A characterization of panconnected graphs satisfying a local ore-type condition

Armen S. Asratian, R. Häggkvist and G. V. Sarkisian

A Characterization of Panconnected Graphs Satisfying a Local Ore-Type Condition

A. S. Asratian
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UMEÅ
S-901 87, UMEÅ, SWEDEN
DEPARTMENT OF MATHEMATICAL CYBERNETICS
YEREVAN STATE UNIVERSITY
YEREVAN, 375049, REPUBLIC OF ARMENIA

R. Häggkvist
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UMEÅ
S-901 87, UMEÅ, SWEDEN

G. V. Sarkisian
DEPARTMENT OF MATHEMATICAL CYBERNETICS
YEREVAN STATE UNIVERSITY
YEREVAN, 375049, REPUBLIC OF ARMENIA

ABSTRACT

It is well known that a graph G of order $p \geq 3$ is Hamilton-connected if $d(u)+d(v) \geq p+1$ for each pair of nonadjacent vertices u and v. In this paper we consider connected graphs G of order at least 3 for which $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| + 1$ for any path uwv with $uv \notin E(G)$, where $N(x)$ denote the neighborhood of a vertex x. We prove that a graph G satisfying this condition has the following properties: (a) For each pair of nonadjacent vertices x, y of G and for each integer $k, d(x,y) \leq k \leq |V(G)| - 1$, there is an $x - y$ path of length k. (b) For each edge xy of G and for each integer k (excepting maybe one $k \in \{3,4\}$) there is a cycle of length k containing xy.

Consequently G is panconnected (and also edge pancyclic) if and only if each edge of

G belongs to a triangle and a quadrangle.

Our results imply some results of Williamson, Faudree, and Schelp.

1. INTRODUCTION

We use Bondy and Murty [6] for terminology and notation not defined here and consider finite simple graphs only. For each vertex u of a graph G we denote by $N(u)$ the set of all vertices of G adjacent to u. The distance between vertices u and v is denoted by $d(u, v)$. A path with x and y as end vertices is called an $x - y$ path. A path is called a Hamilton path if it contains all the vertices of G. A graph G is Hamilton-connected if every two vertices of G are connected by a Hamilton path.

Let G be a graph of order $p \geq 3$. G is called panconnected if for each pair of distinct vertices x and y of G and for each $l, d(x, y) \leq l \leq p - 1$, there is an $x - y$ path of length l. G is called pancyclic if it contains a cycle of length l for each l satisfying $3 \leq l \leq p$. G is called a vertex pancyclic (edge pancyclic) if each vertex (edge) of G lies on a cycle of every length from 3 to p inclusive.

The following results are known.

Theorem 1. (Ore [12]). Let G be a graph of order $p \geq 3$, where $d(u) + d(v) \geq p + 1$ for each pair u, v of nonadjacent vertices. Then G is Hamilton-connected.

Theorem 2. (Williamson [13]). A connected graph of order $p \geq 3$ is panconnected if any of the following two conditions hold:

(a) $d(u) \geq (p + 2)/2$ for each vertex u of G,

(b) $d(u) + d(v) \geq (3p - 2)/2$ for each pair of nonadjacent vertices u, v of G.

Theorem 3. (Faudree and Schelp [8]). If G is a graph of order $p \geq 5$ with $d(u) + d(v) \geq p + 1$ for each pair of nonadjacent vertices u, v then G contains a path of every length from 4 to $n - 1$ inclusive, between any pair of distinct vertices of G.

A shorter proof of Theorem 3 was given by Cai [7]. From results of Bondy [5] and Häggkvist et al. [10] it follows that every graph G satisfying the condition of Theorem 1 is pancyclic. Some other properties of graphs satisfying the condition of Theorem 1 were obtained in [4, 9, 14, 15].

The following generalization of Theorem 1 was found by Asratian et al. [1].

Theorem 4. [1]. Let G be a connected graph of order at least 3 where $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| + 1$ for any path uwv with $uv \notin E(G)$. Then G is Hamilton-connected.

Denote by L the set of all graphs satisfying the condition of Theorem 4. It was proved in [3] that every graph from L is pancyclic, and in [2] it was shown that a graph $G \in L$ is vertex pancyclic if and only if each vertex of G lies on a triangle.

In this paper we show that a graph $G \in L$ has the following properties:

(a) For each pair of nonadjacent vertices x, y of G and for each integer $n, d(x, y) \leq n \leq |V(G)| - 1$, there is an $x - y$ path of length n.

(b) For each edge xy and for each integer $k, 3 \leq n \leq |V(G)|$, (excepting maybe one $k \in \{3,4\}$) there is a cycle of length k containing xy.

This implies that a graph $G \in L$ is panconnected (and also edge pancyclic) if and only if each edge of G lies on a triangle and on a quadrangle.

Note that for each $r \geq 2$ and each $p \geq 3$ there exists a panconnected graph $G_{r,p} \in L$ of order pr with diameter r: its vertex set is $\cup_{i=0}^{r} V_i$ where $V_0, V_1, \ldots, V_r$ are pairwise disjoint sets of cardinality p and two vertices are adjacent if and only if they both belong to $V_i \cup V_{i+1}$ for some $i \in \{0, 1, \ldots, r-1\}$.

2. NOTATION AND PRELIMINARY RESULTS

Let P be a path of G. We denote by $\vec{P}$ the path P with a given orientation and by $\overleftarrow{P}$ the path P with the reverse orientation. If $u, v \in V(P)$, then $u\vec{P}v$ denotes the consecutive vertices of P from u to v in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $v\, \overleftarrow{P}\, u$. We use w^+ to denote the successor of w on $\vec{P}$ and w^- to denote its predecessor. We denote by $N(P)$ the set of vertices v outside P with $N(v) \cap V(P) \neq \emptyset$. If $W \subseteq V(P)$ then $W^+ = \{w^+/w \in W\}$ and $W^- = \{w^-/w \in W\}$.

We will say that a path $\vec{P}$ contains a triangle $a_1a_2a_3a_1$ if $a_1, a_2, a_3 \in V(P), a_1a_3 \in E(G)$ and $a_1^+ = a_2 = a_3^-$. A path $\vec{P}$ containing a triangle Δ is denoted by $\vec{P}^{\Delta}$. The set of all triangles contained in $\vec{P}^{\Delta}$ we denote by $T(\vec{P}^{\Delta})$. We assume that an $x-y$ path $\vec{P}$ has an orientation from x to y. A path on n vertices will be denoted by P_n.

Let A and B be two disjoint subsets of vertices of a graph G. We denote by $\varepsilon(A, B)$ the number of edges in G with one end in A and the other in B.

Proposition 1. [11]. $G \in L$ if and only if for any path uwv with $uv \notin E(G)|N(u) \cap N(v)| \geq |N(w) \setminus (N(u) \cup N(v))| + 1$ holds.

Corollary 1. If $G \in L$ then G is 3-connected and $|N(u) \cap N(v)| \geq 3$ for each pair of vertices u, v with $d(u, v) = 2$.

Proof. Let $d(u, v) = 2$. If $w \in N(u) \cap N(v)$ then $u, v \in N(w) \setminus (N(u) \cup N(v))$ and, by Proposition 1, $|N(u) \cap N(v)| \geq 3$. This implies that G is 3-connected. ∎

Proposition 2. Let $G \in L$ and x, y be two vertices of G with $d(x, y) = l \geq 2$. Then there exists an $x-y$ path P_{l+2}^{Δ}.

Proof. Let $P = u_0u_1 \cdots u_l$ be an $x-y$ path of length $l = d(x, y)$ where $u_0 = x$ and $u_l = y$. If there is a vertex outside P which is adjacent to two consecutive vertices of P then there is an $x-y$ path P_{l+2}^{Δ}. Suppose that there is no such vertex outside P. Since $d(u_0, u_2) = 2$ then, by Proposition 1, we have $|N(u_0) \cap N(u_2)| \geq |N(u_1) \setminus (N(u_0) \cup (u_2))| + 1 \geq 3$. Clearly,

$$N(u_0) \cap V(P) = N(u_0) \cap N(u_2) \cap V(P) = \{u_1\}. \tag{1}$$

Let $N(u_0) \cap N(u_2) = \{w_1, \ldots, w_k\}$ where $k \geq 3$ and $w_1 = u_1$. Furthermore, let $|N(w_1) \cap N(w_2)| = m$. If $w_iw_j \notin E(G)$ for each pair $i, j, 1 \leq i < j \leq k$, then using (1) and

Proposition 1 we obtain

$$m = |N(w_1) \cap N(w_2)| \geq 1 + |N(u_0) \setminus (N(w_1) \cup N(w_2))| \geq k + 1. \tag{2}$$

Furthermore, since $N(w_1) \cap N(w_2) \subseteq N(w_1) = N(w_1) \setminus (N(u_0) \cup N(u_2))$ then $k = |N(u_0) \cap N(u_2)| \geq 1 + |N(w_1) \setminus (N(u_0) \cup N(u_2))| \geq 1 + m$, which contradicts (2). Hence $w_i w_j \in E(G)$ for some pair i, j. Then there is an $x - y$ path $P_{l+2}^{\Delta} = u_0 w_i w_j u_2 \cdots u_l$ with $\Delta = x w_i w_j x$. ∎

Proposition 3. Let $G \in L$ and $xy \in E(G)$. Then there exists an $x - y$ path P_n^{Δ} where $4 \leq n \leq 6$.

Proof. Two cases are possible.

Case 1. *xy does not lie on a triangle.*

Since G is 3-connected we have $d(x) \geq 3$. Let $u_1 x \in E(G)$ and $u_1 \neq y$. Since $d(u_1, y) = 2$ and $|N(y) \cap N(u_1)| \geq 2$ there exists a vertex $u_2 \in N(u_1) \cap N(y), u_2 \neq x$. Consider a path $P = u_0 u_1 u_2 u_3$ where $u_0 = x$ and $u_3 = y$. Clearly, $u_0 u_2, u_1 u_3 \notin E(G), d(u_0, u_2) = 2$ and $u_0 u_3 \in E(G)$. Now we can prove, by repeating the proof of Proposition 2 with (1) changed to $N(u_0) \cap V(P) = N(u_0) \cap N(u_2) \cap V(P) = \{u_1, u_3\}$, that there exists an $u_0 - u_3$ path P_5^{Δ}. Consequently there exists an $x - y$ path P_5^{Δ}, because $x = u_0$ and $y = u_3$.

Case 2. xy lies on a triangle $xyzx$.

Since G is 3-connected we have $d(z) \geq 3$. If there is a vertex $u \in N(z) \setminus \{x, y\}$ such that $ux \in E(G)$ or $uy \in E(G)$ then we have an $x - y$ path P_4^{Δ}.

If no such vertex exists then $ux, uy \notin E(G)$ for each vertex $u \in N(z) \setminus \{x, y\}$. Consider a vertex $w \in N(z) \setminus \{x, y\}$. Then $d(w, x) = 2$ and there is a vertex $u_1 \in (N(x) \cap N(w)) \setminus \{z\}$. Consider a path $P = u_0 u_1 u_2 u_3$ where $u_0 = x, u_2 = w, u_3 = z$. Clearly, $y u_3 \in E(G)$ and $y u_1, y u_2, u_0 u_2, u_1 u_3 \notin E(G)$. Using the same arguments as in Case 1 we will obtain that there is an $u_0 - u_3$ path P_5^{Δ}. Since $x = u_0$ and $y u_3 \in E(G)$ then there is an $x - y$ path P_6^{Δ}. ∎

3. MAIN RESULTS

Theorem 5. Let $G \in L$ and x, y be two distinct vertices of G. If there exists an $x - y$ path P_n^{Δ} such that $4 \leq n \leq |V(G)| - 2$ then there exists an $x - y$ path $P_{n+t}^{\Delta_1}$ where $1 \leq t \leq 2$.

Proof. Since G is connected and $n < |V(G)|$ then $N(P_n^{\Delta}) \neq \emptyset$. For each $v \in N(P_n^{\Delta})$ we denote by W_v the set $N(v) \cap V(P_n^{\Delta})$. Let $U_1 = \{v \in N(P_n^{\Delta}) / |W_v| = 1\}$ and $U_2 = \{v \in N(P_n^{\Delta}) / |W_v| \geq 2 \text{ and } W_v \setminus \{x, y\} \neq \emptyset\}$.

Suppose there does not exist an $x - y$ path $P_{n+t}^{\Delta_1}$, where $1 \leq t \leq 2$. Then the following properties hold.

Property 1. $vw^+ \notin E(G)$ for each $v \in N(P_n^{\Delta})$ and each $w \in W_v \setminus \{y\}$.

Property 2. If $v \in U_1, W_v = \{w\}$ and $w \notin \{x, y\}$ then the set $T(P_n^{\Delta})$ contains the unique triangle $w^- w w^+ w^-$.

Proof. Let $a_1 a_2 a_3 a_1$ be a triangle from the set $T(P_n^{\Delta})$. Suppose $a_2 \neq w$. Since $d(v, w^-) = 2 = d(v, w^+)$ then, by Corollary 1, there exist vertices v_1 and v_2 such that $v_1 \in (N(v) \cap$

$N(w^-)) \setminus V(P_n^\Delta)$ and $v_2 \in (N(v) \cap N(w^+)) \setminus V(P_n^\Delta)$. This gives an $x - y$ path

$$P_{n+2}^{\Delta_1} = \begin{cases} x\vec{P}_n^\Delta w^- v_1 v w \vec{P}_n^\Delta y & \text{if } a_2 \in w^+ \vec{P}_n^\Delta y \\ x\vec{P}_n^\Delta w v v_2 w^+ \vec{P}_n^\Delta y & \text{if } a_2 \in x\vec{P}_n^\Delta w^- \end{cases}$$

with $\Delta_1 = a_1a_2a_3a_1$ such that $V(P_n^\Delta) \subset V(P_{n+2}^{\Delta_1})$, a contradiction. ∎

Property 3. $U_2 \neq \emptyset$.

Proof. Since G is 3-connected then there exists a vertex $v \in N(P_n^\Delta)$ such that $W_v \setminus \{x, y\} \neq \emptyset$. Let $w \in W_v \setminus \{x, y\}$. If $v \notin U_2$ then $v \in U_1$ and, by Property 2, $w^-ww^+w^-$ is the unique triangle in the set $T(P_n^\Delta)$. Since $d(v, w^+) = 2, |W_v| = 1$ and $|N(v) \cap N(w^+)| \geq 3$ then there is a vertex $u \in (N(v) \cap N(w^+)) \setminus V(P_n^\Delta)$. By Property 2, $u \notin U_1$. Therefore $u \in U_2$. ∎

Property 4. Let $v \in U_2$ and Q be a subset of the set $W_v = \{w_1, \ldots, w_p\}$ such that $y \notin Q$. Then

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \geq \sum_{w_i \in Q} (|N(w_i) \setminus (N(v) \cup N(w_i^+))| + 1). \tag{3}$$

Furthermore, if $a_1a_2a_3a_1$ is a triangle from the set $T(P_n^\Delta)$ with $\{a_1, a_2\} \cap Q = \emptyset$ then

$$N(v) \cap N(w_i^+) \subseteq W_v \qquad \text{for each } w_i \in Q \tag{4}$$

and

$$w_i^+ w_j^+ \notin E(G) \text{ for each pair of vertices } w_i, w_j \in Q. \tag{5}$$

Proof. Clearly, (3) follows from Proposition 1. If (4) does not hold for some $w_i \in Q$ then there is a vertex $v_1 \in (N(v) \cap N(w_i^+)) \setminus W_v$ and an $x - y$ path $P_{n+2}^{\Delta_1} = x\vec{P}_n^\Delta w_i v v_1 w_i^+ \vec{P}_n^\Delta y$ with $\Delta_1 = a_1a_2a_3a_1$, a contradiction. So (4) holds. If (5) does not hold then $w_i^+ w_j^+ \in E(G)$ for some pair of vertices $w_i, w_j \in Q$ where $i < j$. Then there is an $x - y$ path $P_{n+1}^{\Delta_1} = x\vec{P}_n^\Delta w_i v w_j \overleftarrow{P}_n^\Delta w_i^+ w_j^+ \vec{P}_n^\Delta y$ with

$$\Delta_1 = \begin{cases} a_1a_2a_3a_1 & \text{if } a_1 \notin w_i^+ \vec{P}_n^\Delta w_j \\ a_3a_2a_1a_3 & \text{otherwise.} \end{cases}$$

a contradiction. So (5) holds. ∎

Property 5. Let $a_1a_2a_3a_1$ be a triangle from the set $T(P_n^\Delta)$. Then $\{a_1, a_2\} \cap W_v \neq \emptyset \neq \{a_2, a_3\} \cap W_v$ for each vertex $v \in U_2$.

Proof. Suppose that $\{a_1, a_2\} \cap W_v = \emptyset$ and let $w_1, \ldots, w_p$ denote the vertices of W_v occurring on P_n^Δ in the order of their indices. Set $Q = \{w_1, \ldots, w_{p-1}\}$. Then, by Property 4, we have (3), (4), and (5). Since w_p can be adjacent to each vertex w_i^+ then

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \leq \varepsilon(Q, Q^+) + p - 1. \tag{6}$$

Furthermore,

$$\sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq \varepsilon(Q, Q^+) + p - 1 \tag{7}$$

since $v \notin Q^+$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \ldots, p-1$. Clearly, (7) is equivalent to

$$\sum_{w_i \in Q} (|N(w_i) \setminus (N(v) \cup N(w_i^+))| + 1) \geq \varepsilon(Q, Q^+) + 2(p-1). \tag{8}$$

But (6) and (8) contradict (3). So $\{a_1, a_2\} \cap W_v \neq \emptyset$.

We can prove $\{a_3, a_2\} \cap W_v \neq \emptyset$ by considering the path $\overleftarrow{P}_n^{\Delta}$ and the triangle $a_3a_2a_1a_3$ and using the above arguments. ∎

Property 6. $|W_v| \geq 3$ for each vertex $v \in U_2$.

Proof. Let $\Delta = a_1a_2a_3a_1$ be a triangle from the set $T(P_n^{\Delta})$. Suppose $W_v = \{w_1, w_2\}$ for some $v \in U_2$ where w_1 and w_2 occur on P_n^{Δ} in the order of their indices. Since $v \in U_2$ then $W_v \setminus \{x, y\} \neq \emptyset$. W.l.o.g. we assume $w_2 \neq y$. Then there is $r \in \{1, 2\}$ such that $w_r^+ \notin \{a_1, a_2, a_3\}$. Since $d(v, w_r^+) = 2$ then $|N(v) \cap N(w_r^+)| \geq 3$ and there exists a vertex $v_1 \in (N(v) \cap N(w_r^+)) \setminus W_v$ together with an $x - y$ path $P_{n+2}^{\Delta} = x\vec{P}_n^{\Delta}w_rvv_1w_r^+\vec{P}_n^{\Delta}y$, a contradiction. So $|W_v| \geq 3$ for each $v \in U_2$. ∎

Property 7. Let $v \in U_2$. Then $a_2 \in W_v$ for each triangle $a_1a_2a_3a_1$ from the set $T(P_n^{\Delta})$.

Proof. Let $w_1, \ldots, w_p$ denote vertices of W_v occurring on P_n^{Δ} in the order of their indices. By Property 6, $p \geq 3$. Suppose $a_2 \notin W_v$ for some triangle $a_1a_2a_3a_1$ from the set $T(P_n^{\Delta})$. Then, by Property 5, $a_1 = w_k, a_3 = w_{k+1}$ and $a_2 = w_k^+ = w_{k+1}^-$ for some $w_k \in W_v$. W.l.o.g. we assume $k < p - 1$. (Otherwise we will consider the path $\overleftarrow{P}_n^{\Delta}$.) Clearly $w_{k+1}^-w_{k+1}^+ \notin E(G)$. Set $Q = W_v \setminus \{w_k, w_p\}$. Then, by Property 4, we have (3), (4), and (5). Since the vertices w_k and w_p can be adjacent to each vertex $w_i^+ \in Q^+$ we have

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \leq \varepsilon(Q, Q^+) + 2(p-2). \tag{9}$$

Furthermore,

$$\sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq \varepsilon(Q, Q^+) + p - 1 \tag{10}$$

because $w_{k+1}^- \notin Q^+, w_{k+1}^- \in N(w_{k+1}) \setminus (N(v) \cup N(w_{k+1}^+))$ and $v \notin Q^+, v \in N(w_i) \setminus (N(w_i^+) \cup N(v))$ for each $w_i \in Q$. Clearly, (10) is equivalent to

$$\sum_{w_i \in Q} (|N(w_i) \setminus (N(v) \cup N(w_i^+))| + 1) \geq \varepsilon(Q, Q^+) + 2(p-2) + 1. \tag{11}$$

But (9) and (11) together contradict (3). ∎

Property 8. Let $v \in U_2$ and $w_1, \ldots, w_p$ denote vertices of W_v occurring on P_n^{Δ} in the order of their indices. Then $w_i^-w_i^+ \in E(G)$ for each $i = 2, \ldots, p-1$.

Proof. Let $\Delta = a_1a_2a_3a_1$ be a triangle from the set $T(P_n^\Delta)$. Then, by Property 7, $a_2 = w_r$ for some $r, 1 \leq r \leq p$. W.l.o.g. we assume $r \leq p-1$. (Otherwise we will consider the path $\overleftarrow{P}_n^\Delta$.) Let us show that

$$\text{if } k < p-1 \text{ and } w_k^- w_k^+ \in E(G) \text{ then } w_{k+1}^- w_{k+1}^+ \in E(G). \tag{12}$$

Set $Q = W_v \setminus \{w_k, w_p\}$. If $w_{k+1}^- w_{k+1}^+ \notin E(G)$ then, by repeating the arguments in the proof of Property 7, we obtain (3), (4), (5), (9), and (11). But (9) and (11) contradict (3). So, $w_i^- w_i^+ \in E(G)$ for each $i, r \leq i \leq p-1$. If $r > 2$ then we will consider the path $\overleftarrow{P}_n^\Delta$. Using the above arguments we obtain $w_i^- w_i^+ \in E(G)$ for each $i, 2 \leq i \leq r-1$.

Now using the above properties we will obtain a contradiction. Let $v \in U_2$ and $w_1, \ldots, w_p$ be vertices of W_v occurring on P_n^Δ in the order of their indices. By Property 8, $w_i^- w_i^+ \in E(G)$ for each $i = 2, \ldots, p-1$. Clearly,

$$d(w_1^+, v) = 2, N(v) \cap N(w_1^+) \subseteq W_v \quad \text{and} \quad |N(v) \cap N(w_1^+)| \geq 3. \tag{13}$$

Hence there is a vertex $w_m \in W_v$ which is adjacent to w_1^+. If $p \geq 4$ then there is an $x-y$ path $\vec{P}_{n+1}^{\Delta_1} = x\vec{P}_n^\Delta w_1 v w_m w_1^+ \vec{P}_n^\Delta w_m^- w_m^+ \vec{P}_n^\Delta y$ with

$$\Delta_1 = \begin{cases} w_2^- w_2 w_2^+ w_2^- & \text{if } m > 2 \\ w_3^- w_3 w_3^+ w_3^- & \text{if } m = 2 \end{cases}$$

a contradiction. So, $p = 3$. From (13) we obtain

$$|N(v) \cap N(w_1^+)| = 3 \quad \text{and} \quad w_1^+ w_i \in E(G) \qquad \text{for } i = 1, 2, 3. \tag{14}$$

Since G is connected and $n \leq |V(G)| - 2$ there is a vertex $u \in N(P_n^\Delta) \setminus \{v\}$. Using Properties 2 and 7 with the vertex u and the triangle $w_2^- w_2 w_2^+ w_2^-$ we obtain $w_2 u \in E(G)$. Clearly, $uv \notin E(G)$. (Otherwise there is an $x-y$ path

$$P_{n+2}^{\Delta_1} = x\vec{P}_n^\Delta w_1 v u w_2 w_1^+ \vec{P}_n^\Delta w_2^- w_2^+ \vec{P}_n^\Delta y$$

with $\Delta_1 = vuw_2v$, a contradiction.) Furthermore, $w_1^+ u \notin E(G)$. (Otherwise there is an $x-y$ path $P_{n+2}^{\Delta_1} = x\vec{P}_n^\Delta w_1 v w_2 u w_1^+ \vec{P}_n^\Delta w_2^- w_2^+ \vec{P}_n^\Delta y$ with $\Delta_1 = w_2 u w_1^+ w_2$, a contradiction.) So, $w_2 \in N(w_1^+) \cap N(v)$ and $u, v, w_2^+ \in N(w_2) \setminus (N(v) \cup N(w_1^+))$. Hence, by Proposition 1, we obtain $|N(v) \cap N(w_1^+)| \geq 4$, which contradicts (14). The proof of Theorem 5 is complete. ∎

Theorem 6. Let $G \in L$. Then, for each edge $xy \in E(G)$ and for each integer, $n, 3 \leq n \leq |V(G)|$, (except maybe one $n \in \{3, 4\}$) there is a cycle of length n containing xy.

Proof. Let $xy \in E(G)$. Since xy lies on a triangle or on a quadrangle (see proof of Proposition 3) it is sufficient to prove that there exists an $x-y$ path P_n for each $n, 5 \leq n \leq |V(G)|$. By Proposition 3 there exists an $x-y$ path P_s^Δ where $4 \leq s \leq 6$. Hence there also exists an $x-y$ path P_{s-1}. Suppose there exist an $x-y$ path P_i for each $i, s-1 \leq i \leq n-1$, and an $x-y$ path P_n^Δ, where $s \leq n \leq |V(G)|-1$.

If $n \leq |V(G)| - 2$ then, by Theorem 5, there exists an $x-y$ path $P_{n+t}^{\Delta_1}$ where $1 \leq t \leq 2$. If $t = 2$ and $\Delta_1 = w^- w w^+ w^-$ then we can obtain an $x-y$ path P_{n+1} by deleting the vertex w from $P_{n+2}^{\Delta_1}$.

Suppose now that $n = |V(G)| - 1$ and let v be the unique vertex outside P_n^Δ. Let $w_1, \ldots, w_p$ be the vertices of W_v occurring on P_n^Δ in the order of their indices. Since G is 3-connected we have $p \geq 3$. If $w_i^+ = w_{i+1}$ for some $i, 1 \leq i \leq p-1$, then there is a Hamilton $x-y$ path. Let $w_i^+ \neq w_{i+1}$ for each $i = 1, \ldots, p-1$. Set $Q = W_v \setminus \{y\}$. Clearly (3) holds. Let us show $w_i^+ w_j^+ \in E(G)$ for some $w_i, w_j \in Q$. Clearly $N(v) \cap N(w_i^+) \subseteq W_v$ for each $w_i \in Q$. If $w_i^+ w_j^+ \notin E(G)$ for each pair of vertices $w_i, w_j \in Q$ then (6), (7), and (8) hold. But (6) and (8) contradict (3). So $w_i^+ w_j^+ \in E(G)$ for some $w_i, w_j \in E(G)$ where $i < j$. Then there is a Hamilton $x-y$ path $P_{n+1} = x\vec{P}_n^\Delta w_i v w_j \overleftarrow{P}_n^\Delta w_i^+ w_j^+ \vec{P}_n^\Delta y$.

Repetition of our argument shows that there is an $x-y$ path P_n for each $n, s \leq n \leq |V(G)|$. This proves the theorem because $4 \leq s \leq 6$. ∎

Using Proposition 2 instead of Proposition 3 and the same arguments as in the proof of Theorem 6 we can prove the following.

Theorem 7. Let $G \in L$ and x, y be two distinct vertices of G with $d(x,y) \geq 2$. Then for each $n, d(x,y)+1 \leq n \leq |V(G)|$, there exists an $x-y$ path P_n.

Clearly, Theorems 6 and 7 imply Theorem 3. Moreover, from Theorem 6 and Theorem 7 we can obtain the following.

Theorem 8. A graph $G \in L$ is panconnected (and also edge pancyclic) if and only if every edge of G lies in a triangle and a quadrangle.

Corollary 2. A graph G satisfying the condition of Theorem 1 is panconnected if and only if each edge of G lies in a triangle and a quadrangle.

It is not difficult to check that in every graph satisfying the condition of Theorem 2 each edge lies on a triangle and a quadrangle. So, Theorem 2 follows from Corollary 2.

Corollary 3. Let G be a connected r-regular graph of order at least 4 where $|N(u) \cup N(v) \cup N(w)| \leq 2r - 1$ for any path uwv with $uv \notin E(G)$. Then G is panconnected unless $r = 2n$ and $G = \bar{K}_{2n-1} \vee nK_2$ where nK_2 denote the union of n disjoint copies of K_2.

Proof. If each edge of G lies in a triangle and a quadrangle then, by Theorem 8, G is panconnected. Now suppose that an edge $e = xy$ does not lie in a triangle or a quadrangle. Let $N(x) = \{y, v_1, \ldots, v_{r-1}\}$. If $N(x) \cap N(y) = \emptyset$ then $|N(y) \cup N(v_1) \cup N(x)| \geq 2r$ because G is r-regular, a contradiction.

So $N(x) \cap N(y) \neq \emptyset$. Without loss of generality we assume that $yv_1 \in E(G)$. Since xy lies in the triangle xyv_1x then, by our assumption, xy does not lie in a quadrangle. Hence $v_1 v_i \notin E(G)$ for each $i = 2, \ldots, r-1$. Let $N(v_1) = \{x, y, u_1, \ldots, u_{r-2}\}$. Since $|N(x) \cup N(v_i) \cup N(v_1)| \leq 2r - 1$ and $\{x, y, u_1, \ldots, u_{r-2}, v_1, \ldots, v_{r-1}\} \subseteq N(x) \cup N(v_i) \cup N(v_1)$ then $|N(x) \cup N(v_i) \cup N(v_1)| = 2r - 1$ for each $i = 2, \ldots, r-1$. This implies that $N(v_i) = \{x, y, u_1, \ldots, u_{r-2}\}$ for each $i = 2, \ldots, r-1$ and $N(y) = \{x, v_1, \ldots, v_{r-1}\}$. If $N(u_j) \setminus \{u_1, \ldots, u_{r-2}, v_1, \ldots, v_{r-1}\} \neq \emptyset$ for some $j, 1 \leq j \leq r-2$, then $|N(u_j) \cup N(v_1) \cup N(x)| \geq 2r$, a contradiction. So, $N(u_j) \subseteq \{u_1, \ldots, u_{r-2}, v_1, \ldots, v_{r-1}\}$ for each $j = 1, \ldots, r-2$. Since G is r-regular we deduce that $r-2$ is an even number and the subgraph induced by the set $\{u_1, \ldots, u_{r-2}\}$ is a 1-factor. So, $r = 2n$ and $G = \bar{K}_{2n-1} \vee nK_2$. ∎

Let, for each vertex w of a graph G, $M_2(w)$ denote the set of vertices v with $d(w,v) \leq 2$.

Corollary 4. Let G be a connected r-regular graph of order at least 4 where $|M_2(w)| \leq 2r - 1$ for each $w \in V(G)$. Then G is panconnected unless $r = 2n$ and $G = \bar{K}_{2n-1} \vee nK_2$.

Proof. Let uwv be a path of G with $uv \notin E(G)$. Clearly, $N(u) \cup N(v) \cup N(w) \subseteq M_2(w)$. Hence, $|M_2(w)| \leq 2r - 1$ implies $|N(u) \cup N(v) \cup N(w)| \leq 2r - 1$. Therefore, by Corollary 3, G is panconnected. ∎

References

[1] A. S. Asratian, O. A. Ambartsumian, and G. V. Sarkisian, Some local conditions for the hamiltonicity and pancyclicity of a graph, *Docl. Acad. Nauk Armenian SSR* **91** (1990), 19–22 (Russian).

[2] A. S. Asratian and G. V. Sarkisian, Some panconnected and pancyclic properties of graphs with a local Ore-type condition, *Graphs and Combinat.,* in press.

[3] A. S. Asratian and G. V. Sarkisian, On cyclic properties of some hamiltonian graphs, *Diskretnaja Matemat.* **3** (1991), 91–104. (Russian.)

[4] K. A. Berman, Proof of a conjecture of Häggkvist on cycles and independent edges, *Discrete Math.* **46** (1983), 9–13.

[5] J. A. Bondy, Pancyclic graphs I, *J. Combin. Theory B* **11** (1971), 80–84.

[6] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* MacMillan, London and Elsevier, New York (1976).

[7] Cai Xiao-tao, A short proof for the Faudree-Schelp theorem on path connected graphs, *J. Graph Theory* **8** (1984), 109–110.

[8] R. J. Faudree and R. H. Schelp, Path connected graphs, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 313–319.

[9] R. Häggkvist, On F-Hamiltonian graphs, in: (eds. J. A. Bondy and U. S. R. Murty, *Graph Theory and Related Topics,* Academic Press, New York (1979), 219–231.

[10] R. Häggkvist, R. J. Faudree, and R. H. Schelp, Pancyclic graphs—connected Ramsey number, *Ars Combinatoria* **11** (1981), 37–49.

[11] A. S. Hasratian and N. K. Khachatrian, Some localization theorems on hamiltonian circuits, *J. Combin. Theory B* **49** (1990), 287–294.

[12] O. Ore, Hamilton connected graphs, *J. Math. Pures Appl.* **42** (1963), 21–27.

[13] J. E. Williamson, Panconnected graphs II, Period. *Math. Hungar.* **8** (1977), 105–116.

[14] Zhang Ke Min and D. A. Holton, On edge-pancyclic graphs, *Soochow J. Math.* **19** (1) (1993), 37–41.

[15] Zhang Ke Min, D. A. Holton, and Sheng Bau, On generalized vertex-pancyclic graphs, *Chinese J. Math.* **21** (1) (1993), 91–98.

Received August 2, 1994