 + 3h_2)(H_2 - h_2)\\
& -\tfrac{1}{24}\varepsilon\tfrac{\partial^4\varphi(*)}{\partial y^4} H_1\{h_2^3 + \tfrac{1}{8}(h_2 + H_2)^3\}
\end{aligned} \qquad (3.18)
$$

$$
\begin{aligned}
\tau_{i,j_\Gamma+1}^y = & \ \tfrac{1}{4}\varepsilon\tfrac{\partial^2\varphi}{\partial y^2} H_1(H_2 - h_2) - \tfrac{1}{24}\varepsilon\tfrac{\partial^3\varphi}{\partial y^3} H_1(h_2 + 3H_2)(H_2 - h_2)\\
& -\tfrac{1}{24}\varepsilon\tfrac{\partial^4\varphi(**)}{\partial y^4} H_1\{H_2^3 + \tfrac{1}{8}(h_2 + H_2)^3\}
\end{aligned} \qquad (3.19)
$$

Where no argument is given the φ derivatives are evaluated in the same grid point as τ^y; the that also near Γ the scheme is inconsistent in the maximum norm, since $h_2 \ll H_2$. Finally, on the Dirichlet boundaries, because (2.6 is exact,

$$\tau_{i,1/2} = \tau_{I+1/2,j} = 0 \tag{3.20}$$

We write

$$e_h = \sum_{k=1}^{4} e^k \tag{3.21}$$

and choose e^1, e^2, e^3 such that $L_h(e^1 + e^2 + e^3)$ equals the first two terms in (3.18), (3.19); e^4 will be estimated with the maximum principle. We choose

$$e^k_{ij} = \psi(x_i, y_j)\mu^k_j \,, \quad k = 1, 2, 3 \tag{3.22}$$

with

$$\begin{array}{rcl} \mu^1_j &=& 0, \quad j \le j_\Gamma; \quad \mu^1_j = H_2^2 - h_2^2, \quad j \ge j_\Gamma + 1 \\ \mu^2_j &=& 0, \quad j \le j_\Gamma; \quad \mu^2_{j_\Gamma+1} = \frac{1}{48}(H_2 + 3h_2)(H_2^2 - h_2^2), \\ \mu^2_j &=& \mu^2_{j-1} + \frac{1}{6}H_2(H_2^2 - h_2^2), \quad j \ge j_\Gamma + 2 \\ \mu^3_j &=& 0, \quad j \le j_\Gamma; \quad \mu^3_j = -\frac{1}{2}(H_2 + h_2)(H_2^2 - h_2^2), \quad j \ge j_\Gamma + 1 \end{array} \tag{3.23}$$

$$\psi_1 = \frac{1}{8}\frac{\partial^2 \varphi}{\partial y^2}, \quad \psi_2 = \frac{\partial^3 \varphi}{\partial y^3}, \quad \psi_3 = \frac{1}{8}\frac{\partial^3 \varphi}{\partial y^3} \tag{3.24}$$

We have

$$\begin{array}{l} L_h\{\psi(x_i, y_j)\mu_j\} = \mu_j L_h \psi(x_i, y_j) \\ -\varepsilon \psi(x_i, y_j)\{(\mu_{j+1} - \mu_j)/K_{j+1/2} - (\mu_j - \mu_{j-1})/K_{j-1/2}\} \\ -\varepsilon \partial \psi(x_i, y_j)/\partial y(\mu_{j+1} - \mu_{j-1}) + O(\frac{1}{2}\mu_j \varepsilon H_2 \frac{\partial^2 \psi(*)}{\partial y^2}) \end{array} \tag{3.25}$$

where $*$ is some point in the grid cels (i, j) or $(i, j \pm 1)$. $L_h \psi$ can be evaluated by (3.4)-(3.19), replacing φ by ψ. After some cumbersome manipulations one finds, to leading order in the mesh sizes,

$$\begin{array}{rcl} L_h e^4 &=& \tau^x + \tilde{\tau}^y, \\ \tilde{\tau}^y_{ij} &=& \tau^y_{ij}, \quad i = 1, ..., I, \quad j \le j_\Gamma - 1 \\ |\tilde{\tau}^y_{ij}| &\le& H_1 H_2^3 C_8, \quad i = 1, ..., I, \quad j_\Gamma \le j \le J - 1 \\ C_8 &=& sup\{|\frac{1}{3}\varepsilon \frac{\partial^4 \varphi}{\partial y^4}| : (x, y) \in X \cup \Omega_c\} \\ |\tilde{\tau}^y_{iJ}| &\le& 9 H_1 H_2^2 C_7 \end{array} \tag{3.26}$$

Note that e^1, e^2, e^3 are $O(H_2^2)$. It remains to estimate e^4.

4 Error estimate with the maximum principle

We have

Proof For every $(i,j) \in G \setminus G_D$, lemma 4.1 applies. Considering every grid point $(i,j) \in G \setminus G_D$ in turn, the theorem follows. □

It is easily seen that L_h is a K-operator in $G \setminus G_D$ with

$$G_D = (i = 1, \dots I, j = 1/2) \cup (i = I + 1/2,\ j = 1, \dots J) \tag{4.11}$$

We have (cf. (3.26))

$$L_h e^4 = \tau^x + \tilde{\tau}^y \tag{4.12}$$

We will contruct a barrier function $\boldsymbol{E}$ such that

$$L_h \boldsymbol{E} \geq |\tau^x| + |\tilde{\tau}^y| \tag{4.13}$$

Inequalities hold pointwise, i.e. $\varphi \leq \psi \iff \varphi_{ij} \leq \psi_{ij},\ \ \forall (i,j) \in G$. It follows that

$$\mathrm{L}_h(\pm e^4 - \boldsymbol{E}) \leq 0 \tag{4.14}$$

and the maximum principle gives, since $e^4 = 0$ on G_D,

$$\pm e^4 - \boldsymbol{E} \leq \max\{-E_{ij} : (i,j) \in G_D\} \tag{4.15}$$

Hence

$$|e^4| - |\boldsymbol{E}| \leq \max\{|E_{ij}| : (i,j) \in G_D\} \tag{4.16}$$

or

$$|e^4| \leq |\boldsymbol{E}| + \max\{|E_{ij}| : (i,j) \in G_D\} \tag{4.17}$$

It remains to construct $\boldsymbol{E}$. Let

$$E(x,y) = B_x E^x(x) + B_y E^y(y) \tag{4.18}$$

with B_x, B_y constants, and

$$\left\{ \begin{array}{lcl} E^x(x) & = & 1 + (1-x)(2 + x + H/2), \quad 0 \leq x \leq 1 - H_1/4 \\ E^x(x) & = & 0, \quad x > 1 - H_1/4 \end{array} \right. \tag{4.19}$$

$$\left\{ \begin{array}{lcl} E^y(y) & = & 0, \quad 0 \leq y \leq h_2/4 \\ E^y(y) & = & 1 + y(2\delta + 3\delta^2 - y)/\delta^2, \quad h_2/4 < y \leq \delta \\ E^y(y) & = & E^y(\delta) + (y - \delta)(3 - y), \quad \delta < y \leq 1 \end{array} \right. \tag{4.20}$$

We find for $j = 1, \dots J$

$$\begin{array}{lcl} L_h E^x_{ij} & > & K_j H_1, \quad i = 1, \dots I-1 \\ L_h E^x_{Ij} & > & K_j \end{array} \tag{4.21}$$

Furthermore, for $i = 1, \dots I$

$$\begin{array}{lcll} L_h E^y_{i1} & > & 2\varepsilon H_1/h_2 & \\ L_h E^y_{ij} & > & 2\varepsilon H_1 h_2/\delta^2, & j = 2, \dots j_\Gamma - 1 \\ L_h E^y_{ij} & > & \frac{3}{2}\varepsilon H_1 H_2, & j = j_\Gamma, \dots J-1 \\ L_h E^y_{ij} & > & \varepsilon H_1, & j = J \end{array} \tag{4.22}$$

Hence, with

$$B_x = H_1 \max\{C_1, C_2, C_3\} \tag{4.23}$$

we have

$$L_h B_x E^x \geq |\tau^x| \tag{4.24}$$

Furthermore, with

$$B_y = \varepsilon^{-1} \max\{h_2^2 C_4/2,\ h_2^2 C_5 \delta^2/2,\ 9H_2^2 C_7,\ 2H_2^2 C_8/3\} \tag{4.25}$$

we have

$$L_h B_y E^y \geq |\tilde{\tau}^y| \tag{4.26}$$

Since $|E^x(x)| \leq 3,\ |E^y(y)| \leq 5$, equation (4.17) gives

$$|e^4| \leq 6B^x + 10B^y \tag{4.27}$$

Furthermore we have, to leading order in the step-sizes,

$$|e^1| \leq \frac{1}{8} H_2^2 \sup\{|\frac{\partial^2 \varphi}{\partial y^2}| : (x,y) \in \Omega_c\} \tag{4.28}$$

$$|e^2| \leq \frac{1}{6} H_2^2 sup\{|\frac{\partial^3 \varphi}{\partial y^3}| : (x,y) \in \Omega_c\} \tag{4.29}$$

$$|e^3| \leq \frac{1}{8} H_2^2 sup\{|\frac{\partial^3 \varphi}{\partial y^3}| : (x,y) \in \Omega_c\} \tag{4.30}$$

Note that for ε fixed

$$e = O(H_1 + H_2^2) \tag{4.31}$$

so that the fact that the scheme is locally inconsistent in the maximum norm does not affect the global error.

5 Dependence of the error on ε

For the analysis of the dependence of the bounds obtained on ε the first term of a matched asymptotic expansion for the solution will be used. In order to show rigorously that higher order terms in this expansion may be neglected a laborious analysis of higher order terms would be needed, from which we will refrain. Numerical experiments will give further validation of the results obtained. For simplicity we assume

$$q \equiv 0 \tag{5.1}$$

For the terminology and the method followed, see the literature on singular perturbation theory cited in section 2. The outer equation is

$$-\partial \varphi_0 / \partial x = 0, \quad \varphi_0(1, y) = f_R(y) \tag{5.2}$$

Its solution, called the outher solution, is given by

$$\varphi_0 = f_R(y) \tag{5.3}$$

The inner or boundary layer solution φ_b that approximates $\varphi - \varphi_0$ is the solution of the following boundary layer problem:

$$\begin{array}{l} -\frac{\partial \varphi_b}{\partial x} - \varepsilon \frac{\partial^2 \varphi_b}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < \infty \\ \varphi_b(1,y) = 0, \quad \varphi_b(x,\infty) = 0, \quad \varphi_b(x,0) = g(x) \end{array} \tag{5.4}$$

with

$$g(x) = f_S(x) - f_R(0) \tag{5.5}$$

In order to avoid a corner singularity it is assumed that $f_S(1) = f_R(0)$. The solution of (5.4) is given by

$$\varphi_b = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon(1-x)}}^{\infty} e^{-\frac{1}{2}t^2} g(x + \frac{y^2}{2\varepsilon t^2}) dt \tag{5.6}$$

A uniformly valid asymptotic approximation to φ is given by $\varphi_b + \varphi_0$. We find, using $g(1) = 0$:

$$\frac{\partial^2 \varphi_b}{\partial y^2} = -\frac{1}{\varepsilon}\sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon(1-x)}}^{\infty} e^{-\frac{1}{2}t^2} g'(x + \frac{y^2}{2\varepsilon t^2}) dt \tag{5.7}$$

Assuming $|g'(x)| \leq M_1, \ 0 < x < 1$ we obtain

$$|\frac{\partial^2 \varphi_b}{\partial y^2}| \leq \frac{1}{\varepsilon} M_1 erfc(y/2\sqrt{\varepsilon}) \tag{5.8}$$

Using the inequality

$$\frac{2/\sqrt{\pi}}{z + \sqrt{z^2+2}} e^{-z^2} < erfc(z) < \frac{2/\sqrt{\pi}}{z + \sqrt{z^2 + 4/\pi}} e^{-z^2} \quad (z \geq 0) \tag{5.9}$$

we may without losing much sharpness replace (5.8) by

$$|\frac{\partial^2 \varphi_b}{\partial y^2}| \leq \frac{4}{\sqrt{\pi\varepsilon}} M_1 \frac{1}{y + \sqrt{y^2 + 12\varepsilon/\pi}} e^{-y^2/4\varepsilon} \tag{5.10}$$

It also necessary to estimate $\partial^4 \varphi_b / \partial y^4$. In order to avoid a corner singularity at (1,0) it is assumed that

$$g'(1) = 0 \tag{5.11}$$

Then

$$\frac{\partial^4 \varphi_b}{\partial y^4} = -\frac{1}{\varepsilon}\sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon(1-x)}}^{\infty} e^{-\frac{1}{2}t^2} g''(x + \frac{y^2}{2\varepsilon t^2}) dt \tag{5.12}$$

Assuming $|g''(x)| \leq M_2$, $0 < x < 1$ we obtain

$$|\frac{\partial^4 \varphi_b}{\partial y^4}| \leq \frac{1}{\varepsilon^2} M_2 erfc(y/2\sqrt{\varepsilon}) \tag{5.13}$$

Again using (5.9) one obtains

$$|\frac{\partial^4 \varphi_b}{\partial y^4}| \leq \frac{4}{\sqrt{\pi\varepsilon^3}} M_2 \frac{1}{y + \sqrt{y^2 + 16\varepsilon/\pi}} e^{-y^2/4\varepsilon} \tag{5.14}$$

Finally, $\partial^3\varphi_b/\partial y^3$ has to be estimated. We obtain

$$\begin{aligned} \frac{\partial^3 \varphi_b}{\partial y^3} &= -\frac{1}{\varepsilon}\sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon(1-x)}}^{\infty} e^{-\frac{1}{2}t^2} \frac{y}{\varepsilon t^2} g''(x + \frac{y^2}{2\varepsilon t^2}) dt \\ &= -\frac{1}{\varepsilon y}\sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon(1-x)}}^{\infty} g'(x + \frac{y^2}{2\varepsilon t^2})(t^2 - 1) e^{-\frac{1}{2}t^2} dt \end{aligned} \tag{5.15}$$

Hence

$$|\frac{\partial^3 \varphi_b}{\partial y^3}| \leq \frac{1}{\varepsilon y} M_1 \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\varepsilon}}^{\infty} |t^2 - 1| e^{-\frac{1}{2}t^2} dt \tag{5.16}$$

Let us consider $e^{(1)}$. Using $\varphi \cong \varphi_b + \varphi_0$ we obtain, using (5.10),

$$sup\{|\frac{\partial^2 \varphi}{\partial y^2}| : (x, y) \in \Omega_c\} \leq M_3 + \frac{2}{\sqrt{\pi\varepsilon}} M_1 \frac{1}{\delta} e^{-\delta^2/4\varepsilon} \tag{5.17}$$

with

$$M_3 = sup\{|f_R''(y)| : 0 < y < 1\} \tag{5.18}$$

For $e^{(2)}$ and $e^{(3)}$ we need, using (5.16) and assuming

$$\delta/\sqrt{2\varepsilon} > 1 \tag{5.19}$$

$$\begin{aligned} sup\{|\frac{\partial^3 \varphi}{\partial y^3}| : (x, y) \in \Omega_c\} &\leq M_4 + \frac{1}{\varepsilon\delta} M_1 \sqrt{\frac{2}{\pi}} \int_{\delta/\sqrt{2\varepsilon}}^{\infty} (t^2 - 1) e^{-\frac{1}{2}t^2} dt \\ &= M_4 + \varepsilon^{-3/2} M_1 \frac{1}{\sqrt{\pi}} \delta e^{-\delta^2/4\varepsilon} \end{aligned} \tag{5.20}$$

where

$$M_4 = sup\{|f'''(y)| : 0 < y < 1\} \tag{5.21}$$

For B_y we need to estimate C_4, C_5, C_7 and C_8. For this is needed:

$$sup\{|\frac{\partial^2 \varphi}{\partial y^2}| : (x, y) \in \Omega_f\} \leq M_1/\varepsilon + M_3 \tag{5.22}$$

$$sup\{|\frac{\partial^4 \varphi}{\partial y^4}| : (x, y) \in \Omega_f\} \leq M_2/\varepsilon^2 + M_5 \tag{5.23}$$

$$sup\{|\frac{\partial^4\varphi}{\partial y^4}| : (x,y) \in X \cup \Omega_c\} \leq M_5 + \varepsilon^{-3/2} M_1 \frac{1}{\sqrt{\pi}} \tilde{\delta} e^{-\tilde{\delta}^2/4\varepsilon} \tag{5.24}$$

$$sup\{|\frac{\partial^3\varphi}{\partial y^3}| : 0 < x < 1,\ 1 - H_2 < y < 1\} \lesssim \tag{5.25}$$

$$M_4 + \frac{1}{\varepsilon} M_1 \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{2\varepsilon}}^{\infty} (t^2 - 1) e^{-\frac{1}{2}t^2} dt = M_4 + \varepsilon^{-3/2} M_1 \frac{1}{\sqrt{\pi}} e^{-1/4\varepsilon}$$

where

$$\tilde{\delta} = \delta - h_2, \quad M_5 = sup\{|f_R^{(4)}(y)| : 0 < y < 1\} \tag{5.26}$$

and where we have assumed

$$\tilde{\delta}/\sqrt{2\varepsilon} > 1 \tag{5.27}$$

Finally, the x-derivatives are $O(1)$.

6 Thickness of the refinement region

We now have to choose δ and h_2 such that the error $\boldsymbol{e}^1 + \boldsymbol{e}^2 + \boldsymbol{e}^3 + \boldsymbol{e}^4$ is uniformly bounded in ε. Choosing

$$\delta = \sqrt{-5\varepsilon \ln \varepsilon} \tag{6.1}$$

we see that

$$\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = O(H_2^2) \tag{6.2}$$

uniformly in ε. Furthermore, to have B_y uniformly bounded it is necessary that

$$h_2 = H_2 \sqrt{-c\varepsilon / \ln \varepsilon} \tag{6.3}$$

with c some positive constant. Then we have

$$\frac{1}{\varepsilon} h_2^2 C_4 / 2 \leq \frac{1}{8} c H_2^2 (M_1 + \varepsilon M_3) / \ln 1/\varepsilon \tag{6.4}$$

$$\frac{1}{2\varepsilon} h_2^2 C_5 \delta^2 \leq \frac{5}{24} c H_2^2 (M_2 + \varepsilon^2 M_5) \tag{6.5}$$

$$\frac{9}{\varepsilon} H_2^2 C_7 \leq \frac{3}{8} H_2^2 (M_4 + \varepsilon^{-3/2} M_1 \frac{1}{\sqrt{\pi}} e^{-1/4\varepsilon}) \tag{6.6}$$

$$\frac{2}{3\varepsilon} H_2^2 C_8 \leq \frac{2}{9} H_2^2 (\frac{1}{\sqrt{\pi}} \varepsilon^{-1/4} \sqrt{-5 \ln \varepsilon} M_1 + M_5) \tag{6.7}$$

In (6.6), the difference between δ and $\tilde{\delta}$ has been neglected, which is asymptotically correct. Uniform boundedness of B_y follows from (6.4) - (6.7). The following error estimate results in the maximum norm:

$$\|e_h\| = O(H_1 + H_2^2) \tag{6.8}$$

uniformly in ε.

Equations (6.1), (6.3) and (6.8) are our main results. Although the goal of uniform accuracy has been achieved, the work increases (slowly) as $\varepsilon \downarrow 0$. If there are n horizontal grid-lines in G_c, then the number of horizontal grid lines n_f in G_f satisfies

$$n_f = O(n \ln 1/\varepsilon) \tag{6.9}$$

However, numerical experience shows that the logarithmic factors in (6.1) and (6.3) come into play only for unrealistically small values of H_2, so that in practice one may work with

$$\delta = O(\sqrt{\varepsilon}), \quad h_2 = O(\sqrt{\varepsilon} H_2) \tag{6.10}$$

Now the work is uniform in ε, whereas numerical experiments still show uniform accuracy.

The analysis of vertex-centered discretization, where the nodes are the vertices of the cells in figure 2.1 rather than the centers, is easier, because an error expansion like (3.21) is not required. By using the maximum principle in a similar way, (6.1), (6.3) and (6.8) may be derived.

It is also possiblle to allow in addition to the parabolic boundary layer at $y = 0$ an ordinary boundary layer at $x = 0$. An additional refinement region of thickness $O(\varepsilon \ln 1/\varepsilon)$ needs to be introduced at $x = 0$ with mesh-size $h_1 = O(H_1 \varepsilon \ln 1/\varepsilon)$. The same method of analysis can be followed.

Because the analysis used the maximum principe, upwind discretization is necessary. But in practice, as will be shown, central discretization also gives good results, with $O(H_1^2 + H_2^2)$ accuracy.

7 Numerical experiments

The following exact solution as assumed:

$$\varphi = \frac{1}{\sqrt{2-x}} \{exp(-\frac{y^2}{4\varepsilon(2-x)}) + exp(-\frac{(2-y)^2}{4\varepsilon(2-x)}\} \tag{7.1}$$

The right-hand-size and boundary conditions in (2.1) and (2.2) are chosen accordingly. Because the solution is extremely smooth in Ω_c it turns out that in G_c the number of cells in the vertical direction can be fixed at 4; the maximum of the error is found to always occur in G_f. We take

$$\delta = 8\sqrt{\varepsilon} \tag{7.2}$$

neglecting the logarithmic factor in (6.1).

Table 7.1 gives results for the cell-centered upwind case. Equation (6.8) is confirmed. Exactly the same results are obtained for $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{7}$, showing ε-uniform accuracy, despite the fact that the logarithmic factors in (6.1) and (6.3) have been neglected, so that the work is dependent of ε. The maximum error occurs in the interior of the boundary layer. Table 7.2 gives results for central discretization. Visual inspection of the results shows no

nx	ny	error * 10^4
8	32	54
32	64	14
128	128	3.6

Table 7.1: Maximum error as function of number of grid-cells for $\varepsilon = 10^{-3}$; cell-centered upwind discretization. nx: horizontal number of cells; ny : vertical number of cells in G_f.

nx	ny	error * 10^4
8	16	92
16	32	28
32	64	7.8
64	128	2.1

Table 7.2: Cell-centered central discretization; $\varepsilon = 10^{-3}$

visible wiggles. The rate of convergence is somewhat worse than the hoped for $O(H_1^2 + H_2^2)$, but here again the same results are obtained for $\varepsilon = 10^{-7}$, showing uniformity in ε.

Table 7.3 gives results for the vertex-centered upwind case. A rate convergence of $O(H_1 + H_2^2)$

nx	ny	error * 10^4
8	32	69
32	64	15
128	128	3.8

Table 7.3: Vertex-centered upwind discretization; $\varepsilon = 10^{-3}$.

is clearly demonstrated. The accuracy is about the same as for cell-centered discretization. Exactly the same results are obtained for $\varepsilon = 10^{-5}$ and 10^{-7}, again showing ε-uniform convergence.

Finally, table 7.4 gives results for vertex-centered central discretization. The rate of convergence is $O(H_1^2 + H_2^2)$. Further refinement shows that the finest grid of table 7.4 is the finest that rounding errors allow. No wiggles are observed. Results for $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-7}$ are virtually identical.

We may conclude that in practice work and accuracy can be made to be uniform in ε, both for cell- and vertex-centered discretization, by using local mesh refinement according to (6.9).

nx	ny	error $* 10^4$
8	16	30
16	32	6.76
32	64	1.88

Table 7.4: Vertex-centered central discretization; $\varepsilon = 10^{-3}$.

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