Perturbations of Markov Chains

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with applications to models of DNA damage and repair

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Abstract

This thesis is concerned with studying the hitting time of an absorbing state on Markov chain models that have a countable state space. For many models it is challenging to study the hitting time directly; I present a perturbative approach that allows one to uniformly bound the difference between the hitting time moment generating functions of two Markov chains in a neighbourhood of the origin. I demonstrate how this result can be applied to both discrete and continuous time Markov chains.

The motivation for this work came from the field of biology, namely DNA damage and repair. Biophysicists have highlighted that the repair process can lead to Double Strand Breaks; due to the serious nature of such an eventuality it is important to understand the hitting time of this event. There is a phase transition in the model that I consider. In the regime of parameters where the process reaches quasi-stationarity before being absorbed I am able to apply my perturbative technique in order to further understand this hitting time.

Declaration

The work in this thesis is based on research carried out in the Probability and Statistics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

I would like to thank my supervisor Ostap Hryniv for his guidance through my graduate studies; his clear interest in my research and in my development as a young researcher is very evident and I know that without his support this project would not have been possible. Additionally I would like to thank the Engineering and Physical Sciences Research Council for providing financial support.

To say the last four years have been challenging is a significant understatement; I count myself very fortunate to have such supportive friends who provided much needed encouragement during the low points of PhD life. In particular I am most grateful for my Christian brothers and sisters who continued to point me to Jesus in whom I have ultimate fulfilment.

This thesis is dedicated to my parents who have supported me unconditionally throughout the last twenty six years.

Contents

Chapter 1

Introduction

This thesis is a contribution to the study of Markov chains. This area of mathematical research began early in the 20th century; Andrey Markov proved the first Markov chain results in his 1906 paper and in due course these processes were named after him[1]. Informally a Markov chain is a process whose future behaviour, given the present state, is independent of the past (this is the Markov property). Markov chains are a popular choice in modelling because many physical systems come close to satisfying the Markov property, in that how they evolve has very little dependence on the past history of the system. Of special interest in applications are Markov chains which exhibit the separation of scales property. These are processes whose state space is separated into regions where transitions between different regions take a long time but when the transition does occur it happens quickly. The transition time (hitting time of one region starting from a neighbouring one) usually has a distribution which is approximately exponential on the appropriate scale, this is due to the fact that there are typically many failed attempts to move from one region to another until the process finally succeeds. Such transition times play a crucial role in understanding many models in physics, chemistry, biology and computer science to name just a few.

My contribution to the field is to have developed a perturbative approach that one can use to compare the hitting time distributions of two Markov chain models,

[1]The interested reader can find a full account of his work in [1].

that are in some sense close to one another. The method I use is based upon the comparison of moment generating functions of the rescaled hitting times, and as a result provides a good control over the closeness of hitting time distributions. In this chapter I will introduce the motivating example that sparked my interest in this area, I will then state the model that I decided to study and finally I state the results that I will prove in this thesis.

1.1 Motivating example

DNA is a long molecule consisting of repeating blocks called bases; bases frequently become damaged[2] but evolution has developed a number of repair mechanisms to restore the DNA code. Based on biological evidence, the authors in [16] suggested that due to the very nature of many repair mechanisms, two such processes running in close vicinity can result in a double strand break which is often fatal to cells. Taking this into account, they introduced a stochastic model to study the occurence of double strand breaks, but then replaced it with a deterministic system of linear differential equations which were obtained in the limit of a continuous space/positive defect density approximation; they then studied the stationary behaviour of this deterministic system. In addition they run a Markov chain Monte Carlo simulation of the original stochastic model.

The analysis in [16] appears to be only applicable to high intensities of damages, which results in a positive fraction of bases being damaged at any given time. Under normal everyday conditions this fraction is much smaller, hence the need for careful investigation of the phenomenon. In this thesis I develop a perurtubative approach for studying moment generating functions of hitting times of Markov chains; I illustrate its power by conducting a rigorous probabilistic analysis of a version of the original stochastic model from [16] for the values of parameters where their assumption does not hold.

[2]A couple of sources of DNA damage are described in [4]; the frequency of DNA damage is startling, the authors of [6] state that every cell in our body experiences $2 \cdot 10^4 - 10^5$ lesions per day! However this is actually very small when compared to the length of the human genome.

1.2 Ring Model

In this section I will formally introduce the DNA damage and repair model that I will study, I will refer to this model as the Ring Model. I think of a DNA string as a sequence of labels from the set $\{0, 1, 2\}$, where 0, 1 and 2 correspond to a undamaged, damaged and critically damaged bases respectively. Let D_{N} be a DNA string with $\mathsf{N} \in \mathbb{N}$ bases that form a closed loop[3]. Also let $\lambda > 0$ and $\mu > 0$ be parameters that control the damage rate and repair rate respectively; furthermore I require that λ and μ are chosen such that $\lambda/\mu \in \mathbb{N}$.

Convention 1.2.1. *Throughout this thesis, in any model that involves parameters* λ, μ *and* N, *I always require that* $\lambda > 0$, $\mu > 0$, $\lambda/\mu \in \mathbb{N}$ *and* $\mathsf{N} \in \mathbb{N}$.

Define a Markov chain $(X_t)_{t \geq 0}$ on the configuration space $S = \{0, 1, 2\}^{D_{\mathsf{N}}}$ such that individual bases in D_{N} evolve independently with rates[4]

$$\begin{aligned} 0 \to 1 &\qquad \text{rate } \lambda/\mathsf{N} \\ 1 \to 0 &\qquad \text{rate } \mu \\ 1 \to 2 &\qquad \text{rate } \lambda/\mathsf{N} \end{aligned}$$

and the initial state, X_0, is the empty configuration, i.e. where all bases are undamaged. Fix a constant $l \in \mathbb{N}$ and call a configuration critically damaged if there is a critically damaged base or there are two damaged bases within distance $2l$ of each other (this is clarified in Example 1.2.2). Define the stopping time T to be the first moment, starting from a non-critically damaged configuration with λ/μ damaged bases[5], that the chain hits a critically damaged configuration. My aim is to study the distribution of T as a function of N, λ, μ and l.

Example 1.2.2. *Consider Figure 1.1 where we zoom in to a piece of DNA and consider the state of* 11 *base pairs. For this example I let* $l = 2$. *The label* 0,

[3] My approach can be applied to both closed or open DNA strings; it is known that for some strains of *E.coli* the DNA forms a closed loop.

[4] Rate λ/N means that the time taken for the transition is exponentially distributed with parameter λ/N.

[5] My results hold for any non-critically damaged starting position with λ/μ damaged bases.

1 or 2 indicates a particular base is undamaged, damaged or critically damaged respectively. The first configuration is a critically damaged configuration because

0	0	2	0	0	0	0	0	0	0	0

0	0	1	0	0	0	1	0	0	0	0

0	0	1	0	0	0	0	1	0	0	0

Figure 1.1

there is a critically damaged base present in the system. The second configuration is a critically damaged configuration because there are two damaged base pairs within distance $2l$ of each other. The third configuration is 'alive' (i.e. not a critically damaged configuration) because there are no critically damaged bases and there is a gap of length at least $2l$ between damaged bases.

1.3 Phase transition

The distribution of T is dependent on whether or not the process reaches quasi-stationarity before hitting a critically damaged configuration. In the case where the process does reach quasi-stationarity, the typical number[6] of damaged bases (before absorption) is λ/μ.

If n points are placed uniformly at random on a unit circle, then the minimal gap size between any two points is of order $1/n^2$ (see page 327 in [7]). Consequently, if one places λ/μ points uniformly at random on a circle of length N then the minimal gap size between any two points will be of order $\frac{\mathsf{N}}{(\lambda/\mu)^2}$. The smallest gap

[6] There are N bases and each gets damaged with rate λ/N so roughly speaking the number of damaged bases increases by 1 with rate λ. Also, each damaged base is repaired with rate μ, therefore if there are k damaged bases then the number of damaged bases decreases by 1 with rate μk. This well known birth death chain has a very concentrated stationary measure about the state λ/μ (see equation (3.8)).

is important because it is this quantity that determines whether the process is in a critically damaged configuration or not.

The critical number of bases is $2l$ (see section 1.2) and so depending whether

$$\frac{\mathsf{N}}{(\lambda/\mu)^2} > 2l \qquad \text{or} \qquad \frac{\mathsf{N}}{(\lambda/\mu)^2} < 2l$$

will significantly impact the survival time distribution. In the former case, the process will typically reach stationarity and survive for a long time being reaching a critically damaged configuration; it is this regime of parameters that I consider in my thesis. Moreover, in this region of the parameter space, a deterministic approximation is not valid and it is therefore important that I carry out a rigorous stochastic analysis of the model.

1.4 Main results

The first result I am able to prove is that one can stochastically sandwich the hitting time T between the survival time of two much simpler Markov chains as follows. Fix a small postitive p and let $\mathsf{N}_p = \lceil 1/p \rceil \in \mathbb{N}$. Also, as in the Ring Model, choose $\lambda > 0$ and $\mu > 0$ such that $\lambda/\mu \in \mathbb{N}$. Define the state space $S' = \{*, 0, 1, 2, \ldots, \mathsf{N}_p\}$, where a number corresponds to the number of damaged bases and the starred state represents a "critically damaged configuration". Consider a Markov chain $(Y_t)_{t\geq 0}$ on S' that evolves with jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } \mu k \\ k &\to k+1 && \text{rate } \lambda(1-d_k) \\ k &\to * && \text{rate } \lambda d_k \end{aligned} \tag{1.1}$$

where

$$d_k = \begin{cases} pk & \text{if } k < \mathsf{N}_p \\ 1 & \text{if } k = \mathsf{N}_p \end{cases}.$$

I will refer to this model as the Projected Model and solely for the purposes of Figure 1.2 I define $\lambda_k = \lambda(1 - d_k)$. In picture form this process looks like:

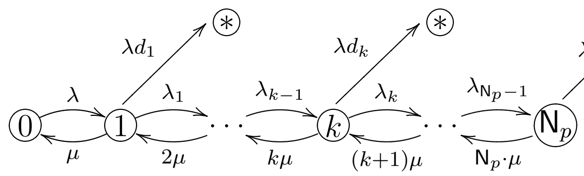

Figure 1.2: Markov Chain Y_t - Projected Model

Definition 1.4.1. *I define two copies of* Y_t *namely* $Y_t^{'}$ *and* $Y_t^{''}$*, with rates as described in equation* (1.1)*, where* $p = \frac{2l+1}{\mathsf{N}}$ *and* $p = \frac{4l+1}{\mathsf{N}}$ *respectively. Also let* $Y_0^{'} = Y_0^{''} = \lambda/\mu$*. Finally, define* T' *and* T'' *to be the hitting times of a starred state for models* $Y_t^{'}$ *and* $Y_t^{''}$ *respectively.*

Theorem 1.4.2. *Let* T*,* T' *and* T'' *be as defined in Section 1.2 and Defintion 1.4.1. We have the following stochastic ordering*

$$T^{''} \preceq T \preceq T^{'} .$$

The proof can be found in Appendix A. In particular this result allows me to estimate the tails of T from above and below, by the appropriate hitting times of processes which are significantly simpler to simulate from. Moreover, in conjunction with Theorem 1.4.4, I deduce that T is bounded above and below by random variables that have exponentially decaying tails.

The result that I will spend most of my thesis proving concerns the limiting behaviour of T' and T'', in fact more interesting than the result itself is the technique I have employed to prove the result which makes use of the intrinsic renewal structure of Y_t.

Convention 1.4.3. *Throughout this thesis, I use* $\mathcal{O}(\cdot)$ *notation to indicate that an expression is uniformly bounded by a constant that does not depend on any of the variables in the argument.*

Theorem 1.4.4. *Fix constants* $0 < \rho_0 < 2/7$ *and* $\kappa_2^* > 0$. *For any* $0 < \bar{v} < 0.5$ *there exists* $\kappa_1^* = \kappa_1^*(\bar{v}) > 0$ *and* $\kappa_3^* = \kappa_3^*(\bar{v}) > 0$ *such that for any* v *that satisfies* $|v| < \bar{v}$, *if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1^*, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{0.2\rho_0}\right] < \kappa_2^* \quad \text{and} \quad \frac{\mu}{\lambda} < \kappa_3^*,$$

then

$$\left|\mathsf{M}_{T'}\left(\frac{2\lambda^2 v}{\mu}\cdot\frac{2l+1}{\mathsf{N}}\right) - \frac{1}{1-2v}\right| = \mathcal{O}\left(\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{2+0.5\rho_0}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-0.25\rho_0)}\right)$$

and

$$\left|\mathsf{M}_{T''}\left(\frac{2\lambda^2 v}{\mu}\cdot\frac{4l+1}{\mathsf{N}}\right) - \frac{1}{1-2v}\right| = \mathcal{O}\left(\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{2+0.5\rho_0}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-0.25\rho_0)}\right).$$

where the implicit constant in the big O in both of the statements is dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*$, $\bar{v}$ *and* l. $\mathsf{M}_{T'}(\cdot)$ *and* $\mathsf{M}_{T''}(\cdot)$ *are the moment generating functions of* T' *and* T'' *respectively.*

Remark 1.4.5. *Notice that the error term in Theorem 1.4.4 is negligible when* λ/μ *is large compared to 1, but small when compared to* $\mathsf{N}^{\frac{1}{2+0.5\rho_0}}$. *Moreover, if* $\rho_0 \to 0$ *then the error terms in Theorem 1.4.4 are optimised, however this does come at a cost; for small* ρ_0 *one needs to take* λ/μ *larger in order to satisfy the condition:*

$$\mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{0.2\rho_0}\right] < \kappa_2^*$$

Recall that $(1-2v)^{-1}$ is the moment generating function of an exponentially distributed random variable with parameter $1/2$. The control we have over the moment generating function in Theorem 1.4.4 when the conditions in Remark 1.4.5 are satisfied is very useful. Not only does it prove that the probability distribution of both T' and T'' (properly scaled) are concentrated near that of a $\mathsf{Exp}(1/2)$ random variable, one can also use the theorem to deduce other useful results, for example large deviation estimates.

Rather than working on the microscopic scale (on the level of individual jumps), I have worked on a mesoscopic scale (on the level of excursions from the typical state λ/μ to the state λ/μ). This has led to a set of criteria on the level of excursions that are required to hold in order to prove the result, this is very general and can be applied to other models where there is a state that is visited frequently.

1.5 Overview of chapters

In Chapter 2 I state and prove a number of key results that I will rely on in later chapters. This includes making use of martingale technology to prove results for the simple symmetric random walk and also taking the well known relation between the expected return time of a state in a Markov chain and its stationary distribution and deriving new relationships that are similar in spirit.

In Chapter 3 I prove results about the discrete time Markov chain $(Z_i)_{i\geq 0}$ on the state space $\{0, 1, 2, \ldots\}$ that evolves with jump probabilities:

$$\mathsf{P}(Z_{i+1} = z_{i+1} | Z_i = z_i) = \begin{cases} \frac{\lambda}{\lambda + \mu z_i}\,, & \text{if } z_{i+1} = z_i + 1 \\ \frac{\mu z_i}{\lambda + \mu z_i}\,, & \text{if } z_{i+1} = z_i - 1 \\ 0\,, & \text{otherwise} \end{cases}$$

This Markov chain underlies many of my models.

In Chapters 4 and 5 I introduce and study Markov chains with an absorbing state which play a crucial role in later chapters by serving as an approximation to more advanced models. The most important model from these chapters is the Discrete Time Constant Killing model, which is introduced at the start of Chapter 5. The key result for this model is Theorem 5.4.4 which can be found on page 54.

In Chapter 6 I introduce the perturbative technique I have developed (see Theorem 6.3.1 on page 59), which is very useful in studying Markov chains which exibit metastable behaviour. I apply this technique to specific discrete and continuous time Markov chains in Chapters 7, 8 and 9, via a series of pairwise comparisons; the main result in each of these chapters is stated in the final section of the chapter. A summary of the models I consider can be found below in Figure 1.3.

In Chapter 10 I link my research back to the motivating example in this chapter by proving Theorem 1.4.4; I also discuss possible directions for future research in this area. Finally in Appendix A I provide a proof of Theorem 1.4.2.

[7] This is the primary notation associated to these models; there is other notation and this is introduced in the relevant chapters.

[8] For models where the holding time is described as $\mathrm{Exp}(\cdot)$, this means that the holding time is exponentially distributed with the respective parameter.

Model name	Notation[7]	Killing probability at state k	Holding time[8] at state k	Chapters in which the model features
Discrete Time Constant Killing	A_d, B_d, T_d	$d = \frac{\lambda}{\mu \mathsf{N}}$	1	5,7
Discrete Time Linear Killing	$A_\epsilon, B_\epsilon, T_\epsilon$	$d_k = \frac{k}{\mathsf{N}}$	1	7,8
Continuous Time Linear Killing	A_d^+, B_d^+, T_d^+	$d_k = \frac{k}{\mathsf{N}}$	$\text{Exp}(2\lambda)$	8,9
Continuous Time Linear Holding	$A_\epsilon^+, B_\epsilon^+, T_\epsilon^+$	$d_k = \frac{k}{\mathsf{N}}$	$\text{Exp}(\lambda + \mu k)$	9,10
Projected Model	T', T''	See section 1.4	See section 1.4	1,10, Appendix A
Ring Model	T	See section 1.2	See section 1.2	1, Appendix A

Figure 1.3: Summary of the main models that appear throughout my thesis.

Chapter 2

Preliminary material: part 1

This chapter contains a variety of results that are required in later chapters. I will reuse the constants C and α throughout the chapter, please do not assume any dependence between the constants that pop up in the different results unless I explicitly make reference to this being the case.

2.1 Simple symmetric random walk results

Let X be a Bernoulli random variable with probability mass function

$$\mathsf{P}(X = x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0 & \text{otherwise} \end{cases}$$

and let $X_1, X_2, \ldots$ be independent copies of X. Also define $S_n = S_0 + \sum_{i=1}^{n} X_i$ for some constant $S_0 \in \mathbb{Z}$. The simple symmetric random walk (SSRW), which is defined by $(S_n)_{n\geq 0}$, is undoubtedly one of the simplest random processes and yet plays an important role in my thesis. Throughout this section, $(S_n)_{n\geq 0}$, will retain the above meaning and as such I will not redefine it in the results that follow. I will start by introducting a number of martingales for the SSRW.

Lemma 2.1.1. *Let $T = \inf\big\{m > 0 : S_m \in \{0, n\}\big\}$ with $0 < S_0 < n$. We have the following martingales with respect to the natural filtration (generated by S_m):*

i. If $M_m = S_m^2 - m$ then $M_{m\wedge T}$ is a martingale.

ii. If $M_m = S_m^4 - 6mS_m^2 + 3m^2 + 2m$ *then* $M_{m\wedge T}$ *is a martingale.*

Proof of Lemma 2.1.1(i). Firstly, using the fact that X_{m+1} is independent of S_m, it follows that

$$\begin{aligned}\mathsf{E}(M_{m+1} - M_m|S_m) &= \mathsf{E}(S_{m+1}^2 - S_m^2 + m - (m+1)|S_m) \\ &= \mathsf{E}((S_m + X_{m+1})^2 - S_m^2 - 1|S_m) \\ &= \mathsf{E}(2S_m X_{m+1} + X_{m+1}^2 - 1|S_m) \\ {}^1 &= 0\end{aligned}$$

Secondly

$$\mathsf{E}(M_{m\wedge T}) \leq \mathsf{E}(|M_{m\wedge T}|) \leq n^2 + \mathsf{E}(T) < \infty$$

because the Markov chain is irreducible and has a finite state space. Consequently $M_{m\wedge T}$ is a martingale.

□

Proof of Lemma 2.1.1(ii). Firstly, using the fact that X_{m+1} is independent of S_m, it follows that

$$\begin{aligned}\mathsf{E}(M_{m+1} - M_m|S_m) = \mathsf{E}(&(S_m + X_{m+1})^4 - S_m^4 + 6mS_m^2 - 6(m+1)(S_m + X_{m+1})^2 \\ &+ 3(m+1)^2 - 3m^2 + 2(m+1) - 2m|S_m) \\ {}^2 = \mathsf{E}(&6S_m^2 X_{m+1}^2 + X_{m+1}^4 - 6S_m^2 - 6X_{m+1}^2 - 6mX_{m+1}^2 \\ &+ 6m + 3 + 2|S_m) \\ {}^3 = \mathsf{E}(&6S_m^2 + 1 - 6S_m^2 - 6 - 6m + 6m + 5|S_m) = 0\end{aligned}$$

Secondly

$$\mathsf{E}(M_{m\wedge T}) \leq \mathsf{E}(|M_{m\wedge T}|) \leq \mathsf{E}(n^4 + 6Tn^2 + 3T^2 + 2T) \leq 12n^4 \cdot \mathsf{E}(T^2) < \infty$$

because the Markov chain is irreducible and has a finite state space. Consequently $M_{m\wedge T}$ is a martingale.

□

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[1] Uses $\mathsf{E}(X_{m+1}) = 0$ and $\mathsf{E}(X_{m+1}^2) = 1$

[2] Uses $\mathsf{E}(X_{m+1}) = \mathsf{E}(X_{m+1}^3) = 0$

[3] Uses $\mathsf{E}(X_{m+1}^2) = \mathsf{E}(X_{m+1}^4) = 1$

The above martingales can be used to prove the following boundary hitting time estimates

Lemma 2.1.2. *Let* $T = \inf\big\{m > 0 : S_m \in \{0, n\}\big\}$*. We have the following results*

i. If $0 \le S_0 \le n$ *then* $\mathsf{P}(S_T = n) = S_0/n$*.*

ii. If $0 \le S_0 \le n$ *then* $\mathsf{E}(T) = S_0(n - S_0)$*.*

iii. If $S_0 = 1$ *then* $\mathsf{E}(T^2) = \mathcal{O}(n^3)$*.*

iv. $\mathsf{P}(T > n^2) \le 0.25$ *uniformly in* $0 \le S_0 \le n$*.*

Proof of Lemma 2.1.2(i). Applying the Optional Stopping Theorem and Dominated Convergence Theorem to the maringale $S_{m\wedge T}$ we have

$$S_0 = \lim_{m\to\infty} \mathsf{E}(S_{m\wedge T}) = \mathsf{E}(\lim_{m\to\infty} S_{m\wedge T}) = \mathsf{E}(S_T) = 0 \cdot \mathsf{P}(S_T = 0) + n \cdot \mathsf{P}(S_T = n)$$

Consequently $\mathsf{P}(S_T = n) = S_0/n$ and $\mathsf{P}(S_T = 0) = 1 - S_0/n$. □

Proof of Lemma 2.1.2(ii). Applying the Optional Stopping Theorem and Dominated Convergence Theorem to the maringale $M_{m\wedge T} = S^2_{m\wedge T} - m \wedge T$ we have

$$S_0^2 = M_0 = \lim_{m\to\infty} \mathsf{E}(M_{m\wedge T}) = \mathsf{E}(\lim_{m\to\infty} M_{m\wedge T}) = \mathsf{E}(M_T)$$

But since

$$\mathsf{E}(M_T) = \mathsf{E}(S_T^2) - \mathsf{E}(T) = n^2\mathsf{P}(S_T = n) - \mathsf{E}(T) = nS_0 - \mathsf{E}(T)$$

where the last equality uses Lemma 2.1.2(i), it follows that $\mathsf{E}(T) = S_0(n - S_0)$. □

Proof of Lemma 2.1.2(iii). Applying the Optional Stopping Theorem and Dominated Convergence Theorem to the maringale

$$M_{m\wedge T} = S^4_{m\wedge T} - 6(m \wedge T)S^2_{m\wedge T} + 3(m \wedge T)^2 + 2(m \wedge T)$$

we have

$$1 = S_0^4 = M_0 = \lim_{m\to\infty} \mathsf{E}(M_{m\wedge T}) = \mathsf{E}(\lim_{m\to\infty} M_{m\wedge T}) = \mathsf{E}(M_T)$$

Combining this with the fact that

$$\mathsf{E}(M_T) = \mathsf{E}(S_T^4 - 6TS_T^2 + 3T^2 + 2T)$$

gives

$$3\mathsf{E}(T^2) \le 1 + \mathsf{E}(6TS_T^2) \le 1 + 6n^2\mathsf{E}(T) = \mathcal{O}(n^3)$$

where the last equality applies Lemma 2.1.2(ii) with $S_0 = 1$. □

Proof of Lemma 2.1.2(iv). I start by applying Markov's inequality to $\mathsf{P}(T > n^2)$ and then making use of Lemma 2.1.2(ii):

$$\mathsf{P}(T > n^2) \le \frac{\mathsf{E}(T)}{n^2} = \frac{S_0(n - S_0)}{n^2}$$

This function of S_0 is largest when $S_0 = 0.5n$

$$\le \frac{0.5n(n - 0.5n)}{n^2} = 0.25$$

□

A slightly more unusual question is one which asks about the hitting time of a particular boundary. Start a SSRW inbetween two absorbing boundary points, what can be said about the moment generating function of the hitting time of a particular boundary on the event that this boundary is the first to be reached (i.e. ignore trajectories that hit the opposite boundary first)? I can prove the following result:

Lemma 2.1.3. *Let $T = \inf\{m \ge 0 : S_m \in \{0, n\}\}$. For any $0 < \alpha \le 0.5$ we have the following results*

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\alpha}{n^2} \cdot T\right)\mathbb{1}_{\{S_T=0\}}\right] \le \frac{1}{n} \cdot \exp\left(\frac{1}{4}\right)$$

and

$$\mathsf{E}_1\left[\exp\left(\frac{\alpha}{n^2} \cdot T\right)\mathbb{1}_{\{S_T=0\}}\right] \le \exp\left(-\frac{1}{2n}\right)$$

Convention 2.1.4. *The subscript next to E defines the starting position of the process.*

Proof. It suffices to prove the lemma for $\alpha = 0.5$. Define

$$g_m = \mathsf{E}_m \left[\exp\left(\beta \cdot T\right) \mathbb{1}_{\{S_T=0\}}\right] \qquad 0 \leq m < n \qquad \beta > 0$$

The g_m quantities satisfy the following system of equations:

$$g_0 = 1 \qquad g_n = 0 \qquad g_m = \gamma \cdot (g_{m-1} + g_{m+1}) \qquad 0 < m < n$$

where $\gamma = \frac{1}{2}\exp\left(\beta\right)$. A simple backward induction shows that

$$g_{n-m} = \varphi_m(0) \cdot g_{n-m-1} \qquad 0 < m < n$$

where $\varphi_m(0)$ denotes the mth iteration of the function $\varphi(x) = \frac{\gamma}{1-\gamma x}$:

$$\varphi_1(x) = \varphi(x) = \frac{\gamma}{1-\gamma x} \qquad \varphi_m(x) = \varphi(\varphi_{m-1}(x)) \qquad 0 < m < n$$

I claim that if $0 < \beta \leq \frac{1}{2n^2}$, then $\varphi_k(0) \leq \frac{k}{k+1} \cdot \exp\left(k\beta\right)$ for all k satisfying $0 < k < n$ (see Lemma 2.1.5 below). Using this result and Lemma 2.3.3 it follows that if $\beta = \frac{1}{2n^2}$ then:

$$g_1 = \varphi_{n-1}(0) \leq \frac{n-1}{(n-1)+1}\exp\left(\frac{n-1}{2n^2}\right) \leq \exp\left(-\frac{1}{n}\right) \cdot \exp\left(\frac{1}{2n}\right) = \exp\left(-\frac{1}{2n}\right)$$

and

$$g_{n-1} = \prod_{k=1}^{n-1} \varphi_k(0) \leq \frac{1}{n}\exp\left(\frac{1}{2n^2} \cdot \frac{1}{2}n(n-1)\right) \leq \frac{1}{n} \cdot \exp\left(\frac{1}{4}\right)$$

This completes the proof. □

Lemma 2.1.5. *If* $0 < \beta \leq \frac{1}{2n^2}$*, then* $\varphi_k(0) \leq \frac{k}{k+1} \cdot e^{k\beta}$ *for all* k *satisfying* $0 < k < n$

Proof. I will prove this result using induction. Noting that the case $k = 1$ is trivial, we assume that $\varphi_k(0) \leq \frac{k}{k+1} \cdot e^{k\beta}$. Then

$$\varphi_{k+1}(0) = \frac{e^{\beta}}{2 - e^{\beta}\varphi_k(0)} \leq \frac{e^{\beta}}{2 - \frac{k}{k+1}e^{\beta(k+1)}} = \frac{(k+1)e^{\beta}}{(k+2) - k(e^{\beta(k+1)} - 1)}$$

$$= \frac{(k+1)e^{\beta}}{k+2} \cdot \frac{1}{1 - \frac{k}{k+2}(e^{\beta(k+1)} - 1)}$$

and so to complete the inductive step it remains to show

$$\left(1 - \frac{k}{k+2}(e^{\beta(k+1)} - 1)\right)^{-1} \leq e^{k\beta}$$

It is straightforward:

$$\left(1 - \frac{k}{k+2}(e^{\beta(k+1)} - 1)\right)^{-1} \leq \left(1 - \frac{k}{k+2} \cdot \frac{(k+1)\beta}{1-(k+1)\beta}\right)^{-1}$$
$$= 1 + \frac{k(k+1)\beta}{k+2-2(k+1)^2\beta}$$
$$= 1 + k\beta \cdot \frac{k+1}{(k+1)+(1-2(k+1)^2\beta)} \leq 1 + k\beta < e^{k\beta}$$

where the first inequality follows from the Lemma 2.3.3 and the second inequality follows from the assumptions[1] of the lemma.

□

Corollary 2.1.6. *By symmetry, the statement in Lemma 2.1.3 is equivalent to the following*

$$\mathsf{E}_1\left[\exp\left(\frac{\alpha}{n^2} \cdot T\right)\mathbb{1}_{\{S_T = n\}}\right] \leq \frac{1}{n} \cdot \exp\left(\frac{1}{4}\right)$$

and

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\alpha}{n^2} \cdot T\right)\mathbb{1}_{\{S_T = n\}}\right] \leq \exp\left(-\frac{1}{2n}\right)$$

I will now state a moderate deviations estimate for the binomial distribution that can be found in [20]:

Lemma 2.1.7. *Let $R_m \sim Bin(m,p)$ be a binomially distributed random variable with parameters m and p and assume c_m is a positive sequence satisfying*

$$c_m \to \infty \quad \textit{and} \quad \frac{c_m}{\sqrt{m}} \to 0 \quad \textit{as } m \to \infty$$

Then for any $x > 0$, we have

$$\lim_{m\to\infty} \frac{1}{c_m^2} \cdot \log\left[\mathsf{P}\left(\left|\frac{R_m - mp}{c_m\sqrt{mp(1-p)}}\right| \geq x\right)\right] = -\frac{x^2}{2}$$

Finally I am in a position to prove a SSRW moderate deviations result:

[1]the conditions on β and k for which the lemma is valid imply that $1 - 2(k+1)^2\beta \geq 0$.

Lemma 2.1.8. *If* $S_0 = 0$ *then there exists constants* $C > 0$ *and* $\gamma > 0$ *such that*

$$\mathsf{P}\left(\max_{i=1,\dots,m} |S_i| > m^{0.75}\right) \leq C \exp\left(-\gamma\sqrt{m}\right)$$

Proof. I start by noting the Etemadi Lemma (Lemma 16.8 from [15]) which gives us the following result

$$\mathsf{P}\left(\max_{i=1,\dots,m} |S_i| > m^{0.75}\right) \leq C_1 \cdot \mathsf{P}(|S_m| > 0.5 \cdot m^{0.75}) \qquad \text{for all } m \in \mathbb{N} \tag{2.1}$$

where C_1 is a constant. This simplifies matters considerably because I now only need to consider the end point of the random walk.

In order that $|S_m| > 0.5 \cdot m^{0.75}$ I require the number of jumps to the right to be bigger than $0.5(m + 0.5 \cdot m^{0.75})$ or less than $0.5(m - 0.5 \cdot m^{0.75})$. The number of jumps to the right is binomially distributed with parameters m and 0.5 and so for convenience I define $R_m \sim \text{Bin}(m, 0.5)$, it follows that

$$\begin{aligned}\mathsf{P}(|S_m| > 0.5 \cdot m^{0.75}) &= \mathsf{P}(R_m > 0.5(m + 0.5 \cdot m^{0.75})) + \mathsf{P}(R_m < 0.5(m - 0.5 \cdot m^{0.75})) \\ &= \mathsf{P}\left(\left|\frac{R_m - 0.5m}{0.25m^{0.75}}\right| \geq 1\right) \qquad (2.2)\end{aligned}$$

I now apply Lemma 2.1.7 with $c_m = 0.5m^{0.25}$ and $x = 1$ to deduce that

$$\lim_{m\to\infty} \frac{4}{\sqrt{m}} \cdot \log\left[\mathsf{P}\left(\left|\frac{R_m - 0.5m}{0.25m^{0.75}}\right| \geq 1\right)\right] = -\frac{1}{2}$$

I now choose an α satisfying $0 < \alpha < \frac{1}{8}$ and there exists $m_0 \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{m}} \cdot \log\left[\mathsf{P}\left(\left|\frac{R_m - 0.5m}{0.25m^{0.75}}\right| \geq 1\right)\right] + \frac{1}{8} \leq \alpha \qquad \text{for all } m > m_0\,.$$

Equivalently

$$\mathsf{P}\left(\left|\frac{R_m - 0.5m}{0.25m^{0.75}}\right| \geq 1\right) \leq \exp\left[\left(\alpha - \frac{1}{8}\right)\sqrt{m}\right] \qquad \text{for all } m > m_0\,,$$

and there exists a constant $C_2 > 0$ such that

$$\mathsf{P}\left(\left|\frac{R_m - 0.5m}{0.25m^{0.75}}\right| \geq 1\right) \leq C_2 \cdot \exp\left[\left(\alpha - \frac{1}{8}\right)\sqrt{m}\right] \qquad \text{for all } m \in \mathbb{N}\,. \tag{2.3}$$

Finally by pulling together equations (2.1), (2.2) and (2.3) we reach the result

$$\mathsf{P}\left(\max_{i=1,\dots,m} |S_i| > m^{0.75}\right) \leq C_1 \cdot C_2 \cdot \exp\left[-\left(\frac{1}{8} - \alpha\right)\sqrt{m}\right].$$

This is the statement of the lemma with $C = C_1 \cdot C_2$ and $\gamma = \frac{1}{8} - \alpha$. ☐

2.2 General Markov chain results

Consider an irreducible and positive recurrent Markov chain, for such a chain there is a well known link between the stationary measure and the expected return time of a state. If one defines B_x to be the first return time for state x and π_x to be the stationary measure of the same state then it holds that

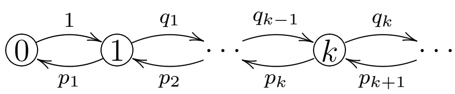
$$\mathsf{E}(B_x) = \frac{1}{\pi_x}$$

Using this result I have derived another compact forumla for the expected return time. This alternative expression proves very useful in Chapter 5.

Define $(U_i)_{i\geq 0}$ to be a positive recurrent Markov chain on the state space $\{0, 1, 2, \dots\}$ evolving according to jump probabilities ($p_k + q_k = 1$ for all $k \in \mathbb{N}$):

$$\mathsf{P}(U_{i+1} = u_{i+1} | U_i = u_i) = \begin{cases} 1 & \text{if } u_{i+1} = 1 \text{ and } u_i = 0 \\ p_{u_i} & \text{if } u_{i+1} = u_i - 1 \text{ and } u_i > 0 \\ q_{u_i} & \text{if } u_{i+1} = u_i + 1 \text{ and } u_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

In picture form it looks like



Lemma 2.2.1. *Consider $(U_i)_{i\geq 0}$ and define $r_k^{\rightarrow}$ to be the total weight of all finite step trajectories starting at the origin and terminating at state k without any intermediate returns to the origin. Also define $\mathsf{E}(\vec{B})$ to be the expected return time of the origin for this Markov chain. Then*

$$\frac{\mathsf{E}(\vec{B})}{2} = 1 + \sum_{k=1}^{\infty} q_k r_k^{\rightarrow}$$

Proof. The following recurrence relation exists between the $r_k^{\rightarrow}$ terms:

$$r_0^{\rightarrow} = p_1 r_1^{\rightarrow}, \qquad r_1^{\rightarrow} = 1 + p_2 r_2^{\rightarrow}, \qquad r_k^{\rightarrow} = q_{k-1} r_{k-1}^{\rightarrow} + p_{k+1} r_{k+1}^{\rightarrow} \quad \text{for } k \geq 2$$

Moreover, the chain is positive recurrent and therefore will return to the origin with probability 1 and so $r_0^{\rightarrow} = 1$. From these equations we can deduce that

$$r_k^{\rightarrow} = \begin{cases} 1 & \text{if } k = 0 \\ \frac{1}{p_k} \prod_{i=1}^{k-1} \frac{q_i}{p_i} & \text{if } k \geq 1 \end{cases}$$

Moreover by solving the detailed balance equations we find that the stationary distribution is

$$\pi_k = \begin{cases} \pi_0 & \text{if } k = 0 \\ \pi_0 \cdot \frac{1}{p_k} \prod_{i=1}^{k-1} \frac{q_i}{p_i} & \text{if } k \geq 1 \end{cases}$$

By straightforward comparison of these displays we see that $r_k^{\rightarrow}$ is proportional to π_k and consequently the $r_k^{\rightarrow}$ terms satisfy the detailed balance equations. Therefore

$$r_0^{\rightarrow} = 1\,, \qquad r_1^{\rightarrow} = 1 + q_1 r_1^{\rightarrow}\,, \qquad r_k^{\rightarrow} = q_{k-1} r_{k-1}^{\rightarrow} + q_k r_k^{\rightarrow} \quad \text{for } k \geq 2$$

Therefore

$$\sum_{k=0}^{\infty} r_k^{\rightarrow} = r_0^{\rightarrow} + r_1^{\rightarrow} + \sum_{k=2}^{\infty} r_k^{\rightarrow} = r_0^{\rightarrow} + r_1^{\rightarrow} + q_1 r_1^{\rightarrow} + 2 \sum_{k=2}^{\infty} q_k r_k^{\rightarrow} = 2 + 2 \sum_{k=1}^{\infty} q_k r_k^{\rightarrow} \quad (2.4)$$

Finally from the relationship between $r_k^{\rightarrow}$ and π_k we also see that

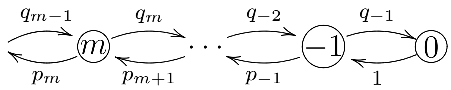
$$\sum_{k=0}^{\infty} r_k^{\rightarrow} = \sum_{k=0}^{\infty} \frac{\pi_k}{\pi_0} = \frac{1}{\pi_0} = \mathsf{E}(\vec{B}) \quad (2.5)$$

Pulling together equations (2.4) and (2.5) gives the desired result. □

The following result considers exactly the same Markov chain as above, but on the negative state space. Define $(V_i)_{i \geq 0}$ to be a positive recurrent Markov chain on the state space $\{0, -1, -2, \dots\}$ evolving according to jump probabilities ($p_{-k} + q_{-k} = 1$ for all $k \in \mathbb{N}$):

$$\mathsf{P}(V_{i+1} = v_{i+1} | V_i = v_i) = \begin{cases} 1 & \text{if } v_{i+1} = -1 \text{ and } v_i = 0 \\ p_{v_i} & \text{if } v_{i+1} = v_i - 1 \text{ and } v_i < 0 \\ q_{v_i} & \text{if } v_{i+1} = v_i + 1 \text{ and } v_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

In picture form it looks like



where m is a negative integer.

Lemma 2.2.2. *Consider $(V_i)_{i\geq 0}$ and define $r_k^{\leftarrow}$ to be the total weight of all finite step trajectories starting at the origin and terminating at state k without any intermediate returns to the origin. Also define $\mathsf{E}(\overleftarrow{B})$ to be the expected return time of the origin for this Markov chain. Then*

$$\frac{\mathsf{E}(\overleftarrow{B})}{2} = \sum_{k=-1}^{-\infty} q_k r_k^{\leftarrow}$$

Remark 2.2.3. *This result is not just the symmetrical image of the previous result, the expression I derive for the expected return time is different.*

Proof. The following recurrence relation exists between the $r_k^{\leftarrow}$ terms:

$$r_0^{\leftarrow} = q_{-1} r_{-1}^{\leftarrow}, \qquad r_{-1}^{\leftarrow} = 1 + q_{-2} r_{-2}^{\leftarrow}, \qquad r_k^{\leftarrow} = q_{k-1} r_{k-1}^{\leftarrow} + p_{k+1} r_{k+1}^{\leftarrow} \quad \text{for } k \leq -2$$

and again the chain is positive recurrent, therefore $r_0^{\leftarrow} = 1$. From these equations we can deduce that

$$r_k^{\leftarrow} = \begin{cases} 1 & \text{if } k = 0 \\ \frac{1}{q_k} \prod_{i=-1}^{k+1} \frac{p_i}{q_i} & \text{if } k \leq -1 \end{cases}$$

Again one can show that $r_k^{\leftarrow} = \frac{\pi_k}{\pi_0}$ and consequently the $r_k^{\leftarrow}$ terms satisfy the detailed balance equations. Therefore

$$r_0^{\leftarrow} = 1 = q_{-1} r_{-1}^{\leftarrow}, \qquad r_k^{\leftarrow} = q_{k-1} r_{k-1}^{\leftarrow} + q_k r_k^{\leftarrow} \quad \text{for } k \leq -1$$

Therefore

$$\sum_{k=0}^{-\infty} r_k^{\leftarrow} = r_0^{\leftarrow} + \sum_{k=-1}^{-\infty} r_k^{\leftarrow} = r_0^{\leftarrow} + q_{-1} r_{-1}^{\leftarrow} + 2\sum_{k=-2}^{-\infty} q_k r_k^{\leftarrow} = 2q_{-1} r_{-1}^{\leftarrow} + 2\sum_{k=-2}^{-\infty} q_k r_k^{\leftarrow}$$

$$= 2\sum_{k=-1}^{-\infty} q_k r_k^{\leftarrow} \tag{2.6}$$

Finally from the relationship between $r_k^{\leftarrow}$ and π_k we also see that

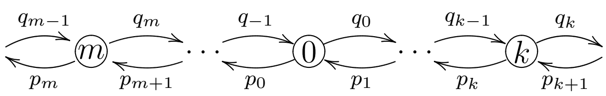
$$\sum_{k=0}^{-\infty} r_k^{\leftarrow} = \sum_{k=0}^{-\infty} \frac{\pi_k}{\pi_0} = \frac{1}{\pi_0} = \mathsf{E}(\overleftarrow{B}) \tag{2.7}$$

Pulling together equations (2.6) and (2.7) gives the desired result. □

I now consider the Markov chain that is formed by joining the above two chains together. The only adjustment I make is to the jump probabilities at the origin. Define $(W_i)_{i\geq 0}$ to be a positive recurrent Markov chain on the state space $\{\dots,-1,0,1,\dots\}$ evolving according to jump probabilities ($p_k + q_k = 1$ for all $k \in \mathbb{Z}$):

$$\mathsf{P}(W_{i+1} = w_{i+1}|W_i = w_i) = \begin{cases} p_{w_i} & \text{if } w_{i+1} = w_i - 1 \\ q_{w_i} & \text{if } w_{i+1} = w_i + 1 \\ 0 & \text{otherwise} \end{cases}$$

In picture form it looks like



Corollary 2.2.4. *Consider $(W_i)_{i\geq 0}$ and define r_k to be the total weight of all finite step trajectories starting at the origin and terminating at state k without any intermediate returns to the origin. Also define $\mathsf{E}(B)$ to be the expected return time of the origin for this Markov chain. It follows that*

$$\frac{\mathsf{E}(B)}{2} = q_0 + \sum_{k=1}^{\infty} (q_k r_k + q_{-k} r_{-k})$$

Proof of Corollary 2.2.4. First of all I apply the Theorem of Total Probability and then I use Lemma 2.2.1 and Lemma 2.2.2:

$$\begin{aligned}\frac{\mathsf{E}(B)}{2} &= q_0 \cdot \frac{\mathsf{E}(\overrightarrow{B})}{2} + p_0 \cdot \frac{\mathsf{E}(\overleftarrow{B})}{2} \\ &= q_0 + q_0 \sum_{k=1}^{\infty} q_k r_k^{\rightarrow} + p_0 \sum_{k=-1}^{-\infty} q_k r_k^{\leftarrow}\end{aligned}$$

By definition of $r_k^{\leftarrow}, r_k^{\rightarrow}$ and r_k it follows that $p_0 \cdot r_k^{\leftarrow} = r_k$ for negative k and $q_0 \cdot r_k^{\rightarrow} = r_k$ for positive k, therefore

$$= q_0 + \sum_{k=1}^{\infty} q_k r_k + \sum_{k=-1}^{-\infty} q_k r_k$$

which completes the proof. □

Finally I present two further identities that relate to moments of B:

Lemma 2.2.5. *Consider the Markov chain $(W_i)_{i \geq 0}$ and define a_m to be the probability that a length m trajectory does not return to the origin. It follows that*

$$\sum_{m=1}^{\infty} a_m = \mathsf{E}(B) - 1 \tag{2.8}$$

and

$$\sum_{m=1}^{\infty} m \cdot a_m = \frac{1}{2}[\mathsf{E}(B^2) - \mathsf{E}(B)] \,. \tag{2.9}$$

Remark 2.2.6. *The lemma still makes sense if $\mathsf{E}(B^2) = \infty$ because the summation on the left hand side of equation* (2.9) *would be infinite too.*

Proof of Lemma 2.2.5. For any fixed k it follows that

$$\sum_{m=1}^{\infty} m^k \cdot a_m = \sum_{m=1}^{\infty} m^k \cdot \mathsf{P}(B > m) = \sum_{m=1}^{\infty} \sum_{n>m} m^k \cdot \mathsf{P}(B = n) = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} m^k \cdot \mathsf{P}(B = n)$$

By substituting $k = 0$ and $k = 1$ into the above equation we reach the desired results:

$$\sum_{m=1}^{\infty} a_m = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \mathsf{P}(B = n) = \sum_{n=1}^{\infty} (n-1) \cdot \mathsf{P}(B = n) = \mathsf{E}(B) - 1$$

$$\sum_{m=1}^{\infty} m \cdot a_m = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} m \cdot \mathsf{P}(B = n) = \sum_{n=1}^{\infty} \frac{1}{2} n(n-1) \cdot \mathsf{P}(B = n) = \frac{1}{2}[\mathsf{E}(B^2) - \mathsf{E}(B)]$$

□

Convention 2.2.7. *When I use these results in later chapters, the hitting time B will have exponential moments, consequently the moment generating function of B is analytic at the origin. As such I will write $\mathsf{E}(B)$ and $\mathsf{E}(B^2)$ as the appropriate derivative of the moment generating function evaluated at* 0*:*

$$\mathsf{E}(B) = \mathsf{M}_B'(0) \quad \textit{and} \quad \mathsf{E}(B^2) = \mathsf{M}_B''(0) \,.$$

2.3 Other results

In this section I will present a selection of standard inequalities that I will refer to throughout this thesis.

Lemma 2.3.1. *If* $x \in \mathbb{R}$ *then* $1 + x \leq e^x$.

Proof. Define $f(x) = e^x - 1 - x$ and note that $f(0) = 0$. By differentiating, we see that $f(x)$ is an increasing function for all $x > 0$ and a decreasing function for all $x < 0$. Consequently $f(x) \geq 0$ for all $x \in \mathbb{R}$. □

Lemma 2.3.2. *If* $x \geq 0$ *then* $e^x - 1 \leq xe^x$.

Proof. Define $f(x) = e^x - 1 - xe^x$ and note that $f(0) = 0$. By differentiating, we see that $f(x)$ is a decreasing function for all $x > 0$. Consequently $f(x) \leq 0$ for all $x \geq 0$. □

Lemma 2.3.3. *If* $x \geq 0$ *then* $1 - x \leq e^{-x}$*. If additionally* $x < 1$ *then* $e^x \leq \frac{1}{1-x}$.

Proof. I will start by proving the first statement. Define $f(x) = 1 - e^{-x} - x$ and note that $f(0) = 0$. By differentiating, we see that $f(x)$ is a decreasing function for all $x > 0$. Consequently $f(x) \leq 0$ for all $x \geq 0$ which is precisely the first part of the lemma. Adding the condition $x < 1$ ensures that I can divide through both sides of the inequality by $1 - x$ without needing to change the sign. □

Lemma 2.3.4. *If* $x > 0$ *then*

$$\frac{1}{1 - e^{-x}} < 1 + \frac{1}{x} .$$

Proof. Start with the known inequality $e^x > 1+x$ which is equivalent to the following

$$e^{-x} < \frac{1}{1+x} \quad \Longleftrightarrow \quad 1 - e^{-x} > \frac{x}{1+x} \quad \Longleftrightarrow \quad \frac{1}{1 - e^{-x}} < \frac{1+x}{x}$$

□

Lemma 2.3.5. *If* $0 \leq x \leq 0.5$ *then* $1 - x \geq e^{-x-x^2}$.

Proof. Take logs of both sides and define $f(x) = \log(1-x) + x + x^2$. Note that $f(0) = 0$. After differentiating, we find that $f'(x) = x \cdot \frac{1-2x}{1-x}$ which is non-negative for all $0 \leq x \leq 0.5$. Consequently $f(x) \geq 0$ for all $0 \leq x \leq 0.5$ as required. □

Lemma 2.3.6. *If* $|x| \leq 1/2$ *then*

$$\left| \frac{1}{1+x} - 1 \right| \leq 2|x| .$$

Proof.

$$\left| \frac{1}{1+x} - 1 \right| = \frac{|x|}{1-x} \leq 2|x|$$

where the inequality holds if $|x| \leq 1/2$. □

Lemma 2.3.7. *If* $0 \leq x \leq 1/2$ *then* $|\log(1-x) + x| \leq x^2$.

Proof.

$$|\log(1-x) + x| = \left| -\sum_{k=2}^{\infty} \frac{x^k}{k} \right| \leq \frac{1}{2} \sum_{k=2}^{\infty} x^k = \frac{1}{2} \cdot \frac{x^2}{1-x} \leq x^2$$

where the last inequality holds if $0 \leq x \leq 1/2$ □

Lemma 2.3.8. *If* $0 \leq x \leq 1/2$ *then* $0 \leq -\log(1-x) \leq 2x$.

Proof.

$$0 \leq -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \leq \sum_{k=1}^{\infty} x^k = \frac{x}{1-x} \leq 2x$$

where the last inequality holds if $0 \leq x \leq 1/2$. □

Lemma 2.3.9. *For any fixed choice of* $\epsilon > 1$ *we have* $|e^x - 1| \leq \epsilon|x|$ *for all* $x < \log(\epsilon)$.

Proof. The result is trivial for $x = 0$ and I will proceed by considering positive and negative x seperately. If $x > 0$ then I want to show $e^x - 1 < \epsilon x$. To do this define $f(x) = e^x - 1 - \epsilon x$, since $f(0) = 0$ and $f(x)$ is a decreasing function for $x < \log(\epsilon)$ I can conclude that $f(x) < 0$ holds for $0 < x < \log(\epsilon)$. Conversely if $x < 0$ then I want to show $1 - e^x < -\epsilon x$. To do this define $g(x) = 1 - e^x + \epsilon x$. Note that $g(0) = 0$ and $g(x)$ is an increasing function for all $x < 0$ (this uses the condition $\epsilon > 1$). Therefore I can conclude that $g(x) < 0$ holds for all $x < 0$. □

Lemma 2.3.10. *If* $0 \leq x < 1$ *then* $\log(1-x) \leq -x$.

Proof. Define $f(x) = \log(1-x) + x$ and note that $f(0) = 0$. By differentiating, we see that $f(x)$ is a decreasing function for all $0 < x < 1$. Consequently $f(x) \leq 0$ for all $0 \leq x < 1$. ☐

Lemma 2.3.11. *If* $\alpha \geq 0$ *and* $0 \leq x \leq \frac{\log(1+\alpha)}{1+\alpha}$ *then*

$$(1-x)^{-1} \leq e^{(1+\alpha)x} .$$

Proof. Since $0 \leq x < 1$ I can multiply through by $1-x$ without changing the sign of the inequality. As such, in order to prove the lemma, I will prove the following equivalent result

$$(1-x) \geq e^{-(1+\alpha)x} .$$

Define $f(x) = e^{-(1+\alpha)x} + x - 1$ and note that $f(0) = 0$. By differentiating, we see that $f(x)$ is a decreasing function for all $x \leq \frac{\log(1+\alpha)}{1+\alpha}$. Consequently $f(x) \leq 0$ for all x satisfying $0 \leq x \leq \frac{\log(1+\alpha)}{1+\alpha}$. ☐

Lemma 2.3.12. *If* $-1 < \alpha < 1$ *and* $x > 1$ *then*

$$\sum_{m=1}^{\infty} m^{\alpha} \cdot \left(1 - \frac{1}{x}\right)^{m} = \mathcal{O}(x^{1+\alpha})$$

where the implicit constant is dependent on the choice of α.

Proof. Firstly I apply Lemma 2.3.3 and then I decompose the sum as follows

$$\sum_{m=1}^{\infty} m^{\alpha} \cdot \left(1 - \frac{1}{x}\right)^{m} \leq \sum_{m=1}^{x} m^{\alpha} \cdot e^{-m/x} + \sum_{m=x}^{\infty} m^{\alpha} \cdot e^{-m/x} \tag{2.10}$$

Now I will show that each of the above terms can be bounded by a constant multiplied by $x^{1+\alpha}$. For the first summation in equation (2.10) I can bound the sum as follows

$$\sum_{m=1}^{x} m^{\alpha} \cdot e^{-m/x} \leq \sum_{m=1}^{x} m^{\alpha} .$$

One can think of the summation, $\sum_{m=1}^{x} m^{\alpha}$, as summing up the area of a sequence of rectangles; it is then straightforward to see that this can be approximated by $\int_1^x y^{\alpha} \mathrm{d}y$. The approximation can be formally written as an inequality by choosing the limits as shown in the formula below (this caters for m^α being either increasing or decreasing):

$$\sum_{m=1}^{x} m^\alpha \leq \int_0^{x+1} y^\alpha \mathrm{d}y = \frac{1}{1+\alpha} \cdot (1+x)^{1+\alpha} \leq \frac{2^{1+\alpha}}{1+\alpha} \cdot x^{1+\alpha}$$

For the second summation in equation (2.10) I decompose the sum as follows

$$\begin{aligned} \sum_{m=x}^{\infty} m^\alpha \cdot e^{-m/x} &= x^\alpha \cdot e^{-1} + \sum_{m=x+1}^{\infty} m^\alpha \cdot e^{-m/x} && (2.11) \\ &= x^\alpha \cdot e^{-1} + \sum_{m=x+1}^{\infty} \frac{1}{m^{1-\alpha}} \cdot m \cdot e^{-m/x} && (2.12) \\ &\leq x^\alpha \cdot e^{-1} + \frac{1}{x^{1-\alpha}} \cdot \sum_{m=x+1}^{\infty} m \cdot e^{-m/x} && (2.13) \end{aligned}$$

Again, one can take the same approach that I used when bounding $\sum_{m=1}^{x} m^\alpha$ in order to bound $\sum_{m=x+1}^{\infty} m \cdot e^{-m/x}$ because $y \cdot e^{-y/x}$ is a decreasing function for $y \geq x$:

$$\sum_{m=x+1}^{\infty} m \cdot e^{-m/x} \leq \int_x^{\infty} y \cdot e^{-y/x} \mathrm{d}y = 2x^2 e^{-1} \qquad (2.14)$$

From equations (2.13) and (2.14) it follows that

$$\sum_{m=x}^{\infty} m^\alpha \cdot e^{-m/x} \leq 3x^{1+\alpha} e^{-1} \,.$$

□

Lemma 2.3.13. *If* $x \leq 0.5\log(2)$ *and* $m \in \mathbb{N}$ *then*

$$0 \leq \left(\frac{1}{1-x}\right)^m - e^{xm} \leq 2x^2 m \cdot \left(1 + e^{2x(m-1)}\right) \,.$$

Proof. Firstly, it is straightforward to show that if $a > b > 0$ then the following holds

$$a^m - b^m = (a-b) \cdot (a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1}) \leq (a-b) \cdot ma^{m-1} \,.$$

I apply this to $a = (1-x)^{-1}$ and $b = e^x$ and note the following result

$$0 \leq \frac{1}{1-x} - e^x = \sum_{k=2}^{\infty} x^k \left(1 - \frac{1}{k!}\right) \leq \frac{x^2}{1-x} \leq 2x^2 \quad \text{for } x \leq 0.5 \,.$$

Therefore

$$0 \leq \left(\frac{1}{1-x}\right)^m - e^{xm} \leq 2x^2 m \cdot \left(\frac{1}{1-x}\right)^{m-1}$$

It remains to observe that by applying Lemma 2.3.11 then

$$\left(\frac{1}{1-x}\right)^{m-1} \leq \begin{cases} e^{2x(m-1)} & \text{if } 0 \leq x \leq 0.5 \log 2 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Therefore the stated result holds. □

Chapter 3

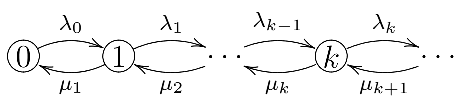
Preliminary material: part 2

3.1 Introduction

In this chapter I will study the Markov chain $(X_i)_{i\geq 0}$ on the state space $\{0, 1, 2, \dots\}$ evolving with jump probabilities:

$$\mathsf{P}(X_{i+1} = x_{i+1} | X_i = x_i) = \begin{cases} \dfrac{\lambda}{\lambda + \mu x_i} & \text{if } x_{i+1} = x_i + 1 \\ \\ \dfrac{\mu x_i}{\lambda + \mu x_i} & \text{if } x_{i+1} = x_i - 1 \end{cases}$$

I will refer to this process as the birth death Markov chain. In picture form it looks like



$$\lambda_k = \frac{\lambda}{\lambda + \mu k} \qquad \mu_k = \frac{\mu k}{\lambda + \mu k}$$

Figure 3.1: Birth death Markov chain

The process can be described as having a single valley, the bottom of which is at state $\frac{\lambda}{\mu}$, and as we move further away from this state the gradient increases. In later chapters I will study processes that are very similar to the birth death chain and I will require estimates on the birth death chain first return time to state $\frac{\lambda}{\mu}$. To this end I make the following definition

$$B = \min\left\{i > 0 : X_i = \frac{\lambda}{\mu}\right\}$$

In this chapter I will prove a number of results concerning the moments of B and the tail probability of B (the results can be found in section 3.3).

Remark 3.1.1. *In addition to my standard assumption that $\lambda/\mu \in \mathbb{N}$, my models in this chapter also rely on λ/μ being a square number.*

3.2 Additional models

I will now introduce two further models, which along with the birth death chain (Figure 3.1), are stochastically ordered (I will define the type of stochastic ordering shortly). I will then state a general Markov chain result which will enable me to state and prove results about B.

3.2.1 Model 1

Let the Markov chain $(Y_t)_{t\geq 0}$ be defined on the state space $\{\dots, -2, -1, 0, 1, 2, \dots\}$ and evolve with jump probabilities:

$$\mathsf{P}(Y_{i+1} = y_{i+1} | Y_i = y_i) = \begin{cases} q_2 & \text{if } y_{i+1} = y_i + 1 \text{ and } y_i \leq \frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}} \\ p_2 & \text{if } y_{i+1} = y_i - 1 \text{ and } y_i \leq \frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}} \\ \frac{1}{2} & \text{if } y_{i+1} = y_i + 1 \text{ and } \frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}} < y_i < \frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}} \\ \frac{1}{2} & \text{if } y_{i+1} = y_i - 1 \text{ and } \frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}} < y_i < \frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}} \\ p_1 & \text{if } y_{i+1} = y_i + 1 \text{ and } y_i \geq \frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}} \\ q_1 & \text{if } y_{i+1} = y_i - 1 \text{ and } y_i \geq \frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}} \end{cases}$$

where

$$p_1 = \frac{\lambda}{\lambda + \mu(\frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}})} = 1 - q_1 < 0.5 \qquad \text{and} \qquad p_2 = \frac{\mu(\frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}})}{\lambda + \mu(\frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}})} = 1 - q_2 < 0.5$$

I require notation for the first return time to state λ/μ so I define

$$B' = \min\left\{i > 0 : Y_i = \frac{\lambda}{\mu}\right\}.$$

In picture form this process looks like

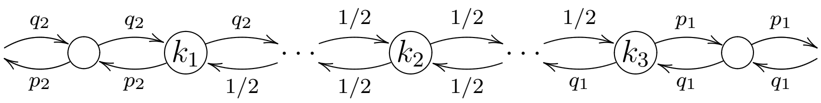


Figure 3.2: $k_1 = \frac{\lambda}{\mu} - \sqrt{\frac{\lambda}{\mu}}$ $\quad k_2 = \frac{\lambda}{\mu}$ $\quad k_3 = \frac{\lambda}{\mu} + \sqrt{\frac{\lambda}{\mu}}$

3.2.2 Model 2

Let the Markov chain $(Z_t)_{t\geq 0}$ be defined on the state space $\{0, 1, 2, \dots\}$ and evolve with jump probabilities:

$$\mathsf{P}(Z_{i+1} = z_{i+1} | Z_i = z_i) = \begin{cases} 1 & \text{if } z_{i+1} = 1 \text{ and } z_i = 0 \\ \frac{1}{2} & \text{if } z_{i+1} = z_i + 1 \text{ and } 0 < z_i < \sqrt{\frac{\lambda}{\mu}} \\ \frac{1}{2} & \text{if } z_{i+1} = z_i - 1 \text{ and } 0 < z_i < \sqrt{\frac{\lambda}{\mu}} \\ p_3 & \text{if } z_{i+1} = z_i + 1 \text{ and } z_i \geq \sqrt{\frac{\lambda}{\mu}} \\ q_3 & \text{if } z_{i+1} = z_i - 1 \text{ and } z_i \geq \sqrt{\frac{\lambda}{\mu}} \end{cases}$$

where $p_3 = \max\{p_1, p_2\} < 0.5$ and $q_3 = \min\{q_1, q_2\} > 0.5$. Again I require notation for the first return time to state 0 so I define

$$B^{''} = \min\{i > 0 : Z_i = 0\} .$$

In picture form this process looks like

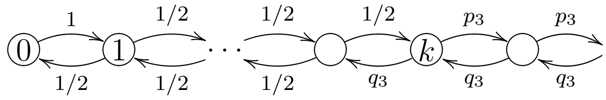


Figure 3.3: $k = \sqrt{\frac{\lambda}{\mu}}$

3.2.3 Results

I will use the following definition of stochastic ordering

Definition 3.2.1. *Let M and N be real valued random variables. I say that N dominates M, and write $M \preccurlyeq N$, if $\mathsf{P}(M > x) \leq \mathsf{P}(N > x)$ for all $x \in \mathbb{R}$.*

Lemma 3.2.2. *Let $X_0 = \lambda/\mu, Y_0 = \lambda/\mu$ and $Z_0 = 0$. We have the following stochastic ordering*

$$B \preccurlyeq B^{'} \preccurlyeq B^{''}$$

Proof. I will prove $B \preccurlyeq B^{'}$ using the maximal coupling. Start one copy of both X_t and Y_t at state λ/μ and run the following one step coupling: if both processes are at different states then run the processes independently for one jump, however if they are both at the same state, say state x, then they move as follows

$$\begin{aligned}\mathsf{P}\left(X_{i+1} = Y_{i+1} = x+1 | X_i = Y_i = x\right)& \\ = \min\left\{\mathsf{P}\left(X_{i+1} = x+1|X_i = x\right), \mathsf{P}\left(Y_{i+1} = x+1|Y_i = x\right)\right\}&\end{aligned}$$

and

$$\begin{aligned}\mathsf{P}\left(X_{i+1} = Y_{i+1} = x-1 | X_i = Y_i = x\right)& \\ = \min\left\{\mathsf{P}\left(X_{i+1} = x-1|X_i = x\right), \mathsf{P}\left(Y_{i+1} = x-1|Y_i = x\right)\right\}&\ .\end{aligned}$$

To ensure that the marginal probabilities match up, if $x \geq \lambda/\mu$ then let

$$\begin{aligned}\mathsf{P}\left(X_{i+1} = x-1, Y_{i+1} = x+1 | X_i = Y_i = x\right)& \\ = \mathsf{P}\left(Y_{i+1} = x+1|Y_i = x\right) - \mathsf{P}\left(X_{i+1} = x+1|X_i = x\right)&\end{aligned}$$

and

$$\mathsf{P}\left(X_{i+1} = x+1, Y_{i+1} = x-1 | X_i = Y_i = x\right) = 0\,.$$

Alternatively if $x \leq \lambda/\mu$ then let

$$\begin{aligned}\mathsf{P}\left(X_{i+1} = x+1, Y_{i+1} = x-1 | X_i = Y_i = x\right)& \\ = \mathsf{P}\left(Y_{i+1} = x-1|Y_i = x\right) - \mathsf{P}\left(X_{i+1} = x-1|X_i = x\right)&\end{aligned}$$

and

$$\mathsf{P}\left(X_{i+1} = x-1, Y_{i+1} = x+1 | X_i = Y_i = x\right) = 0\,.$$

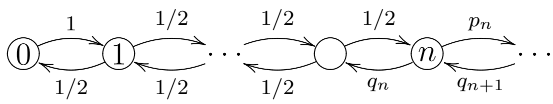
I then repeatedly apply this coupling until one of the processes returns to state λ/μ. A special feature of this coupling is that $|X_t - \lambda/\mu| \le |Y_t - \lambda/\mu|$ for all $t \le \min\{B, B'\}$. Therefore X_t returns to state λ/μ before or at the same time as Y_t with probability one, thus proving that $B \preccurlyeq B'$. A similar coupling can be constructed to prove that $B' \preccurlyeq B''$. □

Finally let me state a general result for a specific class of Markov chains.

Theorem 3.2.3. *Fix $n \in \mathbb{N}$ and let the Markov chain $(W_t)_{t\ge0}$ be defined on the state space $\{0,1,2,\dots\}$ and evolve with jump probabilities:*

$$\mathsf{P}(W_{i+1} = w_{i+1} | W_i = w_i) = \begin{cases} 1 & \textit{if } w_{i+1} = 1 \textit{ and } w_i = 0 \\ \frac{1}{2} & \textit{if } w_{i+1} = w_i + 1 \textit{ and } 0 < w_i < n \\ \frac{1}{2} & \textit{if } w_{i+1} = w_i - 1 \textit{ and } 0 < w_i < n \\ p_{w_i} & \textit{if } w_{i+1} = w_i + 1 \textit{ and } w_i \ge n \\ q_{w_i} & \textit{if } w_{i+1} = w_i - 1 \textit{ and } w_i \ge n \\ 0 & \textit{otherwise} \end{cases}$$

where the p and q terms are chosen such that $\inf_i\{q_i - p_i\} > \epsilon > 0$. In picture form this process looks like



Also define

$$U = min\{i > 0 : W_i = n-1\} \qquad \textit{and} \qquad T = min\{i > 0 : W_i = 0\} \tag{3.1}$$

Then we have the following results

i. If $\mathsf{E}_n(U) = \mathcal{O}(n)$ and $\mathsf{E}_n(U^2) = \mathcal{O}(n^3)$ then $\mathsf{E}_1(T^2) = \mathcal{O}(n^3)$ and $\mathsf{E}_{n-1}(T^2) = \mathcal{O}(n^4)$.

ii. If there exists $\alpha > 0$ *and* $n_0 \in \mathbb{N}$ *such that* $\mathsf{E}_n\left[\exp\left(\frac{\alpha}{n^2}\cdot U\right)\right] \le \exp\left(\frac{1}{4n}\right)$ *for all* $n > n_0$ *then*

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2}\cdot T\right)\right] \le 5\exp\left(\frac{1}{4}\right) \quad \textit{for } 0 < \beta \le \min\left\{\alpha, \frac{1}{2}\right\} \textit{ and } n > n_0\,.$$

Proof of Theorem 3.2.3(i). Denote a trajectory by $\mathcal{X} = (x_0, x_1, x_2, \dots)$ where x_t is the state visited at time t. I say trajectory $\mathcal{X}$ is in $D_{i,j}$ if it starts at state i, finishes at state j and makes no intermediate visits to state 0 or state n,

$$D_{i,j} = \left\{\mathcal{X} : x_0 = i, x_1 \notin \{0,n\}, \dots x_{|\mathcal{X}|-1} \notin \{0,n\}, x_{|\mathcal{X}|} = j\right\} \qquad \text{where } 0 \le i,j \le n\,. \tag{3.2}$$

Similarly I say trajectory $\mathcal{X}$ is in D if it starts at state n, finishes at state $n-1$ and makes no intermediate visits to state $n-1$,

$$D = \left\{\mathcal{X} : x_0 = n, x_1 \ne n-1, \dots x_{|\mathcal{X}|-1} \ne n-1, x_{|\mathcal{X}|} = n-1\right\}. \tag{3.3}$$

The following classes of trajectory are particularly important in this theorem:

- Trajectories that start at state 1 and hit state 0 before state n.
- Trajectories that start at state 1 and hit state n before state 0.
- Trajectories from state n to state $n-1$.
- Trajectories that start at state $n-1$ and hit state 0 before state n.
- Trajectories that start at state $n-1$ and hit state n before state 0.

I will make use of $D_{i,j}$ and D (as defined in equations (3.2) and (3.3)) in the following moment generating functions:

$$\mathsf{M}_{1,0}(u) = \mathsf{E}(e^{u|\mathcal{X}|}\mathbb{1}_{\{\mathcal{X}\in D_{1,0}\}})\,, \qquad \mathsf{M}_{1,n}(u) = \mathsf{E}(e^{u|\mathcal{X}|}\mathbb{1}_{\{\mathcal{X}\in D_{1,n}\}})\,,$$
$$\mathsf{M}_{n-1,0}(u) = \mathsf{E}(e^{u|\mathcal{X}|}\mathbb{1}_{\{\mathcal{X}\in D_{n-1,0}\}})\,, \qquad \mathsf{M}_{n-1,n}(u) = \mathsf{E}(e^{u|\mathcal{X}|}\mathbb{1}_{\{\mathcal{X}\in D_{n-1,n}\}})\,,$$
$$\mathsf{M}_{n,n-1}(u) = \mathsf{E}(e^{u|\mathcal{X}|}\mathbb{1}_{\{\mathcal{X}\in D\}})\,.$$

Observe that $\mathsf{M}_{n,n-1}(u)$ is analytic in a neighbourhood of the origin, this is due to the fact that $\inf_i\{q_i - p_i\}$ is uniformly seperated from zero. All of the other moment generating functions are also analytic in a neighbourhood of the origin and this can be deduced by applying Lemma 2.1.3 and Corollary 2.1.6. This is an important observation because it allows me to take the derivative of all the moment generating functions and evaluate them at the origin.

The conditions of the theorem, $\mathsf{E}_n(U) = \mathcal{O}(n)$ and $\mathsf{E}_n(U^2) = \mathcal{O}(n^3)$, imply

$$\mathsf{M}^{'}_{n,n-1}(u)\ \Big|_{u=0} = \mathcal{O}(n) \quad \text{and} \quad \mathsf{M}^{''}_{n,n-1}(u)\ \Big|_{u=0} = \mathcal{O}(n^3)\,. \tag{3.4}$$

Lemma 2.1.2(ii) implies

$$\mathsf{M}^{'}_{1,0}(u) + \mathsf{M}^{'}_{1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n) \quad \text{and} \quad \mathsf{M}^{'}_{n-1,0}(u) + \mathsf{M}^{'}_{n-1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n)\,.$$

Since the quantities on the left hand side of both equations are positive it follows that

$$\begin{aligned} &\mathsf{M}^{'}_{1,0}(u)\ \Big|_{u=0} = \mathcal{O}(n)\,, && \mathsf{M}^{'}_{1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n)\,, \\ &\mathsf{M}^{'}_{n-1,0}(u)\ \Big|_{u=0} = \mathcal{O}(n)\,, && \mathsf{M}^{'}_{n-1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n)\,. \end{aligned} \tag{3.5}$$

Similarly by applying Lemma 2.1.2(iii), which deals with the second moments, it follows that

$$\begin{aligned} &\mathsf{M}^{''}_{1,0}(u)\ \Big|_{u=0} = \mathcal{O}(n^3)\,, && \mathsf{M}^{''}_{1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n^3)\,, \\ &\mathsf{M}^{''}_{n-1,0}(u)\ \Big|_{u=0} = \mathcal{O}(n^3)\,, && \mathsf{M}^{''}_{n-1,n}(u)\ \Big|_{u=0} = \mathcal{O}(n^3)\,. \end{aligned} \tag{3.6}$$

Finally Lemma 2.1.2(i) implies

$$\begin{aligned} &\mathsf{M}_{1,0}(u)\ \Big|_{u=0} = \frac{n-1}{n}\,, && \mathsf{M}_{1,n}(u)\ \Big|_{u=0} = \frac{1}{n}\,, \\ &\mathsf{M}_{n-1,0}(u)\ \Big|_{u=0} = \frac{1}{n}\,, && \mathsf{M}_{n-1,n}(u)\ \Big|_{u=0} = \frac{n-1}{n}\,. \end{aligned} \tag{3.7}$$

All the above work comes to fruition when we express the moment generating function of the hitting time of state 0 in terms of the moment generating functions defined at the start of the proof

$$\mathsf{E}_1[e^{Tu}] = \mathsf{M}_{1,0}(u) + \mathsf{M}_{1,n}(u)\mathsf{M}_{n,n-1}(u)\mathsf{M}_{n-1,0}(u)\sum_{k=0}^{\infty}\left[\mathsf{M}_{n-1,n}(u)\mathsf{M}_{n,n-1}(u)\right]^k$$

and

$$\mathsf{E}_{n-1}[e^{Tu}] = \mathsf{M}_{n-1,0}(u) + \mathsf{M}_{n-1,n}(u)\mathsf{M}_{n,n-1}(u)\mathsf{M}_{n-1,0}(u)\sum_{k=0}^{\infty}\left[\mathsf{M}_{n-1,n}(u)\mathsf{M}_{n,n-1}(u)\right]^k$$

Differentiating twice and evaluating the expression at $u = 0$ gives us an expression for $\mathsf{E}_1(T^2)$ and $\mathsf{E}_{n-1}(T^2)$ which, when evaluated using equations (3.4), (3.5), (3.6) and (3.7), demonstrates that

$$\mathsf{E}_1(T^2) = \mathcal{O}(n^3) \qquad \text{and} \qquad \mathsf{E}_{n-1}(T^2) = \mathcal{O}(n^4)$$

as required. □

Proof of Theorem 3.2.3(ii). We use the following notation (recall equation (3.1))

$$T = \min\Big\{i > 0 : W_i = 0\Big\}, \quad U = \min\Big\{i > 0 : W_i = n-1\Big\}$$
$$\text{and} \quad V = \min\Big\{i > 0 : W_i \in \{0, n\}\Big\}.$$

The hitting time T exhibits a clear renewal structure (see Figure 3.4) which is

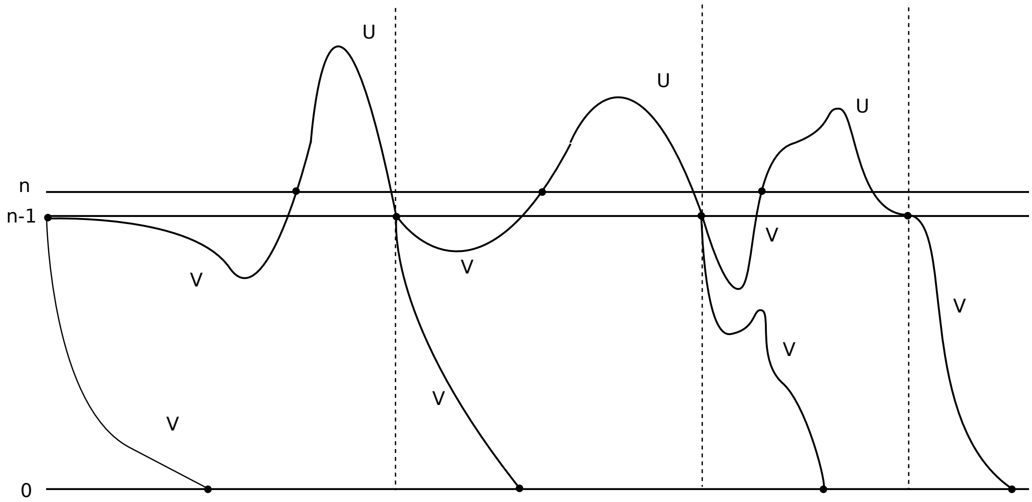


Figure 3.4: Examples of different trajectories

summed up well in the following expression

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2}\cdot T\right)\right] = \frac{\mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2}\cdot V\right)\mathbb{1}_{\{W_V=0\}}\right]}{1 - \mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2}\cdot V\right)\mathbb{1}_{\{W_V=n\}}\right]\cdot \mathsf{E}_n\left[\exp\left(\frac{\beta}{n^2}\cdot U\right)\right]}.$$

Recalling the bounds derivived in Lemma 2.1.3 and Corollary 2.1.6 (pages 13 and 15 respectively) regarding the random variable V and the conditions in the theorem regarding random variable U enables us to upper bound the previous equation

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2}\cdot T\right)\right] \leq \frac{\frac{1}{n}\cdot\exp\left(\frac{1}{4}\right)}{1 - \exp\left(-\frac{1}{2n}\right)\cdot\exp\left(\frac{1}{4n}\right)}$$

Finally Lemma 2.3.4 (page 22) implies that

$$\left[1 - \exp\left(-\frac{1}{2n}\right) \cdot \exp\left(\frac{1}{4n}\right)\right]^{-1} \leq \ (1 + 4n)\,.$$

Consequently

$$\mathsf{E}_{n-1}\left[\exp\left(\frac{\beta}{n^2} \cdot T\right)\right] \leq \frac{\frac{1}{n} \cdot \exp\left(\frac{1}{4}\right)}{1 - \exp\left(-\frac{1}{2n}\right) \cdot \exp\left(\frac{1}{4n}\right)}$$
$$\leq \frac{1}{n} \cdot \exp\left(\frac{1}{4}\right) \cdot (1 + 4n) \leq 5 \exp\left(\frac{1}{4}\right).$$

□

3.3 Birth death Markov chain results

Lemma 3.3.1 (Moments of B)**.** *For some constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$*

i. $C_1\sqrt{\lambda/\mu} < \mathsf{E}_{\lambda/\mu}(B) < C_2\sqrt{\lambda/\mu}$

ii. $\mathsf{E}_{\lambda/\mu}(B^{1.5}) < C_3(\lambda/\mu)$

iii. $\mathsf{E}_{\lambda/\mu}(B^2) < C_4(\lambda/\mu)^{1.5}$

Proof of Lemma 3.3.1(i). The expected return time for the birth death chain is the reciprocal of its stationary measure, π, and I claim that

$$\pi_k = \begin{cases} \frac{\exp(-\lambda/\mu)}{2} & \text{if } k = 0 \\ \frac{\exp(-\lambda/\mu)}{2} \cdot \left[\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} + \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{(k-1)!}\right] & \text{if } k > 0 \end{cases} \tag{3.8}$$

We can verify this claim by checking the detailed balance equations are satisifed and that π is indeed a probability measure. Firstly

$$\pi_0 \frac{\lambda}{\lambda + 0} = \pi_1 \frac{\mu}{\lambda + \mu} \quad \Longleftrightarrow \quad \pi_0 = \pi_0 \left(\frac{\lambda}{\mu} + 1\right) \frac{\mu}{\lambda + \mu} \quad \Longleftrightarrow \quad \pi_0 = \pi_0 \quad \checkmark$$

Secondly for $k > 0$

$$\pi_k \frac{\lambda}{\lambda + \mu k} = \pi_{k+1} \frac{\mu(k+1)}{\lambda + \mu(k+1)}$$
$$\pi_0 \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{(k-1)!} \left(\frac{\lambda}{\mu k} + 1\right) \frac{\lambda}{\lambda + \mu k} = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \left(\frac{\lambda}{\mu(k+1)} + 1\right) \frac{\mu(k+1)}{\lambda + \mu(k+1)}$$
$$\pi_0 \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{(k-1)!} \cdot \frac{\lambda}{\mu k} = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \quad \checkmark$$

Thirdly

$$\sum_{k=0}^{\infty} \pi_k = \frac{\exp(-\lambda/\mu)}{2} \left[1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{(k-1)!} \right]$$
$$= \frac{\exp(-\lambda/\mu)}{2} \left[1 + (\exp(\lambda/\mu) - 1) + \exp(\lambda/\mu)\right] = 1 \quad \checkmark$$

Next, Stirling's approximation [3] tells us that

$$\lim_{k\to\infty} \frac{k!}{k^{k+1/2} \cdot \exp(-k)} = \sqrt{2\pi}$$

Therefore there exist constants C_1 and C_2 such that we have the following bound for all $k \in \mathbb{N}$

$$C_1\sqrt{k} < \frac{k!}{k^k \cdot \exp(-k)} < C_2\sqrt{k}$$

Since $1/\pi_k$ is of this form, that is to say

$$\frac{1}{\pi_k} = \frac{k!}{k^k \cdot \exp(-k)}$$

it follows that

$$C_1\sqrt{\lambda/\mu} < \mathsf{E}_{\lambda/\mu}(B) = \frac{1}{\pi_{\lambda/\mu}} < C_2\sqrt{\lambda/\mu}$$

This completes the proof. □

Proof of Lemma 3.3.1(iii). Consider Model 2 on page 29. This is a specific case of the Markov chain described in Theorem 3.2.3 with $n = \sqrt{\lambda/\mu}$. If I wish to apply Theorem 3.2.3(i) I need to show that the condition regarding U is satisfied. Note that

$$p_3 \cdot q_3 = \max\{p_1, p_2\} \cdot \min\{q_1, q_2\} < \frac{1}{4}\left(1 - \frac{1}{9\lambda/\mu}\right)$$

Any trajectory that reaches state $n-1$ after $2m+1$ steps makes $m+1$ left steps and m right steps, therefore the probability of such a trajectory can be bounded above by $(p_3 \cdot q_3)^m$. By using the Ballot Theorem[10] to count the number of such trajectories it follows that

$$\mathsf{E}_n(U) \le \sum_{m=1}^{\infty} (2m+1) \left[\frac{1}{4}\left(1 - \frac{1}{9\lambda/\mu}\right)\right]^m \cdot \binom{2m+1}{m} \cdot \frac{1}{2m+1}$$
$$\le C \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(1 - \frac{1}{9\lambda/\mu}\right)^m = \mathcal{O}\left(\sqrt{\lambda/\mu}\right)$$

where the last equality uses Lemma 2.3.12 with $\alpha = -0.5$ (page 24). Similarly

$$\mathsf{E}_n(U^2) \le \sum_{m=1}^{\infty} (2m+1)^2 \left[\frac{1}{4}\left(1 - \frac{1}{9\lambda/\mu}\right)\right]^m \cdot \binom{2m+1}{m} \cdot \frac{1}{2m+1}$$
$$\le C \sum_{m=1}^{\infty} \sqrt{m}\left(1 - \frac{1}{9\lambda/\mu}\right)^m = \mathcal{O}\left((\lambda/\mu)^{1.5}\right)$$

where the last equality uses Lemma 2.3.12 with $\alpha = 0.5$ (page 24). Consequently letting $n = \sqrt{\lambda/\mu}$ and applying Theorem 3.2.3(i) to Model 2 gives

$$\mathsf{E}_1((B'')^2) = \mathcal{O}((\lambda/\mu)^{1.5}),$$

which implies

$$\mathsf{E}_0((B'')^2) = \mathcal{O}((\lambda/\mu)^{1.5}).$$

But since B is stochastically dominated by B'' (Lemma 3.2.2) we have the desired result:

$$\mathsf{E}_{\lambda/\mu}((B)^2) = \mathcal{O}((\lambda/\mu)^{1.5}).$$

□

Proof of Lemma 3.3.1(ii). The Cauchy-Schwarz inequality [17],[2] states that for random variables, X and Y, we have

$$\mathsf{E}(|XY|) \le \sqrt{\mathsf{E}(|X|^2) \cdot \mathsf{E}(|Y|^2)}.$$

Applying this result with $X = B$ and $Y = B^{0.5}$ gives

$$\mathsf{E}_{\lambda/\mu}(B^{1.5}) \le \sqrt{\mathsf{E}_{\lambda/\mu}(B^2) \cdot \mathsf{E}_{\lambda/\mu}(B)} \le C_3 \frac{\lambda}{\mu}$$

where the last inequality uses Lemma 3.3.1(i) and Lemma 3.3.1(iii). □

Lemma 3.3.2. *There exist constants $\alpha > 0$ and $C > 0$ such that for any $A \in \mathbb{N}$ we have that*

$$\mathsf{P}_{\lambda/\mu}\left(B > A \cdot \frac{\lambda}{\mu}\right) \le C \cdot \exp(-\alpha A)$$

Remark 3.3.3. *Simulations suggest that a more precise version of the above estimate holds:*

$$\mathsf{P}_{\lambda/\mu}\left(B > A \cdot \frac{\lambda}{\mu}\right) \le \sqrt{\frac{\mu}{\lambda}} \cdot \frac{C}{\sqrt{A}} \cdot \exp(-\alpha A)$$

However Lemma 3.3.2 is sufficent for my purposes and simpler to prove so I will stick with it!

Proof of Lemma 3.3.2. Consider Model 2 on page 29. This is a specific case of the Markov chain described in Theorem 3.2.3 with $n = \sqrt{\lambda/\mu}$. If I wish to apply Theorem 3.2.3(ii) I need to show that the condition regarding U is satisfied. The moment generating function for the time it takes a simple asymmetric random walk to move one step with the drift is a standard result and can be found in [10]. Applying it to Model 2 (page 29) we obtain

$$\mathsf{E}_n\left[\exp\left(\frac{\alpha}{n^2}\cdot U\right)\right] = \frac{2q_3\exp\left(\frac{\alpha}{n^2}\right)}{1+\sqrt{1-4p_3q_3\exp\left(\frac{2\alpha}{n^2}\right)}} = \frac{1-\sqrt{1-4p_3q_3\exp\left(\frac{2\alpha}{n^2}\right)}}{2p_3\exp\left(\frac{\alpha}{n^2}\right)}$$

where

$$q_3 = \min\{q_1, q_2\} = \frac{n^2+n}{2n^2+n} = \frac{n+1}{2n+1} = 1-p_3 \qquad \text{and} \qquad n = \sqrt{\lambda/\mu}$$

I need to verify that I can find $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$\mathsf{E}_n\left[\exp\left(\frac{\alpha}{n^2}\cdot U\right)\right] < \exp\left(\frac{1}{4n}\right)$$

This expression is equivalent to

$$\left[1-4p_3q_3\exp\left(\frac{2\alpha}{n^2}\right)\right]^{1/2} > \frac{1-\frac{p_3}{q_3}\exp\left(\frac{1}{2n}\right)}{1+\frac{p_3}{q_3}\exp\left(\frac{1}{2n}\right)} \tag{3.9}$$

which I claim holds if $\alpha \leq 1/16$. In Lemma 3.3.4 and Lemma 3.3.5 below I prove that the following inequalities hold for $\alpha \leq 1/16$, $0 \leq \epsilon \leq 1$, $0 \leq \delta \leq 1$ and n large enough

$$\left[1-4p_3q_3\exp\left(\frac{2\alpha}{n^2}\right)\right]^{1/2} \geq \frac{1-\epsilon}{2\sqrt{2}n} \quad \text{and} \quad \frac{1-\frac{p_3}{q_3}\exp\left(\frac{1}{2n}\right)}{1+\frac{p_3}{q_3}\exp\left(\frac{1}{2n}\right)} \leq \frac{1+\delta}{3n}.$$

Thus choosing ϵ and δ such that

$$\frac{1-\epsilon}{2\sqrt{2}} > \frac{1+\delta}{3}$$

we deduce that equation (3.9) holds. Therefore going ahead and applying Theorem 3.2.3(ii) to Model 2, we find that for $0 < \beta \leq 1/16$ and λ/μ sufficiently large

$$\mathsf{E}_{\sqrt{\lambda/\mu}-1}\left[\exp\left(\beta\cdot\frac{\mu}{\lambda}\cdot B''\right)\right] \leq 5\exp\left(\frac{1}{4}\right).$$

But since the hitting time of state 0 starting from state $\sqrt{\lambda/\mu} - 1$ stochastically dominates the return time of state 0, it follows that

$$\mathsf{E}_0\left[\exp\left(\beta \cdot \frac{\mu}{\lambda} \cdot B''\right)\right] \leq 5\exp\left(\frac{1}{4}\right).$$

By using the exponential Markov inequality and then recalling that B is stochastically dominated by B'' (Lemma 3.2.2) we obtain

$$\begin{aligned}
\mathsf{P}_{\lambda/\mu}\left(B > A \cdot \frac{\lambda}{\mu}\right) &\leq \mathsf{E}_{\lambda/\mu}\left[\exp\left(\beta \cdot \frac{\mu}{\lambda} \cdot B\right)\right] \cdot \exp(-\beta A) \\
&\leq \mathsf{E}_0\left[\exp\left(\beta \cdot \frac{\mu}{\lambda} \cdot B''\right)\right] \cdot \exp(-\beta A) \\
&\leq 5\exp\left(\frac{1}{4}\right) \cdot \exp(-\beta A)
\end{aligned}$$

for $0 < \beta \leq 1/16$ and λ/μ sufficently large. □

Lemma 3.3.4. *If $\alpha < 1/16$ and n is large enough then*

$$\left[1 - 4pq\exp\left(\frac{2\alpha}{n^2}\right)\right]^{1/2} \geq \frac{1-\epsilon}{2\sqrt{2}n} \quad \textit{where } 0 < \epsilon < 1 \textit{ and } q = \frac{n+1}{2n+1} = 1-p$$

Proof.

$$\begin{aligned}
4pq\exp\left(\frac{2\alpha}{n^2}\right) &= \left[1 - \frac{1}{(2n+1)^2}\right] \cdot \exp\left(\frac{2\alpha}{n^2}\right) \leq \exp\left\{-\frac{1}{(2n+1)^2} + \frac{2\alpha}{n^2}\right\} \\
&= \exp\left\{-\frac{1}{n^2}\left(\frac{n^2}{(2n+1)^2} - 2\alpha\right)\right\} \leq \exp\left\{-\frac{1}{8n^2}\right\}
\end{aligned}$$

Therefore

$$\begin{aligned}
\left[1 - 4pq\exp\left(\frac{2\alpha}{n^2}\right)\right]^{1/2} &> \left[1 - \exp\left\{-\frac{1}{8n^2}\right\}\right]^{1/2} \\
&= \frac{1}{2\sqrt{2}n} \cdot \left[8n^2\left(1 - \exp\left\{-\frac{1}{8n^2}\right\}\right)\right]^{1/2} > \frac{1-\epsilon}{2\sqrt{2}n}
\end{aligned}$$

The last inequality relies on the fact the square bracketed term tends to 1 as n tends to infinity. □

Lemma 3.3.5. *If n is large enough then*

$$\frac{1 - \frac{p}{q}\exp\left(\frac{1}{2n}\right)}{1 + \frac{p}{q}\exp\left(\frac{1}{2n}\right)} \leq \frac{1+\delta}{3n} \quad \textit{where } 0 < \delta < 1 \textit{ and } q = \frac{n+1}{2n+1} = 1-p$$

Proof. Observe the following (the first inequality uses Lemma 2.3.5)

$$\begin{aligned}\frac{p}{q}\exp\left(\frac{1}{2n}\right) &= \left(1-\frac{1}{n+1}\right)\cdot\exp\left(\frac{1}{2n}\right) \geq \exp\left\{\frac{1}{2n}-\frac{1}{n+1}-\frac{1}{(n+1)^2}\right\} \\ &= \exp\left\{\frac{(n+1)^2-2n(n+1)-2n}{2n(n+1)^2}\right\} = \exp\left\{-\frac{1}{2n}\cdot\frac{n^2+2n-1}{(n+1)^2}\right\} \\ &\geq \exp\left\{-\frac{1+\delta}{2n}\right\}\end{aligned}$$

Therefore if n is large enough

$$\frac{1-\frac{p}{q}\exp\left(\frac{1}{2n}\right)}{1+\frac{p}{q}\exp\left(\frac{1}{2n}\right)} \leq \frac{1-\exp\left\{-\frac{1+\delta}{2n}\right\}}{1+\exp\left\{-\frac{1+\delta}{2n}\right\}} \leq \frac{2}{3}\cdot\left(1-\exp\left\{-\frac{1+\delta}{2n}\right\}\right) \leq \frac{1+\delta}{3n}$$

where the last inequality uses Lemma 2.3.3. □

Corollary 3.3.6. *Let $X_0 = \lambda/\mu$. There exists constants $\alpha > 0$ and $C > 0$ such that for any $t \in \mathbb{C}$ satisfying $\Re(t) < (0.75\alpha)/(\lambda/\mu)$ we have*

$$|\mathsf{M}_B(t)| = |\mathsf{E}_{\lambda/\mu}(\exp(tB))| \leq C\cdot\frac{\lambda}{\mu}\cdot\frac{\exp(0.75\alpha)}{1-\exp(-0.25\alpha)}$$

Proof of Corollary 3.3.6. Let $t = u + iv$ where $u, v \in \mathbb{R}$

$$\begin{aligned}|\mathsf{M}_B(t)| &\leq \sum_{k=0}^{\infty} |\exp\{(u+iv)k\}|\cdot\mathsf{P}_{\lambda/\mu}(B=k) \\ &\leq \sum_{k=0}^{\infty}\exp\{uk\}\cdot\mathsf{P}_{\lambda/\mu}(B\geq k)\end{aligned}$$

Now split up the sum by grouping the first $\frac{\lambda}{\mu}$ terms together, then group the next $\frac{\lambda}{\mu}$ terms together and so on. By taking a uniform estimate for each grouping and then applying Lemma 3.3.2 (page 37) it follows that

$$\begin{aligned}&\leq \sum_{j=0}^{\infty}\exp\left\{u(j+1)\cdot\frac{\lambda}{\mu}\right\}\cdot\mathsf{P}_{\lambda/\mu}\left(B\geq j\cdot\frac{\lambda}{\mu}\right)\cdot\frac{\lambda}{\mu} \\ &\leq C\cdot\frac{\lambda}{\mu}\cdot\exp\left\{u\cdot\frac{\lambda}{\mu}\right\}\sum_{j=0}^{\infty}\exp\left\{uj\cdot\frac{\lambda}{\mu}-\alpha j\right\} \\ &= \frac{C\cdot\frac{\lambda}{\mu}\cdot\exp\left\{u\cdot\frac{\lambda}{\mu}\right\}}{1-\exp\left(u\frac{\lambda}{\mu}-\alpha\right)}\end{aligned}$$

Because the last expression is an increasing function of u

$$|\mathsf{M}_B(t)| \leq \frac{\lambda}{\mu}\cdot\frac{C\cdot\exp\{0.75\alpha\}}{1-\exp\{-0.25\alpha\}} \qquad \text{for } t:\Re(t)=u<\frac{0.75\alpha}{\lambda/\mu}$$

□

Chapter 4

Limit theorems

4.1 Introduction

In this chapter I will introduce several Markov chains where all of the states, bar one, constitute an irreducible closed class; the remaining state is absorbing. I will refer to such processes as Markov chains with killing. The random variable of interest is the hitting time of the absorbing state and I will state limiting results for this hitting time.

Remark 4.1.1. *I use the concept of death, in the context of Markov chain models with killing, to mean that the process has reached the absorbing state.*

4.2 Notation

Let $(X_i)_{i\geq 0}$ be a Markov chain with an absorbing state which I will call $*$. We say that the process X_i has died once it reaches this absorbing state. Also, let $X_0 = x_0$ be the starting state of the Markov chain. I shall adopt the following notation:

- T_d - number of jumps until death starting from state x_0.
- A_d - number of jumps until death starting from state x_0 without any returns to state x_0 (trajectories that return to state x_0 before death contribute to B_d, which is defined below).

- B_d - number of jumps until the first return to state x_0 (trajectories that die before returning to state x_0 contribute to A_d).
- $a^d_{m,k}$ - weight of all m step trajectories that end at state k, don't return to the state x_0 and don't die ($a^d_{0,k} = 0$ for all k and $a^d_{m,x_0} = 0$ for all m).
- $a^d_m = \sum_{k\geq 1} a^d_{m,k}$ - weight of all m step trajectories that don't return to the state x_0 and don't die ($a^d_0 = 0$).
- $r^d_k = \sum_{m\geq 1} a^d_{m,k}$ - weight of all trajectories that end at state k, don't return to the state x_0 and don't die ($r^d_{x_0} = 0$).
- b^d_m - weight of all m step trajectories that return to the state x_0 for the first time on the m-th step ($b^d_0 = 0$).

Furthermore I define the following moment generating functions:

$$\mathsf{M}_{T_d}(v) = \mathsf{E}\Big(\exp\left(v\cdot T_d\right)\Big)\,,\quad \mathsf{M}_{B_d}(v) = \mathsf{E}\Big(\exp\left(v\cdot B_d\right)\Big)\,,\quad \mathsf{M}_{A_d}(v) = \mathsf{E}\Big(\exp\left(v\cdot A_d\right)\Big)$$

Example 4.2.1. $\{A_d = 10\}$ *means death occurs on the eleventh jump without any returns to state* x_0*,* $\{B_d = 4\}$ *means the first return to state* x_0 *occurs on the fourth jump.*

4.3 Results

Lemma 4.3.1. *Let the Markov chain* $(X_i)_{i\geq 0}$ *be defined on the state space* $\{*, 0, 1, 2, \dots\}$ *and evolve with jump probabilities:*

$$\mathsf{P}(X_{i+1} = x_{i+1}|X_i = x_i) = \begin{cases} 1-d & \text{if } x_{i+1} = 1 \text{ and } x_i = 0 \\ d & \text{if } x_{i+1} = * \text{ and } x_i = 0 \\ p_{x_i}(1-d) & \text{if } x_{i+1} = x_i - 1 \text{ and } x_i > 0 \\ q_{x_i}(1-d) & \text{if } x_{i+1} = x_i + 1 \text{ and } x_i > 0 \\ d & \text{if } x_{i+1} = * \text{ and } x_i > 0 \\ 1 & \text{if } x_{i+1} = * \text{ and } x_i = * \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq d \leq 1$ *and the* p *and* q *terms are chosen such that* $\inf_i\{p_i - q_i\} > \epsilon > 0$. *In picture form this process looks like*

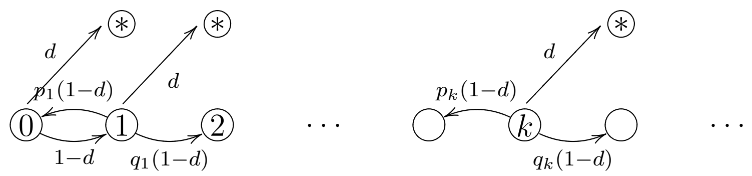


Figure 4.1: Markov chain

Let $x_0 = 0$ *and we have the following result*

$$\lim_{d\to 0} \mathsf{P}(T_d \cdot d > y) = e^{-y}$$

Remark 4.3.2. *In the proofs that follow I will use the notation* b^0_m, $a^0_{m,k}$, a^0_m, B_0 *and* A_0. *These functions/random variables relate to the chain* X_i *when* $d = 0$ *(i.e. no killing).*

Proof. We can write the moment generating function of the survival time as follows

$$\mathsf{M}_{T_d}(vd) = \mathsf{E}\left[e^{vdT_d}\right] = \frac{\mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)} \tag{4.1}$$

First I consider the numerator

$$\mathsf{M}_{A_d}(vd) = de^{vd} + \sum_{m=1}^{\infty} a^d_m \cdot d \cdot e^{vd(m+1)} = de^{vd} + \sum_{m=1}^{\infty} a^0_m (1-d)^m \cdot d \cdot e^{vd(m+1)}$$

and use the Monotone Convergence Theorem to switch the order of limits and summation in

$$\lim_{d\to 0} \frac{\mathsf{M}_{A_d}(vd)}{d} = \lim_{d\to 0} e^{vd} + \sum_{m=1}^{\infty} \lim_{d\to 0} \left[a^0_m (1-d)^m \cdot e^{vd(m+1)}\right] = 1 + \sum_{m=1}^{\infty} a^0_m = \mathsf{E}(B_0) \tag{4.2}$$

where the last equality uses Lemma 2.2.5 (page 21). Now take the denominator and again by applying the Monotone Convergence Theorem we find

$$\lim_{d\to 0} \mathsf{M}_{B_d}(vd) = \lim_{d\to 0} \left[\sum_{m=1}^{\infty} b^d_m \cdot e^{vdm}\right] = \sum_{m=1}^{\infty} b^0_m \cdot \lim_{d\to 0} \left[(1-d)^m \cdot e^{vdm}\right] = 1$$

This presents a problem because

$$\lim_{d\to 0}\left[\frac{1-\mathsf{M}_{B_d}(vd)}{d}\right]=\frac{0}{0}$$

To resolve this we find the derivative and apply L'Hôpital's rule

$$\frac{\partial}{\partial d}\mathsf{M}_{B_d}(vd)=\sum_{m=1}^{\infty}b_m^0\cdot\left[-m(1-d)^{m-1}e^{vdm}+mv(1-d)^m e^{vdm}\right]\xrightarrow[d\to 0]{}(v-1)\mathsf{E}(B_0)$$

Therefore

$$\lim_{d\to 0}\left[\frac{1-\mathsf{M}_{B_d}(vd)}{d}\right]=\lim_{d\to 0}\left[-\frac{\partial}{\partial d}\mathsf{M}_{B_d}(vd)\right]=(1-v)\mathsf{E}(B_0) \tag{4.3}$$

The condition in the lemma, $\inf_i\{p_i-q_i\}>\epsilon>0$, ensures that $\mathsf{E}(B_0)$ is finite. Combining equations (4.1), (4.2) and (4.3) gives

$$\lim_{d\to 0}\mathsf{M}_{T_d}(vd)=\frac{\mathsf{E}(B_0)}{(1-v)\mathsf{E}(B_0)}=\frac{1}{1-v}$$

This implies that $T_d\cdot d\to \text{Exp}(1)$ as $d\to 0$ in distribution therefore the lemma follows. □

Remark 4.3.3. *This turns out to be a trivial result since it is clear that T_d is nothing other than a geometric random variable*[1] *with parameter d for any choice of p_k and q_k. However taking into account the extra conditions specified in the lemma allows me to develop a more general method to prove the result that is useful when I consider other less trivial Markov chains models with killing.*

Lemma 4.3.4. *Let the Markov chain $(Y_i)_{i\geq 0}$ be defined on the state space $\{*,0,1,2,\dots\}$ and evolve with jump probabilities:*

$$\mathsf{P}(Y_{i+1}=y_{i+1}|Y_i=y_i)=\begin{cases}1-d & \text{if } y_{i+1}=1 \text{ and } y_i=0\\ d & \text{if } y_{i+1}=* \text{ and } y_i=0\\ p_{x_i} & \text{if } y_{i+1}=y_i-1 \text{ and } y_i>0\\ q_{x_i}(1-d) & \text{if } y_{i+1}=y_i+1 \text{ and } y_i>0\\ d\cdot q_{x_i} & \text{if } y_{i+1}=* \text{ and } y_i>0\\ 1 & \text{if } y_{i+1}=* \text{ and } y_i=*\\ 0 & \text{otherwise}\end{cases}$$

[1]This follows from the fact that on every jump the probability of dying is d.

where $0 \leq d \leq 1$ *and the* p *and* q *terms are chosen such that* $\inf_i\{p_i - q_i\} > \epsilon > 0$. *In picture form this process looks like*

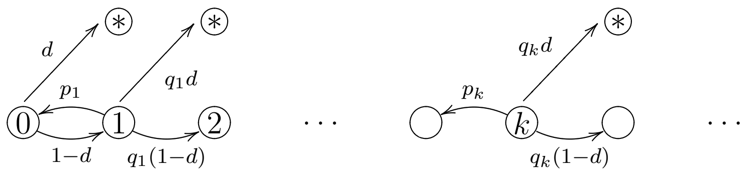


Figure 4.2: Markov chain

Let $y_0 = 0$ *and we have the following result*

$$\lim_{d\to 0} \mathsf{P}(T_d \cdot d > y) = e^{-y/2}$$

Proof. We can write the moment generating function of the survival time as follows

$$\mathsf{M}_{T_d}(vd) = \mathsf{E}[\exp{(vdT_d)}] = \frac{\mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)}$$

Following a similar line of reasoning as in the previous theorem we see

$$\lim_{d\to 0} \frac{\mathsf{M}_{A_d}(vd)}{d} = 1 + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a^0_{m,k} \cdot q_k = \underbrace{1 + \sum_{k=1}^{\infty} r^0_k \cdot q_k = \frac{\mathsf{E}(B_0)}{2}}_{\text{Uses Lemma 2.2.1 (page 17)}}$$

and

$$\lim_{d\to 0}\left[\frac{1 - \mathsf{M}_{B_d}(vd)}{d}\right] = \sum_{m=1}^{\infty} b^0_m \left[\frac{m}{2} - vm\right] = \left(\frac{1}{2} - v\right)\mathsf{E}(B_0)$$

Consequently

$$\lim_{d\to 0} \mathsf{M}_{T_d}(vd) = \frac{\mathsf{E}(B_0)}{(1-2v)\mathsf{E}(B_0)} = \frac{1}{1-2v}$$

This implies that $T_d \cdot d \to \text{Exp}(0.5)$ as $d \to 0$ in distribution therefore the lemma follows. □

Chapter 5

Reference model

5.1 Introduction

We now move to a model where there is a deeper relationship between the left and right jump probabilities and the killing probability. As a result it does not make sense to prove a limit theorem because the limiting object does not exist in the same way that it did in previous models. Instead I determine the order of magnitude of the difference between the moment generating function of the survival time of the model and the limiting distribution.

5.2 Model

In this chapter I will consider the following model.

5.2.1 Discrete Time Constant Killing model

Define $(X_i)_{i\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \dots\}$ evolving according to jump probabilities:

$$\mathsf{P}(X_{i+1} = x_{i+1} | X_i = x_i) = \begin{cases} 1-d & \text{if } x_{i+1} = 1 \text{ and } x_i = 0 \\ d & \text{if } x_{i+1} = * \text{ and } x_i = 0 \\ p_{x_i} & \text{if } x_{i+1} = x_i - 1 \text{ and } x_i > 0 \\ q_{x_i}(1-d) & \text{if } x_{i+1} = x_i + 1 \text{ and } x_i > 0 \\ d \cdot q_{x_i} & \text{if } x_{i+1} = * \text{ and } x_i > 0 \\ 1 & \text{if } x_{i+1} = * \text{ and } x_i = * \\ 0 & \text{otherwise} \end{cases}$$

where

$$q_{x_i} = \frac{\lambda}{\lambda + \mu x_i} \qquad p_{x_i} = \frac{\mu x_i}{\lambda + \mu x_i} \qquad d = \frac{\lambda}{\mu \mathsf{N}}$$

In picture form it looks like

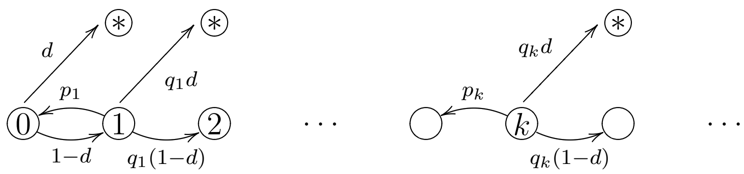


Figure 5.1: Markov chain X_i - Discrete Time Constant Killing model

From this point onwards, I will use the following notation exclusively for this Markov chain[1].

- T_d - number of jumps until death starting from state λ/μ.

[1]The same notation was used for my Markov chains in Chapter 4, however from this point onwards in my thesis, the notation will solely be used to refer to the Markov chain introduced at the beginning of Section 5.2.1.

- A_d - number of jumps until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_d, which is defined below).

- B_d - number of jumps until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_d).

- $a^d_{m,k}$ - weight of all m step trajectories that end at state k, do not return to the state λ/μ and do not die ($a^d_{0,k} = 0$ for all k and $a^d_{m,\lambda/\mu} = 0$ for all m).

- $a^d_m = \sum_{k\geq 1} a^d_{m,k}$ - weight of all m step trajectories that do not return to the state λ/μ and do not die ($a^d_0 = 0$).

- $r^d_k = \sum_{m\geq 1} a^d_{m,k}$ - weight of all trajectories that end at state k, do not return to the state λ/μ and do not die ($r^d_{\lambda/\mu} = 0$).

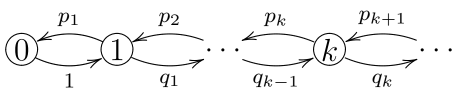
- b^d_m - weight of all m step trajectories that return to the state λ/μ for the first time on the m-th step ($b^d_0 = 0$).

- Let $\mathbb{P}_d$ be a matrix consisting of the one step transition probabilities for this Markov chain.

Furthermore I define the following moment generating functions:

$$\mathsf{M}_{T_d}(v) = \mathsf{E}\big(\exp(v \cdot T_d)\big), \quad \mathsf{M}_{B_d}(v) = \mathsf{E}\big(\exp(v \cdot B_d)\big), \quad \mathsf{M}_{A_d}(v) = \mathsf{E}\big(\exp(v \cdot A_d)\big)$$

Convention 5.2.1. *In the results that follow in this and later chapters I will use the notation b^0_m, $a^0_{m,k}$, a^0_m, B_0 and A_0. These functions/random variables relate to the chain X_i when $d = 0$ (i.e. no killing), which is shown below*



$$q_k = \frac{\lambda}{\lambda + \mu k} \qquad p_k = \frac{\mu k}{\lambda + \mu k}$$

This chain is the process that I studied in Chapter 3, as such I can make use of the estimates I derived in that chapter. For example

$$\sum_{m=1}^{\infty} m \cdot b_m^0 = \mathsf{E}(B_0) = \mathsf{E}_{\lambda/\mu}(B) < C_2\sqrt{\lambda/\mu}$$

where $\mathsf{E}_{\lambda/\mu}(B)$ *is notation that I used in Chapter 3 for the expected return time to the state* λ/μ*, before I introduced killing into my Markov chains. And the inequality in the equation follows from Lemma 3.3.1 (page 35).*

Convention 5.2.2. *The moment generating function of* B_0 *is analytic at the origin. As such I will write* $\mathsf{E}(B_0)$ *and* $\mathsf{E}(B_0^2)$ *as the appropriate derivative of the moment generating function evaluated at* 0*:*

$$\mathsf{E}(B_0) = \mathsf{M}'_{B_0}(0) \quad \textit{and} \quad \mathsf{E}(B_0^2) = \mathsf{M}''_{B_0}(0)\,.$$

5.3 Intermediate results

Lemma 5.3.1. *The following results hold*

$$b_m^d = b_m^0 \cdot (1-d)^{0.5m} \qquad \textit{and} \qquad a_{m,k}^d = a_{m,k}^0 \cdot (1-d)^{0.5(m+k-\frac{\lambda}{\mu})}$$

Proof. I observe that each jump to the right picks up a factor of $(1-d)$ and so if one knows the length of a trajectory and its end point (as is the case for b_m^d and $a_{m,k}^d$) then one can factor out the effect that the killing has. In the case of b_m^d, we are considering an m step trajectory that finishes up where it starts, therefore half the jumps were to the left and half to the right. This means that it picks up a factor of $(1-d)^{0.5m}$ and so we can write $b_m^d = b_m^0 \cdot (1-d)^{0.5m}$. In the case of $a_{m,k}^d$, we are considering an m step trajectory that finishes up at state k; as such it makes $0.5(m+k-\lambda/\mu)$ jumps to the right and in so doing, picks up a factor of $(1-d)^{0.5(m+k-\frac{\lambda}{\mu})}$. □

Corollary 5.3.2. *The following results hold*

$$b_m^d \le b_m^0\,, \qquad a_{m,k}^d \le a_{m,k}^0 \qquad \textit{and} \qquad a_m^d \le a_m^0$$

Proof. The first two statements follow immediately from Lemma 5.3.1. The last statement follows from the middle statement by summing over k on both sides of the inequality and using the following definition

$$a_m^d = \sum_{k\geq 1} a_{m,k}^d\,.$$

□

5.4 Main results

All the proofs in this section rely on a condition of the form

$$\left(\frac{\lambda}{\mu}\right)^2 \cdot \frac{1}{\mathsf{N}} < \kappa\,.$$

One consequence of this is

$$\left(\frac{\lambda}{\mu}\right)^\theta \cdot \frac{1}{\mathsf{N}} < \kappa \quad \text{for any } \theta < 2\,.$$

For example

$$d = \frac{\lambda}{\mu \mathsf{N}} < \kappa\,.$$

I will readily use this fact below without additional comment.

Lemma 5.4.1. *For any v such that $|v| < 0.5$, if $(\lambda/\mu)^2 < 0.5\mathsf{N}$ then*

$$\left|\frac{\mathsf{M}_{A_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \frac{1}{2}\right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is a pure constant.

Proof of Lemma 5.4.1. I will start by noting a simple consequence of Corollary 2.2.4 (page 20) which follows by decomposing r_k, in the notation of Corollary 2.2.4, as $r_k = \sum_{m=1}^{\infty} a_{m,k}^0$:

$$\frac{1}{2}\mathsf{M}'_{B_0}(0) = \frac{1}{2} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a_{m,k}^0 \cdot q_k \tag{5.1}$$

Also we can express $\mathsf{M}_{A_d}(vd)$ by using the law of total probability and Lemma 5.3.1:

$$\begin{aligned}
\mathsf{M}_{A_d}(vd) &= \frac{d}{2}\cdot e^{vd} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^d_{m,k}\cdot d\cdot q_k \cdot e^{vd(m+1)} \\
&= \frac{d}{2}\cdot e^{vd} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^0_{m,k}(1-d)^{(m+k-\frac{\lambda}{\mu})/2}\cdot d\cdot q_k \cdot e^{vd(m+1)} \qquad (5.2)\\
&= \frac{d}{2}\cdot e^{vd} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^0_{m,k}\cdot d\cdot q_k \cdot \exp\left(vd(m+1) + \frac{m+k-\frac{\lambda}{\mu}}{2}\log(1-d)\right)
\end{aligned}$$

Also note that

$$|e^{vd}-1| \le |vd|e^{|vd|} \le 0.5d\cdot e^{0.5\cdot 0.5} \le d \qquad (5.3)$$

Pulling together equations (5.1) and (5.2) and then applying (5.3) gives

$$\begin{aligned}
\left|\mathsf{M}_{A_d}(vd) - \frac{d}{2}\mathsf{M}'_{B_0}(0)\right| \le \frac{d^2}{2} \\
+ \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^0_{m,k}\cdot d\cdot q_k \left|\exp\left(vd(m+1) + \frac{m+k-\frac{\lambda}{\mu}}{2}\log(1-d)\right) - 1\right|
\end{aligned}$$

I need to bound the term in absolute value bars in the above equation. In order to make use of the inequality $|\exp(x)-1| < \epsilon|x|$, which holds if $x < \log(\epsilon)$ and $\epsilon > 1$ (see Lemma 2.3.9 on page 23), I need to check that

$$vd(m+1) + 0.5\cdot\left(m+k-\frac{\lambda}{\mu}\right)\log(1-d)$$

can be bounded above by a pure constant for all $|v| < 0.5$. By applying Lemma 2.3.10 (page 24) to bound $\log(1-d)$ we obtain:

$$\begin{aligned}
vd(m+1) + \frac{m+k-\frac{\lambda}{\mu}}{2}\cdot\log(1-d) &\le vd(m+1) - \frac{m+k}{2}\cdot d + \frac{\lambda/\mu}{2}\cdot d \\
&\le md\left(v-\frac{1}{2}\right) + vd + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\frac{1}{\mathsf{N}} \\
&\le 0 + 0.5\cdot 0.5 + 0.5\cdot 0.5 = 0.5
\end{aligned}$$

where the last inequality applies the conditions in the lemma. Therefore

$$\begin{aligned}
&\left|\mathsf{M}_{A_d}(vd) - \frac{1}{2}\mathsf{M}'_{B_0}(0)d\right| \\
&\le \frac{d^2}{2} + d\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^0_{m,k}q_k\cdot e^{0.5}\left|vd(m+1) + \frac{m+k-\frac{\lambda}{\mu}}{2}\log(1-d)\right| \\
&\le \frac{d^2}{2} + d\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a^0_{m,k}q_k\cdot e^{0.5}\left(|2vdm| + \left|\frac{m}{2}\log(1-d)\right| + \left|\frac{k-\frac{\lambda}{\mu}}{2}\log(1-d)\right|\right)
\end{aligned}$$

Next I apply Lemma 2.3.8 (page 23) to bound $|\log(1-d)|$ and I use the fact that a length m trajectory can not venture more than m steps away from the starting position to bound $|k - \lambda/\mu|$

$$\begin{aligned}
&\leq \frac{d^2}{2} + d \cdot e^{0.5} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} a^0_{m,k} \cdot (2vdm + md + md) \\
&= \frac{d^2}{2} + 2d^2(1+v) \cdot e^{0.5} \sum_{m=1}^{\infty} m \cdot a^0_m \\
&\leq \frac{d^2}{2} + 2d^2(1+0.5) \cdot e^{0.5} \cdot \mathsf{M}''_{B_0}(0) = \mathcal{O}(d^2(\lambda/\mu)^{3/2})
\end{aligned}$$

where the last inequality applies Lemma 2.2.5 (page 21) and the last equality applies the bound in Lemma 3.3.1(iii) (page 35). In summary, we can conclude that for any $|v| < 0.5$:

$$\left| \mathsf{M}_{A_d}(vd) - \frac{1}{2}\mathsf{M}'_{B_0}(0)d \right| = \mathcal{O}(d^2(\lambda/\mu)^{3/2})$$

Finally dividing through by $\mathsf{M}'_{B_0}(0)d$ and making use of the lower bound in Lemma 3.3.1(i) (page 3.3.1) the result follows. □

Lemma 5.4.2. *For any v such that $|v| < 0.5$, if $(\lambda/\mu) < 0.5\mathsf{N}$ then*

$$\left| \frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \left(\frac{1}{2} - v\right) \right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is a pure constant.

Remark 5.4.3. *One can rewrite the statement from Lemma 5.4.2 without the big O by introducing a constant $C > 0$ as follows*

$$\left| \frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \left(\frac{1}{2} - v\right) \right| \leq C \cdot \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}$$

Therefore

$$\frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} \geq \left(\frac{1}{2} - v\right) - C\left(\frac{\lambda}{\mu}\right)^2 \cdot \frac{1}{\mathsf{N}}$$

Also recall Lemma 3.3.1 (page 35), which states that for some constants $C_1 > 0$ and $C_2 > 0$ we have

$$C_1\sqrt{\lambda/\mu} < \mathsf{E}_{\lambda/\mu}(B) = \mathsf{M}'_{B_0}(0) < C_2\sqrt{\lambda/\mu}\,.$$

Finally choose $\bar{v}$ such that $0 < \bar{v} < 0.5$. It follows that for any λ, μ, N and v chosen such that

$$|v| < \bar{v} \quad \textit{and} \quad \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}} < \frac{1}{C} \cdot \frac{0.5 - \bar{v}}{2}$$

then

$$\begin{aligned}\frac{1 - \mathsf{M}_{B_d}(vd)}{C_1 \cdot d\sqrt{\lambda/\mu}} > \frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} &\geq \left(\frac{1}{2} - v\right) - C\left(\frac{\lambda}{\mu}\right)^2 \cdot \frac{1}{\mathsf{N}} \\ &\geq \left(\frac{1}{2} - \bar{v}\right) - \frac{1}{2} \cdot \left(\frac{1}{2} - \bar{v}\right) = \frac{1}{2} \cdot \left(\frac{1}{2} - \bar{v}\right) > 0\end{aligned}$$

This means as long as $|v|$ is uniformly seperated from 0.5 and $\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}$ is sufficiently small (where the smallness depends on $\bar{v}$) then

$$\frac{1 - \mathsf{M}_{B_d}(vd)}{d\sqrt{\lambda/\mu}}$$

is uniformly seperated from 0 for all v that satisfy $|v| < \bar{v}$. This will be a particularly useful result in Chapters 7, 8 and 9.

Proof of Lemma 5.4.2. Firstly observe that by using the law of total probability and Lemma 5.3.1

$$\mathsf{M}_{B_d}(vd) = \sum_{m=1}^{\infty} b_m^d \exp(mvd) = \sum_{m=1}^{\infty} b_m^0 (1-d)^{m/2} \exp(mvd) = \mathsf{M}_{B_0}(w_d)$$

where $w_d = vd + 0.5\log(1-d)$. Secondly Taylor's theorem tells us

$$\mathsf{M}_{B_0}(w_d) = \mathsf{M}_{B_0}(0) + \mathsf{M}'_{B_0}(0)w_d + \int_0^{w_d} (w_d - u)\mathsf{M}''_{B_0}(u)\mathrm{d}u$$

Rearranging the above equation and applying the triangle inequality gives

$$\left|\frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \left(\frac{1}{2} - v\right)\right| \leq \left|\frac{1}{\mathsf{M}'_{B_0}(0)d}\int_0^{w_d} (w_d - u)\mathsf{M}''_{B_0}(u)\mathrm{d}u\right| + \left|\frac{1}{2} + \frac{\log\sqrt{1-d}}{d}\right|$$

Now I will estimate both of the terms in the right hand side of the above equation. Since $w_d < (v - 0.5)d < 0$ for all $v < 0.5$ and $\mathsf{M}''_{B_0}(u)$ is an increasing function of u it follows that

$$\begin{aligned}&\left|\frac{1}{\mathsf{M}'_{B_0}(0)d}\int_0^{w_d} (w_d - u)\mathsf{M}''_{B_0}(u)\mathrm{d}u\right| \\ &\qquad\leq \frac{\mathsf{M}''_{B_0}(0)}{\mathsf{M}'_{B_0}(0)d} \cdot \frac{w_d^2}{2} \leq \frac{\mathsf{M}''_{B_0}(0)}{\mathsf{M}'_{B_0}(0)} \cdot \frac{d(|v|+1)^2}{2} = \mathrm{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)\end{aligned}$$

where the second inequality holds for $d < 0.5$ and the last equality holds using the bounds in Lemma 3.3.1 (page 35). Finally we use Lemma 2.3.7 to deduce that

$$\left|\frac{1}{2} + \frac{\log\sqrt{1-d}}{d}\right| < \frac{d}{2} = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

for $d < 0.5$ which completes the proof. □

Finally we pull together the previous two lemmas in order to prove the following result:

Theorem 5.4.4. *Consider $(X_i)_{i\geq 0}$ and let $X_0 = \lambda/\mu$. Fix $0 < \bar{v} < 0.5$. For any v such that $|v| < \bar{v}$, if $(\lambda/\mu)^2 < \kappa_1\mathsf{N}$ and $\kappa_1 = \kappa_1(\bar{v})$ is sufficiently small then*

$$\left|\mathsf{M}_{T_d}(vd) - \frac{1}{1-2v}\right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is dependent on $\bar{v}$.

Proof of Theorem 5.4.4.

$$\mathsf{M}_{T_d}(vd) = \frac{\mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)} = \frac{0.5 + \epsilon_1}{0.5 - v + \epsilon_2} = \frac{1}{1-2v} \cdot \frac{1 + 2\epsilon_1}{1 + (0.5-v)^{-1}\epsilon_2} \tag{5.4}$$

where

$$\epsilon_1 = \frac{\mathsf{M}_{A_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \frac{1}{2} = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

and

$$\epsilon_2 = \frac{1 - \mathsf{M}_{B_d}(vd)}{\mathsf{M}'_{B_0}(0)d} - \left(\frac{1}{2} - v\right) = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right)$$

By choosing κ_1 small enough so that $|\epsilon_2| < 0.5(0.5-\bar{v})$ implies $|(0.5-v)^{-1}\epsilon_2| < 0.5$. Consequently by applying Lemma 2.3.6 (page 23) we deduce

$$\left|\frac{1+2\epsilon_1}{1-2v}\left(\frac{1}{1+(0.5-v)^{-1}\epsilon_2} - 1\right)\right| \leq \frac{4}{(1-2\bar{v})^2}|(1+2\epsilon_1)\epsilon_2| = \mathrm{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right) \tag{5.5}$$

Also

$$\left|\frac{1+2\epsilon_1}{1-2v} - \frac{1}{1-2v}\right| \leq \frac{2|\epsilon_1|}{1-2\bar{v}} = \mathrm{O}\left(\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}}\right) \tag{5.6}$$

By using equation (5.4) and the triangle equality:

$$\begin{aligned}
\left|\mathsf{M}_{T_d}(vd) - \frac{1}{1-2v}\right| &= \left|\frac{1}{1-2v} \cdot \frac{1+2\epsilon_1}{1+(0.5-v)^{-1}\epsilon_2} - \frac{1}{1-2v}\right| \\
&= \left|\frac{1+2\epsilon_1}{1-2v}\left(\frac{1}{1+(0.5-v)^{-1}\epsilon_2} - 1\right) + \frac{1+2\epsilon_1}{1-2v} - \frac{1}{1-2v}\right| \\
&\leq \left|\frac{1+2\epsilon_1}{1-2v}\left(\frac{1}{1+(0.5-v)^{-1}\epsilon_2} - 1\right)\right| + \left|\frac{1+2\epsilon_1}{1-2v} - \frac{1}{1-2v}\right|
\end{aligned}$$

The result follows by applying equations (5.5) and (5.6). □

Chapter 6

Perturbation technique

6.1 Overview

In Chapters 4 and 5 I introduced a number of Markov chain models with an absorbing state; the random variable of interest being the time until absorption. For these models I was able to prove results directly, however, in other cases this is not possible; in Chapters 7, 8 and 9 we will see examples of such models. I will use perturbation techniques in a similar manner to [8], in order to compare processes that are not 'solvable' directly to processes that are.

The perturbative technique for moment generating functions of additive functionals of finite state Markov chains, developed in [8], was based upon 1) the positivity of the spectral gap for the transition matrix of the Markov chain in question (or exponential decay to equilibrium) and 2) smallness of the perturbation compared to the gap.

For the models here in the context of DNA damage and repair, once the parameters $\mathsf{N}, l, \lambda, \mu$ are fixed, the perturbation is uniquely determined and cannot be made uniformly small in the whole state space. Additional complications arise from the fact that the hitting time in question is not uniformly bounded and thus the moment generating functions involves summation over trajectories of all possible lengths which requires a careful control of the error terms.

This is best explained with an example. Let $\mathbb{P}$ be the one step transtion matrix for a Markov chain and let $(\mathbb{P})_{i,j}$ be the probability of moving from state i to state j

in one step. Let us define a perturbed version of this Markov chain to have transtion matrix $\mathbb{Q}$ which is defined by $(\mathbb{Q})_{i,j} = (\mathbb{P})_{i,j}(1+\epsilon_{i,j})$. A naive application of the approach in [8] to this example would require a uniform smallness of $|\epsilon_{i,j}|$ compared to the spectral gap of the transition matrix $\mathbb{P}$.

6.2 A simple example

Consider the following two Markov chains with killing where the starred states are absorbing and I am interested in how long it takes until absorption. In the first process $d > 0$ is constant but in the second process it is replaced by a state dependent function $d_k > 0$. I now make two crucial assumptions: Firstly, I assume that p_k and q_k are chosen such that the processes spend most of their time near some state m, prior to being absorbed. Secondly, I assume that the differences $|d_k - d|$ are all very small quantities for k that is close to m. It is then natural to expect that if $d > 0$ is small enough so that the typical absorption time at a starred state is large, then a similar property should hold for the process with state dependent killing (d_k).

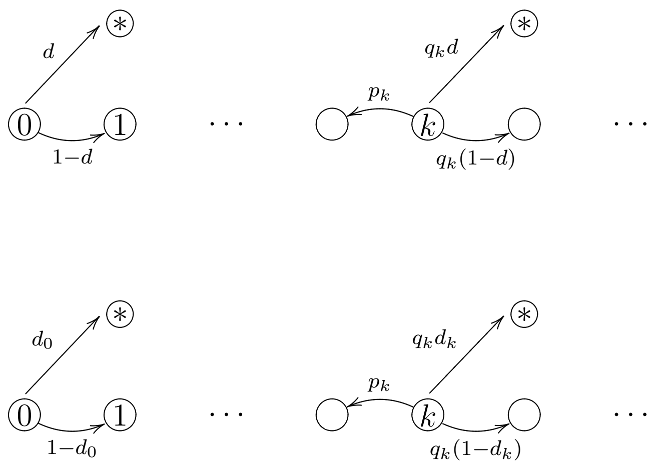


Figure 6.1

My aim is to prove that the killing time probability distribution in both models are close to one another; to do this I will show that the difference between the moment generating functions of the killing times in the respective models is small. In the models arising in analysis of DNA damage and repair that we are interested in here, the differences $|d_k - d|$ are small for k near m, but are large otherwise. It is therefore necessary to employ a different strategy to that which is employed in [8].

6.3 My strategy

Rather than working on the microscopic scale (on the level of individual jumps), I will work on a mesoscopic scale (on the level of excursions from a typical state to a typical state). I will demonstrate that this leads to a set of criteria on the level of excursions that are required to hold in order to prove the closeness of the respective moment generating functions. In many cases this will overcome the problem I explained above due to the fact that the largeness of $|d_k - d|$ for atypical states k is compensated for by the likelihood (or should I say unlikelihood) of ever reaching such states in a single excursion. For now assume that state m is a typical state and for the first of the two processes in Figure 6.1 I define:

- T_d - number of jumps until death starting from state m.
- A_d - number of jumps until death starting from state m but without any returns to m (trajectories that return to state m before death contribute to B_d).
- B_d - number of jumps until the first return to state m (trajectories that die before returning to state m contribute to A_d).

And for the second process in Figure 6.1 I define:

- T_d^* - number of jumps until death starting from state m.
- A_d^* - number of jumps until death starting from state m but without any returns to m (trajectories that return to state m before death contribute to B_d^*).
- B_d^* - number of jumps until the first return to state m (trajectories that die before returning to state m contribute to A_d^*).

Theorem 6.3.1. *Assume the existence of a constant* $v_0 > 0$ *such that* $\mathsf{M}_{B_d}(v_0) = 1$ *and fix* $\bar{v}$ *so that* $0 < \bar{v} < v_0$*. For any* $\epsilon > 0$ *and* $K > 0$ *there exists* $\delta > 0$ *such that for* v *that satisfies* $|v| < \bar{v}$*, if*

$$\left|\frac{\mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v)}{1 - \mathsf{M}_{B_d}(v)}\right| < \delta\,, \qquad \left|\frac{\mathsf{M}_{A_d^*}(v) - \mathsf{M}_{A_d}(v)}{1 - \mathsf{M}_{B_d}(v)}\right| < \delta \quad \textit{and} \quad \left|\frac{\mathsf{M}_{A_d}(v)}{1 - \mathsf{M}_{B_d}(v)}\right| < K$$

then

$$|\mathsf{M}_{T_d^*}(v) - \mathsf{M}_{T_d}(v)| < \epsilon$$

Remark 6.3.2. *In Theorem 6.3.1 I assume the existence of a constant* $v_0 > 0$ *such that* $\mathsf{M}_{B_d}(v_0) = 1$*; for any models with killing there does exist such a constant however this is not guarenteed in models that do not contain any killing.*

Proof. I will start by deriving an expression for

$$\mathsf{M}_{T_d^*}(v) - \mathsf{M}_{T_d}(v)$$

The natural way to express $\mathsf{M}_{T_d}(v)$ and $\mathsf{M}_{T_d^*}(v)$ is via

$$\mathsf{M}_{T_d}(v) = \frac{\mathsf{M}_{A_d}(v)}{1 - \mathsf{M}_{B_d}(v)} \qquad\qquad \mathsf{M}_{T_d^*}(v) = \frac{\mathsf{M}_{A_d^*}(v)}{1 - \mathsf{M}_{B_d^*}(v)}$$

This follows from the fact that a trajectory in either model can be decomposed into excursions from a typical state m to itself and the final open trajectory. However, in order to compare $\mathsf{M}_{T_d^*}(v)$ and $\mathsf{M}_{T_d}(v)$ on the level of excursions I found it useful to further decompose $\mathsf{M}_{B_d^*}(v)$ as follows

$$\mathsf{M}_{B_d^*}(v) = \underbrace{\mathsf{M}_{B_d}(v)}_{\substack{\text{unperturbed}\\ \text{excursion}}} + \underbrace{\mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v)}_{\substack{\text{perturbed}\\ \text{excursion}}}$$

I now introduce a shading scheme whereby excursions are black lines if they are unperturbed, dotted lines if they are perturbed and dashed lines if the process dies.

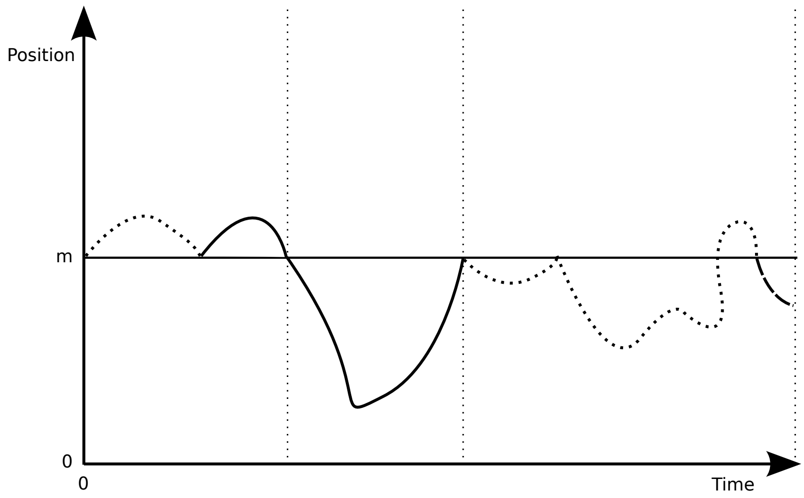


Figure 6.2

In order that I take into account all shadings I make use of the intrinsic renewal structure and introduce the cutting rule whereby I cut after every unperturbed excursion (the dashed vertical lines in Figure 6.2 indicate renewal moments in this example). In so doing I group together any perturbed excursions that occur prior to the next unperturbed excursion, the length of such a trajectory has moment generating function:

$$\left[1+\frac{\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v)}{1-(\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v))}\right]\cdot\mathsf{M}_{B_d}(v)$$

I also need to take into account the death event, to be consistent with the above definition I group together the death excursion with all the perturbed excursions that occured since the last renewal, the length of such a trajectory has moment generating function:

$$\left[1+\frac{\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v)}{1-(\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v))}\right]\cdot\mathsf{M}_{A_d^*}(v)$$

Consequently

$$\mathsf{M}_{T_d^*}(v)=\frac{(1+G)\overline{H}}{1-(1+G)F}\qquad\text{and}\qquad\mathsf{M}_{T_d}(v)=\frac{H}{1-F}$$

where

$$F=\mathsf{M}_{B_d}(v)\qquad G=\frac{\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v)}{1-(\mathsf{M}_{B_d^*}(v)-\mathsf{M}_{B_d}(v))}\qquad H=\mathsf{M}_{A_d}(v)\qquad\overline{H}=\mathsf{M}_{A_d^*}(v)$$

Therefore

$$\mathsf{M}_{T_d^*}(v) - \mathsf{M}_{T_d}(v) = \frac{\mathsf{M}_{A_d^*}(v)}{1-\mathsf{M}_{B_d^*}(v)} - \frac{\mathsf{M}_{A_d}(v)}{1-\mathsf{M}_{B_d}(v)} \tag{6.1}$$

$$= \frac{(1+G)\overline{H}}{1-(1+G)F} - \frac{H}{1-F} \tag{6.2}$$

$$= \frac{(1+G)\overline{H}}{1-(1+G)F} - \frac{\overline{H}}{1-F} + \frac{\overline{H}-H}{1-F} \tag{6.3}$$

$$= \frac{G\overline{H}}{(1-(1+G)F)(1-F)} + \frac{\overline{H}-H}{1-F} \tag{6.4}$$

$$= \frac{G}{(1-(1+G)F)} \cdot \left(\frac{\overline{H}-H}{(1-F)} + \frac{H}{(1-F)} \right) + \frac{\overline{H}-H}{1-F} \tag{6.5}$$

I will now explain how we can deduce the result $|\mathsf{M}_{T_d^*}(v) - \mathsf{M}_{T_d}(v)| < \epsilon$, using the conditions of the theorem. The last two conditions of the theorem,

$$\left| \frac{\mathsf{M}_{A_d^*}(v) - \mathsf{M}_{A_d}(v)}{1-\mathsf{M}_{B_d}(v)} \right| < \delta \qquad \text{and} \qquad \left| \frac{\mathsf{M}_{A_d}(v)}{1-\mathsf{M}_{B_d}(v)} \right| < K\,,$$

are equivalent to

$$\left| \frac{\overline{H}-H}{(1-F)} \right| < \delta \qquad \text{and} \qquad \left| \frac{H}{(1-F)} \right| < K\,. \tag{6.6}$$

In order to deduce the smallness of the final term

$$\frac{G}{1-(1+G)F}$$

the reasoning is a little more convoluted. Firstly

$$\left| \frac{\mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v)}{1-\mathsf{M}_{B_d}(v)} \right| < \delta \qquad \Longrightarrow \qquad \left| \mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v) \right| < \delta$$

and we can apply this to bound G as follows

$$|G| = \left| \frac{\mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v)}{1-(\mathsf{M}_{B_d^*}(v) - \mathsf{M}_{B_d}(v))} \right| < \frac{\delta}{1-\delta}\,.$$

Secondly, we have $F = \mathsf{M}_{B_d}(v) < \mathsf{M}_{B_d}(\bar{v})$. Taking into account these two facts and choosing δ small enough such that

$$\left(1 + \frac{\delta}{1-\delta}\right) \cdot \mathsf{M}_{B_d}(\bar{v}) < \bar{\delta} < 1$$

it follows that $(1+G)F < \bar{\delta} < 1$. Therefore

$$\left| \frac{G}{(1-(1+G)F)} \right| \le \frac{\delta}{(1-\delta)(1-\bar{\delta})}\,. \tag{6.7}$$

Finally, pulling together equations (6.6) and (6.7) gives

$$
\begin{aligned}
|\mathsf{M}_{T_d^*}(v) - \mathsf{M}_{T_d}(v)| &\leq \left|\frac{G}{(1-(1+G)F)}\right| \cdot \left(\left|\frac{\overline{H}-H}{(1-F)}\right| + \left|\frac{H}{(1-F)}\right|\right) + \left|\frac{\overline{H}-H}{1-F}\right| \\
&\leq \frac{\delta \cdot (\delta + K)}{(1-\delta)(1-\bar{\delta})} + \delta
\end{aligned}
$$

and this can be bounded by ϵ as long as δ is small enough. □

This technique is very general and can be applied to many models where there is a state that is visited frequently. Moreover this method is equally applicable to discrete and continuous time models as I will demonstrate shortly.

Chapter 7

Discrete time perturbation

7.1 Introduction

I have studied a number of different Markov chain models (with killing) up to this point and they have all had one thing in common; results concerning the survival time have been proved directly. The complexity of many models does now allow results to be proved directly and as I explained in Chapter 6 my aim has been to develop a technique to allow results to be proved for more complicated models that are small pertubations of simpler models. I will demostrate how one can use pertubation techniques to compare two discrete time Markov chain models and show that the survival times in both models are close to one another.

7.2 Models

In this chapter I will compare the following two models. The Discrete Time Constant Killing model is the same model that I introduced and studied in Chapter 5, all the notation remains the same and I repeat it here purely for ease of reading. The Discrete Time Linear Killing model is similar in spirit to the Discrete Time Constant Killing model, the main difference being that the killing is linear in the state (d is replaced by $d_k = k/\mathsf{N}$), consequently the Discrete Time Linear Killing model has a finite state space. The survival time is close in both models because the processes spend most of their time near state $k = \lambda/\mu$ where $d_k = d$.

7.2.1 Discrete Time Constant Killing model

Define $(X_i)_{i\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \dots\}$ evolving according to jump probabilities:

$$\mathsf{P}(X_{i+1} = x_{i+1} | X_i = x_i) = \begin{cases} 1-d & \text{if } x_{i+1} = 1 \text{ and } x_i = 0 \\ d & \text{if } x_{i+1} = * \text{ and } x_i = 0 \\ p_{x_i} & \text{if } x_{i+1} = x_i - 1 \text{ and } x_i > 0 \\ q_{x_i}(1-d) & \text{if } x_{i+1} = x_i + 1 \text{ and } x_i > 0 \\ d \cdot q_{x_i} & \text{if } x_{i+1} = * \text{ and } x_i > 0 \\ 1 & \text{if } x_{i+1} = * \text{ and } x_i = * \\ 0 & \text{otherwise} \end{cases}$$

where

$$q_{x_i} = \frac{\lambda}{\lambda + \mu x_i} \qquad p_{x_i} = \frac{\mu x_i}{\lambda + \mu x_i} \qquad d = \frac{\lambda}{\mu \mathsf{N}}$$

In picture form it looks like

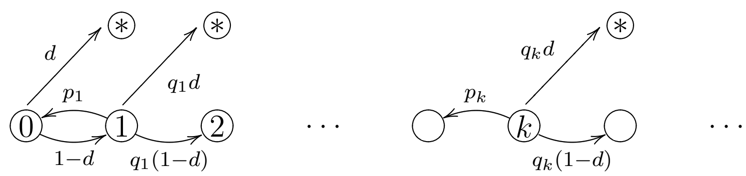


Figure 7.1: Markov chain X_i - Discrete Time Constant Killing model

For this specific Markov chain I recall the following notation:

- T_d - number of jumps until death starting from state λ/μ.
- A_d - number of jumps until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_d, which is defined below).
- B_d - number of jumps until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_d).

- $a^d_{m,k}$ - weight of all m step trajectories that end at state k, do not return to the state λ/μ and do not die ($a^d_{0,k} = 0$ for all k and $a^d_{m,\lambda/\mu} = 0$ for all m).
- $a^d_m = \sum_{k\geq 1} a^d_{m,k}$ - weight of all m step trajectories that do not return to the state λ/μ and do not die ($a^d_0 = 0$).
- $r^d_k = \sum_{m\geq 1} a^d_{m,k}$ - weight of all trajectories that end at state k, do not return to the state λ/μ and do not die ($r^d_{\lambda/\mu} = 0$).
- b^d_m - weight of all m step trajectories that return to the state λ/μ for the first time on the m-th step ($b^d_0 = 0$).
- Let $\mathbb{P}_d$ be a matrix consisting of the one step transition probabilities for this Markov chain.

Furthermore I recall the following moment generating functions:

$$\mathsf{M}_{T_d}(v) = \mathsf{E}\big(\exp{(v \cdot T_d)}\big)\,, \quad \mathsf{M}_{B_d}(v) = \mathsf{E}\big(\exp{(v \cdot B_d)}\big)\,, \quad \mathsf{M}_{A_d}(v) = \mathsf{E}\big(\exp{(v \cdot A_d)}\big)$$

7.2.2 Discrete Time Linear Killing model

Define $(Y_i)_{i\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \dots \mathsf{N}\}$ evolving according to jump probabilities:

$$\mathsf{P}(Y_{i+1} = y_{i+1} | Y_i = y_i) = \begin{cases} p_{y_i} & \text{if } y_{i+1} = y_i - 1 \text{ and } y_i > 0 \\ q_{y_i}(1 - d_{y_i}) & \text{if } y_{i+1} = y_i + 1 \text{ and } y_i > 0 \\ d_{y_i} \cdot q_{y_i} & \text{if } y_{i+1} = * \text{ and } y_i > 0 \\ 1 & \text{if } y_{i+1} = * \text{ and } y_i = * \\ 0 & \text{otherwise} \end{cases}$$

where

$$q_{y_i} = \frac{\lambda}{\lambda + \mu y_i} \qquad p_{y_i} = \frac{\mu y_i}{\lambda + \mu y_i} \qquad d_{y_i} = \frac{y_i}{\mathsf{N}}$$

In picture form it looks like

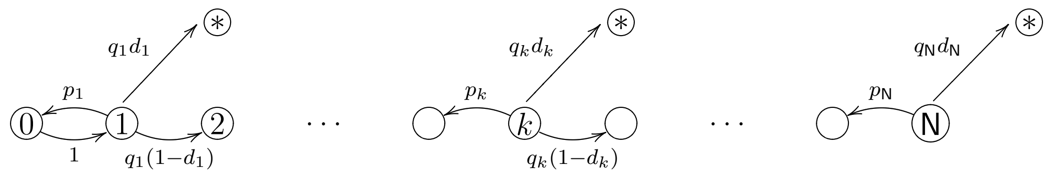


Figure 7.2: Markov chain Y_i - Discrete Time Linear Killing model

For this specific Markov chain I introduce the following notation:

- T_ϵ - number of jumps until death starting from state λ/μ.
- A_ϵ - number of jumps until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_ϵ, which is defined below).
- B_ϵ - number of jumps until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_ϵ).
- $a^\epsilon_{m,k}$ - weight of all m step trajectories that end at state k, do not return to the state λ/μ and do not die ($a^\epsilon_{0,k} = 0$ for all k and $a^\epsilon_{m,\lambda/\mu} = 0$ for all m).
- $a^\epsilon_m = \sum_{k\geq 1} a^\epsilon_{m,k}$ - weight of all m step trajectories that do not return to the state λ/μ and do not die ($a^\epsilon_0 = 0$).
- $r^\epsilon_k = \sum_{m\geq 1} a^\epsilon_{m,k}$ - weight of all trajectories that end at state k, do not return to the state λ/μ and do not die ($r^\epsilon_{\lambda/\mu} = 0$).
- b^ϵ_m - weight of all m step trajectories that return to the state λ/μ for the first time on the m-th step ($b^\epsilon_0 = 0$).
- Let $\mathbb{P}_\epsilon$ be a matrix consisting of the one step transition probabilities for this Markov chain.

Furthermore I define the following moment generating functions:

$$\mathsf{M}_{T_\epsilon}(v) = \mathsf{E}\big(\exp(v\cdot T_\epsilon)\big), \quad \mathsf{M}_{B_\epsilon}(v) = \mathsf{E}\big(\exp(v\cdot B_\epsilon)\big), \quad \mathsf{M}_{A_\epsilon}(v) = \mathsf{E}\big(\exp(v\cdot A_\epsilon)\big)$$

7.2.3 Additional notation

Finally I need to define trajectory probabilities and a few important classes of trajectory.

Definition 7.2.1. *Given a trajectory $\mathcal{X} = (x_0, x_1, \ldots, x_m)$ I define $\mathbb{P}_d(\mathcal{X})$ to be the probability of the trajectory $\mathcal{X}$ with respect to the Discrete Time Constant Killing model, that is*

$$\mathbb{P}_d(\mathcal{X}) = \prod_{j=0}^{m-1} (\mathbb{P}_d)_{x_j, x_{j+1}}$$

Similarly I define $\mathbb{P}_\epsilon(\mathcal{X})$ to be the probability of the trajectory $\mathcal{X}$ with respect to the Discrete Time Linear Killing model

$$\mathbb{P}_\epsilon(\mathcal{X}) = \prod_{j=0}^{m-1} (\mathbb{P}_\epsilon)_{x_j, x_{j+1}}$$

Definition 7.2.2. *Given a trajectory $\mathcal{X} = (x_0, x_1, \ldots, x_m)$ I define $\mathbb{P}_0(\mathcal{X})$ to be the probability of the trajectory $\mathcal{X}$ with respect to the Discrete Time Constant Killing model with $d = 0$, that is*

$$\mathbb{P}_0(\mathcal{X}) = \mathbb{P}_d(\mathcal{X})\Big|_{d=0}.$$

Remark 7.2.3. *I will also take this opportunity to recall the notation that I introduced in Chapter 5. In a similar manner to Definition 7.2.2, the following functions and random variables: b^0_m, $a^0_{m,k}$, a^0_m, B_0 and A_0 relate to the Discrete Time Constant Killing when $d = 0$ (i.e. no killing).*

I am interested in two types of trajectory, i) trajectories that start at state λ/μ, have a particular length and have no intermediate visits to the state λ/μ and ii) trajectories that start at state λ/μ, have a particular length, have no intermediate visits to the state λ/μ and do not deviate by more than a given distance away from the initial state.

Definition 7.2.4. *I say trajectory $\mathcal{X}$ has length m (and write $|\mathcal{X}| = m$) if and only if $\mathcal{X}$ is of the form $\mathcal{X} = (x_0 = \frac{\lambda}{\mu}, x_1 \notin \{\frac{\lambda}{\mu}, *\}, \ldots, x_{m-1} \notin \{\frac{\lambda}{\mu}, *\}, x_m \notin \{*\})$*

Definition 7.2.5. *Take a trajectory $\mathcal{X}$ such that $|\mathcal{X}| = m$. I say trajectory $\mathcal{X}$ deviates less than n from the starting position (and write $||\mathcal{X}|| < n$) if and only if the following conditions are satisfied*

$$|x_1 - x_0| < n\,, \qquad |x_2 - x_0| < n\,, \quad \dots\,, \quad |x_m - x_0| < n$$

Let me link this new notation to the notation that was introduced at the beginning of the chapter:

$$b_m^d = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_d(\mathcal{X})\,, \quad a_m^d = \sum_{\substack{|\mathcal{X}|=m \\ x_m \neq \frac{\lambda}{\mu}}} \mathbb{P}_d(\mathcal{X})\,, \quad a_{m,k}^d = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = k}} \mathbb{P}_d(\mathcal{X})\,, \tag{7.1}$$

$$b_m^\epsilon = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_\epsilon(\mathcal{X})\,, \quad a_m^\epsilon = \sum_{\substack{|\mathcal{X}|=m \\ x_m \neq \frac{\lambda}{\mu}}} \mathbb{P}_\epsilon(\mathcal{X})\,, \quad a_{m,k}^\epsilon = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = k}} \mathbb{P}_\epsilon(\mathcal{X})\,. \tag{7.2}$$

7.3 Intermediate results

In order to compare the difference between the moment generating functions of the survival time for the Discrete Time Constant Killing and Discrete Time Linear Killing models I need to be able to bound quantites like

$$b_m^d - b_m^\epsilon = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_d(\mathcal{X}) - \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_\epsilon(\mathcal{X}) \tag{7.3}$$

Due to the fact that any trajectory that doesn't move to the right of state N exists in both models, I will bound quantites like that in equation (7.3) by comparing trajectories on a one to one basis (I will deal with trajectories that move to the right of state N seperately). A natural requirement for such an approach is to find a relatively uniform bound for $\mathbb{P}_d(\mathcal{X}) - \mathbb{P}_\epsilon(\mathcal{X})$ that only depends on a few simple characteristics of $\mathcal{X}$. I will start with a lemma that relates one step transition probabilities.

Lemma 7.3.1. *Recall that $\mathbb{P}_d$ and $\mathbb{P}_\epsilon$ are the transition matrices for the Discrete Time Constant Killing and Discrete Time Linear Killing models respectively. The* *following relation holds*

$$(\mathbb{P}_\epsilon)_{i,j} = (\mathbb{P}_d)_{i,j}(\mathbb{Q})_{i,j} \qquad 0 \le i,j \le \mathsf{N} \quad \textit{or } j = *$$

where

$$(\mathbb{Q})_{i,j} = \begin{cases} 1 + \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu} & \textit{if } j = i+1 \\ \frac{i}{\lambda/\mu} & \textit{if } j = * \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Looking at the respective Markov chain diagrams, we see that the only jump probabilities that differ are 'up one' jumps and the jump to a starred state. So in order to prove the lemma I just need to verify the following two statements

$$\begin{aligned} & (\mathbb{P}_\epsilon)_{i,i+1} = (\mathbb{P}_d)_{i,i+1}\left(1 + \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu}\right) \\ \Leftrightarrow \quad & \frac{\lambda}{\lambda + \mu i}\left(1 - \frac{i}{\mathsf{N}}\right) = \frac{\lambda}{\lambda + \mu i}(1-d)\left(1 + \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu}\right) \\ \Leftrightarrow \quad & \frac{\mathsf{N} - i}{\mathsf{N} - \lambda/\mu} = 1 + \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu} \\ \Leftrightarrow \quad & \frac{\mathsf{N} - i - \mathsf{N} + \lambda/\mu}{\mathsf{N} - \lambda/\mu} = \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu} \qquad \checkmark \end{aligned}$$

$$(\mathbb{P}_\epsilon)_{i,*} = (\mathbb{P}_d)_{i,*} \cdot \frac{i}{\lambda/\mu} \quad \Leftrightarrow \quad \frac{\lambda}{\lambda + \mu i} \cdot \frac{i}{\mathsf{N}} = \frac{\lambda}{\lambda + \mu i} \cdot \frac{\lambda}{\mu \mathsf{N}} \cdot \frac{i}{\lambda/\mu} \quad \checkmark$$

□

Lemma 7.3.1 enables me to prove a number of useful results, but let me first express $(\mathbb{Q})_{i,j}$ in a different format. Define $\epsilon_{i,j}$ as follows

$$\epsilon_{i,j} = \begin{cases} \frac{\lambda/\mu - i}{\mathsf{N} - \lambda/\mu} & \text{if } j = i+1 \\ \frac{i}{\lambda/\mu} - 1 & \text{if } j = * \\ 0 & \text{otherwise} \end{cases}$$

where $0 \le i,j \le \mathsf{N}$ or $j = *$. Using this new notation, it follows that

$$(\mathbb{Q})_{i,j} = (1 + \epsilon_{i,j}). \tag{7.4}$$

Lemma 7.3.2. *If $\frac{\lambda}{\mu} < \mathsf{N}$ then for any trajectory $\mathcal{X}$ such that $|\mathcal{X}| = m \in \mathbb{N}$ and $||\mathcal{X}|| < m^\alpha$ ($\alpha > 0$) it follows that*

$$|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| \leq \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}} \exp\left(\frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}}\right)$$

Proof. I start by considering a trajectory, $\mathcal{X}$, that moves beyond state N and I make two observations. Firstly $|\mathcal{X}| = m \geq \mathsf{N} - \frac{\lambda}{\mu}$ and secondly $\mathbb{P}_\epsilon(\mathcal{X}) = 0$. As a result

$$\frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}} \exp\left(\frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}}\right) \geq 1$$

and the statement in the lemma follows immediately. Next I consider trajectories that do not travel beyond state N. I let $A \subseteq \mathbb{Z}$ and so when I write $A \subseteq [0, m-1]$ it means that A is a subset of $\{0, 1, \ldots, m-1\}$. I now use Definition 7.2.1, Lemma 7.3.1 and equation (7.4) to express $\mathbb{P}_\epsilon(\mathcal{X})$ as follows:

$$\mathbb{P}_\epsilon(\mathcal{X}) = \mathbb{P}_d(\mathcal{X}) \prod_{j=0}^{m-1} (1 + \epsilon_{x_j, x_{j+1}}) = \sum_{A \subseteq [0, m-1]} \mathbb{P}_d(\mathcal{X}) \prod_{a \in A} \epsilon_{x_a, x_{a+1}} \tag{7.5}$$

Remark 7.3.3. *Summing over $A \subseteq [0, m-1]$ includes $A = \emptyset$, the convention being that an empty product is equal to* 1.

By using the expression for $\mathbb{P}_\epsilon(\mathcal{X})$ in equation (7.5) and then applying the condition in the lemma, $||\mathcal{X}|| < m^\alpha$, it follows that

$$\begin{aligned}
|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| &\leq \sum_{\substack{A \subseteq [0, m-1]: \\ A \neq \emptyset}} \mathbb{P}_d(\mathcal{X}) \prod_{a \in A} |\epsilon_{x_j, x_{j+1}}| \\
&\leq \mathbb{P}_d(\mathcal{X}) \sum_{\substack{A \subseteq [0, m-1]: \\ A \neq \emptyset}} \left(\frac{m^\alpha}{\mathsf{N} - \frac{\lambda}{\mu}}\right)^{|A|} \\
&= \mathbb{P}_d(\mathcal{X}) \left[\left(1 + \frac{m^\alpha}{\mathsf{N} - \frac{\lambda}{\mu}}\right)^m - 1\right]
\end{aligned}$$

Finally I apply Lemma 2.3.1 and Lemma 2.3.2 (page 22)

$$\leq \mathbb{P}_d(\mathcal{X}) \left[\exp\left(\frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}} \right) - 1 \right]$$
$$\leq \mathbb{P}_d(\mathcal{X}) \frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}} \exp\left(\frac{m^{1+\alpha}}{\mathsf{N} - \frac{\lambda}{\mu}} \right)$$

□

Additionally we have the following result which doesn't have any dependence on $||\mathcal{X}||$:

Lemma 7.3.4. *If* $\frac{\lambda}{\mu} < \mathsf{N}$ *then for any trajectory* $\mathcal{X}$ *such that* $|\mathcal{X}| = m \in \mathbb{N}$ *it follows*

$$|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| \leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}} \right)^m$$

Proof. For any trajectory whose first jump is to the right we have

$$\mathbb{P}_d(\mathcal{X}) \geq \mathbb{P}_\epsilon(\mathcal{X}) \geq 0\,.$$

This implies $|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| \leq \mathbb{P}_d(\mathcal{X})$, which is trivially less than the right hand side of the equation in the lemma. For any trajectory whose first jump is to the left, we have $\mathbb{P}_\epsilon(\mathcal{X}) \geq \mathbb{P}_d(\mathcal{X}) \geq 0$. Moreover, I use Definition 7.2.1, Lemma 7.3.1 and equation (7.4) to derive an expression for $\mathbb{P}_\epsilon(\mathcal{X})$ that I can bound from above:

$$\mathbb{P}_\epsilon(\mathcal{X}) = \mathbb{P}_d(\mathcal{X}) \prod_{j=0}^{m-1} (1 + \epsilon_{x_j, x_{j+1}}) \leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}} \right)^m \tag{7.6}$$

where the inequality uses the fact that $\lambda/\mu - x_j \leq \lambda/\mu$ for all $j \in \{0, 1, \ldots, m-1\}$. Consequently

$$|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| \leq \mathbb{P}_\epsilon(\mathcal{X}) \leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}} \right)^m$$

This completes the proof. □

Lemma 7.3.5. *For any trajectory* $\mathcal{X}$ *it follows*[1]*:* $\mathbb{P}_d(\mathcal{X}) \leq \mathbb{P}_0(\mathcal{X})$.

[1]This result is along the same vein as Lemma 5.3.1 and Corollary 5.3.2.

Proof. Let $\mathcal{X}$ be an m step trajectory, k steps of which are jumps to the right. Each of these jumps to the right picks up a factor of $(1-d)$, therefore

$$\mathbb{P}_d(\mathcal{X}) = \mathbb{P}_0(\mathcal{X}) \cdot (1-d)^k \leq \mathbb{P}_0(\mathcal{X})\,.$$

□

Lemma 7.3.6. *If* $\frac{\lambda}{\mu \mathsf{N}} < \kappa < 1$ *then for any trajectory* $\mathcal{X}$ *such that* $|\mathcal{X}| = m$ *it follows*

$$\mathbb{P}_\epsilon(\mathcal{X}) \leq \mathbb{P}_d(\mathcal{X}) \cdot e^{(1-\kappa)^{-1} \cdot dm} \tag{7.7}$$

Proof. For any trajectory whose first jump is to the right we have $\mathbb{P}_\epsilon(\mathcal{X}) \leq \mathbb{P}_d(\mathcal{X})$ which implies the desired result. For any trajectory whose first jump is to the left,

$$\mathbb{P}_\epsilon(\mathcal{X}) \leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}}\right)^m ,$$

as demonstrated in equation (7.6). This can be further bounded by pulling out a factor of N from the denominator of the main fraction and applying the condition in the lemma, $\lambda/(\mu\mathsf{N}) < \kappa < 1$,

$$\leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\lambda}{\mu \mathsf{N}} \cdot \frac{1}{1-\kappa}\right)^m .$$

Finally by applying Lemma 2.3.1 (page 22) and recalling the definition $d = \lambda/(\mu\mathsf{N})$

$$\leq \mathbb{P}_d(\mathcal{X}) \cdot e^{(1-\kappa)^{-1} \cdot dm} .$$

□

Corollary 7.3.7. *If* $\frac{\lambda}{\mu \mathsf{N}} < \kappa < 1$ *then*

$$b_m^\epsilon \leq b_m^d \cdot e^{(1-\kappa)^{-1} \cdot dm}, \quad a_m^\epsilon \leq a_m^d \cdot e^{(1-\kappa)^{-1} \cdot dm} \quad \text{and} \quad a_{m,k}^\epsilon \leq a_{m,k}^d \cdot e^{(1-\kappa)^{-1} \cdot dm}$$

Proof. All three bounds are a straightforward application of Lemma 7.3.6 and the relationships in equations (7.1) and (7.2). For example

$$b_m^\epsilon = \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_\epsilon(\mathcal{X}) \leq \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \frac{\lambda}{\mu}}} \mathbb{P}_d(\mathcal{X}) \cdot e^{(1-\kappa)^{-1} \cdot dm} = b_m^d \cdot e^{(1-\kappa)^{-1} \cdot dm} .$$

The other estimates are similar. □

7.4 Main results

All the proofs in this section rely on a condition of the form

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \cdot \frac{1}{\mathsf{N}} < \kappa < 1$$

for some $\rho_0 > 0$. One consequence of this is

$$\left(\frac{\lambda}{\mu}\right)^{\theta} \cdot \frac{1}{\mathsf{N}} < \kappa < 1 \quad \text{for any } \theta < 2 + \rho_0 \,. \tag{7.8}$$

In particular, when $\theta = 1$, we have $d = \lambda/(\mu\mathsf{N}) < \kappa < 1$. This implies

$$\frac{1}{\mathsf{N} - \lambda/\mu} \leq \frac{1}{\mathsf{N}(1-\kappa)} \,. \tag{7.9}$$

I will readily use these facts in the argument below without additional comment.

Lemma 7.4.1. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $\bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $v < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \ , \quad \mathsf{N} \exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \ ,$$
$$0 < \rho < \min\left\{\rho_0 \, , \frac{1}{7}\right\} \quad \text{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4} \, , \rho\right\}$$

then

$$\begin{aligned}\frac{|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|}{d\sqrt{\lambda/\mu}} &\leq \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot (1-\kappa_1)^{-1} \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1} \\ &+ \left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \cdot C \cdot (1-\kappa_1)^{-1} \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1} \\ &+ \left(\frac{\mu}{\lambda}\right)^{0.5-\rho} \cdot C \cdot \kappa_2 \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1} + \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot C \cdot \kappa_2\end{aligned}$$

Therefore

$$\frac{|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 and $\bar{v}$.

Proof. It is necessary to express $\mathsf{M}_{B_d}(vd)$ and $\mathsf{M}_{B_\epsilon}(vd)$ in such a way that I am able to bound the difference between the two. To this end I apply the law of total probability

$$\mathsf{M}_{B_d}(vd) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m = \lambda/\mu}} \mathbb{P}_d(\mathcal{X}) e^{vdm}$$

and

$$\mathsf{M}_{B_\epsilon}(vd) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) e^{vdm} .$$

I start by decomposing $|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|$ into four terms as follows

$$|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)| \leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$$

where

$$\Sigma_1 = \sum_{m=1}^{D} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| \tag{7.10}$$

$$\Sigma_2 = \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}||\leq m^{0.75}}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| \tag{7.11}$$

$$\Sigma_3 = \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}||> m^{0.75}}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| \tag{7.12}$$

$$\Sigma_4 = \sum_{m>E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| \tag{7.13}$$

where

$$D = \left(\frac{\lambda}{\mu}\right)^{0.5} \quad \text{and} \quad E = \left(\frac{\lambda}{\mu}\right)^{1+\rho} .$$

In order to prove Lemma 7.4.1 I will individually bound each of the above expressions.

I start with Σ_1, here the pertubation is small because a short trajectory does not have the time required to reach a far away state where the pertubation is large. I bound the inner sum by applying Lemma 7.3.2 (page 70) with $\alpha = 1$ and then I apply equation (7.9):

$$\begin{aligned}\sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| &\leq \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} e^{vdm} \cdot \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^2}{\mathsf{N} - \frac{\lambda}{\mu}} \cdot e^{\frac{m^2}{\mathsf{N}-\frac{\lambda}{\mu}}} \\ &\leq b_m^d \cdot \frac{m^2}{\mathsf{N}} \cdot (1-\kappa_1)^{-1} \cdot \exp\left(vdm + \frac{m^2}{\mathsf{N}} \cdot (1-\kappa_1)^{-1}\right)\end{aligned}$$

Now I return to the full sum and after taking into account the above expression I take uniform upper bounds for all terms next to b^d_m:

$$\Sigma_1 \leq \frac{\lambda}{\mu \mathsf{N}} \cdot (1-\kappa_1)^{-1} \cdot \exp\left(vd \left(\frac{\lambda}{\mu}\right)^{0.5} + \frac{\lambda}{\mu \mathsf{N}} \cdot (1-\kappa_1)^{-1} \right) \sum_{m=1}^{D} b^d_m$$

The remaining summation is clearly bounded by one and after applying the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \quad \text{and} \quad v < \bar{v}\,,$$

it follows

$$\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} \leq \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot (1-\kappa_1)^{-1} \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1}$$

Secondly I consider m-step trajectories that do not deviate from the starting position by more than $m^{0.75}$, they contribute to Σ_2. I bound the inner sum by applying Lemma 7.3.2 (page 70) with $\alpha = 0.75$ and then I apply equation (7.9):

$$\begin{aligned}
\sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu\\ ||\mathcal{X}||\leq m^{0.75}}} e^{vdm} \left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| &\leq \sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu\\ ||\mathcal{X}||\leq m^{0.75}}} e^{vdm} \cdot \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^{1.75}}{\mathsf{N}-\frac{\lambda}{\mu}} \cdot e^{\frac{m^{1.75}}{\mathsf{N}-\frac{\lambda}{\mu}}} \\
&\leq \sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu}} \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^{1.75}}{\mathsf{N}(1-\kappa_1)} \cdot e^{vdm+\frac{m^{1.75}}{\mathsf{N}}\cdot(1-\kappa_1)^{-1}} \\
&= [b^d_m \cdot m^{1.5}] \cdot \frac{m^{0.25}}{\mathsf{N}} \cdot (1-\kappa_1)^{-1} \cdot e^{vdm+\frac{m^{1.75}}{\mathsf{N}}\cdot(1-\kappa_1)^{-1}}
\end{aligned}$$

Now I return to the full sum and after taking into account the above expression I take uniform upper bounds for all terms next to $[b^d_m \cdot m^{1.5}]$:

$$\Sigma_2 \leq \frac{\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{0.25(1+\rho)}}{1-\kappa_1} \cdot \exp\left(vd\left(\frac{\lambda}{\mu}\right)^{1+\rho} + \frac{(1-\kappa_1)^{-1}}{\mathsf{N}} \cdot \left(\frac{\lambda}{\mu}\right)^{1.75(1+\rho)} \right) \cdot \sum_{m=D+1}^{E} b^d_m \cdot m^{1.5}$$

Next by using the fact $b^d_m < b^0_m$ (see Corollary 5.3.2) and then bounding the summation by applying Lemma 3.3.1(ii) (page 35) we obtain

$$\leq C \cdot \frac{\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{0.25(1+\rho)+1}}{1-\kappa_1} \cdot \exp\left(vd\left(\frac{\lambda}{\mu}\right)^{1+\rho} + \frac{(1-\kappa_1)^{-1}}{\mathsf{N}} \cdot \left(\frac{\lambda}{\mu}\right)^{1.75(1+\rho)} \right)$$

After applying the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1\,, \qquad v < \bar{v}\,, \qquad \rho < \rho_0 \quad \text{and} \quad \rho < \frac{1}{7}\,,$$

it follows

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \leq \left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \cdot C \cdot (1-\kappa_1)^{-1} \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1}$$

Thirdly I consider longer trajectories that explore a wide section of the state space and which contribute to Σ_3. I bound the inner sun by applying Lemma 7.3.4 (page 71), equation (7.9) (page 73), Lemma 2.3.1 (page 22) and Lemma 7.3.5 (page 71) in that order:

$$\begin{aligned}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu\\ ||\mathcal{X}||>m^{0.75}}} e^{vdm}\,|\mathbb{P}_\epsilon(\mathcal{X})-\mathbb{P}_d(\mathcal{X})| &\leq \sum_{\substack{|\mathcal{X}|=m:\\ ||\mathcal{X}||>m^{0.75}}} e^{vmd}\cdot\mathbb{P}_d(\mathcal{X})\cdot\left(1+\frac{\frac{\lambda}{\mu}}{\mathsf{N}-\frac{\lambda}{\mu}}\right)^m \\ &\leq \sum_{\substack{|\mathcal{X}|=m:\\ ||\mathcal{X}||>m^{0.75}}} e^{vmd}\cdot\mathbb{P}_d(\mathcal{X})\cdot\left(1+\frac{\lambda}{\mu\mathsf{N}}\cdot\frac{1}{1-\kappa_1}\right)^m \\ &\leq e^{dm(v+(1-\kappa_1)^{-1})}\cdot\sum_{\substack{|\mathcal{X}|=m:\\ ||\mathcal{X}||>m^{0.75}}}\mathbb{P}_0(\mathcal{X})\end{aligned}$$

The remaining summation is the probability that an m-step trajectory on the birth death chain (Figure 3.1), reaches a state that is a distance of $m^{0.75}$ away from the starting position, at some point during the trajectory. This can be bounded above by the probability of the same event, but this time on a simple symmetric random walk. This is because a simple symmetric random walk is likely to explore more of the state space due to the fact that, unlike the birth death chain, there is no drift pulling it back to state λ/μ. I then apply the simple symmetric random walk moderate deviation estimate (Lemma 2.1.8, page 16):

$$\leq C\cdot e^{-\gamma\cdot m^{0.5}}\cdot e^{dm(v+(1-\kappa_1)^{-1})}$$

By taking uniform estimates for all terms, multiplying and dividing by N and rearranging terms we obtain:

$$\begin{aligned}\Sigma_3 &\leq \sum_{m=D+1}^{E}\left[C\cdot e^{-\gamma\cdot m^{0.5}}\cdot e^{dm(v+(1-\kappa_1)^{-1})}\right] \\ &\leq C\cdot\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{1+\rho}\cdot\mathsf{N}\exp\left[-\gamma\left(\frac{\lambda}{\mu}\right)^{0.25}\right]\cdot\exp\left[d\left(\frac{\lambda}{\mu}\right)^{1+\rho}\cdot(v+(1-\kappa_1)^{-1})\right]\end{aligned}$$

After applying the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}}<\kappa_1\,,\quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right]<\kappa_2\,,\quad v<\bar{v}\,,\quad \rho<\rho_0\quad\text{and}\quad \rho_1<\frac{1}{4}\,,$$

it follows

$$\frac{\Sigma_3}{d\sqrt{\lambda/\mu}} \leq C \cdot \left(\frac{\mu}{\lambda}\right)^{0.5-\rho} \cdot \kappa_2 \cdot e^{(\bar{v}+(1-\kappa_1)^{-1})\kappa_1}$$

Fourthly I consider long excursions which contribute to Σ_4. I apply Lemma 7.3.4 (page 71), Lemma 2.3.1 and equation (7.9), in that order, to bound the terms in the summation:

$$|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| \leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}}\right)^m \leq \mathbb{P}_d(\mathcal{X}) \cdot e^{dm(1-\kappa_1)^{-1}}$$

By applying this bound, then multiplying and dividing by $e^{\delta m}$ and using $e^{-\delta m} \leq e^{-\delta E}$ (due to the fact that I am summing over $m > E$) we obtain

$$\begin{aligned}\sum_{m>E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} e^{vdm} |\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})| &\leq \sum_{m>E} e^{dm(v+(1-\kappa_1)^{-1})} \cdot b_m^d \\ &\leq e^{-\delta E} \sum_{m=1}^{\infty} e^{dm(v+(1-\kappa_1)^{-1})+\delta m} \cdot b_m^0 \\ &= e^{-\delta E} \cdot \mathsf{M}_{B_0}(d(v+(1-\kappa_1)^{-1})+\delta)\end{aligned}$$

If we choose[2] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$\kappa_1(\bar{v} + (1-\kappa_1)^{-1}) + 0.375\alpha < 0.75\alpha$$

holds, then due to the fact that $\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}} < \kappa_1$ and $v < \bar{v}$ (from the conditions of the lemma), it follows that

$$\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}} \cdot \left(v + (1-\kappa_1)^{-1}\right) + 0.375\alpha < 0.75\alpha \quad \Leftrightarrow \quad d(v + (1-\kappa_1)^{-1}) + \delta < \frac{0.75\alpha}{\lambda/\mu}$$

Consequently one can apply Corollary 3.3.6 (page 40) which implies

$$\mathsf{M}_{B_0}(d(v+(1-\kappa_1)^{-1})+\delta) \leq C \cdot \lambda/\mu\,.$$

Applying this bound gives

$$\Sigma_4 \leq \exp\left[-0.375\alpha\left(\frac{\lambda}{\mu}\right)^\rho\right] \cdot C \cdot \lambda/\mu = \mathsf{N}\exp\left[-0.375\alpha\left(\frac{\lambda}{\mu}\right)^\rho\right] \cdot C \cdot \lambda/\mu \cdot \frac{1}{\mathsf{N}}$$

[2]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

Therefore by applying the conditions in the lemma, $\mathsf{N}\exp\left[-(\lambda/\mu)^{\rho_1}\right] < \kappa_2$ and $\rho_1 < \rho$, it follows

$$\frac{\Sigma_4}{d\sqrt{\lambda/\mu}} \leq C \cdot \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot \kappa_2$$

This completes the proof. □

Lemma 7.4.2. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $\bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $v < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1 \ , \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \ ,$$
$$0 < \rho < \min\left\{\frac{\rho_0}{2}, \frac{1}{3}\right\} \quad \textit{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

then

$$\begin{aligned}\frac{|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)|}{d\sqrt{\lambda/\mu}} &\leq C \cdot (1-\kappa_1)^{-1} \cdot \frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot e^{(2\bar{v}+(1-\kappa_1)^{-1})\kappa_1} \\ &+ e^{\bar{v}\kappa_1} \cdot C \cdot \kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5} + e^{2\bar{v}\kappa_1}\left(\frac{\mu}{\lambda}\right)^{0.5} + e^{2\bar{v}\kappa_1}\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \\ &+ e^{2\bar{v}\kappa_1} \cdot C \cdot \kappa_2 \cdot \left(\frac{\lambda}{\mu}\right)^{0.5+2\rho} \cdot \frac{1}{\mathsf{N}} + e^{\bar{v}\kappa_1} \cdot C \cdot (\kappa_2)^2 \cdot \left(\frac{\mu}{\lambda}\right)^{2.5}\end{aligned}$$

This implies

$$\frac{|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{2+2\rho}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 and $\bar{v}$.

Proof. Note the following

$$\begin{aligned}|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)| &= \left|\sum_{m=1}^{\infty} e^{vd(m+1)}\sum_{k=1}^{\infty}(a^\epsilon_{m,k}q_kd_k - a^d_{m,k}q_kd)\right| &(7.14)\\ &\leq \sum_{m=1}^{\infty} e^{vd(m+1)}\sum_{k=1}^{\infty}|a^\epsilon_{m,k}q_kd_k - a^d_{m,k}q_kd_k| &(7.15)\\ &+ \sum_{m=1}^{\infty} e^{vd(m+1)}\sum_{k=1}^{\infty}|a^d_{m,k}q_kd_k - a^d_{m,k}q_kd| &(7.16)\\ &\leq \sum_{m=1}^{\infty} e^{vd(m+1)}\sum_{k=1}^{\infty}|a^\epsilon_{m,k} - a^d_{m,k}|d &(7.17)\\ &+ \sum_{m=1}^{\infty} e^{vd(m+1)}\sum_{k=1}^{\infty}a^d_{m,k}|d_k - d| &(7.18)\end{aligned}$$

where the second inequality uses the following facts

$$q_k d_k = \frac{\lambda/\mu}{\lambda/\mu + k} \cdot \frac{k}{\mathsf{N}} = \frac{k}{\lambda/\mu + k} \cdot \frac{\lambda/\mu}{\mathsf{N}} \leq \frac{\lambda/\mu}{\mathsf{N}} = d \quad \text{and} \quad q_k = \frac{\lambda/\mu}{\lambda/\mu + k} \leq 1$$

Let

$$D = \left(\frac{\lambda}{\mu}\right)^{0.5} \quad \text{and} \quad E = \left(\frac{\lambda}{\mu}\right)^{1+\rho} .$$

I will now deal with expression (7.17). Recalling the definitions of $\mathbb{P}_\epsilon(\mathcal{X})$ and $\mathbb{P}_d(\mathcal{X})$ (page 67), I can rewrite the expression as follows

$$\sum_{m=1}^{\infty} e^{vd(m+1)} \sum_{k=1}^{\infty} |a^\epsilon_{m,k} - a^d_{m,k}| d \leq d \sum_{m=1}^{\infty} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \Big| \mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X}) \Big| = \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = d \sum_{m=1}^{E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \Big| \mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X}) \Big| \tag{7.19}$$

$$\Sigma_2 = d \sum_{m>E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \Big| \mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X}) \Big| \tag{7.20}$$

I will now bound equations (7.19) and (7.20). Starting with shorter trajectories which contribute to Σ_1, I bound the inner sum by applying Lemma 7.3.2 (page 70) with $\alpha = 1$ and then I make use of equation (7.9):

$$\sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \Big| \mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X}) \Big| \leq a^d_{m,k} \cdot \frac{m^2}{\mathsf{N}(1-\kappa_1)} \cdot e^{(1-\kappa_1)^{-1} \cdot \frac{m^2}{\mathsf{N}}}$$

I now return to the full sum and take uniform upper bounds of all the terms in the summation with the exception of $a^d_{m,k}$:

$$\Sigma_1 \leq \frac{d}{\mathsf{N}(1-\kappa_1)} \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \exp\left[vd \left(\frac{\lambda}{\mu}\right)^{1+\rho} + vd + \frac{1}{\mathsf{N}(1-\kappa_1)} \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \right] \sum_{m=1}^{E} \sum_{k=1}^{\infty} a^d_{m,k}$$

By making use of Lemma 2.2.5 (page 21) and Lemma 3.3.1 (page 35) one can deduce

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} a^0_{m,k} \leq \mathsf{E}_{\lambda/\mu}(B_0) \leq C\sqrt{\lambda/\mu}$$

Therefore by applying this result together with the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 , \quad v < \bar{v} \quad \text{and} \quad \rho < \frac{\rho_0}{2} ,$$

it follows

$$\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} \leq C \cdot (1-\kappa_1)^{-1} \cdot \frac{1}{\mathsf{N}} \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot e^{(2\bar{v}+(1-\kappa_1)^{-1})\kappa_1}$$

Moving onto longer trajectories, I bound the terms in Σ_2 by applying Lemma 7.3.4 (page 71), equation (7.9) (page 73), Lemma 2.3.1 (page 22) and Corollary 5.3.2 (page 49) in that order:

$$\begin{aligned}\left|\mathbb{P}_\epsilon(\mathcal{X}) - \mathbb{P}_d(\mathcal{X})\right| &\leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\frac{\lambda}{\mu}}{\mathsf{N} - \frac{\lambda}{\mu}}\right)^m \\ &\leq \mathbb{P}_d(\mathcal{X}) \cdot \left(1 + \frac{\lambda}{\mu \mathsf{N}} \cdot \frac{1}{1-\kappa_1}\right)^m \\ &\leq \mathbb{P}_0(\mathcal{X}) \cdot e^{dm(1-\kappa_1)^{-1}}\end{aligned}$$

By applying this result it follows:

$$\begin{aligned}\Sigma_2 &\leq d \sum_{m>E} e^{vd(m+1)} \sum_{k=0}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \mathbb{P}_0(\mathcal{X}) \cdot e^{dm(1-\kappa_1)^{-1}} \\ &= de^{vd} \sum_{m>E} e^{(v+(1-\kappa_1)^{-1})dm} \cdot a_m^0\end{aligned}$$

Next I multiply and divide the terms of the sum by $e^{\delta m}$ and apply $e^{-\delta m} \leq e^{-\delta E}$ (due to the fact that I am summing over $m > E$), I also use the fact that $a_m^0 = \sum_{j=m+1}^{\infty} b_j^0$

$$\leq de^{vd} \cdot e^{-\delta E} \cdot \sum_{m=0}^{\infty} e^{(v+(1-\kappa_1)^{-1})dm+\delta m} \cdot \sum_{j=m+1}^{\infty} b_j^0$$

I now change the order of summation

$$= de^{vd} \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} \sum_{m=0}^{j-1} e^{(v+(1-\kappa_1)^{-1})dm+\delta m} \cdot b_j^0$$

Finally I take uniform bounds over the terms involving m and apply the inequality $jd < e^{jd}$

$$\begin{aligned}&\leq de^{vd} \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} j \cdot e^{(v+(1-\kappa_1)^{-1})dj+\delta j} \cdot b_j^0 \\ &\leq e^{vd} \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} e^{(v+1+(1-\kappa_1)^{-1})dj+\delta j} \cdot b_j^0 \\ &= e^{vd} \cdot \mathsf{N} \cdot e^{-\delta E} \cdot \frac{1}{\mathsf{N}} \cdot \mathsf{M}_{B_0}((v+1+(1-\kappa_1)^{-1})d+\delta)\end{aligned}$$

If we choose[3] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$[\kappa_1(\bar{v} + 1 + (1 - \kappa_1)^{-1}) + 0.375\alpha] < 0.75\alpha$$

holds then one can apply Corollary 3.3.6 which implies

$$\mathsf{M}_{B_0}((v + 1 + (1 - \kappa_1)^{-1})d + \delta) \leq C \cdot \lambda/\mu\,.$$

Therefore by applying this result together with the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad v < \bar{v} \quad \text{and} \quad \rho_1 < \rho\,,$$

it follows

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \leq C \cdot e^{\bar{v}\kappa_1} \cdot \kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5}$$

I will now deal with expression (7.18).

$$\sum_{m=1}^{\infty} e^{vd(m+1)} \sum_{k=1}^{\infty} a_{m,k}^{d} |d_k - d| = \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6$$

where

$$\Sigma_3 = \sum_{m=1}^{D} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \tag{7.21}$$

$$\Sigma_4 = \sum_{m=D+1}^{E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k \\ ||\mathcal{X}||<m^{0.75}}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \tag{7.22}$$

$$\Sigma_5 = \sum_{m=D+1}^{E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k \\ ||\mathcal{X}||>m^{0.75}}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \tag{7.23}$$

$$\Sigma_6 = \sum_{m>E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \tag{7.24}$$

Recall that

$$D = \left(\frac{\lambda}{\mu}\right)^{0.5} \quad \text{and} \quad E = \left(\frac{\lambda}{\mu}\right)^{1+\rho}.$$

[3]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

I will now bound these four expressions. I start by considering Σ_3 and use the fact that the distance between the start and end position of a trajectory can not be greater than the length of the trajectory, consequently $|\lambda/\mu - k| \leq m$ which in turn implies $|d_k - d| \leq \frac{m}{\mathsf{N}}$, therefore

$$\Sigma_3 \leq \sum_{m=1}^{D} e^{vd(m+1)} \sum_{k=0}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot \frac{m}{\mathsf{N}}$$

$$= \sum_{m=1}^{D} e^{vd(m+1)} \cdot a_m^d \cdot \frac{m}{\mathsf{N}}$$

Now I use Corollary 5.3.2 to bound $a_m^d < a_m^0$ and then I take uniform upper bounds for all terms next to a_m^0, it follows

$$< \exp\left(vd \left(\frac{\lambda}{\mu}\right)^{0.5} + vd \right) \cdot \left(\frac{\lambda}{\mu}\right)^{0.5} \frac{1}{\mathsf{N}} \cdot \sum_{m=1}^{\infty} a_m^0$$

Therefore by applying Lemma 2.2.5 and Lemma 3.3.1 to bound the remaining summation and using the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \quad \text{and} \quad v < \bar{v}\,,$$

it follows

$$\frac{\Sigma_3}{d\sqrt{\lambda/\mu}} \leq e^{2\bar{v}\kappa_1} \left(\frac{\mu}{\lambda}\right)^{0.5}$$

Secondly I consider equation Σ_4, here I am summing over trajectories that satisfy $|\mathcal{X}| = m$ and $||\mathcal{X}|| < m^{0.75}$, this implies that I can use the bound $|d_k - d| \leq \frac{m^{0.75}}{\mathsf{N}}$ therefore

$$\Sigma_4 \leq \sum_{m=D+1}^{E} e^{vd(m+1)} \sum_{k=0}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k \\ ||\mathcal{X}||<m^{0.75}}} \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^{0.75}}{\mathsf{N}}$$

$$\leq \sum_{m=D+1}^{E} e^{vd(m+1)} \sum_{k=0}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot \frac{m^{0.75}}{\mathsf{N}}$$

$$= \sum_{m=D+1}^{E} e^{vd(m+1)} \cdot a_m^d \cdot \frac{m^{0.75}}{\mathsf{N}}$$

Now I use Corollary 5.3.2 to bound $a_m^d < a_m^0$ and then I take uniform upper bounds for all terms next to a_m^0, it follows

$$< \exp\left(vd \left(\frac{\lambda}{\mu}\right)^{1+\rho} + vd \right) \cdot \left(\frac{\lambda}{\mu}\right)^{0.75(1+\rho)} \frac{1}{\mathsf{N}} \cdot \sum_{m=1}^{\infty} a_m^0$$

Therefore by applying Lemma 2.2.5 and Lemma 3.3.1 to bound the remaining summation and using the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,, \quad v < \bar{v} \quad \text{and} \quad \rho < \rho_0\,,$$

it follows

$$\frac{\Sigma_4}{d\sqrt{\lambda/\mu}} \leq e^{2\bar{v}\kappa_1}\left(\frac{\mu}{\lambda}\right)^{0.25(1-3\rho)}$$

Thirdly I consider equation Σ_5. Once again I use the fact that $|d_k - d| \leq \frac{m}{\mathsf{N}}$

$$\begin{aligned}\Sigma_5 &= \sum_{m=D+1}^{E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m:\\ x_m=k\\ ||\mathcal{X}||>m^{0.75}}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \\ &\leq \sum_{m=D+1}^{E} e^{vd(m+1)} \cdot \frac{m}{\mathsf{N}} \cdot \sum_{\substack{|\mathcal{X}|=m:\\ ||\mathcal{X}||>m^{0.75}}} \mathbb{P}_0(\mathcal{X})\end{aligned}$$

Next I apply Lemma 2.1.8 (page 16) and then I uniformly upper bound all the remaining terms

$$\begin{aligned}&\leq \sum_{m=D+1}^{E} e^{vd(m+1)} \cdot \frac{m}{\mathsf{N}} \cdot C \cdot e^{-\gamma m^{0.5}} \\ &\leq C \cdot \exp\left[v \cdot \frac{1}{\mathsf{N}} \cdot \left(\frac{\lambda}{\mu}\right)^{2+\rho} + vd - \gamma \cdot \left(\frac{\lambda}{\mu}\right)^{0.25}\right] \cdot \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}} \\ &= C \cdot \exp\left[v \cdot \frac{1}{\mathsf{N}} \cdot \left(\frac{\lambda}{\mu}\right)^{2+\rho} + vd\right] \cdot \mathsf{N}\exp\left[-\gamma \cdot \left(\frac{\lambda}{\mu}\right)^{0.25}\right] \cdot \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}^2}\end{aligned}$$

After applying the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad v < \bar{v}\,, \quad \rho < \rho_0 \quad \text{and} \quad \rho_1 < \frac{1}{4}\,,$$

it follows

$$\frac{\Sigma_5}{d\sqrt{\lambda/\mu}} \leq C \cdot e^{2\bar{v}\kappa_1} \cdot \kappa_2 \cdot \left(\frac{\lambda}{\mu}\right)^{0.5+2\rho} \cdot \frac{1}{\mathsf{N}}$$

Fourthly and finally I come to equation Σ_6. The method of bounding this equation is very similar to that which I employed to bound equation Σ_2:

$$\begin{aligned}
\Sigma_6 &= \sum_{m>E} e^{vd(m+1)} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot |d_k - d| \\
&\leq e^{vd} \sum_{m>E} e^{vdm} \sum_{k=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_d(\mathcal{X}) \cdot \frac{m}{\mathsf{N}} \\
&\leq e^{vd-\delta E} \cdot \sum_{m=0}^{\infty} e^{vdm+\delta m} \cdot \frac{m}{\mathsf{N}} \cdot a_m^0
\end{aligned}$$

I now use the fact that $a_m^0 = \sum_{j=m+1}^{\infty} b_j^0$ and then change the order of summation

$$\begin{aligned}
&= e^{vd-\delta E} \cdot \sum_{m=0}^{\infty} e^{vdm+\delta m} \cdot \frac{m}{\mathsf{N}} \cdot \sum_{j=m+1}^{\infty} b_j^0 \\
&= e^{vd-\delta E} \cdot \sum_{j=1}^{\infty} \sum_{m=0}^{j-1} e^{vdm+\delta m} \cdot \frac{m}{\mathsf{N}} \cdot b_j^0 \\
&\leq e^{vd-\delta E} \cdot \sum_{j=1}^{\infty} e^{vdj+\delta j} \cdot \frac{j^2}{\mathsf{N}} \cdot b_j^0
\end{aligned}$$

I now multiple and divide by d^2 and then use the inequality $(dj)^2 < e^{2dj}$

$$\begin{aligned}
&= \mathsf{N} \cdot \left(\frac{\mu}{\lambda}\right)^2 \cdot e^{vd-\delta E} \cdot \sum_{j=1}^{\infty} (dj)^2 \cdot e^{vdj+\delta j} \cdot b_j^0 \\
&\leq \mathsf{N} \cdot \left(\frac{\mu}{\lambda}\right)^2 \cdot e^{vd-\delta E} \cdot \sum_{j=1}^{\infty} e^{(v+2)dj+\delta j} \cdot b_j^0 \\
&= \mathsf{N} \cdot \left(\frac{\mu}{\lambda}\right)^2 \cdot e^{vd-\delta E} \mathsf{M}_{B_0}(d(v+2)+\delta)
\end{aligned}$$

If we choose[4] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$[\kappa_1(\bar{v}+2) + 0.375\alpha] < 0.75\alpha$$

[4]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

holds, then due to the fact that $\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}} < \kappa_1$ and $v < \bar{v}$ (from the conditions of the lemma), it follows that

$$\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\mathsf{N}} \cdot (v+2) + 0.375\alpha < 0.75\alpha \quad \Leftrightarrow \quad d(v+2) + \delta < \frac{0.75\alpha}{\lambda/\mu}$$

Consequently one can apply Corollary 3.3.6 (page 40) which implies

$$\mathsf{M}_{B_0}(d(v+2)+\delta) \leq C \cdot \lambda/\mu\,.$$

Therefore

$$\frac{\Sigma_6}{d\sqrt{\lambda/\mu}} \leq C \cdot e^{vd} \cdot \Big[\mathsf{N}\exp\left(-0.5\delta E\right)\Big]^2 \cdot \left(\frac{\mu}{\lambda}\right)^{2.5}$$

Finally by applying the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad v < \bar{v} \quad \text{and} \quad \rho_1 < \rho\,,$$

it follows

$$\frac{\Sigma_6}{d\sqrt{\lambda/\mu}} \leq C \cdot e^{\bar{v}\kappa_1} \cdot (\kappa_2)^2 \cdot \left(\frac{\mu}{\lambda}\right)^{2.5}$$

By adding up all the components we reach the result stated in the lemma. □

7.5 Application

Theorem 7.5.1. *Fix contants $\rho_0 > 0$ and $\kappa_2^* > 0$. For any choice of $0 < \bar{v} < 0.5$ there exists $\kappa_1^* = \kappa_1^*(\bar{v}) > 0$ and $\kappa_3^* = \kappa_3^*(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1^*\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2^*\,, \quad \frac{\mu}{\lambda} < \kappa_3^*\,,$$

$$0 < \rho < \min\left\{\frac{\rho_0}{2}, \frac{1}{7}\right\} \quad \text{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

then

$$|\mathsf{M}_{T_\epsilon}(vd) - \mathsf{M}_{T_d}(vd)| = \mathcal{O}\left(\frac{1}{\mathsf{N}}\left(\frac{\lambda}{\mu}\right)^{2+2\rho}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and $\bar{v}$.

Proof of Theorem 7.5.1. I begin by fixing constants: $\rho_0 > 0$, $\kappa_2^* > 0$ and $0 < \bar{v} < 0.5$. Next I choose $\bar{\kappa}_1$ such that $\kappa_1 = \bar{\kappa}_1$ and $\kappa_2 = \kappa_2^*$ satisfy the conditions of Lemmas 7.4.1 and 7.4.2. Consequently, when I require

$$\frac{|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|}{d\sqrt{\lambda/\mu}} \quad \text{and} \quad \frac{|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)|}{d\sqrt{\lambda/\mu}}$$

to be sufficiently small, I need only concern myself with the smallness of

$$\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \quad \text{and} \quad \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \frac{1}{\mathsf{N}}. \tag{7.25}$$

This is due to the fact that all other terms that appear in the main statement of Lemmas 7.4.1 and 7.4.2 are dependent on quantities that have already been fixed. I also observe that both expressions in equation (7.25) can be made as small as one wishes by choosing κ_1^* and κ_3^* appropriately.

Let us now return to the statement of this theorem. In order to compare $\mathsf{M}_{T_\epsilon}(vd)$ and $\mathsf{M}_{T_d}(vd)$ I start by re-writing $\mathsf{M}_{T_\epsilon}(vd) - \mathsf{M}_{T_d}(vd)$ in terms of excursions and Chapter 6 provides the machinery to be able to do this. Equation (6.5) (page 61) is a key result and enables one to deduce the following

$$\begin{aligned}\mathsf{M}_{T_\epsilon}(vd) - \mathsf{M}_{T_d}(vd) = {} & \frac{F}{1 - \mathsf{M}_{B_d}(vd)F} \cdot \left(\frac{\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)} + \mathsf{M}_{T_d}(vd)\right) \\ & + \frac{\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)}\end{aligned} \tag{7.26}$$

where

$$F = \frac{\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)} \cdot \frac{1}{1 - (\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd))}.$$

Next, in order to use Lemma 7.4.1 and Lemma 5.4.2 (and Remark 5.4.3) to bound equation (7.26), whenever either of the expressions

$$\frac{\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)} \quad \text{or} \quad \frac{\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)}{1 - \mathsf{M}_{B_d}(vd)}$$

appear in (7.26) I divide the numerator and denominator of such fractions by $d\sqrt{\lambda/\mu}$ and then apply the aforementioned results. Now by choosing κ_1^* and κ_3^* small enough, one can uniformly seperate the denominators in equation (7.26) away from zero for all $|v| < \bar{v}$. This allows us to bound equation (7.26) as follows

$$|\mathsf{M}_{T_\epsilon}(vd) - \mathsf{M}_{T_d}(vd)| \leq C\left(\frac{|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|}{d\sqrt{\lambda/\mu}} \cdot \frac{|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)|}{d\sqrt{\lambda/\mu}} + \frac{|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)|}{d\sqrt{\lambda/\mu}} \cdot |\mathsf{M}_{T_d}(vd)| + \frac{|\mathsf{M}_{A_\epsilon}(vd) - \mathsf{M}_{A_d}(vd)|}{d\sqrt{\lambda/\mu}}\right)$$

where C is a constant dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*, \bar{\kappa}_1$ and $\bar{v}$. Finally by applying Lemma 7.4.1, Lemma 7.4.2 and Theorem 5.4.4 we reach the stated result.

□

Chapter 8

Discrete to continuous time perturbation

8.1 Introduction

In this chapter I compare a discrete model and continuous model and to make sure they are comparable I scale the random variables appropriately.

8.2 Models

I will compare the following two models; the Discrete Time Linear Killing model is the same model that I introduced and studied in Chapter 7.

8.2.1 Discrete Time Linear Killing model

Define $(Y_i)_{i\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \ldots \mathsf{N}\}$ evolving according to jump probabilities:

$$\mathsf{P}(Y_{i+1} = y_{i+1} | Y_i = y_i) = \begin{cases} p_{y_i} & \text{if } y_{i+1} = y_i - 1 \text{ and } y_i > 0 \\ q_{y_i}(1 - d_{y_i}) & \text{if } y_{i+1} = y_i + 1 \text{ and } y_i > 0 \\ d_{y_i} \cdot q_{y_i} & \text{if } y_{i+1} = * \text{ and } y_i > 0 \\ 1 & \text{if } y_{i+1} = * \text{ and } y_i = * \\ 0 & \text{otherwise} \end{cases}$$

where

$$q_{y_i} = \frac{\lambda}{\lambda + \mu y_i} \qquad p_{y_i} = \frac{\mu y_i}{\lambda + \mu y_i} \qquad d_{y_i} = \frac{y_i}{\mathsf{N}}$$

In picture form it looks like

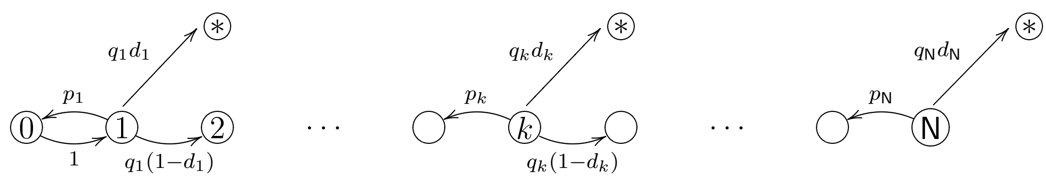


Figure 8.1: Markov chain Y_i - Discrete Time Linear Killing model

For this specific Markov chain I recall the following notation:

- T_ϵ - number of jumps until death starting from state λ/μ.
- A_ϵ - number of jumps until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_ϵ, which is defined below).
- B_ϵ - number of jumps until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_ϵ).
- $a^\epsilon_{m,k}$ - weight of all m step trajectories that end at state k, do not return to the state λ/μ and do not die ($a^\epsilon_{0,k} = 0$ for all k and $a^\epsilon_{m,\lambda/\mu} = 0$ for all m).

- $a_m^\epsilon = \sum_{k\geq 1} a_{m,k}^\epsilon$ - weight of all m step trajectories that do not return to the state λ/μ and do not die ($a_0^\epsilon = 0$).

- $r_k^\epsilon = \sum_{m\geq 1} a_{m,k}^\epsilon$ - weight of all trajectories that end at state k, do not return to the state λ/μ and do not die ($r_{\lambda/\mu}^\epsilon = 0$).

- b_m^ϵ - weight of all m step trajectories that return to the state λ/μ for the first time on the m-th step ($b_0^\epsilon = 0$).

- Let $\mathbb{P}_\epsilon$ be a matrix consisting of the one step transition probabilities for this Markov chain.

Furthermore I recall the following moment generating functions:

$$\mathsf{M}_{T_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot T_\epsilon)\big), \quad \mathsf{M}_{B_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot B_\epsilon)\big), \quad \mathsf{M}_{A_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot A_\epsilon)\big)$$

8.2.2 Continuous Time Linear Killing model

Define $(Z_t)_{t\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \ldots, \mathsf{N}\}$ evolving with jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } 2\lambda \cdot p_k \\ k &\to k+1 && \text{rate } 2\lambda \cdot q_k(1-d_k) \\ k &\to * && \text{rate } 2\lambda \cdot q_k d_k \end{aligned}$$

where

$$q_k = \frac{\lambda}{\lambda + \mu k} \qquad p_k = \frac{\mu k}{\lambda + \mu k} \qquad d_k = \frac{k}{\mathsf{N}}$$

In picture form it looks like

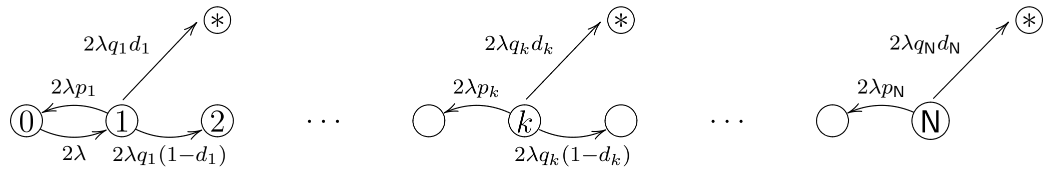


Figure 8.2: Markov chain Z_t - Continuous Time Linear Killing model

For this specific Markov chain we have the following notation:

- T_d^+ - time until death starting from state λ/μ.
- A_d^+ - time until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_d^+).
- B_d^+ - time until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_d^+).

Furthermore I define the following moment generating functions:

$$\mathsf{M}_{T_d^+}(v) = \mathsf{E}\big(\exp(v \cdot T_d^+)\big), \quad \mathsf{M}_{B_d^+}(v) = \mathsf{E}\big(\exp(v \cdot B_d^+)\big), \quad \mathsf{M}_{A_d^+}(v) = \mathsf{E}\big(\exp(v \cdot A_d^+)\big)$$

Remark 8.2.1. *An alternative but equivalent way to describe the process Z_t is that it jumps with probabilities as shown in Figure 9.2 (which incidentially is the same as the Discrete Time Linear Killing model) however there is a holding time at each state which is exponentially distributed with parameter 2λ.*

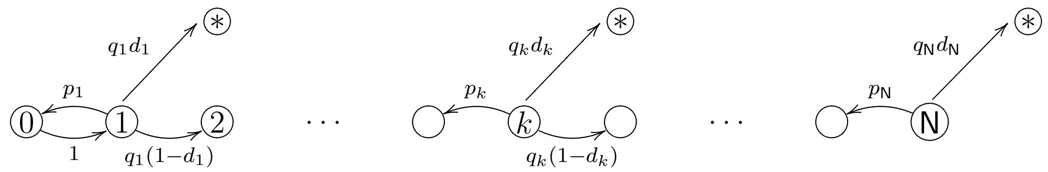


Figure 8.3: Jump chain of Z_t

8.3 Main results

Lemma 8.3.1. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $0 < \bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 , \quad \mathsf{N} \exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \quad \text{and} \quad 0 < \rho_1 < \rho < \rho_0$$

then

$$\frac{\left|\mathsf{M}_{B_d^+}(2\lambda v d) - \mathsf{M}_{B_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} \leq 2\left(\frac{\lambda}{\mu}\right)^{1.5+\rho} \frac{1}{\mathsf{N}} \cdot \bar{v}^2 \cdot (1 + e^{2\bar{v}\kappa_1}) + 4\bar{v}^2 C \cdot \kappa_2 \cdot \left(\frac{\lambda}{\mu}\right)^{0.5} \cdot \frac{1}{\mathsf{N}}$$

This implies

$$\frac{\left|\mathsf{M}_{B_d^+}(2\lambda v d) - \mathsf{M}_{B_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{1.5+\rho}\frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 *and* $\bar{v}$.

Proof. It is necessary to express $\mathsf{M}_{B_d^+}(2\lambda vd)$ and $\mathsf{M}_{B_\epsilon}(vd)$ in such a way that I am able to bound the difference between the two. To this end I apply the law of total probability and condition on the trajectory, it follows

$$\mathsf{M}_{B_\epsilon}(vd) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) e^{vdm} = \sum_{m=1}^{\infty} b_m^\epsilon e^{vdm}$$

For B_d^+, a continuous time process, once we have this conditioned on the trajectory the time that said trajectory takes will be the sum of independent exponentially distributed random variables. The moment generating function of $X \sim \text{Exp}(\sigma)$ is

$$\mathsf{E}\left(e^{X\cdot u}\right) = \frac{\sigma}{\sigma - u},$$

therefore

$$\mathsf{M}_{B_d^+}(2\lambda vd) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \left(\frac{2\lambda}{2\lambda - 2\lambda vd}\right)^m = \sum_{m=1}^{\infty} b_m^\epsilon \left(\frac{1}{1-vd}\right)^m.$$

The first condition in the lemma implies $d \leq \kappa_1$, therefore by choosing κ_1 such that

$$\bar{v} \cdot \kappa_1 < 0.5 \log 2$$

it follows that

$$vd < 0.5 \log 2 \quad \text{for } v < \bar{v}$$

Consequently one can apply Lemma 2.3.13 to bound the difference between the two moment generating functions as follows

$$0 \leq \mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd) \leq \sum_{m=1}^{\infty} b_m^\epsilon \cdot 2(vd)^2 m \cdot (1 + e^{2vd(m-1)})$$

I define

$$E = \left(\frac{\lambda}{\mu}\right)^{1+\rho}$$

and will decompose the previous expression as follows

$$|\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)| \leq \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{m=1}^{E} b_m^\epsilon \cdot 2(vd)^2 m \cdot (1 + e^{2vd(m-1)}) \tag{8.1}$$

$$\Sigma_2 = \sum_{m>E} b_m^\epsilon \cdot 2(vd)^2 m \cdot (1 + e^{2vd(m-1)}) \tag{8.2}$$

I will now bound from above each of the above sums. I start with Σ_1 and by uniformly bounding all the terms next to b_m^ϵ it follows:

$$\Sigma_1 \leq 2(vd)^2 E \cdot (1 + e^{2vd(E-1)}) \cdot \sum_{m=1}^{E} b_m^\epsilon$$

The remaining summation is clearly bounded from above by one. Additionally by using the conditions in the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1\,, \quad |v| < \bar{v} \quad \text{and} \quad \rho < \rho_0\,,$$

it follows

$$\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} \leq \left(\frac{\lambda}{\mu}\right)^{1.5+\rho} \frac{1}{\mathsf{N}} \cdot 2\bar{v}^2 \cdot (1 + e^{2\bar{v}\kappa_1})$$

Next consider Σ_2, I use the inequalities

$$md \leq e^{md} \quad \text{and} \quad b_m^\epsilon < b_m^0 \cdot e^{(1-\kappa_1)^{-1}dm}$$

to bound the terms in the sum (the second inequality can be proved by applying Lemma 7.3.7 and Lemma 5.3.2):

$$b_m^\epsilon \cdot 2(vd)^2 m \cdot (1 + e^{2vd(m-1)}) \leq b_m^0 \cdot e^{(1-\kappa_1)^{-1}dm} \cdot 2v^2 d e^{dm} \cdot (1 + e^{2vd(m-1)})$$

Therefore

$$\Sigma_2 \leq \sum_{m>E} \left[b_m^0 \cdot e^{(1-\kappa_1)^{-1}dm} \cdot 2v^2 d e^{dm} \cdot (1 + e^{2vd(m-1)}) \right]$$

By multiplying and dividing the terms of the summation by $e^{\delta m}$ and using $e^{-\delta m} < e^{-\delta E}$ (due to the fact I am summing over $m > E$) we obtain

$$\leq 2v^2 d \cdot e^{-\delta E} \cdot \left[\mathsf{M}_{B_0}\Big(d(1-\kappa_1)^{-1} + d + \delta\Big) + \mathsf{M}_{B_0}\Big(d(1-\kappa_1)^{-1} + d + 2vd + \delta\Big) \right]$$

I choose[1] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$\kappa_1((1-\kappa_1)^{-1} + 1 + 2\bar{v}) + 0.375\alpha < 0.75\alpha$$

holds. This allows me to apply Corollary 3.3.6 (page 40) which implies

$$\mathsf{M}_{B_0}(d(1-\kappa_1)^{-1}+d+\delta) \le C\cdot\lambda/\mu \quad\text{and}\quad \mathsf{M}_{B_0}(d(1-\kappa_1)^{-1}+d+2vd+\delta) \le C\cdot\lambda/\mu\,.$$

Applying this bound gives

$$\Sigma_2 \le 2v^2 d\cdot e^{-\delta E}\cdot 2C\cdot\frac{\lambda}{\mu} = 4v^2 C\cdot d^2\cdot \mathsf{N}e^{-\delta E}$$

Therefore by using the conditions in the lemma,

$$\mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,,\quad |v| < \bar{v} \quad\text{and}\quad \rho_1 < \rho\,,$$

it follows

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \le 4\bar{v}^2 C\cdot\kappa_2\cdot\left(\frac{\lambda}{\mu}\right)^{0.5}\cdot\frac{1}{\mathsf{N}}$$

This completes the proof. □

Lemma 8.3.2. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $0 < \bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,,\quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \quad\textit{and}\quad 0<\rho_1<\rho<\rho_0$$

then

$$\frac{\left|\mathsf{M}_{A_d^+}(2\lambda v d) - \mathsf{M}_{A_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} \le 8C\cdot\bar{v}^2\cdot e^{2\bar{v}\kappa_1+(1-\kappa_1)^{-1}\kappa_1}\cdot\kappa_1\cdot\left(\frac{\lambda}{\mu}\right)^{1+\rho}\cdot\frac{1}{\mathsf{N}} + 8\bar{v}^2\cdot C\cdot\kappa_2\cdot\left(\frac{\lambda}{\mu}\right)^{0.5}\cdot\frac{1}{\mathsf{N}}$$

This implies

$$\frac{\left|\mathsf{M}_{A_d^+}(2\lambda v d) - \mathsf{M}_{A_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{1+\rho}\cdot\frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 and $\bar{v}$.

[1]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

Proof. By applying the law of total probability and condition on the trajectory, it follows

$$\mathsf{M}_{A_\epsilon}(vd) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot q_k d_k \cdot e^{vd(m+1)}$$

and

$$\mathsf{M}_{A_d^+}(2\lambda vd) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot q_k d_k \cdot \left(\frac{1}{1-vd}\right)^{m+1}.$$

By choosing κ_1 such that $\bar{v} \cdot \kappa_1 < 0.5 \log 2$ I can bound the two momeng generating functions by applying Lemma 2.3.13 in exactly the same way as I did in Lemma 8.3.1 (the previous result). Additionally I make use of the fact that $0 \leq q_k d_k \leq d$:

$$\begin{aligned} 0 \leq \mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd) &\leq \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot d \cdot 2(vd)^2(m+1) \cdot (1+e^{2vdm}) \\ &\leq 4v^2d^3 \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot (m+1) \cdot e^{2\bar{v}dm} \end{aligned}$$

I define

$$E = \left(\frac{\lambda}{\mu}\right)^{1+\rho}$$

and will decompose the previous expression as follows

$$\left|\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)\right| \leq \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = 4v^2d^3 \sum_{m=0}^{E}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot (m+1) \cdot e^{2\bar{v}dm} \tag{8.3}$$

$$\Sigma_2 = 4v^2d^3 \sum_{m>E}\sum_{k=1}^{\infty} a^{\epsilon}_{m,k} \cdot (m+1) \cdot e^{2\bar{v}dm} \tag{8.4}$$

I will now bound from above each of the above sums. Firstly I consider Σ_1 and start by using Lemma 7.3.7 (page 72) and Corollary 5.3.2 (page 49) to bound $a^{\epsilon}_{m,k}$:

$$a^{\epsilon}_{m,k} \leq a^{0}_{m,k} \cdot e^{(1-\kappa_1)^{-1}dm},$$

then I uniformly upper bound all the terms next to $a^0_{m,k}$:

$$\Sigma_1 \leq 4v^2d^3 \cdot (E+1) \cdot e^{2\bar{v}dE + (1-\kappa_1)^{-1}dE} \cdot \sum_{m=0}^{E}\sum_{k=1}^{\infty} a^{0}_{m,k}$$

The remaining summation can be bounded by applying Lemma 2.2.5 (page 21) and Lemma 3.3.1 (page 35) as follows:

$$\sum_{m=0}^{E}\sum_{k=1}^{\infty} a^0_{m,k} \leq \sum_{m=0}^{\infty} a^0_m = \mathsf{E}_{\lambda/\mu}(B_0) \leq C \cdot \sqrt{\lambda/\mu}$$

Finally by applying the bounds in the condition of the lemma,

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1\,, \quad |v| < \bar{v} \quad \text{and} \quad \rho < \rho_0\,,$$

it follows that

$$\begin{aligned}\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} &\leq 8C \cdot \bar{v}^2 \cdot e^{2\bar{v}\kappa_1 + (1-\kappa_1)^{-1}\kappa_1} \cdot \left(\frac{\lambda}{\mu}\right)^{3+\rho} \cdot \frac{1}{\mathsf{N}^2} \\ &\leq 8C \cdot \bar{v}^2 \cdot e^{2\bar{v}\kappa_1 + (1-\kappa_1)^{-1}\kappa_1} \cdot \kappa_1 \cdot \left(\frac{\lambda}{\mu}\right)^{1+\rho} \cdot \frac{1}{\mathsf{N}}\end{aligned}$$

Next consider Σ_2, firstly I will upper bound the terms in the summation

$$\begin{aligned}4v^2d^3 \cdot a^\epsilon_{m,k} \cdot (m+1) \cdot e^{2\bar{v}dm} &\leq 8v^2d^3 m \cdot a^\epsilon_{m,k} \cdot e^{2\bar{v}dm} \\ &\leq 8v^2d^2 \cdot a^\epsilon_{m,k} \cdot e^{(1+2\bar{v})dm} \\ &\leq 8v^2d^2 \cdot a^0_{m,k} \cdot e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm}\end{aligned}$$

where the last inequality applies Lemma 7.3.7 (page 72) and Corollary 5.3.2 (page 49). Therefore

$$\begin{aligned}\Sigma_2 &\leq 8v^2d^2 \cdot \sum_{m>E}\sum_{k=1}^{\infty} a^0_{m,k} \cdot e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm} \\ &\leq 8v^2d^2 \cdot \sum_{m>E} a^0_m \cdot e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm}\end{aligned}$$

I multiple and divide by $e^{\delta m}$ and due to the fact that I am summing over $m > E$ I can bound $e^{-\delta m} \leq e^{-\delta E}$

$$\leq 8v^2d^2 \cdot e^{-\delta E} \cdot \sum_{m=0}^{\infty} a^0_m \cdot e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm+\delta m}$$

Now I use the fact that $a^0_m = \sum_{j=m+1}^{\infty} b^0_j$ and then change the order of summation

$$\begin{aligned}&\leq 8v^2d^2 \cdot e^{-\delta E} \cdot \sum_{m=0}^{\infty} e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm+\delta m} \cdot \sum_{j=m+1}^{\infty} b^0_j \\ &= 8v^2d^2 \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty}\sum_{m=0}^{j-1} e^{((1-\kappa_1)^{-1}+1+2\bar{v})dm+\delta m} \cdot b^0_j\end{aligned}$$

Finally I can use $m \leq j$ to bound the terms within the summation, and then apply the inequality $jd < e^{jd}$

$$\leq 8v^2 d^2 \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} j \cdot e^{((1-\kappa_1)^{-1}+1+2\bar{v})dj+\delta j} \cdot b_j^0$$

$$\leq 8v^2 d \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} e^{((1-\kappa_1)^{-1}+2+2\bar{v})dj+\delta j} \cdot b_j^0$$

$$= \frac{8v^2 d}{\mathsf{N}} \cdot \mathsf{N} e^{-\delta E} \cdot \mathsf{M}_{B_0}\left[((1-\kappa_1)^{-1} + 2 + 2\bar{v})d + \delta\right]$$

If we choose[2] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$\kappa_1((1-\kappa_1)^{-1} + 2 + 2\bar{v}) + 0.375\alpha < 0.75\alpha$$

holds then one can apply Corollary 3.3.6 which implies

$$\mathsf{M}_{B_0}[((1-\kappa_1)^{-1} + 2 + 2\bar{v})d + \delta] \leq C \cdot \lambda/\mu\,.$$

Therefore by applying the conditions in the lemma,

$$\mathsf{N} \exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad |v| < \bar{v} \quad \text{and} \quad \rho_1 < \rho\,,$$

it follows

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \leq 8\bar{v}^2 \cdot C \cdot \kappa_2 \cdot \left(\frac{\lambda}{\mu}\right)^{0.5} \cdot \frac{1}{\mathsf{N}}$$

This completes the proof. □

8.4 Application

Theorem 8.4.1. *Fix contants $\rho_0 > 0$ and $\kappa_2^* > 0$. For any choice of $0 < \bar{v} < 0.5$ there exists $\kappa_1^* = \kappa_1^*(\bar{v}) > 0$ and $\kappa_3^* = \kappa_3^*(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1^*\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2^*\,, \quad \frac{\mu}{\lambda} < \kappa_3^*\,,$$

$$0 < \rho < \min\left\{\frac{\rho_0}{2}, \frac{1}{7}\right\} \quad \textit{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

[2]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

then

$$\left|\mathsf{M}_{T_d^+}(2\lambda vd) - \mathsf{M}_{T_\epsilon}(vd)\right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{1.5+\rho}\frac{1}{\mathsf{N}}\right)$$

where the implicit constant in the big O is dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*$ *and* $\bar{v}$.

Proof. I begin by fixing constants: $\rho_0 > 0$, $\kappa_2^* > 0$ and $0 < \bar{v} < 0.5$. Next I choose $\bar{\kappa}_1$ such that $\kappa_1 = \bar{\kappa}_1$ and $\kappa_2 = \kappa_2^*$ satisfy the conditions of Lemmas 7.4.1, 8.3.1 and 8.3.2. Consequently, when I require

$$\frac{\left|\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}}, \quad \frac{\left|\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} \quad \text{and} \quad \frac{\left|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)\right|}{d\sqrt{\lambda/\mu}}$$

to be sufficiently small, I need only concern myself with the smallness of

$$\left(\frac{\lambda}{\mu}\right)^{1.5+\rho}\frac{1}{\mathsf{N}} \quad \text{and} \quad \left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}. \tag{8.5}$$

This is due to the fact that all other terms that appear in the main statement of Lemmas 7.4.1, 8.3.1 and 8.3.2 are dependent on quantities that have already been fixed. I also observe that both expressions in equation (8.5) can be made as small as one wishes by choosing κ_1^* and κ_3^* appropriately.

Let us now return to the statement of this theorem. In order to compare $\mathsf{M}_{T_d^+}(2\lambda vd)$ and $\mathsf{M}_{T_\epsilon}(vd)$ I start by re-writing $\mathsf{M}_{T_d^+}(2\lambda vd) - \mathsf{M}_{T_\epsilon}(vd)$ in terms of excursions and Chapter 6 provides the machinery to be able to do this. Equation (6.5) (page 61) is a key result and enables one to deduce the following

$$\mathsf{M}_{T_d^+}(2\lambda vd) - \mathsf{M}_{T_\epsilon}(vd) = \frac{F}{1-\mathsf{M}_{B_\epsilon}(vd)F}\cdot\left(\frac{\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)}{1-\mathsf{M}_{B_\epsilon}(vd)} + \mathsf{M}_{T_\epsilon}(vd)\right) + \frac{\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)}{1-\mathsf{M}_{B_\epsilon}(vd)} \tag{8.6}$$

where

$$\begin{aligned} F &= \frac{\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)}{1-\mathsf{M}_{B_\epsilon}(vd)}\cdot\frac{1}{1-(\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd))} \\ &= \frac{\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)}{(1-\mathsf{M}_{B_d}(vd)) + (\mathsf{M}_{B_d}(vd) - \mathsf{M}_{B_\epsilon}(vd))}\cdot\frac{1}{1-(\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd))} \end{aligned}$$

Next, in order to use Lemma 8.3.1, Lemma 7.4.1 and Lemma 5.4.2 to bound equation (8.6), I divide the appropriate numerators and denominators of fractions by $d\sqrt{\lambda/\mu}$

and then apply the aforementioned results. Now by choosing κ_1^* and κ_3^* small enough, one can uniformly seperate all the denominators in equation (8.6) away from zero for all $|v| < \bar{v}$. This allows us to write equation (8.6) as

$$|\mathsf{M}_{T_d^+}(2\lambda vd) - \mathsf{M}_{T_\epsilon}(vd)| \leq C \left(\frac{|\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)|}{d\sqrt{\lambda/\mu}} \cdot \frac{|\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)|}{d\sqrt{\lambda/\mu}} + \frac{|\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)|}{d\sqrt{\lambda/\mu}} \cdot |\mathsf{M}_{T_\epsilon}(vd)| + \frac{|\mathsf{M}_{A_d^+}(2\lambda vd) - \mathsf{M}_{A_\epsilon}(vd)|}{d\sqrt{\lambda/\mu}} \right)$$

where C is a constant dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*, \bar{\kappa}_1$ and $\bar{v}$. Finally by applying Lemma 8.3.2, Lemma 8.3.1, Theorem 7.5.1 and Theorem 5.4.4 we reach the stated result.

□

Chapter 9

Continuous time perturbation

9.1 Introduction

In this chapter I will compare the survival time of two continuous time Markov chain models with killing. Please recall the notation introduced in sections 7.2.1, 7.2.2 and 7.2.3 which can be found in chapter 7. In particular recall that d is defined as

$$d = \frac{\lambda}{\mu \mathsf{N}}\,.$$

9.2 Models

I will compare the following two models; the Continuous Time Linear Killing model is the same model that I introduced and studied in Chapter 8.

9.2.1 Continuous Time Linear Killing model

Define $(Z_t)_{t\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \ldots, \mathsf{N}\}$ evolving with jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } 2\lambda \cdot p_k \\ k &\to k+1 && \text{rate } 2\lambda \cdot q_k(1-d_k) \\ k &\to * && \text{rate } 2\lambda \cdot q_k d_k \end{aligned}$$

where

$$q_k = \frac{\lambda}{\lambda + \mu k} \qquad p_k = \frac{\mu k}{\lambda + \mu k} \qquad d_k = \frac{k}{\mathsf{N}}$$

In picture form it looks like

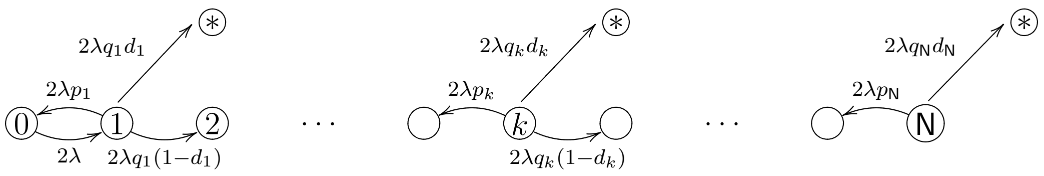


Figure 9.1: Markov chain Z_t - Continuous Time Linear Killing model

For this specific Markov chain I recall the following notation:

- T_d^+ - time until death starting from state λ/μ.
- A_d^+ - time until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B_d^+).
- B_d^+ - time until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A_d^+).

Furthermore I recall the following moment generating functions:

$$\mathsf{M}_{T_d^+}(v) = \mathsf{E}\big(\exp\left(v \cdot T_d^+\right)\big)\,, \quad \mathsf{M}_{B_d^+}(v) = \mathsf{E}\big(\exp\left(v \cdot B_d^+\right)\big)\,, \quad \mathsf{M}_{A_d^+}(v) = \mathsf{E}\big(\exp\left(v \cdot A_d^+\right)\big)$$

Remark 9.2.1. *An alternative but equivalent way to describe the process Z_t is that it jumps with probabilities as shown in Figure 9.2 (which incidentially is the same as the Discrete Time Linear Killing model) however there is a holding time at each state which is exponentially distributed with parameter 2λ.*

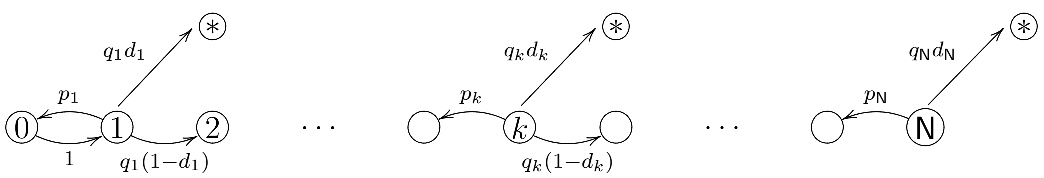


Figure 9.2: Jump chain of Z_t

Remark 9.2.2. *$q_k d_k$ can be bounded, uniformly in k, as follows*

$$q_k d_k = \frac{\lambda/\mu}{\lambda/\mu + k} \cdot \frac{k}{\mathsf{N}} = \frac{k}{\lambda/\mu + k} \cdot \frac{\lambda/\mu}{\mathsf{N}} \leq \frac{\lambda/\mu}{\mathsf{N}} = d\,.$$

9.2.2 Continuous Time Linear Holding model

Define $(W_t)_{t\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \ldots, \mathsf{N}\}$ evolving with jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } \mu k \\ k &\to k+1 && \text{rate } \lambda(1-d_k) \\ k &\to * && \text{rate } \lambda \cdot d_k \end{aligned}$$

where $d_k = \frac{k}{\mathsf{N}}$. In picture form it looks like

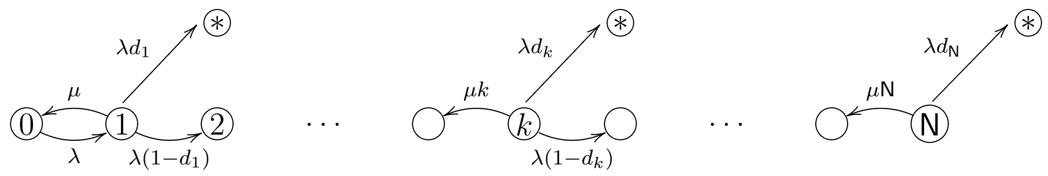


Figure 9.3: Markov chain W_t - Continuous Time Linear Holding model

For this specific Markov chain I will introduce the following notation:

- T^+_ϵ - time until death starting from state λ/μ.
- A^+_ϵ - time until death starting from state λ/μ but without any returns (trajectories that return to state λ/μ before death contribute to B^+_ϵ).
- B^+_ϵ - time until the first return to state λ/μ (trajectories that die before returning to state λ/μ contribute to A^+_ϵ).

Furthermore I define the following moment generating functions:

$$\mathsf{M}_{T^+_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot T^+_\epsilon)\big)\,, \quad \mathsf{M}_{B^+_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot B^+_\epsilon)\big)\,, \quad \mathsf{M}_{A^+_\epsilon}(v) = \mathsf{E}\big(\exp(v \cdot A^+_\epsilon)\big)$$

Remark 9.2.3. *Again an equivalent way to describe the process W_t is that it jumps with probabilities as shown in Figure 9.2 (same as the Discrete Time Linear Killing model) however there is a holding time, which at state k, is exponentially distributed with parameter $\lambda + \mu k$.*

9.3 Intermediate results

I will now introduce some key results which crop up repeatedly in the argument below.

Definition 9.3.1. *Define ϵ_x as follows*

$$\epsilon_x = \frac{vd \cdot \left(\frac{\lambda}{\mu} - x\right)}{\frac{\lambda}{\mu} + x - 2vd \cdot \frac{\lambda}{\mu}} \tag{9.1}$$

where $x \in \{0, 1, 2, \ldots, \mathsf{N}\}$.

Lemma 9.3.2. *If $|2vd| < \kappa < 1$ then*

$$|\epsilon_x| \leq \frac{|v|}{1-\kappa} \cdot \frac{|\lambda/\mu - x|}{\mathsf{N}} \tag{9.2}$$

Proof. Note that the denominator of $|\epsilon_x|$ can be bounded as follows

$$\begin{aligned} \left|\frac{\lambda}{\mu} + x - 2vd \cdot \frac{\lambda}{\mu}\right| &> \left|\frac{\lambda}{\mu} + x\right| - \left|2vd \cdot \frac{\lambda}{\mu}\right| \\ &\geq \left|\frac{\lambda}{\mu}\right| - \left|2vd \cdot \frac{\lambda}{\mu}\right| \\ &= \frac{\lambda}{\mu} \cdot (1 - |2vd|) \\ &> \frac{\lambda}{\mu}(1-\kappa) \end{aligned}$$

where the last inequality applies the condition of the lemma. Therefore

$$|\epsilon_x| \leq \frac{|vd| \cdot \left|\frac{\lambda}{\mu} - x\right|}{\left|\frac{\lambda}{\mu} + x - 2vd \cdot \frac{\lambda}{\mu}\right|} \leq \frac{|vd| \cdot \left|\frac{\lambda}{\mu} - x\right|}{\frac{\lambda}{\mu}(1-\kappa)} = \frac{|v|}{1-\kappa} \cdot \frac{\left|\frac{\lambda}{\mu} - x\right|}{\mathsf{N}}$$

□

Remark 9.3.3. *Recall definition 7.2.4; a trajectory $\mathcal{X}$ has length m (which is written $|\mathcal{X}| = m$) if and only if $\mathcal{X}$ is of the form*

$$\mathcal{X} = \left(x_0 = \frac{\lambda}{\mu}, x_1 \notin \left\{ \frac{\lambda}{\mu}, * \right\}, \ldots, x_{m-1} \notin \left\{ \frac{\lambda}{\mu}, * \right\}, x_m \notin \{*\} \right).$$

Lemma 9.3.4. *Consider a trajectory $\mathcal{X}$ such that $|\mathcal{X}| = m$. Let $\mathcal{X} = (x_0, x_1, \ldots, x_m)$. If $|2vd| < \kappa < 1$ then*

$$\left| \prod_{i=0}^{m} (1 + \epsilon_{x_i}) - 1 \right| \leq e^{|v|(1-\kappa)^{-1} d(m+1)} \tag{9.3}$$

Proof. I will start by recalling Definition 9.3.1:

$$\epsilon_x = \frac{vd \cdot \left(\frac{\lambda}{\mu} - x \right)}{\frac{\lambda}{\mu} + x - 2vd \cdot \frac{\lambda}{\mu}}.$$

In order to prove the lemma, it is necessary to consider the cases v positive and v negative seperately, I will start by assuming $v > 0$. For any trajectory whose first jump is to the right, and taking into account $2vd < 1$ (from the conditions of the lemma), we have $-1 \leq \epsilon_{x_i} \leq 0$ for $0 \leq i \leq m$. Therefore

$$0 \leq \prod_{i=0}^{m} (1 + \epsilon_{x_i}) \leq 1$$

which implies the desired result. For any trajectory whose first jump is to the left, and taking into account $2vd < 1$ (from the conditions of the lemma), we see $\epsilon_{x_i} \geq 0$ for $0 \leq i \leq m$. Moreover ϵ_{x_i} is maximised when $x_i = 0$. Consequently

$$1 \leq \prod_{i=0}^{m} (1 + \epsilon_{x_i}) \leq \left(1 + \frac{v}{1-\kappa} \cdot \frac{\lambda}{\mu \mathsf{N}} \right)^{m+1}$$

Finally by applying Lemma 2.3.1:

$$\leq e^{v(1-\kappa)^{-1} d(m+1)}$$

This completes the case for $v > 0$. Now I assume that $v < 0$ and for a trajectory whose first jump is to the left we have $-1 \leq \epsilon_{x_i} \leq 0$ for $0 \leq i \leq m$. Therefore

$$0 \leq \prod_{i=0}^{m} (1 + \epsilon_{x_i}) \leq 1$$

which implies the desired result. For any trajectory whose first jump is to the right

$$0 \leq \epsilon_{x_i} \leq \frac{vd\big(\frac{\lambda}{\mu} - x_i\big)}{\frac{\lambda}{\mu}(1-\kappa) + x_i} = \frac{-vd\big(x_i - \frac{\lambda}{\mu}\big)}{\frac{\lambda}{\mu}(1-\kappa) + x_i} \leq \frac{-vd\big(x_i - 0\big)}{0 + x_i} = -vd \quad \text{for } 0 \leq i \leq m$$

Therefore

$$1 \leq \prod_{i=0}^{m} (1 + \epsilon_{x_i}) \leq \prod_{i=0}^{m} (1 - vd) = (1 - vd)^{m+1} \leq e^{-vd(m+1)}$$

the final inequality applies Lemma 2.3.1. This completes the case for $v < 0$, and by using absolute value bars, as shown in the right hand side of equation (9.3), ensures the result holds for v positive and negative. □

9.4 Main results

Lemma 9.4.1. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $0 < \bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \ , \quad \mathsf{N} \exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \ ,$$

$$0 < \rho < \min\left\{\frac{1}{7}, \rho_0\right\} \quad \text{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

then

$$\begin{aligned}\frac{\left|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}} \leq\ & \frac{\bar{v}}{1-\kappa_1} \cdot \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot \exp\left(1.5\bar{v}\kappa_1 + \bar{v}(1-\kappa_1)^{-1}\kappa_1\right) \\ & + \frac{\bar{v} \cdot C}{1-\kappa_1} \cdot e^{((1-\kappa_1)^{-1} + 1.5\bar{v})\kappa_1 + \bar{v}(1-\kappa_1)^{-1}\kappa_1} \cdot \left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \\ & + C \cdot \kappa_2 \cdot e^{1.5\bar{v}\kappa_1 + (1+\bar{v})(1-\kappa_1)^{-1}\kappa_1} \cdot \left(\frac{\mu}{\lambda}\right)^{0.5-\rho} \\ & + C \cdot \kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5}\end{aligned}$$

This implies

$$\frac{\left|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 and $\bar{v}$.

Proof. It is necessary to express $\mathsf{M}_{B_d^+}(2\lambda vd)$ and $\mathsf{M}_{B_\epsilon^+}(2\lambda vd)$ in such a way that I am able to bound the difference between the two. To this end I apply the law of total probability and condition on the trajectory because then the time the trajectory takes will just be the sum of independent exponentially distributed random variables. Recalling that the moment generating function of $X \sim \text{Exp}(\sigma)$ is

$$\mathsf{E}(e^{X \cdot u}) = \frac{\sigma}{\sigma - u},$$

it follows

$$\mathsf{M}_{B_\epsilon^+}(u) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \prod_{i=0}^{m-1} \frac{\lambda + \mu x_i}{\lambda + \mu x_i - u}$$

and

$$\mathsf{M}_{B_d^+}(u) = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \prod_{i=0}^{m-1} \frac{2\lambda}{2\lambda - u}.$$

Therefore

$$\begin{aligned}
&\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd) \\
&\quad = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \left[\left(\prod_{i=0}^{m-1} \frac{\lambda + \mu x_i}{\lambda + \mu x_i - 2\lambda vd} \right) - \left(\prod_{i=0}^{m-1} \frac{2\lambda}{2\lambda - 2\lambda vd} \right) \right] \\
&\quad = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot \left(\frac{1}{1 - vd} \right)^m \cdot \left[\prod_{i=0}^{m-1} (1 + \epsilon_{x_i}) - 1 \right] \\
&\quad = \sum_{m=1}^{\infty} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot \left(\frac{1}{1 - vd} \right)^m \cdot \left[\sum_{\substack{A \subseteq [0,m-1]: \\ A \neq \emptyset}} \prod_{a \in A} \epsilon_{x_a} \right]
\end{aligned}$$

where ϵ_x is as defined earlier in the chapter (Definition 9.3.1). I now define

$$D = \left(\frac{\lambda}{\mu} \right)^{0.5} \quad \text{and} \quad E = \left(\frac{\lambda}{\mu} \right)^{1+\rho}$$

and using Lemma 2.3.11 to bound $(1 - vd)^{-1}$ I decompose the previous expression as follows

$$|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)| \leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$$

where

$$\Sigma_1 = \sum_{m=1}^{D} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \sum_{\substack{A\subseteq[0,m-1]: \\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm} \cdot \prod_{a\in A} |\epsilon_{x_a}| \tag{9.4}$$

$$\Sigma_2 = \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}||\leq m^{0.75}}} \sum_{\substack{A\subseteq[0,m-1]: \\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm} \cdot \prod_{a\in A} |\epsilon_{x_a}| \tag{9.5}$$

$$\Sigma_3 = \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}||> m^{0.75}}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm} \cdot \left| \prod_{i=0}^{m-1} (1+\epsilon_{x_i}) - 1 \right| \tag{9.6}$$

$$\Sigma_4 = \sum_{m>E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm} \cdot \left| \prod_{i=0}^{m-1} (1+\epsilon_{x_i}) - 1 \right| \tag{9.7}$$

Remark 9.4.2. *Recall definition 7.2.5; a trajectory $\mathcal{X}$ with length m, deviates less than n from the starting position (which is written $||\mathcal{X}|| < n$) if and only if the following conditions are satisfied*

$$|x_1 - x_0| < n\,, \qquad |x_2 - x_0| < n\,, \quad \ldots\,, \quad |x_m - x_0| < n\,.$$

I will now bound each of the above sums. I start with Σ_1 and proceed to bound the inner summation. Firstly I apply Lemma 9.3.2 and then I use the fact that a length m trajectory can not venture further away from state λ/μ by distance m

$$\prod_{a\in A} |\epsilon_{x_a}| \leq \prod_{a\in A} \frac{|v|\cdot|\lambda/\mu - x_a|}{(1-\kappa_1)\mathsf{N}} \leq \left(\frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^{|A|}$$

Now I return to the full sum, the first inequality applies the above bound and the last inequality applies Lemma 2.3.1

$$\begin{aligned}
\Sigma_1 &\leq \sum_{m=1}^{D} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu}} \sum_{\substack{A\subseteq[0,m-1]: \\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm} \cdot \left(\frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^{|A|} \\
&= \sum_{m=1}^{D} b_m^\epsilon \cdot e^{1.5vdm} \cdot \left[\left(1 + \frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^m - 1\right] \\
&\leq \sum_{m=1}^{D} b_m^\epsilon \cdot e^{1.5vdm} \cdot \left[\exp\left(|v|(1-\kappa_1)^{-1}\cdot\frac{m^2}{\mathsf{N}}\right) - 1\right]
\end{aligned}$$

Finally by using Lemma 2.3.2 (page 22) to bound the term in square brackets and then taking uniform upper bounds to bound all the terms next to b^{ϵ}_m we obtain

$$\leq \sum_{m=1}^{D} b^{\epsilon}_m \cdot e^{1.5vdm} \cdot \frac{|v| \cdot m^2}{(1-\kappa_1)\mathsf{N}} \cdot \exp\left(\frac{|v| \cdot m^2}{(1-\kappa_1)\mathsf{N}}\right)$$

$$\leq e^{1.5vdD} \cdot \frac{|v| \cdot D^2}{(1-\kappa_1)\mathsf{N}} \cdot \exp\left(\frac{|v| \cdot D^2}{(1-\kappa_1)\mathsf{N}}\right) \cdot \sum_{m=1}^{D} b^{\epsilon}_m$$

The remaining summation is clearly bounded from above by one. Additionally by applying the bounds in the condition of the lemma

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \quad \text{and} \quad |v| < \bar{v}\,,$$

it follows that

$$\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} \leq \frac{\bar{v}}{1-\kappa_1} \cdot \left(\frac{\mu}{\lambda}\right)^{0.5} \cdot \exp\left(1.5\bar{v}\kappa_1 + \bar{v}(1-\kappa_1)^{-1}\kappa_1\right)$$

Moving onto expression Σ_2, I apply Lemma 9.3.2 and then I use the fact that I am summing over trajectories that do not venture further away from state λ/μ by distance $m^{0.75}$

$$\prod_{a \in A} |\epsilon_{x_a}| \leq \prod_{a \in A} \frac{|v| \cdot |\lambda/\mu - x_a|}{(1-\kappa_1)\mathsf{N}} \leq \left(\frac{|v| \cdot m^{0.75}}{(1-\kappa_1)\mathsf{N}}\right)^{|A|}$$

Now putting everything together gives

$$\begin{aligned}
\Sigma_2 &\leq \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}|| \leq m^{0.75}}} \sum_{\substack{A \subseteq [0,m-1]: \\ A \neq \emptyset}} \mathbb{P}_{\epsilon}(\mathcal{X}) e^{1.5vdm} \cdot \left(\frac{|v| \cdot m^{0.75}}{(1-\kappa_1)\mathsf{N}}\right)^{|A|} \\
&= \sum_{m=D+1}^{E} \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}|| \leq m^{0.75}}} \mathbb{P}_{\epsilon}(\mathcal{X}) \cdot e^{1.5vdm} \cdot \left[\left(1 + \frac{|v| \cdot m^{0.75}}{(1-\kappa_1)\mathsf{N}}\right)^m - 1\right] \\
&\leq \sum_{m=D+1}^{E} b^{\epsilon}_m \cdot e^{1.5vdm} \cdot \left[\exp\left(|v|(1-\kappa_1)^{-1} \cdot \frac{m^{1.75}}{\mathsf{N}}\right) - 1\right]
\end{aligned}$$

Next I use Lemma 7.3.7 (page 72) to bound b^{ϵ}_m and Lemma 2.3.2 (page 22) to bound the term in square brackets

$$\begin{aligned}
&\leq \sum_{m=D+1}^{E} b^{d}_m \cdot e^{(1-\kappa_1)^{-1}dm + 1.5vdm} \cdot \frac{|v| \cdot m^{1.75}}{(1-\kappa_1)\mathsf{N}} \cdot \exp\left(\frac{|v| \cdot m^{1.75}}{(1-\kappa_1)\mathsf{N}}\right) \\
&= \sum_{m=D+1}^{E} \left[b^{d}_m \cdot m^{1.5}\right] \cdot \frac{|v| \cdot m^{0.25} \cdot e^{(1-\kappa_1)^{-1}dm + 1.5vdm}}{(1-\kappa_1)\mathsf{N}} \cdot \exp\left(\frac{|v| \cdot m^{1.75}}{(1-\kappa_1)\mathsf{N}}\right)
\end{aligned}$$

Then taking uniform upper bounds for all the terms next to $b_m^d \cdot m^{1.5}$ we obtain

$$\leq e^{(1-\kappa_1)^{-1}dE+1.5vdE} \cdot \frac{|v| \cdot E^{0.25}}{(1-\kappa_1)\mathsf{N}} \cdot \exp\left(\frac{|v| \cdot E^{1.75}}{(1-\kappa_1)\mathsf{N}}\right) \sum_{m=D+1}^{E} b_m^d \cdot m^{1.5}$$

Now I use Lemma 5.3.2 which gives us $b_m^d < b_m^0$ and then we apply Lemma 3.3.1 (page 35) in order to bound the remaining summation

$$\sum_{m=D+1}^{E} b_m^d \cdot m^{1.5} \leq C \cdot \frac{\lambda}{\mu}$$

Additionally by applying the bounds in the condition of the lemma

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0} \frac{1}{\mathsf{N}} < \kappa_1 \,, \quad |v| < \bar{v} \,, \quad \rho < \rho_0 \quad \text{and} \quad \rho < \frac{1}{7} \,,$$

it follows that

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \leq \bar{v} \cdot C \cdot (1-\kappa_1)^{-1} \cdot e^{((1-\kappa_1)^{-1}+1.5\bar{v})\kappa_1+\bar{v}(1-\kappa_1)^{-1}\kappa_1} \cdot \left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}$$

Moving onto expression Σ_3, one can bound the contents of the summation by applying Lemma 9.3.4

$$\left|\prod_{i=0}^{m-1} (1+\epsilon_{x_i}) - 1\right| \leq e^{|v|(1-\kappa_1)^{-1}dm}$$

Now taking into account the above bound

$$\Sigma_3 \leq \sum_{m=D+1}^{E} e^{1.5vdm+|v|(1-\kappa_1)^{-1}dm} \cdot \sum_{\substack{|\mathcal{X}|=m: \\ x_m=\lambda/\mu \\ ||\mathcal{X}||>m^{0.75}}} \mathbb{P}_\epsilon(\mathcal{X})$$

then by using Lemma 7.3.6 and Lemma 7.3.5 (page 71) to bound $\mathbb{P}_\epsilon(\mathcal{X})$ it follows

$$\leq \sum_{m=D+1}^{E} e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm} \cdot \sum_{\substack{|\mathcal{X}|=m: \\ ||\mathcal{X}||>m^{0.75}}} \mathbb{P}_0(\mathcal{X})$$

Next I use Lemma 2.1.8 (page 16) to bound the inner summation, then I take a uniform upper bound for all the terms that remain

$$\leq \sum_{m=D+1}^{E} e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm} \cdot C \cdot e^{-\gamma \cdot m^{0.5}}$$

$$\leq E \cdot e^{1.5vdE+(1+|v|)(1-\kappa_1)^{-1}dE} \cdot C \cdot e^{-\gamma \cdot D^{0.5}}$$

Finally I multiple and divide by N

$$= \left(\frac{\lambda}{\mu}\right)^{1+\rho} \cdot \frac{1}{\mathsf{N}} \cdot \exp\left(1.5vd\left(\frac{\lambda}{\mu}\right)^{1+\rho} + \frac{(1+|v|)d}{1-\kappa_1}\left(\frac{\lambda}{\mu}\right)^{1+\rho}\right) \cdot C \cdot \mathsf{N}\exp\left(-\gamma \cdot \left(\frac{\lambda}{\mu}\right)^{0.25}\right)$$

By applying the bounds in the condition of the lemma

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad |v| < \bar{v}\,, \quad \rho < \rho_0 \quad \text{and} \quad \rho_1 < \frac{1}{4}\,,$$

it follows that

$$\frac{\Sigma_3}{d\sqrt{\lambda/\mu}} \leq \left(\frac{\mu}{\lambda}\right)^{0.5-\rho} \cdot e^{1.5\bar{v}\kappa_1 + (1+\bar{v})(1-\kappa_1)^{-1}\kappa_1} \cdot C \cdot \kappa_2$$

Moving onto expression Σ_4, one can bound the contents of the summation by applying Lemma 9.3.4

$$\left|\prod_{i=0}^{m-1}(1+\epsilon_{x_i}) - 1\right| \leq e^{|v|(1-\kappa_1)^{-1}dm}$$

Now I return to the full sum and after taking into account the above inequality I use Lemma 7.3.6 and Lemma 7.3.5 (page 71) to bound $\mathbb{P}_\epsilon(\mathcal{X})$

$$\begin{aligned}
\Sigma_4 &\leq \sum_{m>E}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot e^{1.5vdm+|v|(1-\kappa_1)^{-1}dm} \\
&\leq \sum_{m>E}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=\lambda/\mu}} \mathbb{P}_0(\mathcal{X}) \cdot e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm} \\
&= \sum_{m>E} b_m^0 \cdot e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm}
\end{aligned}$$

Next I multiple and divide by $e^{\delta m}$ and due to the fact that I am summing over $m > E$ I can bound $e^{-\delta m} < e^{-\delta E}$

$$\begin{aligned}
&\leq e^{-\delta E} \cdot \sum_{m=1}^{\infty} b_m^0 \cdot e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm+\delta m} \\
&= \mathsf{N}\exp\left[-\delta\left(\frac{\lambda}{\mu}\right)^{1+\rho}\right] \cdot \frac{1}{\mathsf{N}} \cdot \mathsf{M}_{B_0}(1.5vd + (1+|v|)(1-\kappa_1)^{-1}d + \delta)
\end{aligned}$$

Finally I choose[1] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$\kappa_1(1.5\bar{v} + (1+\bar{v})(1-\kappa_1)^{-1}) + 0.375\alpha < 0.75\alpha$$

[1]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

holds. This allows me to apply Corollary 3.3.6 (page 40) which implies

$$\mathsf{M}_{B_0}(1.5vd + (1+|v|)(1-\kappa_1)^{-1}d + \delta) \leq C \cdot \lambda/\mu\,.$$

By applying this result and using a condition of the lemma

$$\mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \quad \text{and} \quad \rho_1 < \rho\,,$$

it follows that

$$\frac{\Sigma_4}{d\sqrt{\lambda/\mu}} \leq C \cdot \kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5}$$

This completes the proof. □

Lemma 9.4.3. *Fix constants $\rho_0 > 0$ and $\kappa_2 > 0$. For any $0 < \bar{v} < 0.5$ there exists $\kappa_1 = \kappa_1(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2 \quad \textit{and} \quad 0 < \rho_1 < \rho < \frac{\rho_0}{2}$$

then

$$\frac{\left|\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}} \leq 2\bar{v}(1-\kappa_1)^{-1} \cdot \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}} \cdot e^{3\bar{v}\kappa_1 + (1+2\bar{v})(1-\kappa_1)^{-1}\cdot\kappa_1} + C\kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5}$$

This implies

$$\frac{\left|\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}} = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.5}\right)$$

where the implicit constant in the big O is dependent on κ_1, κ_2 and $\bar{v}$.

Proof. It is necessary to express $\mathsf{M}_{A_d^+}(2\lambda vd)$ and $\mathsf{M}_{A_\epsilon^+}(2\lambda vd)$ in such a way that I am able to bound the difference between the two. To this end I apply the law of total probability and condition on the trajectory because then the time the trajectory takes will just be the sum of independent exponentially distributed random variables. Recalling that the moment generating function of $X \sim \mathrm{Exp}(\sigma)$ is

$$\mathsf{E}(e^{X\cdot u}) = \frac{\sigma}{\sigma - u}\,,$$

it follows

$$\mathsf{M}_{A_\epsilon^+}(u) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot \prod_{i=0}^{m} \frac{\lambda+\mu x_i}{\lambda+\mu x_i - u}$$

and

$$\mathsf{M}_{A_d^+}(u) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot \prod_{i=0}^{m} \frac{2\lambda}{2\lambda - u}.$$

Therefore

$$\begin{aligned}
&\mathsf{M}_{A_\epsilon^+}(2\lambda v d) - \mathsf{M}_{A_d^+}(2\lambda v d)\\
&\quad= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot \left[\prod_{i=0}^{m} \frac{\lambda+\mu x_i}{\lambda+\mu x_i - 2\lambda v d} - \prod_{i=0}^{m} \frac{2\lambda}{2\lambda - 2\lambda v d}\right]\\
&\quad= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot \left(\frac{1}{1-vd}\right)^{m+1} \cdot \left[\prod_{i=0}^{m} (1+\epsilon_{x_i}) - 1\right]\\
&\quad= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \sum_{\substack{A\subseteq[0,m]:\\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot \left(\frac{1}{1-vd}\right)^{m+1} \cdot \prod_{a\in A} \epsilon_{x_a}
\end{aligned}$$

where ϵ_x is as defined at the beginning of the chapter (Definition 9.3.1). I now define

$$E = \left(\frac{\lambda}{\mu}\right)^{1+\rho}$$

and using Lemma 2.3.11 to bound $(1-vd)^{-1}$ I decompose the previous expression as follows

$$|\mathsf{M}_{A_\epsilon^+}(2\lambda v d) - \mathsf{M}_{A_d^+}(2\lambda v d)| \leq \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \sum_{\substack{A\subseteq[0,m]:\\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot e^{1.5vd(m+1)} \cdot \prod_{a\in A} |\epsilon_{x_a}| \tag{9.8}$$

$$\Sigma_2 = \sum_{m>E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot e^{1.5vd(m+1)} \cdot \left|\prod_{i=0}^{m} (1+\epsilon_{x_i}) - 1\right| \tag{9.9}$$

I will now bound each of the above sums. I start with Σ_1 and one can bound the contents of the summation as follows. Firstly I apply Lemma 9.3.2 and then I use the fact that a lengh m trajectory can not venture further away from state λ/μ by distance m, (i.e. $|\lambda/\mu - x_a| \leq m$):

$$\prod_{a\in A} |\epsilon_{x_a}| \leq \prod_{a\in A} \frac{|v|\cdot|\lambda/\mu - x_a|}{(1-\kappa_1)\mathsf{N}} \leq \left(\frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^{|A|}$$

Now putting everything together and bounding $q_k d_k < d$ (remark 9.2.2), we obtain

$$\begin{aligned}
\Sigma_1 &\leq \sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}}\sum_{\substack{A\subseteq[0,m]:\\ A\neq\emptyset}} \mathbb{P}_\epsilon(\mathcal{X})\cdot q_k d_k \cdot e^{1.5vd(m+1)}\cdot\left(\frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^{|A|}\\
&= \sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) e^{1.5vd(m+1)}\cdot q_k d_k\cdot\left[\left(1+\frac{|v|\cdot m}{(1-\kappa_1)\mathsf{N}}\right)^{m+1} - 1\right]\\
&\leq \sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) e^{1.5vd(m+1)}\cdot d\left[\exp\left(|v|(1-\kappa_1)^{-1}\cdot\frac{2m^2}{\mathsf{N}}\right) - 1\right]
\end{aligned}$$

I use Lemma 2.3.2 (page 22) to bound the term in square brackets. Additionally I use Lemma 7.3.6 and Lemma 7.3.5 (page 71) to bound $\mathbb{P}_\epsilon(\mathcal{X})$

$$\begin{aligned}
&\leq d\sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X})\cdot e^{1.5vd(m+1)}\cdot\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\cdot\exp\left(\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\right)\\
&\leq d\sum_{m=1}^{E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_0(\mathcal{X})\cdot e^{1.5vd(m+1)+(1-\kappa_1)^{-1}\cdot dm}\cdot\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\cdot\exp\left(\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\right)\\
&= d\sum_{m=1}^{E} a_m^0\cdot e^{1.5vd(m+1)+(1-\kappa_1)^{-1}\cdot dm}\cdot\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\cdot\exp\left(\frac{2|v|\cdot m^2}{(1-\kappa_1)\mathsf{N}}\right)
\end{aligned}$$

Then by uniformly bounding all the terms next to a_m^0 we obtain

$$\leq d\cdot e^{1.5vd(E+1)+(1-\kappa_1)^{-1}\cdot dE}\cdot\frac{2|v|\cdot E^2}{(1-\kappa_1)\mathsf{N}}\cdot\exp\left(\frac{2|v|\cdot E^2}{(1-\kappa_1)\mathsf{N}}\right)\cdot\sum_{m=1}^{E} a_m^0$$

The remaining summation can be bounded using Lemma 2.2.5 and Lemma 3.3.1

$$\sum_{m=1}^{E} a_m^0 \leq \sum_{m=0}^{\infty} a_m^0 = \mathsf{E}_{\lambda/\mu}(B_0) \leq C\cdot\sqrt{\lambda/\mu}$$

Additionally by applying the bounds in the condition of the lemma

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1\,,\quad |v| < \bar{v}\quad\text{and}\quad \rho < \frac{\rho_0}{2}\,,$$

it follows that

$$\frac{\Sigma_1}{d\sqrt{\lambda/\mu}} \leq 2\bar{v}(1-\kappa_1)^{-1} \cdot \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}} \cdot e^{3\bar{v}\kappa_1+(1+2\bar{v})(1-\kappa_1)^{-1}\cdot\kappa_1}$$

Moving onto expression Σ_2, one can bound the contents of the summation by applying Lemma 9.3.4

$$\left|\prod_{i=0}^{m}(1+\epsilon_{x_i})-1\right| \leq e^{|v|(1-\kappa_1)^{-1}d(m+1)}$$

Now putting everything together and bounding $q_k d_k < d$ (remark 9.2.2), we obtain

$$\begin{aligned}
\Sigma_2 &\leq \sum_{m>E}\sum_{k=1}^{\infty}\sum_{\substack{|\mathcal{X}|=m:\\ x_m=k}} \mathbb{P}_\epsilon(\mathcal{X}) \cdot q_k d_k \cdot e^{1.5vdm+|v|(1-\kappa_1)^{-1}dm} \\
&\leq \sum_{m>E} a_m^\epsilon \cdot d \cdot e^{1.5vdm+|v|(1-\kappa_1)^{-1}dm} \\
&\leq \sum_{m>E} a_m^0 \cdot d \cdot e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm}
\end{aligned}$$

where the last inequality applies Lemma 7.3.7 (page 72) and Corollary 5.3.2 (page 49). Then multiplying and dividing by $e^{\delta m}$ and using $e^{-\delta m} \leq e^{-\delta E}$ (due to the fact that I am summing over $m > E$) we obtain

$$\leq e^{-\delta E} \cdot \sum_{m>E} a_m^0 \cdot d \cdot e^{1.5vdm+(1+|v|)(1-\kappa_1)^{-1}dm+\delta m}$$

Next I use the fact that $a_m^0 = \sum_{j=m+1}^{\infty} b_j^0$, and then change the order of summation

$$\begin{aligned}
&\leq d \cdot e^{-\delta E} \cdot \sum_{m=0}^{\infty} e^{(1.5v+(1+|v|)(1-\kappa_1)^{-1})dm+\delta m} \cdot \sum_{j=m+1}^{\infty} b_j^0 \\
&= d \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty}\sum_{m=0}^{j-1} e^{(1.5v+(1+|v|)(1-\kappa_1)^{-1})dm+\delta m} \cdot b_j^0
\end{aligned}$$

Finally take uniform bounds over the terms involving m and apply the inequality $jd < e^{jd}$

$$\begin{aligned}
&\leq d \cdot e^{-\delta E} \cdot \sum_{j=1}^{\infty} j \cdot e^{(1.5v+(1+|v|)(1-\kappa_1)^{-1})dj+\delta j} \cdot b_j^0 \\
&\leq e^{-\delta E} \cdot \sum_{j=1}^{\infty} e^{(1.5v+1+(1+|v|)(1-\kappa_1)^{-1})dj+\delta j} \cdot b_j^0 \\
&= \mathsf{N} \cdot e^{-\delta E} \cdot \frac{1}{\mathsf{N}} \cdot \mathsf{M}_{B_0}((1.5v+1+(1+|v|)(1-\kappa_1)^{-1})d+\delta)
\end{aligned}$$

If we choose[2] $\delta = \frac{0.375\alpha}{\lambda/\mu}$ and κ_1 small enough so that

$$\kappa_1(1.5\bar{v} + 1 + (1+\bar{v})(1-\kappa_1)^{-1}) + 0.375\alpha < 0.75\alpha$$

holds then one can apply Corollary 3.3.6 which implies

$$\mathsf{M}_{B_0}((1.5v + 1 + (1+|v|)(1-\kappa_1)^{-1})d + \delta) \leq C \cdot \lambda/\mu\,.$$

Therefore by applying this result together with the conditions in the lemma,

$$\mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2\,, \quad |v| < \bar{v} \quad \text{and} \quad \rho_1 < \rho\,,$$

it follows

$$\frac{\Sigma_2}{d\sqrt{\lambda/\mu}} \leq C \cdot \kappa_2 \cdot \left(\frac{\mu}{\lambda}\right)^{0.5}$$

This completes the proof. □

9.5 Application

Theorem 9.5.1. *Fix contants $\rho_0 > 0$ and $\kappa_2^* > 0$. For any choice of $0 < \bar{v} < 0.5$ there exists $\kappa_1^* = \kappa_1^*(\bar{v}) > 0$ and $\kappa_3^* = \kappa_3^*(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \kappa_1^*\,, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \kappa_2^*\,, \quad \frac{\mu}{\lambda} < \kappa_3^*\,,$$

$$0 < \rho < \min\left\{\frac{\rho_0}{2}, \frac{1}{7}\right\} \quad \textit{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

then

$$\left|\mathsf{M}_{T_\epsilon^+}(2\lambda v d) - \mathsf{M}_{T_d^+}(2\lambda v d)\right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{2+2\rho}\cdot\frac{1}{\mathsf{N}}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and $\bar{v}$.

[2]The constant α is taken directly from Lemma 3.3.2, during the proof of which I derive that $0 < \alpha < 1/16$ is sufficient for the lemma to hold.

Proof. I begin by fixing constants: $\rho_0 > 0$, $\kappa_2^* > 0$ and $0 < \bar{v} < 0.5$. Next I choose $\bar{\kappa}_1$ such that $\kappa_1 = \bar{\kappa}_1$ and $\kappa_2 = \kappa_2^*$ satisfy the conditions of Lemmas 7.4.1, 8.3.1, 9.4.1 and 9.4.3. Consequently, when I require

$$\frac{\left|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}}, \quad \frac{\left|\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)\right|}{d\sqrt{\lambda/\mu}},$$

$$\frac{\left|\mathsf{M}_{B_d^+}(2\lambda vd) - \mathsf{M}_{B_\epsilon}(vd)\right|}{d\sqrt{\lambda/\mu}} \quad \text{and} \quad \frac{\left|\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d}(vd)\right|}{d\sqrt{\lambda/\mu}}$$

to be sufficiently small, I need only concern myself with the smallness of

$$\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \quad \text{and} \quad \left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{1}{\mathsf{N}} \tag{9.10}$$

This is due to the fact that all other terms that appear in the main statement of Lemmas 7.4.1, 8.3.1, 9.4.1 and 9.4.3 are dependent on quantities that have already been fixed. I also observe that both expressions in equation (9.10) can be made as small as one wishes by choosing κ_1^* and κ_3^* appropriately.

Let us now return to the statement of this theorem. In order to compare $\mathsf{M}_{T_\epsilon^+}(2\lambda vd)$ and $\mathsf{M}_{T_d^+}(2\lambda vd)$ I start by re-writing $\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \mathsf{M}_{T_d^+}(2\lambda vd)$ in terms of excursions and Chapter 6 provides the machinery to be able to do this. Equation (6.5) (page 61) is a key result and enables one to deduce the following

$$\begin{aligned}\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \mathsf{M}_{T_d^+}(2\lambda vd) &= \frac{F}{1 - \mathsf{M}_{B_d^+}(2\lambda vd)F} \\ &\quad \cdot \left(\frac{\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)}{1 - \mathsf{M}_{B_d^+}(2\lambda vd)} + \mathsf{M}_{T_d^+}(2\lambda vd)\right) \\ &\quad + \frac{\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)}{1 - \mathsf{M}_{B_d^+}(2\lambda vd)}\end{aligned} \tag{9.11}$$

where

$$\begin{aligned}F &= \frac{\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)}{1 - \mathsf{M}_{B_d^+}(2\lambda vd)} \cdot \frac{1}{1 - (\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd))} \\ &= \frac{\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)}{(1 - \mathsf{M}_{B_d}(vd)) + (\mathsf{M}_{B_d}(vd) - \mathsf{M}_{B_\epsilon}(vd)) + (\mathsf{M}_{B_\epsilon}(vd) - \mathsf{M}_{B_d^+}(2\lambda vd))} \\ &\quad \cdot \frac{1}{1 - (\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd))}\end{aligned}$$

Next, in order to use Lemma 9.4.1, Lemma 8.3.1, Lemma 7.4.1 and Lemma 5.4.2 to bound equation (9.11), I divide the appropriate numerators and denominators of fractions by $d\sqrt{\lambda/\mu}$ and then apply the aforementioned results. Now by choosing κ_1^* and κ_3^* small enough, one can uniformly seperate all the denominators in equation (9.11) away from zero for all $|v| < \bar{v}$. This allows us to write equation (9.11) as

$$\begin{aligned}&\left|\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \mathsf{M}_{T_d^+}(2\lambda vd)\right| \\ &\quad \leq C\left(\frac{|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)|}{d\sqrt{\lambda/\mu}} \cdot \frac{|\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)|}{d\sqrt{\lambda/\mu}}\right. \\ &\quad \left. + \frac{|\mathsf{M}_{B_\epsilon^+}(2\lambda vd) - \mathsf{M}_{B_d^+}(2\lambda vd)|}{d\sqrt{\lambda/\mu}} \cdot |\mathsf{M}_{T_d^+}(2\lambda vd)| + \frac{|\mathsf{M}_{A_\epsilon^+}(2\lambda vd) - \mathsf{M}_{A_d^+}(2\lambda vd)|}{d\sqrt{\lambda/\mu}}\right)\end{aligned}$$

where C is a constant dependent on $\kappa_1^*, \kappa_2^*, \kappa_3^*, \bar{\kappa}_1$ and $\bar{v}$. Finally by applying Lemma 9.4.3, Lemma 9.4.1, Theorem 8.4.1, Theorem 7.5.1 and Theorem 5.4.4 we reach the stated result.

□

9.6 Summary of perturbation chapters

We can now summarise the outcome of Chapters 5, 7, 8 and 9 with the following result:

Theorem 9.6.1. *Fix contants $\rho_0 > 0$ and $\bar{\kappa}_2^* > 0$. For any choice of $0 < \bar{v} < 0.5$ there exists $\bar{\kappa}_1^* = \bar{\kappa}_1^*(\bar{v}) > 0$ and $\bar{\kappa}_3^* = \bar{\kappa}_3^*(\bar{v}) > 0$ such that for any v that satisfies $|v| < \bar{v}$, if*

$$\left(\frac{\lambda}{\mu}\right)^{2+\rho_0}\frac{1}{\mathsf{N}} < \bar{\kappa}_1^*, \quad \mathsf{N}\exp\left[-\left(\frac{\lambda}{\mu}\right)^{\rho_1}\right] < \bar{\kappa}_2^*, \quad \frac{\mu}{\lambda} < \bar{\kappa}_3^*,$$

$$0 < \rho < \min\left\{\frac{\rho_0}{2}, \frac{1}{7}\right\} \quad \textit{and} \quad 0 < \rho_1 < \min\left\{\frac{1}{4}, \rho\right\}$$

then

$$\left|\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \frac{1}{1-2v}\right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{2+2\rho}\cdot\frac{1}{\mathsf{N}}\right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)}\right)$$

where the implicit constant in the big O is dependent on $\bar{\kappa}_1^*, \bar{\kappa}_2^*, \bar{\kappa}_3^*$ and $\bar{v}$.

Remark 9.6.2. *One can formulate Theorem 9.6.1 without introducing the variables ρ and ρ_1 as follows: If $0 < \rho_0 \leq \frac{2}{7}$ then $\min\left\{\frac{\rho_0}{2}, \frac{1}{7}\right\} = \frac{\rho_0}{2}$. Consequently choosing $\rho = \frac{\rho_0}{4}$ and $\rho_1 = \frac{\rho_0}{5}$ meets the criteria as set out in Theorem 9.6.1.*

Proof of Theorem 9.6.1. I begin by fixing constants: $\rho_0 > 0$, $\bar{\kappa_2^*} > 0$ and $0 < \bar{v} < 0.5$. Next I choose $\bar{\kappa}_1$ such that $\kappa_1 = \bar{\kappa}_1$ and $\kappa_2 = \bar{\kappa_2^*}$ satisfy the conditions of Lemmas 7.4.1, 7.4.2, 8.3.1, 8.3.2, 9.4.1 and 9.4.3. Finally I choose $\bar{\kappa_1^*}$ and $\bar{\kappa_3^*}$ such that $\kappa_1^* = \bar{\kappa_1^*}$, $\kappa_2^* = \bar{\kappa_2^*}$ and $\kappa_3^* = \bar{\kappa_3^*}$ satisfy the conditions of Theorems 7.5.1, 8.4.1 and 9.5.1.

Let us now return to the statement of this theorem. The following decomposition follows from the triangle inequality

$$\left|\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \frac{1}{1-2v}\right| \leq \left|\mathsf{M}_{T_\epsilon^+}(2\lambda vd) - \mathsf{M}_{T_d^+}(2\lambda vd)\right| + \left|\mathsf{M}_{T_d^+}(2\lambda vd) - \mathsf{M}_{T_\epsilon}(vd)\right| + \left|\mathsf{M}_{T_\epsilon}(vd) - \mathsf{M}_{T_d}(vd)\right| + \left|\mathsf{M}_{T_d}(vd) - \frac{1}{1-2v}\right|$$

Recall that each of the components on the right hand side of the above equation have been bounded - see Theorems 5.4.4, 7.5.1, 8.4.1 and 9.5.1. The stated result follows directly from these theorems. □

Chapter 10

Conclusion

I will start by proving Theorem 1.4.4 which was presented in the introduction of my thesis. I will then summarise the work I have done in my thesis and finally I will comment on further research that can be done to extend my work.

10.1 Final proof

Recall the Continuous Time Linear Holding model, $(W_t)_{t\geq 0}$ (defined in Chapter 9), this is a Markov chain on the state space $\{*, 0, 1, 2, \ldots, \mathsf{N}\}$ which evolves with jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } \mu k \\ k &\to k+1 && \text{rate } \lambda(1-d_k) \\ k &\to * && \text{rate } \lambda \cdot d_k \end{aligned}$$

where $d_k = \frac{k}{\mathsf{N}}$. The main result I have proved for $(W_t)_{t\geq 0}$ can be found in Theorem 9.6.1 (page 117). I will now introduce a simple variation of this model by rescaling the constant N, in what follows I will replace N by N/c. I will use the ceiling function to ensure that there are no problems when N/c is not an integer.

Define $(V_t)_{t\geq 0}$ to be a Markov chain on the state space $\{*, 0, 1, 2, \ldots, \lceil \mathsf{N}/c \rceil\}$ that evolves with the following jump rates:

$$\begin{aligned} k &\to k-1 && \text{rate } \mu k \\ k &\to k+1 && \text{rate } \lambda(1-d_k) \\ k &\to * && \text{rate } \lambda \cdot d_k \end{aligned}$$

where

$$d_k = \begin{cases} \dfrac{c \cdot k}{\mathsf{N}} & \text{if } k < \lceil \mathsf{N}/c \rceil \\ 1 & \text{if } k = \lceil \mathsf{N}/c \rceil \end{cases}.$$

Let the process start from the state λ/μ and define $T_\epsilon^\times$ to be the hitting time of a starred state. By making the same rescaling of N in the statement of Theorem 9.6.1 we can conclude the following result under the same conditions of the theorem:

$$\left| \mathsf{M}_{T_\epsilon^\times}(2\lambda v d \cdot c) - \frac{1}{1-2v} \right| = \mathcal{O}\left(\left(\frac{\lambda}{\mu}\right)^{2+2\rho} \cdot \frac{c}{\mathsf{N}} \right) + \mathcal{O}\left(\left(\frac{\mu}{\lambda}\right)^{0.25(1-\rho)} \right)$$

By choosing c appropriately[1], the Markov chain $(V_t)_{t\geq 0}$ (above) is identical to the Projected Model, $(Y_t)_{t\geq 0}$, when p is replaced by either $\frac{2l+1}{\mathsf{N}}$ or $\frac{4l+1}{\mathsf{N}}$ (the Projected Model was defined in section 1.4 of Chapter 1). As such Theorem 1.4.4 follows immediately.

10.2 Summary of thesis

In this thesis I have studied a particular class of Markov chain models with killing; the primary focus of which was to address the fact that for many such models it is not possible to analyse quantities of interest (e.g. hitting times) directly. My approach was to write the model which was not analytically tractable as a perturbation of a model which is 'solvable'; I hoped to be able to bound the difference between the models in some useful sense. This brought its own share of problems, firstly any technique known to me required one to be able to make the perturbation uniformly small which was not possible in my case. Secondly, due to the fact that the killing time is an unbounded random variable, a study of the difference between the models would require me to take into account the vast number of trajectorys of varying lengths and it was unclear how I would do this.

By running simulations I noticed that in a Markov chain of interest, the process typically returned to the state λ/μ many times before being killed; consequently I used the strong Markov property to decompose the trajectory into excursions. I

[1] If $p = \frac{2l+1}{\mathsf{N}}$ then choose $c = 2l+1$, and if $p = \frac{4l+1}{\mathsf{N}}$ then choose $c = 4l+1$.

was able to derive a set of conditions on excursions, under which I could compare the difference between the moment generating functions of the hitting times for two different models. This technique provided one way around both of the problems I described in the previous paragraph. By comparing a number of discrete and continuous time models using this technique I have proved results for two models which stochastically sandwich the survival time of a biologically motivated model of DNA damage and repair.

10.3 Further work

A natural extension to this work would be to develop the perturbation theory further in order to prove a result for the model that I introduced in Section 1.2. Beyond this, one might wish to investigate the model that was introduced in [16]; the main difference between this model and the model in Section 1.2 is that rather than the repair process starting instantaneously after the base becomes damaged, the repair only commences if there is an available repair enzyme (the number of repair enzymes is finite). If the number of repair enzymes is sufficently large then one would expect that the survival time in both models to be close and there are strong indications to suggest that perturbation techniques can once again come to the rescue!

Appendix A

Appendix

Recall the Ring Model, $(X_t)_{t\geq 0}$, that was introduced in Chapter 1. A base has a label from the set $\{0, 1, 2\}$, where 0, 1 and 2 correspond to a undamaged, damaged and critically damaged base respectively. Let D_N be a DNA string with N bases that form a closed loop. The Markov chain $(X_t)_{t\geq 0}$ lives on the configuration space $S = \{0, 1, 2\}^{D_\mathsf{N}}$ and individual bases in D_N evolve independently with rates

$$\begin{aligned} 0 &\to 1 \qquad \text{rate } \lambda/\mathsf{N} \\ 1 &\to 0 \qquad \text{rate } \mu \\ 1 &\to 2 \qquad \text{rate } \lambda/\mathsf{N} \end{aligned}$$

and the initial state, X_0, is a non-critically damaged configuration where there are λ/μ damaged bases. I say that a configuration is critically damaged if there is a critically damaged base or there are two damaged bases within distance $2l$ of each other. T is the first moment that the Ring Model hits a critically damaged configuration.

Remark A.0.1. *An alternative but equivalent way to describe the Ring Model is that if it is at a configuration with k damaged bases and there are m bases where an arrival would result in a critically damaged configuration then the number of damaged bases changes with the following rates*

$$\begin{aligned} k &\to k+1 \qquad \textit{rate } \lambda \\ k &\to k-1 \qquad \textit{rate } \mu k \end{aligned}$$

If there was a repair then uniformly at random we decrease the label of one of the damaged states by one. If there was an arrival then uniformly at random we increase *the label of one of the states by one, in which case the process is still alive with probability* $1 - \frac{m}{\mathsf{N}}$ *and critically damaged with probability* $\frac{m}{\mathsf{N}}$.

Remark A.0.2. *If a configuration has* k *damaged bases and there are* m *bases where an arrival would result in a critically damaged configuration then* m *and* k *satisfy the following relationship:*

$$(2l+1)k \leq m \leq (4l+1)k\,.$$

The upper bound is obtained on a class of configurations that have particular property, namely all the defects are spaced out so that there is at least a 4l gap in-between damaged bases. On the other hand, in any configuration for which all the defects are packed together as closely as possible without being critically damaged, m *(in the relation above) will be close to the lower bound.*

Also recall the Projected Model, $(Y_t)_{t\geq 0}$, which was introduced in Chapter 1. This process lives on the state apce $S' = \{*, 0, 1, 2, \ldots, \mathsf{N}_p = \lceil 1/p \rceil\}$ and evolves with jump rates:

$$\begin{array}{ll} k \to k-1 & \text{rate } \mu k \\ k \to k+1 & \text{rate } \lambda(1-d_k) \\ k \to * & \text{rate } \lambda d_k \end{array}$$

where

$$d_k = \begin{cases} pk & \text{if } k < \mathsf{N}_p \\ 1 & \text{if } k = \mathsf{N}_p \end{cases}.$$

I consider two copies of Y_t namely $Y_t^{'}$ and $Y_t^{''}$, with $p = \frac{2l+1}{\mathsf{N}}$ and $p = \frac{4l+1}{\mathsf{N}}$ respectively. Also let $Y_0^{'} = Y_0^{''} = \lambda/\mu$. I define T' and T'' to be the respective hitting times of a starred state and I will now prove Theorem 1.4.2 which states that we have the following stochastic ordering

$$T^{''} \preceq T \preceq T^{'}.$$

Proof of Theorem 1.4.2. I will start by proving $T^{''} \preceq T$. Let $X_t = \sigma$ where σ is a configuration that is not critically damaged, has k damaged bases and m bases where an arrival would result in a critically damaged configuration. Also let $Y_t = k$. The fact I am considering the situation where the number of defects in the main model is equal to the state of the Projected Model is on purpose. I will demonstrate that the following one step coupling can be applied repeatedly as is necessary:

If $k < \lceil \frac{\mathsf{N}}{4l+1} \rceil$ then

- With rate μk there is a repair in both models.
- With rate λ there is an arrival in both models, after which:
  - Both processes become critically damaged with probability $\frac{m}{\mathsf{N}}$.
  - With probability $\frac{(4l+1)k}{\mathsf{N}} - \frac{m}{\mathsf{N}}$ only the Projected Model becomes critically damaged.
  - Both processes survive with probability $1 - \frac{(4l+1)k}{\mathsf{N}}$.

If $k = \lceil \frac{\mathsf{N}}{4l+1} \rceil$ then

- With rate μk there is a repair in both models.
- With rate λ there is an arrival in both models, after which:
  - Both processes become critically damaged with probability $\frac{m}{\mathsf{N}}$.
  - With probability $1 - \frac{m}{\mathsf{N}}$ only the Projected Model becomes critically damaged.

It is straighforward to check that the above jump rates and probabilities respect the marginal jump rates of both models. Initially the coupling can be applied with $k = \lambda/\mu$. After one event (arrival or repair) using the above coupling either both models are critically damaged, just the Projected Model is critically damaged or both models survive. In the latter case the state of the Projected Model equals the number of defects in the main model - this means the same one step coupling can be applied again. Under this coupling the Projected Model will become critically damaged before or at the same time that the full model becomes critically damaged. Therefore $T^{''} \preceq T$.

To prove $T \preceq T^{'}$ the argument follows in exactly the same way, however we use the following one step coupling:
If $k < \lceil \frac{\mathsf{N}}{2l+1} \rceil$ then

- With rate μk there is a repair in both models.
- With rate λ there is an arrival in both models, after which:
  - Both processes become critically damaged with probability $\dfrac{(2l+1)k}{\mathsf{N}}$.
  - With probability $\dfrac{m}{\mathsf{N}} - \dfrac{(2l+1)k}{\mathsf{N}}$ only the main model becomes critically damaged.
  - Both processes survive with probability $1 - \dfrac{m}{\mathsf{N}}$.

If $k = \lceil \frac{\mathsf{N}}{2l+1} \rceil$ then

- With rate μk there is a repair in both models.
- With rate λ there is an arrival in both models which results in both processes becoming critically damaged.

These jump rates and probabilities respect the marginal jump rates of both models. Initially $k = \lambda/\mu$ and after one event (arrival or repair) using the above coupling either both models are critically damaged, just the main model is critically damaged or both models survive. In the latter case the state of the Projected Model equals the number of defects in the main model - this means the same one step coupling can be applied again. Under this coupling the main model will become critically damaged before or at the same time that the full model becomes critically damaged. Therefore $T \preceq T'$. This completes the proof. □

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