Simultaneous Trading in 'Lit' and Dark Pools

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Abstract

We consider an optimal trading problem over a finite period of time during which an investor has access to both a standard exchange and a dark pool. We take the exchange to be an order-driven market and propose a continuous-time setup for the best bid price and the market spread, both modelled by Lévy processes. Effects on the best bid price arising from the arrival of limit buy orders at more favourable prices, the incoming market sell orders potentially walking the book, and deriving from the cancellations of limit sell orders at the best ask price are incorporated in the proposed price dynamics. A permanent impact that occurs when 'lit' pool trades cannot be avoided is built in, and an instantaneous impact that models the slippage, to which all lit exchange trades are subject, is also considered. We assume that the trading price in the dark pool is the mid-price and that no fees are due for posting orders. We allow for partial trade executions in the dark pool, and we find the optimal trading strategy in both venues. Since the mid-price is taken from the exchange, the dynamics of the limit order book also affects the optimal allocation of shares in the dark pool. We propose a general objective function and we show that, subject to suitable technical conditions, the value function can be characterised by the unique continuous viscosity solution to the associated partial integro-differential equation. We present two explicit examples of the price and the spread models, derive the associated optimal trading strategy numerically, and describe the numerical method used in the appendix. We discuss the various degrees of the agent's risk aversion and further show that roundtrips—i.e. posting the remaining inventory in the dark pool at every point in time—are not necessarily beneficial.

Keywords: Stochastic control, optimal trading strategies, Hamilton-Jacobi-Bellman equation, viscosity solutions, limit order book, market impact, dark pools.

1 Introduction

The focus in this paper is put on trading in 'lit' and dark pools. We study how an investor's optimal trading programme is affected by the availability of a dark pool as an alternative venue with its own set rules. Since the trade execution price in the dark pool is related to the best bid and ask prices in the standard exchange (often referred to as the 'lit' pool or 'lit' exchange), we propose at the same time novel microstructure models for the price dynamics in the lit exchange. In this paper we assume the exchange to be an order-driven market and put forward continuous-time models for the best bid price and for the bid-ask spread. The mid-price is of course recovered by adding half the spread to the best bid price. We capture the permanent price impact by market activity in the drift function and in the compound Poisson processes. The spread process is constructed similarly so as to obtain the same structure for the best ask price when compared to the best bid price process. Optimal trading in LOBs, of which microstructural features are modelled, is for example also treated in (i) Cartea and Jaimungal [10] who model the spot price via a hidden Markov chain to capture the switches between price regimes, (ii) Cartea et al. [12] who model the deviation of mid-price from its long-term mean via a jump-diffusion process, and (iii) Obizhaeva and Wang [31], and Alfonsi et al. [1] who propose trading models by taking into account shape functions for the LOB. Research in optimal execution in an order-driven market has its roots in the papers by Bertsimas and Lo [4], and Almgren and Chriss [2]. More recent contributions include, e.g., Pemy and Zhang [33], Pemy et al. [34], Gatheral and Schied [19], Brigo and Di Graziano [6], Moazeni et al. [30], Cartea et al. [11], and Ishitani and Kato [27]. Cartea and Jaimungal [9] consider a continuous-time, jump-diffusion mid-price model and explicitly take into account the impact of the market activity on the mid-price. In the same context, we also mention the works by Guéant et al. [20] and Guilbaud et Pham [22], who treat optimal liquidation via limit orders. A dark pool is an alternative trading venue where buy and sell orders are not publicly displayed so that the participants' identity is not revealed. Other advantages of trading in a dark pool include (i) the possibility of submitting large orders without impacting the market price, and (ii) trading at more favourable conditions—the dark pool trading price lies between the best bid and the best ask prices quoted in the standard exchange. (For simplicity, we will assume that the dark pool utilises the standard exchange mid-price.) On the other hand, dark pools do not guarantee execution of the trade for it is subject to the availability of a trade counterparty. Dark pools might also reduce market liquidity since large and institutional investors, in order to protect private information, resort to these alternative trading venues where orders remain hidden to the general market participants. Most of the research effort in dark pools has been devoted to the impact of dark pools on market welfare. A game-theoretic approach has often been used to model the two competing trading venues, so to find equilibrium prices and optimal strategies of the market players, also including the effect of private information and adverse selection on trading. This branch of research includes the early work by Hendershott and Mendelson [23], who extend the Kyle [29] model to capture the dynamics of the interplay between investors, dark pool and standard exchange, followed by Degryse et al. [16], Buti et al. [7] and Daniëls et al. [15]. The work by Ye [37] and Zhu [39] focus on the effects on price discovery due to the migration of order flows from the exchanges to the dark pools, when unobservable liquidity is added to the model.

Kratz and Schöneborn [28] consider trading in the dark pool within the classical field of optimal liquidation. They model the LOB mid-price by an exogenous square-integrable martingale and regard the dark pool as a complete-or-zero-execution venue where the arrival of trading counterparties follows a Poisson process. In this context, we also refer to work by Horst and Naujokat [25], in which the authors find the optimal strategy when trading in an illiquid market. The agent under consideration seeks to minimise the deviation from a given target while submitting market orders in the standard exchange and "passive orders" in a dark pool. The execution of dark-pool orders is modelled via Poisson processes. As we shall see, we incorporate such a feature in the model presented in Section 3.

The goal in this paper is to find the optimal trading strategy when utilising standard exchange and the services of a dark pool. Since the smoothness of the value function is not guaranteed on the whole domain, there are no classical solutions to the optimisation problem, in general. Therefore we resort to a weaker notion, the viscosity solutions, and we show that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman partial integro-differential equation (HJB PIDE) associated with the optimisation problem. There is a vast literature on the theory of viscosity solution of HJB equations of which state variables are driven by general Markov processes. We here refer in particular to Pham [35, 36], and Fleming and Soner [17] for technical details. Our results concerning the exchange are coherent with the ones obtained by Chevalier et al. [13]. Indeed we find that the optimal strategy depends on the book's dynamics. Furthermore, we show that this is also the case in the presence of a dark pool.

The paper is organised as follows: after the introduction, in Section 2 we discuss the optimal trading problem and present our LOB model. Then we formulate the optimisation problem when only standard exchanges are available to an investor. We consider a generalised objective function and derive the HJB PIDE for the control problem. In Section 3, we introduce the dark pool by modifying the setup presented in the previous section. We propose an objective function that accounts for trading in the alternative pool and obtain the associated HJB PIDE. In Section 4, we provide two numerical examples of the optimal strategy and we focus on the roles played by the parameters of the explicit model. We conclude with Section 5 by summarising the contributions offered in this paper. In order for the present work to be self-contained, we collect the mathematical conditions, the propositions and the related proofs at the basis of the viscosity property of the value function in the appendix. We further include the pseudocode to numerically solve the HJB equations presented herein.

2 Price model and optimal trading in a standard exchange

We take the view of an investor who seeks to liquidate a sizeable amount of shares over a finite period of time $[t,T]$, where the initial time t lies in the interval $[0,T)$. As far as an optimal liquidation strategy is concerned—which in our setup involves the submission of market sell orders only—it suffices to model the best bid (or the best ask in the case of an optimal acquisition), since we consider a temporary price impact to account for the slippage effect. In the short run, the best bid price at time $u \in [t,T]$ depends on its initial state and on the cumulated market activity up to time u.

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\{\mathscr{F}_u\}_{0\le u\le T}$ satisfying the usual conditions. In what follows, we assume that the task is to liquidate an amount $X_t = x$ of shares, where $0 \le x < \infty$ and write for the inventory dynamics

$$\mathrm{d}X_u = -\nu_u \mathrm{d}u, \tag{2.1}$$

where the rate of trading $\nu := \{\nu_u\}_{t\le u\le T}$ is the control process of the stochastic optimisation problem presented in Section 2.1. We model the best bid price process by

$$\mathrm{d}S^b_u = \mu^b\big(u, S^b_{u^-}, \nu_u\big)\mathrm{d}u + \sum_{i=1}^{2} h^b_i\left(u, S^b_{u^-}\right)\mathrm{d}\tilde{J}^{b,i}_u, \tag{2.2}$$

and the spread process by

$$\mathrm{d}\Delta_u = \mu^\Delta\left(u, \Delta_{u^-}, \nu_u\right)\mathrm{d}u + \sum_{i=1}^{2} h^\Delta_i\left(u, \Delta_{u^-}\right)\mathrm{d}\tilde{J}^{\Delta,i}_u + \sum_{i=1}^{2} h^{b,\Delta}_i\left(u, S^b_{u^-}, \Delta_{u^-}\right)\mathrm{d}\tilde{J}^{b,i}_u, \tag{2.3}$$

with initial values $S^b_t = s^b$ and $\Delta_t = \Delta$. We introduce four compensated compound Poisson processes defined by

$$\tilde{J}^{b,i}_u = \sum_{j=1}^{N^{b,i}_u} z^{b,i}_j - \lambda^b_i u\,\mathbb{E}\big[z^{b,i}_1\big], \qquad \tilde{J}^{\Delta,i}_u = \sum_{j=1}^{N^{\Delta,i}_u} z^{\Delta,i}_j - \lambda^\Delta_i u\,\mathbb{E}\big[z^{\Delta,i}_1\big],$$

where $i = 1,2$. In the above notation, $\{N^{b,i}_u\}$ and $\{N^{\Delta,i}_u\}$ are independent Poisson processes with intensity λ^b_i and λ^Δ_i, respectively. The jump sizes are modelled by sequences of i.i.d. random variables $z^{b,i}_j$ and $z^{\Delta,i}_j$, where j=1,2,. . . ,. We say that $\{\nu_u\}$ is admissible if (i) it is progressively measurable, (ii) it is such that $\mathbb{E}\left[\int_t^T \left|\mu^b\left(u, S^b_u, \nu_u\right)\right|^2 + \left|\mu^\Delta\left(u, \Delta_u, \nu_u\right)\right|^2 \mathrm{d}u\right] < \infty$, and (iii) it belongs to a compact set $\mathscr{V} = [0,N] \subset [0,\infty)$. The functions μ^b, μ^Δ, h^b_i, h^Δ_i and $h^{b,\Delta}_i$ can be chosen such that both the best bid price and the spread are positive and in ways to reproduce market features. In Equation (2.2) we consider positive jumps which model the incoming limit buy orders at a more favourable price, and negative jumps to account for cancellations of limit buy orders and market sell orders which walk the LOB. Explicit examples will be provided in Section 4. Equation (2.3) models the spread process, of which width is affected by both, the market activity on the ask side and on the bid side of the book. In particular, when the best bid price increases, the spread process should decrease by the same amount, assuming that the ask price remains unchanged. We therefore include the jump processes of the bid price in the dynamics of the spread process through the functions $h_i^{b,\Delta}$. The activity on the ask side is accounted for by the functions h_i^{Δ}. We further include the trading rate ν_u in the drift of the spread process. This is financially justified for ν_u has an impact on the best bid price—which decreases—and thus it has an opposite effect on the spread. The best ask price,

$$S_u^a = S_u^b + \Delta_u, \tag{2.4}$$

shares the same model structure as the best bid price given that the spread process $\{\Delta_u\}$ has the form (2.3). Although we view the optimisation problem from the perspective of liquidating orders, the model proposed via Equations (2.1), (2.2), and (2.3) can be adapted to the case of optimal acquisition. Last, we define the cash (wealth) process by

$$\mathrm{d}Y_u = \mu^y(u, S_u^b, \nu_u)\mathrm{d}u,$$

with initial cash $Y_t = y$. The function μ^y models the instantaneous gains made by the investor through selling shares, possibly taking into account the temporary price impact of trades.

2.1 Optimal control problem

An investor seeks to liquidate x shares over a finite period of time $[t, T]$. The goal is to maximise the total reward, while minimising the unfavourable impact of executed trades on the best bid price $\{S_u^b\}$. We assume that the investor trades by means of market sell orders only, and we introduce a stopping time τ,

$$\tau := \inf\{u \geq t \,|\, X_u \leq 0\} \wedge T, \tag{2.5}$$

that describes the first time the inventory is depleted, should such an event occur before the terminal date T. For notational simplicity, in this section we introduce the space $\mathcal{O} = [0, x] \times \mathbb{R}_+ \times \mathbb{R}$ and let the vector of the state variables be defined by $\boldsymbol{X}_u = \left(X_u, S_u^b, Y_u\right) \in \mathcal{O}$, with initial values $\boldsymbol{x} = (x, s^b, y) \in \mathcal{O}$. We propose a general objective function of the form

$$V(t, \boldsymbol{x}) = \sup_{\nu \in \mathcal{V}} \mathbb{E}\left[\int_t^{\tau} \mathrm{e}^{-r(u-t)} f(u, \boldsymbol{X}_u, \nu_u)\,\mathrm{d}u + \mathrm{e}^{-r(\tau-t)} g(\boldsymbol{X}_\tau)\right], \tag{2.6}$$

where $r \geq 0$ is a discount rate and $\mathbb{E}[\cdot]$ is the expectation given the initial state of the system $(t, \boldsymbol{x}) \in [0, T) \times \mathcal{O}$. We interpret the discount factor as the urgency of the agent to liquidate and enter in possession of the revenues deriving from the sales. That is, even a risk-neutral investor ($\gamma = 0$) may prefer to liquidate faster as a result of his preference for a more immediate liquidity. The function $f : [0, T] \times \mathcal{O} \times \mathcal{V} \to \mathbb{R}$ may have several interpretations. For example it may represent the gains made from the shares sale, or it may correspond to a mean-variance criterion. The function $g : \mathcal{O} \to \mathbb{R}$ may represent the terminal reward obtained by a block trade liquidation of the remaining inventory at time T. However, g may also be a penalty resulting from failing to liquidate the whole inventory by time T. We let $\boldsymbol{p} := (p_1, p_2, p_3) \in \mathbb{R}^3$, and we define the operator $\mathcal{H}$ by

$$\mathcal{H}(t, \boldsymbol{x}, \boldsymbol{p}) = \sup_{v \in \mathcal{V}} \left\{ f(t, \boldsymbol{x}, v) - v p_1 + \mu^b \left(t, s^b, v\right) p_2 + \mu^y(t, s^b, v) p_3 \right\}. \tag{2.7}$$

Furthermore, for polynomially bounded functions $\varphi \in C^{1,1}([0, T] \times \mathcal{O})$, we define the operator $\mathcal{B}_b$ by

$$\mathcal{B}_b(t, \boldsymbol{x}, \varphi) = \sum_{i=1}^{2} \lambda_i^b \mathbb{E}^{(z^{b,i})} \left[\varphi\left(t, x, s^b + h_i^b\left(t, s^b\right) z^{b,i}, y\right) - \varphi(t, \boldsymbol{x}) - h_i^b\left(t, s^b\right) z^{b,i} \frac{\partial \varphi}{\partial s^b}(t, \boldsymbol{x}) \right], \tag{2.8}$$

where $\mathbb{E}^{(z^{b,i})}$ is the expectation taken with respect to the random variable $z^{b,i}$. Standard arguments from dynamic programming (DP) suggest that the value function of the optimal control problem (2.6) satisfies the following HJB PIDE:

$$rV(t, \boldsymbol{x}) - \frac{\partial V}{\partial t}(t, \boldsymbol{x}) - \mathcal{H}(t, \boldsymbol{x}, D_x V) - \mathcal{B}_b(t, \boldsymbol{x}, V) = 0, \tag{2.9}$$

on $[0, T) \times \mathcal{O}$, with terminal condition $V(\tau, \boldsymbol{x}) = g(\boldsymbol{x})$ and boundary condition $V(u, 0, s^b, y) = g(0, s^b, y)$. The meaning of the boundary condition is that if there are no shares to liquidate, the agent's duty is terminated and there are no further actions to be taken. The same holds at time τ, if $X_\tau = 0$. Since one cannot guarantee the smoothness of $V(t, \boldsymbol{x})$ on the whole domain, one cannot discuss the solution of the HJB PIDE in the classical sense. We resort to the weaker notion of viscosity solutions and, in the appendix, we show the aforementioned property of the value function V.

An optimal trading strategy—similar to the one obtained by Bertsimas and Lo [4]—whereby one liquidates at a constant rate, can be reproduced within the present setup as follows: (i) consider profit maximisation from trading for a risk-neutral investor as the objective function, and (ii) require the bid price process (2.2) be a (local) martingale. Under such assumptions, the optimal trading strategy is independent of the LOB dynamics (cfr. Figure 3, left). By contrast, the solutions to the concrete examples of the liquidation problems treated in the next section confirm the view—also in agreement with the work by Chevalier et al. [13]—that the book dynamics is crucial when dealing with optimal execution. Further details can be found in Section 4 (Figures 2, 3 and 4).

3 Dark pool

We introduce the possibility for the investor to trade in a standard exchange and in a dark pool at the same time. The investor faces a trade-off between a costly and sure trade execution in the exchange and the alternative of a more remunerative but uncertain execution in the dark pool.

We take this opportunity to emphasise the differences between the recent literature—e.g. Kratz and Schöneborn [28], and Horst and Naujokat [25]—and the present work. Firstly, we model both the best bid price and the spread process, thus allowing for a more realistic description of market features, for there is no such thing as a mid-price in the market. Furthermore, by modelling both the bid price and the spread process, we also include the effect of the sell side of the LOB on the optimal strategy, which is justified since the two sides of the LOB are known to be highly correlated. We also include a permanent impact by the trades on the market prices. We do not require for the price processes to be martingales, as it is well known that they mean-revert to their long-term value. We further consider the possibility of partial execution in the dark pool, and we do not restrict the problem to full liquidation by T. Finally we propose a class of models that can be adapted to (i) market-specific features, e.g. mean-reversion, seasonality, and intra-day patterns of the price processes, as well as to (ii) agent-specific preferences of calculating the P&L and/or measuring utility and risk-aversion. Such generalisations come at a cost and, in particular, we are not in the position to provide analytical trading strategies. As specified earlier in this paper, we assume that the trades in the dark pool take place at the mid-price $\{S_u^m\}$ given by

$$S_u^m := S_u^b + \tfrac{1}{2}\Delta_u \,. \tag{3.1}$$

The purpose of this section is to find the optimal trading strategy for each venue at any time $u \in [t, T)$. We denote the optimal order size in the dark pool by $\{\eta_u\}$. Thus, the control process will be given by a vector-valued process $\boldsymbol{\nu} = \{\boldsymbol{\nu}_u\}_{t\le u\le T}$ defined by $\boldsymbol{\nu}_u := (\nu_u, \eta_u)$, where $\{\nu_u\}$ is progressively measurable, $\{\eta_u\}$ is predictable and is such that $\eta_u \in \mathcal{N} := [0, X_u]$. We modify the inventory and the wealth dynamics since they now also depend on the dark pool activity. Along the lines of Horst and Naujokat [25], we model the dark-pool execution part by a jump process, but we choose a compound Poisson process to account for partial execution of the submitted order. We thus write

$$\mathrm{d}X_u = -\nu_u \mathrm{d}u - \eta_u \mathrm{d}J_u^y, \tag{3.2}$$

for the combined inventory and

$$dY_u = \mu^y(u, S^b_u, \nu_u)du + h^y(u, S^b_{u^-}, \Delta_{u^-}, \eta_u)dJ^y_u \tag{3.3}$$

for the wealth arising from simultaneously trading in the lit and the dark pools. In the above, $J^y_u := \sum_{j=1}^{N^y_u} z^y_j$ is a compound Poisson process with intensity λ^y and i.i.d. random variables z^y_j supported on $[0,1]$, which model the executed portion of the submitted order. To simplify the notation, we introduce the space $\mathcal{O} := [0,x] \times \mathbb{R}^2_+ \times \mathbb{R}$ and we define a vector of state variables $\boldsymbol{X}_u = \left(X_u, S^b_u, \Delta_u, Y_u\right) \in \mathcal{O}$ with initial values $\boldsymbol{x} = (x, s^b, \Delta, y) \in \mathcal{O}$. Next we consider a generalised optimisation problem of the form

$$V(t,\boldsymbol{x}) = \sup_{\boldsymbol{\nu} \in \mathscr{X}} \mathbb{E}\left[\int_t^\tau e^{-r(u-t)} f_1(u, \boldsymbol{X}_u, \nu_u)\, du + e^{-r(\tau-t)} g_1(\boldsymbol{X}_\tau)\right], \tag{3.4}$$

where τ is defined by (2.5), $\mathscr{X} := \mathscr{V} \times \mathscr{N}$, and $\mathbb{E}[\cdot]$ is the expectation given the initial state of the system $(t, \boldsymbol{x} \in [0,T] \times \mathcal{O})$. The function f_1 may play the role of a running gain or penalty criterion. We include the lit-pool trading rate ν in f_1 as it may reflect a penalisation for the information leakage to which publicly-displayed orders are subject to. The function g_1 is the terminal reward function which includes the terminal cash and a penalty for the remainder of the inventory at time τ. The HJB PIDE for trading simultaneously in the lit and the dark pool is analogous to the one presented in Section 2.1. We skip the details at this stage, but write the exact formulation in the appendix for completion and ease of comparison with the one for trading optimally only in the standard exchange. We further note that one can recover the setting presented in Section 2.1 by specifying $z^y_i \equiv 0$. We emphasise here that Kratz and Schöneborn [28] model the dark pool as a complete-or-zero-execution venue, where the placed order is either fully executed, when a suitable counterparty is found, or it expires unexecuted. Their optimal strategy in the single-asset case—which consists of placing small portions of the inventory in the exchange and all the remainder in the dark pool—can be obtained as a special case of the present model (see Figure 14).

4 Explicit price models and numerical computation of the trading strategy

In this section we investigate in detail two particular examples of the best bid price and the spread processes defined by Equations (2.2) and (2.3). In the numerical solutions that follow, the chosen parameters are for illustrative purposes. An analysis of real data is beyond the scope of the present work.

4.1 Mean-reverting model

As well-known by practitioners and as also taken into account in much of the current literature, see e.g. Cartea et al. [12], and Fodra and Pham [18], the LOB mid-price mean-reverts quickly to its long-term mean. One may be critical of such a price model for both, the price and the spread processes can take negative values with positive probability. Empirical evidence suggest though that prices follow a mean-reverting pattern. We therefore consider a mean-reverting price model capable of incorporating observable market features so to plan the liquidation strategy accordingly, see Figure 8. Thus we choose the following dynamics for the best bid price and the spread processes:

$$\mathrm{d}S_u^b = \kappa^b \left[\bar{S} - S_u^b - \mu^b \nu_u\right]\mathrm{d}u + \mathrm{d}J_u^{b,1} - \mathrm{d}J_u^{b,2}, \tag{4.1}$$

$$\mathrm{d}\Delta_u = \kappa^\Delta \left[\bar{\Delta} - \Delta_u + \mu^\Delta \nu_u\right]\mathrm{d}u + \mathrm{d}J_u^{\Delta,1} - \mathrm{d}J_u^{\Delta,2} - \mathrm{d}J_u^{b,1} + \mathrm{d}J_u^{b,2}, \tag{4.2}$$

where, for $i = 1, 2$, $J_u^{b,i} = \sum_{j=1}^{N_u^{b,i}} z_j^{b,i}$ and $J_u^{\Delta,i} = \sum_{j=1}^{N_u^{\Delta,i}} z_j^{\Delta,i}$ are independent compound Poisson processes with intensities λ_i^b and λ_i^Δ, respectively. For $j = 1, 2, \ldots$, we let $z_j^{b,i}$ and $z_j^{\Delta,i}$ be sequences of non-negative i.i.d. random variables with bounded supports Z_i and Y_i, respectively. In (4.1) we interpret $\mathrm{d}J^{b,1}$ as the change in the best bid price due to the submission of limit buy orders at a more favourable price, whereas $\mathrm{d}J^{b,2}$ models the changes due to incoming market sell orders which walk the book and the effect of cancellations of limit sell orders posted at the best price. The quantities μ^b and μ^Δ model the permanent impact to which both, the best bid price and the spread in the standard exchange are subject to, after a trade takes place. In Figure 1, we plot a simulation of the best ask, the mid, and the best bid prices given respectively by Equations (2.4), (3.1), and (4.1).

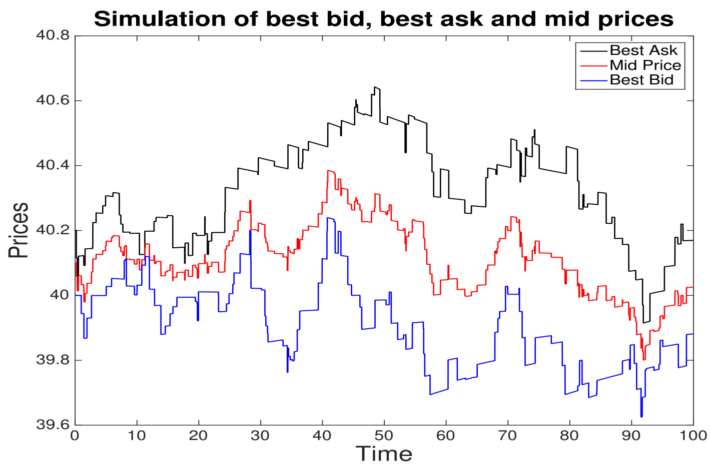

Figure 1: Simulation of the best ask, best bid and mid prices for a time frame of 100 seconds. We set $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.5$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta = 0.2$, $\bar{\Delta} = 0.1$, $\bar{S} = 40.1$, $\kappa^b = \kappa^\Delta = 2 \times 10^{-5}$.

An assumption that is usually made, see e.g. Almgren and Chriss [2], is that the investor will get the trade orders executed at a price that includes an instantaneous impact commensurate to the liquidation rate ν_u. This feedback effect is commonly referred to as "temporary impact". The temporarily impacted best bid price $\hat{S}_u^b$ and the wealth process are given by

$$\hat{S}_u^b = S_u^b - \beta \nu_u,$$

$$\mathrm{d}Y_u = \nu_u \hat{S}_u^b \mathrm{d}u + \eta_u S_{u^-}^m \mathrm{d}J_u^y, \tag{4.3}$$

where S_u^m is defined by Equation (3.1). Next we consider an investor who wants to optimally liquidate his portfolio by placing sell orders in both, the standard market and the dark pool. The inventory process of the investor is here modelled by (3.2). The maximised expected return derived by the shares sale is obtained by solving the optimisation problem

$$V(t, \boldsymbol{x}) = \sup_{\nu \in \mathscr{X}} \mathbb{E}\left[Y_\tau + (S_\tau^b - \alpha X_\tau) X_\tau - \gamma \int_t^\tau X_u^2 \mathrm{d}u \right], \tag{4.4}$$

where τ is defined by (2.5) and $\gamma \in \mathbb{R}_+ \cup 0$. In the considered performance criterion, we allow for a maximisation of the terminal cash Y_τ together with the terminal value of the portfolio $S_\tau^b X_\tau$ and a penalty for a non-zero inventory level at time τ given by $-\alpha X_\tau^2$, where $\alpha > 0$. The integral term, as in Cartea et al. [8, 9], penalises for the inventory holding over the whole period in which the strategy is applied. It is shown in the aforementioned papers that such a term can also be related to both, the variance of the portfolio value and the ambiguity aversion to the mid-price-drift. (In such a case, γ represents the risk-aversion parameter.) The associated HJB PIDE is given by

$$\begin{aligned}
\sup_{\nu \in \mathscr{X}} \Bigg\{ -\gamma x^2 + \frac{\partial V}{\partial t}(t, \boldsymbol{x}) + \kappa^b \left[\bar{S} - s^b - \mu^b \nu \right] \frac{\partial V}{\partial s^b}(t, \boldsymbol{x}) + \kappa^\Delta \left[\bar{\Delta} - \Delta + \mu^\Delta \nu \right] \frac{\partial V}{\partial \Delta}(t, \boldsymbol{x}) \\
- \nu \frac{\partial V}{\partial x}(t, \boldsymbol{x}) + \nu (s^b - \beta \nu) \frac{\partial V}{\partial y}(t, \boldsymbol{x}) \\
+ \lambda^y \mathbb{E}^{(z^y)} \left[V\left(t, x - n z^y, s^b, \Delta, y + n z^y \left(s^b + \frac{\Delta}{2} \right) \right) - V(t, \boldsymbol{x}) \right] \\
+ \lambda_1^b \mathbb{E}^{(z^{b,1})} \left[V\left(t, x, s^b + z^{b,1}, \Delta - z^{b,1}, y \right) - V(t, \boldsymbol{x}) \right] \\
+ \lambda_2^b \mathbb{E}^{(z^{b,2})} \left[V\left(t, x, s^b - z^{b,2}, \Delta + z^{b,2}, y \right) - V(t, \boldsymbol{x}) \right] \\
+ \lambda_1^\Delta \mathbb{E}^{(z^{\Delta,1})} \left[V\left(t, x, s^b, \Delta + z^{\Delta,1}, y \right) - V(t, \boldsymbol{x}) \right] \\
+ \lambda_2^\Delta \mathbb{E}^{(z^{\Delta,2})} \left[V\left(t, x, s^b, \Delta - z^{\Delta,2}, y \right) - V(t, \boldsymbol{x}) \right] \Bigg\} = 0,
\end{aligned} \tag{4.5}$$

with terminal condition $V(\tau, \boldsymbol{x}) = y + (s^b - \alpha x)x$ and boundary condition $V(u, 0, s^b, \Delta, y) = y$. We first show the optimal trading strategy, which we obtain numerically by solving (4.5), when only a standard exchange is available to the investor. This is achieved by imposing $z_i^y \equiv 0$, so to consider a dark pool that never executes orders (the dark pool is not available to the investor, that is). Equation (4.5) is solved by the finite-difference method; the pseudocode is provided in the appendix. The following figures show how the optimal strategy varies depending on whether the mid-price process is a sub-, super-, or (local) martingale.

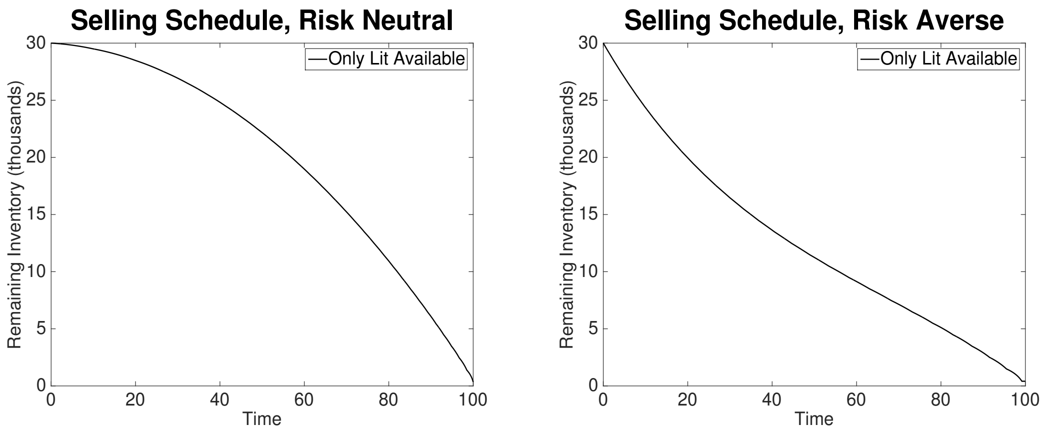

Figure 2: Optimal strategy when the LOB best bid price is a submartingale. We set $\gamma = 0$ (risk-neutral investor, left panel) , $\gamma = 0.01$ (risk-averse investor, right panel).

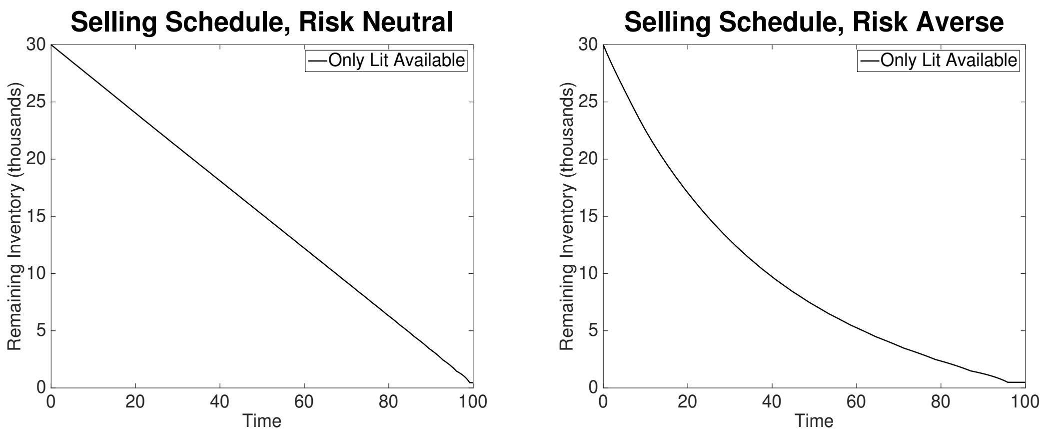

Figure 3: Optimal strategy when the LOB best bid price is a martingale. We set $\gamma = 0$ (risk-neutral investor, left panel) , $\gamma = 0.01$ (risk-averse investor, right panel).

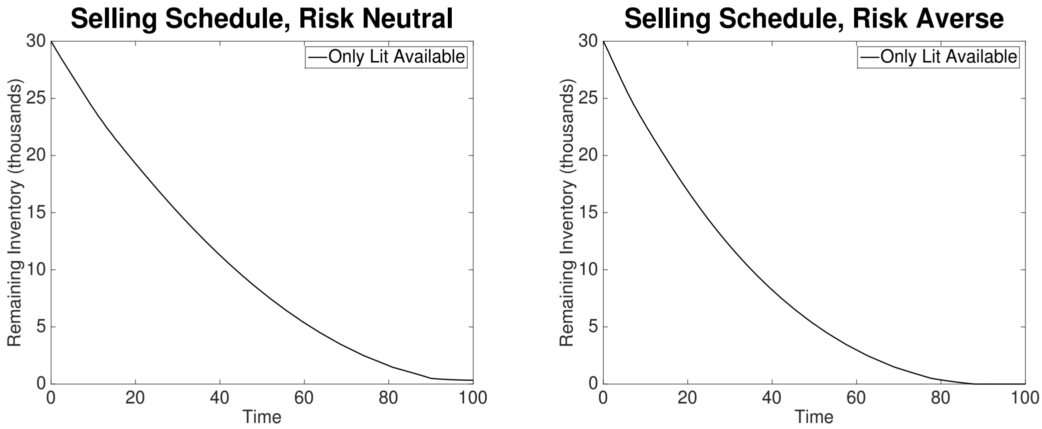

Figure 4: Optimal strategy when the LOB best bid price is a supermartingale. We set $\gamma = 0$ (risk-neutral investor, left panel) , $\gamma = 0.01$ (risk-averse investor, right panel).

In Figure 5, we show the mean inventory evolution, throughout the liquidation period, for a risk-neutral investor (i.e. $\gamma = 0$). We obtain the optimal trading strategy by solving the HJB PIDE numerically and we plot the case of multiple executions (occurring at $\tau_1 = 30$, $\tau_2 = 40$ and $\tau_3 = 50$, right panel). We emphasise that: τ_1, τ_2 and τ_3 are fixed arbitrarily—for the sake of illustration only—after a complete solution has been found. The dotted line shows the evolution of the inventory when there is no dark-pool execution over the entire period (although the dark pool is available to the investor). The dashed line shows the results for partial execution in the dark pool, while the solid line shows the results when the posted order in the dark pool is fully executed. We first look at the role played by α.

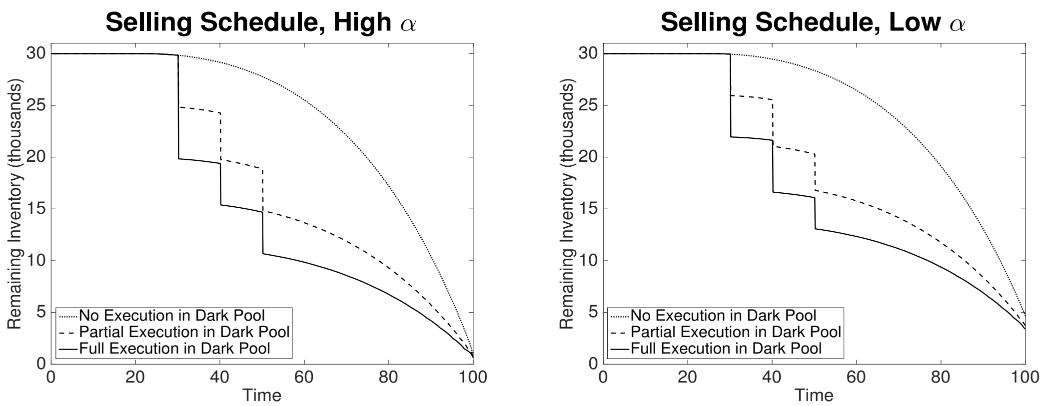

Figure 5: Optimal selling strategy displayed as a function of the remaining inventory. We set $\gamma = 0.0001$, $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta = 0.1$, $\bar{\Delta} = 0.1$, $\bar{S} = 40$, $\kappa^b = \kappa^\Delta = 0.02$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0, 1]$, $\lambda^y = 0.1$, $\mu^\Delta = \mu^b = 0.01$. Left panel: $\alpha = 6$. Right panel: $\alpha = 0.5$.

The parameter α models the terminal penalty for failing to liquidate the whole inventory by time T. Higher values of α incentivise the agent to increase the selling rate in the standard exchange and the size of the dark pool posting (left panel). On the contrary, smaller values of α allow the agent to retain a larger portion of inventory by terminal date T. The important feature of the resulting strategy is that it may be suboptimal to place all the remaining shares in the dark pool at every point in time. This is due to the LOB dynamics specified above. In Figure 6, we show how the strategy changes for a risk-averse investor ($\gamma > 0$). A higher risk aversion (right panel) incentivises the agent to liquidate faster as he is more sensitive to potential movements of the market price, while a lower degree of risk aversion is reflected in a slower liquidation (left panel).

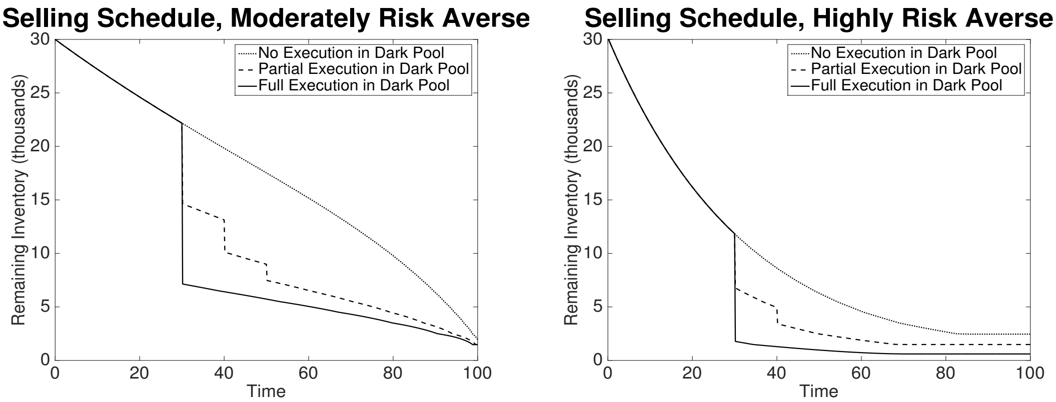

Figure 6: Optimal selling strategy displayed as a function of the remaining inventory. We set $\alpha = 2$, $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta = 0.1$, $\bar{\Delta} = 0.1$, $\bar{S} = 40$, $\kappa^b = \kappa^\Delta = 0.02$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0, 1]$, $\lambda^y = 0.1$, $\mu^\Delta = \mu^b = 0.01$. Left panel: $\gamma = 0.01$. Right panel: $\gamma = 0.1$.

In Figure 7 we show the sensitivity of the strategy with respect to the permanent price impact. Depending on the market conditions, it may be worth waiting before posting orders in the standard exchange (when there is a high permanent impact) and hope to be executed in the dark pool, while accelerating the lit pool trading is optimal for the case of a more liquid market (less permanent price impact).

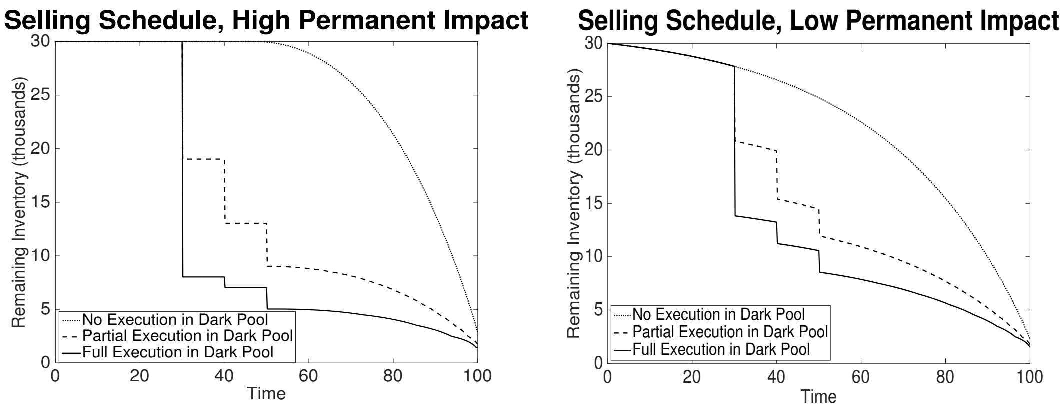

Figure 7: Optimal selling strategy displayed as a function of the remaining inventory. We set $\alpha = 2$, $\gamma = 0.0001$, $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta = 0.1$, $\bar{\Delta} = 0.1$, $\bar{S} = 40$, $\kappa^b = \kappa^\Delta = 0.02$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0, 1]$, $\lambda^y = 0.1$. Left panel: $\mu^\Delta = \mu^b = 0.01$. Right panel: $\mu^\Delta = \mu^b = 0.0001$.

The speed of mean reversion of the best bid price (i.e. κ^b) has a different impact on the selling schedule, depending on the initial value of the bid price s^b and, in particular, whether it is higher or lower than its long-term mean $\bar{S}$. In the top panels of Figure 8, we set $s^b > \bar{S}$, which implies that, on average, the price decreases so as to approach its long-term mean. The agent is thus incentivised to liquidate faster at the beginning since the price is higher than it is supposed to be. This feature is more evident when the speed of mean reversion κ^b is higher (top-left panel). In fact, the price reverts faster and the agent increases his liquidation speed so to exploit the opportunity of selling at a higher price. In the bottom panels, the starting price is lower than its long-term mean, i.e. $s^b < \bar{S}$, and thus the agent waits for it to revert so to liquidate at a more favourable price. Unlike before, a higher κ^b reduces the liquidation speed. This is rather intuitive: the agent is willing to wait for the price to increase, which, on average, takes less time than the case when κ^b is low.

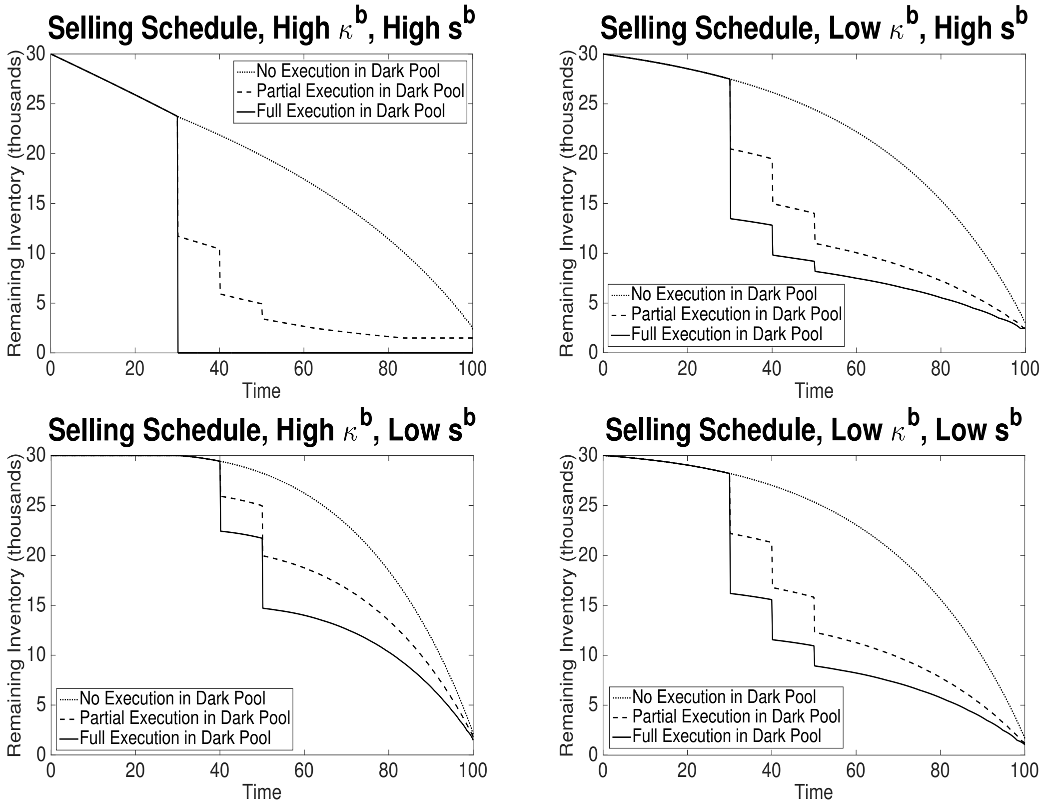

Figure 8: Optimal selling strategy displayed as a function of the remaining inventory. We set $\kappa^\Delta = 0.01$, $\beta = 0.01$, $x = 30,000$, $s = 40$, $\Delta = 1$, $\bar{\Delta} = 0.5$, $\lambda^{b,1} = \lambda^{b,2} = \lambda^{\Delta,1} = \lambda^{\Delta,2} = 0.2$, $\mu = 0.01$, $\lambda^y = 0.1$, $z^y \sim U[0,1]$, $\alpha = 2$, $\phi = 0.001$, $r = 0.01$, $T = 10$. In the top panels we set $\bar{S} = 35$, $\kappa^b = 0.1$ (left), $\kappa^b = 0.0005$ (right). In the bottom panels we set $\bar{S} = 45$, $\kappa^b = 0.1$ (left), $\kappa^b = 0.0005$ (right).

Roundtrips in the dark pool are not necessarily beneficial, that is, the agent may not wish to post all the remaining inventory in the dark pool. We have investigated such a feature by looking at fixed times τ, but now we plot the whole optimal strategy ν and η (i.e.) as a function of the bid price and the spread processes. This confirms our intuition that the agent may at times retain part of the inventory, depending on the market conditions. In Figure 9 we show the optimal strategy for both the lit and dark pools in the case that $\lambda_1^b > \lambda_2^b$, while in Figure 10 we consider the case where $\lambda_1^b < \lambda_2^b$. When the price is expected to increase (Figure 9) we notice that both strategies are reduced in size compared to Figure 10. This is coherent with the intuition that the agent may wish to exploit the opportunity of a price increase by waiting to liquidate the inventory. We further see that the quantity posted in both venues increases as we approach the terminal date T. Take, e.g., the left panel in Figure 9. The bottom surface is the optimal trading rate in the standard exchange at the beginning of the trading period ($t = 0$) with 30,000 shares to liquidate. As t increases, the respective surface lifts up indicating that having less time remaining implies posting higher quantities in the venue—we assume that the inventory $x = 30,000$ at each time. The analogous holds for the right panel, in which we plot the dark pool strategy.

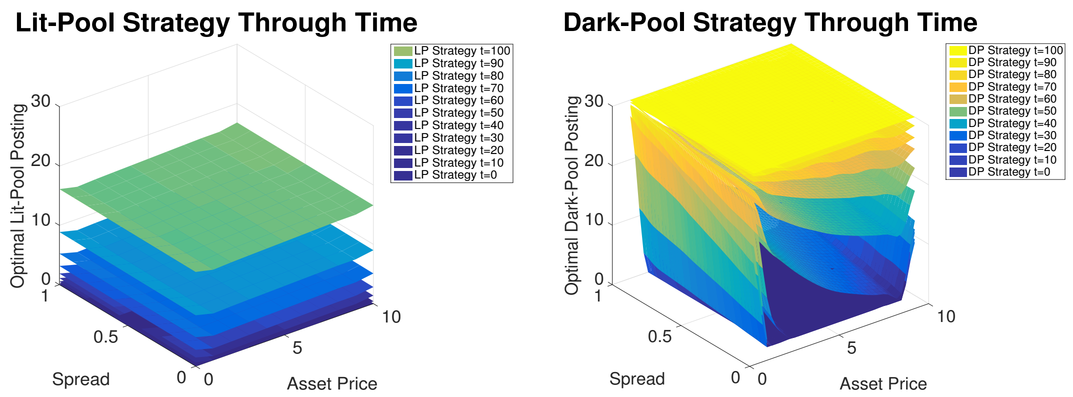

Figure 9: Optimal lit and dark pool strategies. We set $X_t = 30,000$ for each t, $\alpha = 2$, $\gamma = 0.0001$, $\lambda_1^b = 0.5$, $\lambda_2^b = 0.1$, $\lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $\bar{\Delta} = 0.1$, $\bar{S} = 40$, $\kappa^b = \kappa^\Delta = 0.02$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0,1]$, $\lambda^y = 0.1$, $\mu^\Delta = \mu^b = 0.001$.

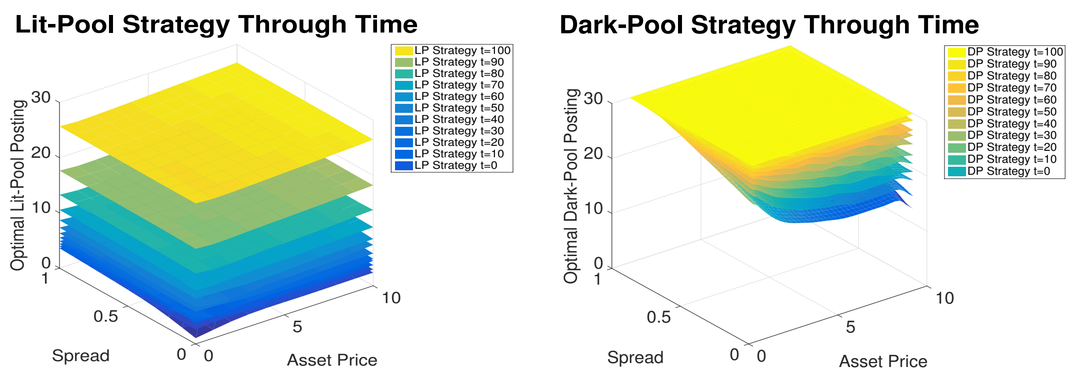

Figure 10: Optimal lit and dark pool strategies. We set $X_t = 30,000$ for each t, $\alpha = 2$, $\gamma = 0.0001$, $\lambda_1^b = 0.1$, $\lambda_2^b = 0.5$, $\lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $\bar{\Delta} = 0.1$, $\bar{S} = 40$, $\kappa^b = \kappa^\Delta = 0.02$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0,1]$, $\lambda^y = 0.1$, $\mu^\Delta = \mu^b = 0.001$.

4.2 Geometric Lévy model

Next we propose an exponential model for the best bid price and the spread process, so to ensure their positivity at every point in time, while upholding all other assumptions made in Section 4.1. In particular, all the jump processes considered herein are taken to be independent. We set

$$\frac{\mathrm{d}S_u^b}{S_{u^-}^b} = (\mu - \mu^b \nu_u)\mathrm{d}u + \mathrm{d}J_u^{b,1} - \mathrm{d}J_u^{b,2}, \tag{4.6}$$

$$\frac{\mathrm{d}\Delta_u}{\Delta_{u^-}} = \mu^\Delta \nu_u \mathrm{d}u + \mathrm{d}J_u^{\Delta,1} - \mathrm{d}J_u^{\Delta,2} - \mathrm{d}J_u^{b,1} + \mathrm{d}J_u^{b,2}. \tag{4.7}$$

Here, μ^b and μ^Δ are the coefficients of the permanent price impact caused by the lit pool orders submitted by the agent, and μ is a constant drift coefficient which may model the market trend. In order to ensure positivity of both, the price and the spread processes, we recall that the random variables $z^{b,i}$ and $z^{\Delta,i}$ are almost surely positive. Furthermore we impose $z^{b,i} < 1$ and $z^{\Delta,2} < 1$, almost surely. All other quantities are defined as in the previous example. In Figure 11, we plot a simulation of the best ask, the mid and the best bid prices.

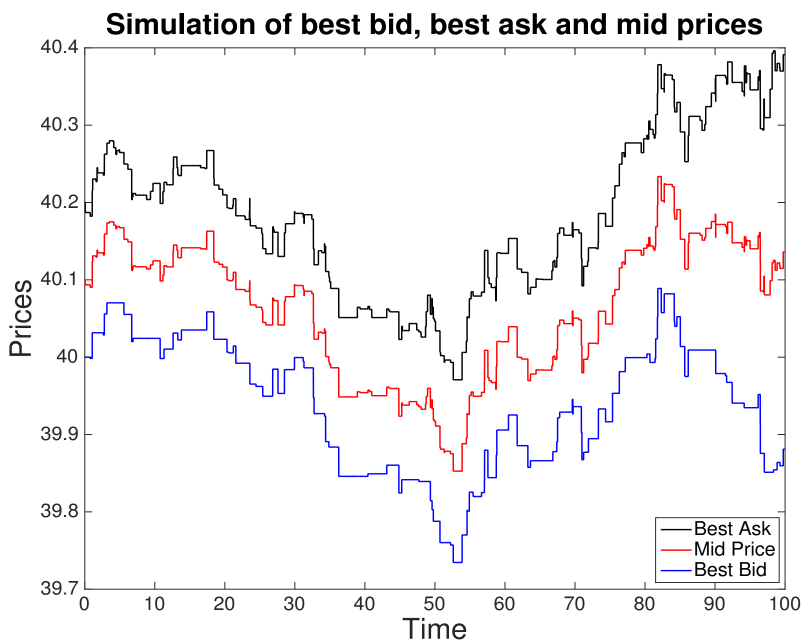

Figure 11: Simulation of the best ask, best bid and mid prices for a time frame of 100 seconds. We set $\mu = 0$, $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.5$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta = 0.2$.

By considering the same problem treated in Section 4.1, Equation (4.4), the associated HJB

PIDE is given by

$$\sup_{\nu\in\mathscr{X}}\Bigg\{\frac{\partial V}{\partial t}(t,\boldsymbol{x})+(\mu-\mu^b\nu)s^b\frac{\partial V}{\partial s^b}(t,\boldsymbol{x})+\mu^\Delta\nu\Delta\frac{\partial V}{\partial\Delta}(t,\boldsymbol{x})+\nu(s^b-\beta\nu)\frac{\partial V}{\partial y}(t,\boldsymbol{x})$$
$$-\gamma x^2-\nu\frac{\partial V}{\partial x}(t,\boldsymbol{x})+\lambda^y\mathbb{E}^{(z^y)}\left[V\left(t,x-nz^y,s^b,\Delta,y+nz^y\left(s^b+\frac{\Delta}{2}\right)\right)-V(t,\boldsymbol{x})\right]$$
$$+\lambda_1^b\mathbb{E}^{(z^{b,1})}\Big[V\big(t,x,s^b(1+z^{b,1}),\Delta(1-z^{b,1}),y\big)-V(t,\boldsymbol{x})\Big]$$
$$+\lambda_2^b\mathbb{E}^{(z^{b,2})}\Big[V\big(t,x,s^b(1-z^{b,2}),\Delta(1+z^{b,2}),y\big)-V(t,\boldsymbol{x})\Big]$$
$$+\lambda_1^\Delta\mathbb{E}^{(z^{\Delta,1})}\Big[V\big(t,x,s^b,\Delta(1+z^{\Delta,1}),y\big)-V(t,\boldsymbol{x})\Big]$$
$$+\lambda_2^\Delta\mathbb{E}^{(z^{\Delta,2})}\Big[V\big(t,x,s^b,\Delta(1-z^{\Delta,2}),y\big)-V(t,\boldsymbol{x})\Big]\Bigg\}=0$$

with terminal condition $V(\tau,\boldsymbol{x})=y+(s^b-\alpha x)x$ and boundary condition $V(u,0,s,\Delta,y)=y$.

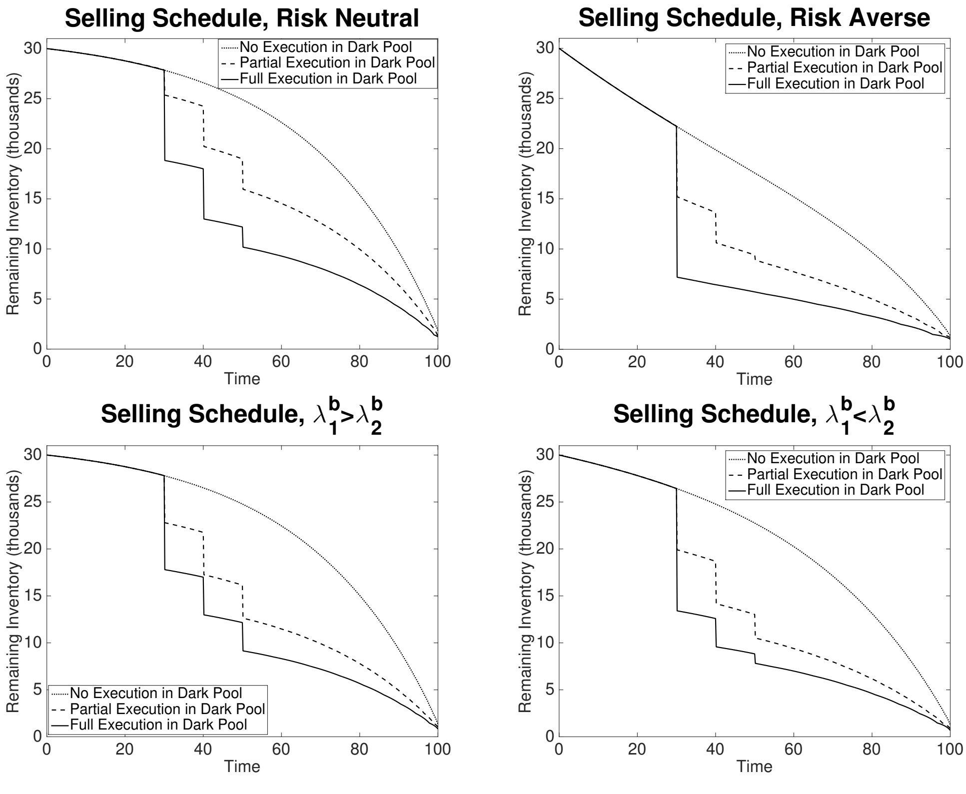

Figure 12: Optimal selling strategy displayed as a function of the remaining inventory. We set $z^{b,i},z^{\Delta,i}\sim U[0,0.1]$, $s^b=40$, $\Delta=0.3$, $\mu=0$, $\mu^b=0,\mu^\Delta=0.0001$, $\beta=1\times10^{-5}$, $\alpha=2$, $z^y\sim U[0,1]$, $\lambda^y=0.1$. Top panels: $\lambda_1^b=\lambda_2^b=\lambda_1^\Delta=\lambda_2^\Delta=0.3$ and $\gamma=0$ (left), $\gamma=0.01$ (right). Bottom panels: $\gamma=0.0001$ and $\lambda_1^b=\lambda_2^\Delta=0.5$, $\lambda_2^b=\lambda_1^\Delta=0.1$ (left), $\lambda_1^b=\lambda_2^\Delta=0.1$, $\lambda_2^b=\lambda_1^\Delta=0.5$.

In Figure 12 we show the inventory evolution for the case of a risk-neutral and of a risk-averse investor (top panels). Furthermore, we show the optimal trading trajectories for an increasing and a decreasing bid price (bottom panels). We notice that an increasing bid price (i.e. $\lambda_1^b > \lambda_2^b$) causes the agent to decelerate trading in the standard exchange so to benefit from the future favourable market trend. The opposite holds for a decreasing bid price. As in Section 4.1, we show the lit and dark pools full trading strategies as a function of the best bid price and the spread. In Figure 13 we show both, the case of an increasing bid price (top panels) and the case of a decreasing bid price (bottom panels). For Figures 12 and 13 similar considerations as for the mean-reverting model apply.

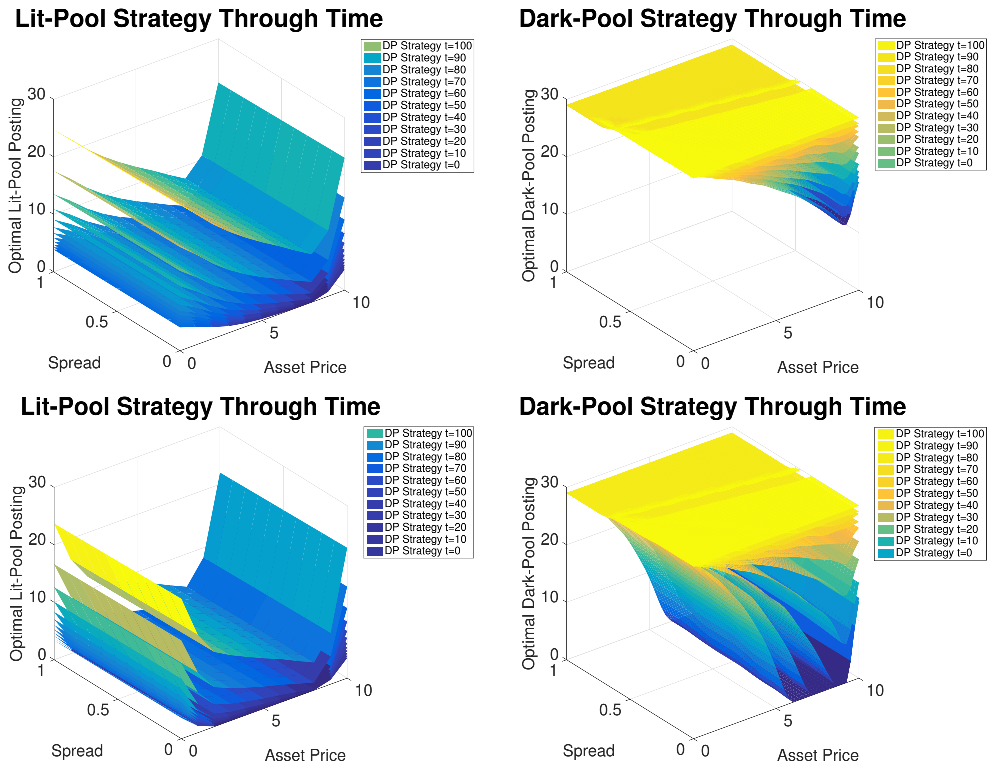

Figure 13: Optimal lit and dark pool strategies. We set $X_t = 30{,}000$ for each t, $\alpha = 2$, $\gamma = 0.0001$, $\lambda_1^b = 0.1$, $\lambda_2^b = 0.5$, $\lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $\beta = 1 \times 10^{-5}$, $z^y \sim U[0,1]$, $\lambda^y = 0.1$, $\mu = 0$, $\mu^\Delta = \mu^b = 0.001$. Top panels: $\lambda_1^b = \lambda_2^\Delta = 0.5$, $\lambda_2^b = \lambda_1^\Delta = 0.1$. Bottom panels: $\lambda_1^b = \lambda_2^\Delta = 0.1$, $\lambda_2^b = \lambda_1^\Delta = 0.5$.

Similar features to the Kratz and Schöneborn [28] optimal trading strategy in the single-asset case, can be recovered in the present framework by (i) modelling $\{S_u^b\}$ by a (local) martingale, for example by specifying $\mu^b = 0$ and $\mu = -\lambda_1^b \mathbb{E}[z^{b,1}] + \lambda_2^b \mathbb{E}[z^{b,2}]$, (ii) setting $\Delta_u \equiv 0 \ \ \forall\, u \in [t, T]$, and (iii) setting the random variables $z_i^y \equiv 1$ so to avoid partial execution. In Figure 14, we provide the results for the above choices in the case of a risk-neutral investor, $\gamma = 0$, (left picture) and of a risk-averse investor, $\gamma > 0$, (right picture). The shape of the policy shares the same features of the results obtained by Kratz and Schöneborn [28] (see Figure 4.1, page 17) without the constraint that total liquidation has to be achieved by T.

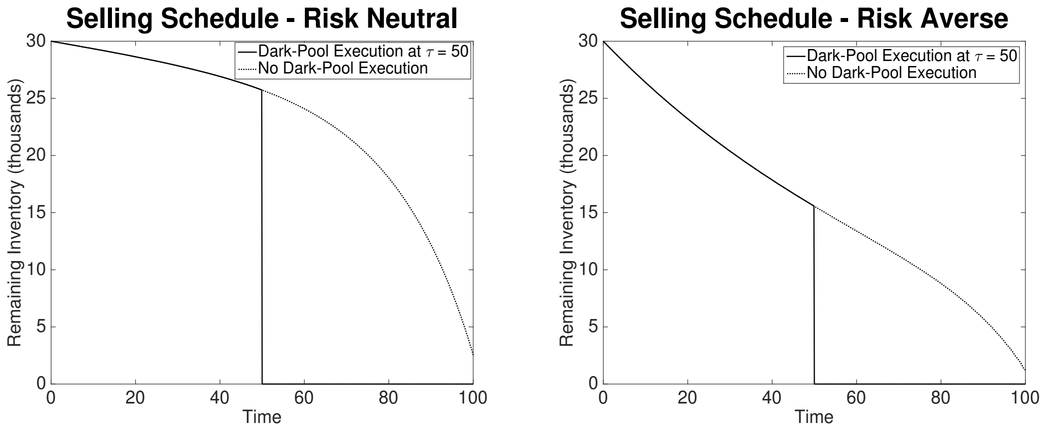

Figure 14: Optimal selling strategy displayed as a function of the remaining inventory. We set $\lambda_1^b = \lambda_2^b = \lambda_1^\Delta = \lambda_2^\Delta = 0.2$, $z^{b,i}, z^{\Delta,i} \sim U[0, 0.1]$, $s^b = 40$, $\Delta \equiv 0$, $\kappa^b = \mu^b = 0$, $\mu = 0$, $\beta = \alpha = 1 \times 10^{-5}$. Left panel $\gamma = 0$, and right panel $\gamma = 0.01$.

5 Conclusions

In this paper we consider an optimal trading problem in the situation where an investor has access to both, a standard exchange and a dark pool. We propose novel continuous-time dynamics for the best bid price and the market spread, thus obtaining the mid-price, in an order-driven market, and capture the impact of the market activity in the compound Poisson processes. In particular we show how the best bid price changes due to the submission of limit buy orders at a more advantageous price, to the incoming market sell orders which may walk the book, or due to cancellations of limit sell orders posted at the best bid price. The orders in the dark pool are executed at the current mid-price in line with the available liquidity. We consider a partial execution setting so to account for various liquidity levels in the dark pool. We observe that the LOB dynamics plays an important role in determining the optimal trading strategy, for both the lit ad the dark pool. This is expected, for the price traded in the dark pool is taken from the lit pool. In fact, the mid-price is influenced by the LOB dynamics, and so must be the optimal orders placement in the dark pool. We obtain the optimal trading strategies first in the case that only a standard exchange is available, and then also for when one may trade simultaneously in a dark pool. We discuss in detail the cases in which the market price dynamics is modelled by (i) a mean-reverting jump process and (ii) a geometric Lévy process, and pay particular attention to properties usually less emphasised in the literature, e.g. price dynamics which model realistic market features, the permanent price impact in simultaneous lit and dark pool trading strategies, and the effect of the sell side of the book in a liquidation problem. We employ the dynamic programming approach to obtain the optimal trading policies and we characterise—by means of standard arguments—the value function by the unique viscosity solution of the HJB PIDE associated to the optimal control problem.

Although we treat a more general problem in Section 3, the considered models in Section 4 are closer to the classical optimal liquidation settings, making comparisons between the models rather straightforward. The detailed model examples are also used to show that so-called round-trips, whereby an agent decides to post all the remaining inventory in the dark pool, are not generally beneficial. The discussion extends to the behaviour of market agents who have various levels of risk-appetite and who will find different strategies of simultaneous trading in lit and dark pools more attractive.

6 Appendix

6.1 Pseudocode for solving Equation (4.5) numerically

In this section we briefly show the numerical scheme adopted to solve Equation (4.5) so to find the agent's optimal strategy. This is similar to the one proposed by Guilbaud and Pham [22] and Bian et al. [5], and adapted to the particular model considered in this paper. We here provide the pseudocode for the mean-reverting model, the one for the geometric Lévy model is analogous.

We create an equally-spaced grid for (i) the time axis such that $t_{i+1} - t_i = \delta t, \ \ \forall\ i = 0, 1, 2, \dots, n-1$, (ii) the inventory, where $x_{j+1} - x_j = \delta x \ \ \forall\ j = 0, 1, 2, \dots, m-1$, (iii) the price, where $s_{k+1} - s_k = \delta s \ \ \forall\ k = 0, 1, 2, \dots, p-1$, (iv) the spread, where $\Delta_{\ell+1} - \Delta_\ell = \delta\Delta \ \ \forall\ \ell = 0, 1, 2, \dots, q-1$, and (v) the cash process where $y_{u+1} - y_u = \delta y \ \ \forall\ u = 0, 1, 2, \dots, o-1$. In this section we make use of the notation $c_1 = \kappa^b(\bar{S} - s_k - \mu^b \nu)$, $c_2 = -\nu$, $c_3 = \kappa^\Delta(\bar{\Delta} - \Delta_\ell + \mu^\Delta \nu)$, $c_4 = \nu(s_k - \beta\nu)$, and note that such coefficients are variables and depend on both, the state variables and the control values. We write $V^{i,j,k,\ell,u} = V(t_i, x_j, s_k, \Delta_\ell, y_u)$ and we consider an upwind discretisation scheme for the numerical derivatives:

$$V_{(x)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell,u} - V^{i,j-1,k,\ell,u}}{\delta x}$$

$$V_{(s)}^{i,j,k,\ell,u} := \frac{V^{i,j,k+1,\ell,u} - V^{i,j,k,\ell,u}}{\delta s}, \text{ if } c_1 \geq 0, \text{ and } V_{(s)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell,u} - V^{i,j,k-1,\ell,u}}{\delta s}, \text{ if } c_1 < 0$$

$$V_{(\Delta)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell+1,u} - V^{i,j,k,\ell,u}}{\delta\Delta}, \text{ if } c_3 \geq 0, \text{ and } V_{(\Delta)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell,u} - V^{i,j,k,\ell-1,u}}{\delta\Delta}, \text{ if } c_3 < 0$$

$$V_{(y)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell,u+1} - V^{i,j,k,\ell,u}}{\delta y}, \text{ if } c_4 \geq 0, \text{ and } V_{(s)}^{i,j,k,\ell,u} := \frac{V^{i,j,k,\ell,u} - V^{i,j,k,\ell,u-1}}{\delta y}, \text{ if } c_4 < 0.$$

Moreover, we discretise the distribution of $\{z^{b,1}, z^{b,2}, z^{\Delta,1}, z^{\Delta,2}\}$ such that they are all supported in a finite set, say, $I = 0, 1, 2, \dots, \bar{z}$ which denotes the number of nodes the price and the spread move with probability $p(z_\iota)$, where $\iota \in I$, associated with every state. We thus reduce the space grids of s^b and Δ, of which indices are in the sets $\mathbb{S} := \bar{z}, \bar{z}+1, \dots, p-\bar{z}$ and $\mathbb{D} := \bar{z}, \bar{z}+1, \dots, q-\bar{z}$, respectively. We further reduce the space grid y to be $\mathbb{Y} := \bar{y}, \bar{y}+1, \ldots, o-\bar{y}$, where $\bar{y} := \arg\min_{\xi=0,1,2,\ldots,o} |y_\xi - x_m(s_p - \alpha x_m)|$.

Pseudocode

for all j,k,ℓ,u
  Set $V^{n,j,k,\ell,u} = (y_u + x_j(s_k - \alpha x_j))$.
end
for *i=n-1,n-2,...,0*
  for all $j \in \mathbb{X}, k \in \mathbb{S}, \ell \in \mathbb{D}, u \in \mathbb{Y}$
   ** Compute*

$$
\begin{aligned}
V^{i,j,k,\ell,u} = F_{\delta t}(V_{\delta t}) = V^{i+1,j,k,\ell,u} \\
+\delta t \Big(\sup_\nu \Big\{ c_1 V_{(s)}^{i+1,j,k,\ell,u} + c_2 V_{(x)}^{i+1,j,k,\ell,u} + c_3 V_{(\Delta)}^{i+1,j,k,\ell,u} + c_4 V_{(y)}^{i+1,j,k,\ell,u} \Big\} \\
-\gamma x_j^2 + \lambda^y \sup_{\eta \in [0,x_j]} \sum_\iota p(z_\iota^y) \Big[V^{i+1,j-\zeta_{1,\iota},k,\ell,u+\zeta_{2,\iota}} - V^{i+1,j,k,\ell,u} \Big] \\
+\lambda_1^b \sum_\iota p(z_\iota^{b,1}) \Big[V^{i+1,j,k+z_\iota^{b,1},\ell-z_\iota^{b,1},u} - V^{i+1,j,k,\ell,u} \Big] \\
+\lambda_2^b \sum_\iota p(z_\iota^{b,2}) \Big[V^{i+1,j,k-z_\iota^{b,2},\ell+z_\iota^{b,2},u} - V^{i+1,j,k,\ell,u} \Big] \\
+\lambda_1^\Delta \sum_\iota p(z_\iota^{\Delta,1}) \Big[V^{i+1,j,k,\ell+z_\iota^{\Delta,1},u} - V^{i+1,j,k,\ell,u} \Big] \\
+\lambda_2^\Delta \sum_\iota p(z_\iota^{\Delta,2}) \Big[V^{i+1,j,k,\ell-z_\iota^{\Delta,2},u} - V^{i+1,j,k,\ell,u} \Big] \Big),
\end{aligned}
$$

   where $\zeta_{1,\iota} := \arg\min_{\xi=0,1,2,\ldots,m-j} |x_\xi - \eta z_\iota^y|$,
   and $\zeta_{2,\iota} := \arg\min_{\xi=0,1,2,\ldots,o-u} |y_\xi - \eta z_\iota^y (s_k + \Delta_\ell/2)|$.
  end
 ** Compute* $V^{i,j,k,\ell,u}$ *for* $k = 0,1,\ldots,\bar{z}-1,\ k = p-\bar{z}+1,\ldots,p,$
  $\ell = 0,1,\ldots,\bar{z}-1,\ \ell = q-\bar{z}+1,\ldots,q,$
  $u = 0,1,\ldots,\bar{y}-1,\ u = o-\bar{y}+1,\ldots,o$ *by interpolation.*
end

A theorem by Barles and Souganidis [3] ensures that any scheme which is monotone, stable and consistent, converges to the viscosity solution, provided that a comparison principle holds. In particular, we say that a scheme is monotone if $F_{\delta t}(W_{\delta t}) \leq F_{\delta t}(U_{\delta t})$ for $W_{\delta t} \leq U_{\delta t}$, where the operator F is defined in the pseudocode, above. A scheme is stable if there exist a constant K such that $||V_{\delta t}|| \leq K$. A numerical scheme is consistent if the discretisation converges to the PDE—as the mesh tends to zero—for any continuous differentiable test function. We prove the viscosity solution property of the value function and the comparison principle in the remainder of the appendix. We now provide the conditions sufficient for a simplified version of the above scheme to converge, and recall that, heuristically, a finer time grid is the key for the non-explosive nature of an explicit finite-difference scheme. There is no loss of generality by considering the simplified system presented below. This is because, while a non-simplified version would include more terms, the analysis remains the same for the larger system. Let us assume that, for $\bar{\delta}^b, \bar{\delta}^\Delta > 0$, the random variables $z^{b,1} \in \{0, \bar{\delta}^b\}$ and $z^{b,2} \in \{-\bar{\delta}^b, 0\}$ with $p(z^{b,i} \neq 0) = p^{b,i}$, the random variables $z^{\Delta,1} \in \{0, \bar{\delta}^\Delta\}$ and $z^{\Delta,2} \in \{-\bar{\delta}^\Delta, 0\}$ with $p(z^{\Delta,i} \neq 0) = p^{\Delta,i}$. Furthermore, we assume that $z^y = \{0, 1/2, 1\}$ with associated probabilities p^y_0, $p^y_{1/2}$, p^y_1. Furthermore, we performe the change of variables $\tau = T - t$. One then obtains:

$$
\begin{aligned}
V^{i+1,j,k,\ell,u} = V^{i,j,k,\ell,u} + \delta t \sup_{v,n_{x,y}} \Big(& \tilde{c}_1 \delta s V^{i,j,k,\ell,u}_{(s)} + \tilde{c}_2 \delta x V^{i,j,k,\ell,u}_{(x)} + \tilde{c}_3 \delta \Delta V^{i,j,k,\ell,u}_{(\Delta)} + \tilde{c}_4 \delta y V^{i,j,k,\ell,u}_{(y)} \\
& + \tilde{c}_5 \Big[V^{i,j-\frac{1}{2}n_x,k,\ell,u+\frac{1}{2}n_y} - V^{i,j,k,\ell,u} \Big] + \tilde{c}_6 \Big[V^{i,j-n_x,k,\ell,u+n_y} - V^{i,j,k,\ell,u} \Big] \\
& + \tilde{c}_7 \Big[V^{i,j,k+\bar{\delta}^b,\ell-\bar{\delta}^b,u} - V^{i,j,k,\ell,u} \Big] + \tilde{c}_8 \Big[V^{i,j,k-\bar{\delta}^b,\ell+\bar{\delta}^b,u} - V^{i,j,k,\ell,u} \Big] \\
& + \tilde{c}_9 \Big[V^{i,j,k,\ell+\bar{\delta}^\Delta,u} - V^{i,j,k,\ell,u} \Big] + \tilde{c}_{10} \Big[V^{i,j,k,\ell-\bar{\delta}^\Delta,u} - V^{i,j,k,\ell,u} \Big] \Big),
\end{aligned}
\tag{6.1}
$$

where $\tilde{c}_1 = c_1/\delta s$, $\tilde{c}_2 = c_2/\delta x$, $\tilde{c}_3 = c_3/\delta\Delta$, $\tilde{c}_4 = c_4/\delta y$, $\tilde{c}_5 = \lambda^y p^y_{\frac{1}{2}}$, $\tilde{c}_6 = \lambda^y p^y_1$, $\tilde{c}_7 = \lambda^{b,1} p^{b,1}$, $\tilde{c}_8 = \lambda^{b,2} p^{b,2}$, $\tilde{c}_9 = \lambda^{\Delta,1}$ and $\tilde{c}_{10} = \lambda^{\Delta,2} p^{\Delta,2}$. Furthermore, the values n_x and n_y model the inventory and cash movement of the agent when the dark-pool execution takes place. Since the scheme has variable coefficients ($c_{1,2,3,4}$), we need to make sure that the properties hold for every value that such coefficients may take. After a few simplifying steps, one finds that the monotonicity property of the scheme in Equation (6.1) is satisfied if

$$
\delta t \leq \frac{1}{\sum_{i=1}^{10} \tilde{c}^m_i}. \tag{6.2}
$$

where $\tilde{c}^m_i = \max\{|\tilde{c}^{min}_i|, |\tilde{c}^{max}_i|\}$ is the largest possible absolute value (i.e. worst-case scenario) of each coefficient taking into account (i) the possible values of the state variables embedded in the coefficients $\tilde{c}_i$ and (ii) all the possible values that the controls may take.

For the stability property, we perform a standard Von Neumann analysis. We define the error $\epsilon(t)$ by the difference between the exact solution of the discretised equation (6.1) and the numerical solution of the same equation. Since both solutions should satisfy equation (6.1), the error should also satisfy (6.1). We make use of the following (standard) ansatz for the error $\epsilon(t)$

$$
\epsilon(t) = \mathrm{e}^{at} \mathrm{e}^{i(j_m x + k_m s + \ell_m \Delta + u_m y)}, \tag{6.3}
$$

where i is the imaginary unit and j_m, k_m, ℓ_m and u_m are wavenumbers. We define the error amplification factor $g(t) = \epsilon(t + \delta t)/\epsilon(t)$ and recall that, for a scheme to be stable in the Von Neumann sense, we must have $|g(t)| \leq 1$. By substituting Equation (6.3) in Equation (6.1) and by making use of Euler's formulae, one can rewrite the amplification factor in the form

$$
g(t) = 1 + \sum_{i=1}^{10} \alpha_m (1 - \cos(\theta_m)) + i \sum_{i=1}^{10} c_m \sin(\theta_m), \tag{6.4}
$$

where $\alpha_m = \delta t|\tilde{c}_i|$, $c_m = \delta t \tilde{c}_i$ and θ_m are arbitrary phase angles such that $-\pi \leq \theta_m \leq 1$. Such an amplification factor has been studied in Hindmarsh et al. [24], who prove that Equation (6.2) is also a sufficient condition for stability. Finally, the consistency property is satisfied by noticing that $\lim_{\delta t,\delta x,\delta s,\delta\Delta,\delta y\to 0}(F_{t+\delta t}(\phi_{\delta t}) - F_t(\phi_t))/\delta t$ tends to the reduced version of the HJB Equation (4.5), provided that the simplifying assumptions made in this section are also made for the non-discretised equation. We thus conclude that a sufficient condition for the scheme to converge to the viscosity solution is given by Equation (6.2).

6.2 Dark pool optimal control problem

We give the details of the optimal control problem treated in Section 3. For $\boldsymbol{p} \in \mathbb{R}^4$ with components (p_1, p_2, p_3, p_4), we define the operators $\mathscr{H}_1$, $\mathscr{B}_{b,\Delta}$ and $\mathscr{B}_\Delta$ by

$$\mathscr{H}_1(t,\boldsymbol{x},\boldsymbol{p}) = \sup_{v\in\mathscr{V}} \left\{ f_1(t,\boldsymbol{x},v) - v p_1 + \mu^b\left(t,s^b,v\right) p_2 + \mu^\Delta(t,\Delta,v) p_3 + \mu^y(t,s^b,\Delta,v) p_4 \right\},$$

$$\begin{aligned}\mathscr{B}_{b,\Delta}(t,\boldsymbol{x},\varphi) = \sum_{i=1}^{2} \lambda_i^b \mathbb{E}^{(z^{b,i})}\Big[&\varphi\big(t,x,s^b + h_i^b\big(t,s^b\big)z^{b,i}, \Delta + h_i^{b,\Delta}\big(t,s^b,\Delta\big)z^{b,i}, y\big) - \varphi\big(t,\boldsymbol{x}\big) \\ &- h_i^b\big(t,s^b\big)z^{b,i}\frac{\partial\varphi}{\partial s^b}\big(t,\boldsymbol{x}\big) - h_i^{b,\Delta}\big(t,s^b,\Delta\big)z^{b,i}\frac{\partial\varphi}{\partial\Delta}\big(t,\boldsymbol{x}\big)\Big],\end{aligned} \tag{6.5}$$

and

$$\mathscr{B}_\Delta(t,\boldsymbol{x},\varphi) = \sum_{i=1}^{2} \lambda_i^\Delta \mathbb{E}^{(z^{\Delta,i})}\Big[\varphi\big(t,x,s^b,\Delta + h_i^\Delta\big(t,\Delta\big)z^{\Delta,i},y\big) - \varphi\big(t,\boldsymbol{x}\big) - h_i^\Delta\big(t,\Delta\big)z^{\Delta,i}\frac{\partial\varphi}{\partial\Delta}\big(t,\boldsymbol{x}\big)\Big]. \tag{6.6}$$

We also define the operator $\mathscr{B}_y$ by

$$\mathscr{B}_y(t,\boldsymbol{x},\varphi) = \sup_{n\in\mathscr{N}} \lambda^y \mathbb{E}^{(z^y)}\Big[\varphi(t, x - nz^y, s^b, \Delta, y + h^y(t,s^b,\Delta,n)z^y) - \varphi(t,\boldsymbol{x})\Big]. \tag{6.7}$$

Standard dynamic programming arguments suggest that the HJB equation associated to the optimisation problem in (3.4) is a PIDE of the form

$$rV(t,\boldsymbol{x}) - \frac{\partial V}{\partial t}(t,\boldsymbol{x}) - \mathscr{H}_1(t,\boldsymbol{x},D_{\boldsymbol{x}}V) - \mathscr{B}_{b,\Delta}(t,\boldsymbol{x},V) - \mathscr{B}_\Delta(t,\boldsymbol{x},V) - \mathscr{B}_y(t,\boldsymbol{x},V) = 0, \tag{6.8}$$

on $[0,T)\times\mathscr{O}$, with terminal condition $V(\tau,\boldsymbol{x}) = g_1(\boldsymbol{x})$ and boundary condition $V(u,0,s^b,\Delta,y) = g_1(0,s^b,\Delta,y)$.

Definition 6.1. (Pham [35]) *A continuous function $V : [0,T)\times\mathcal{O}\to\mathbb{R}$ is a viscosity subsolution (resp. supersolution) of the HJB Equation (6.8) if*

$$r\phi(\bar{t},\bar{\boldsymbol{x}})-\frac{\partial\phi}{\partial t}(\bar{t},\bar{\boldsymbol{x}})-\mathcal{H}_1(\bar{t},\bar{\boldsymbol{x}},D_{\boldsymbol{x}}\phi)-\mathcal{B}_{b,\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)-\mathcal{B}_{\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)-\mathcal{B}_{y}(\bar{t},\bar{\boldsymbol{x}},\phi)\le 0$$

(resp. ≥ 0) for each $\phi\in\mathcal{C}^{1,1}([0,T)\times\mathcal{O})\cap PB$ such that $V(t,\boldsymbol{x})-\phi(t,\boldsymbol{x})$ attains its maximum (resp. minimum) at $(\bar{t},\bar{\boldsymbol{x}})\subset[0,T)\times\mathcal{O}$. A continuous function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We here assume that the real-valued function

$$\psi(t,\boldsymbol{y},\boldsymbol{v})\in\left\{\mu^b(\cdot),\ h^b_{1,2}(\cdot),\ \mu^{\Delta}(\cdot),\ h^{\Delta}_{1,2}(\cdot),\ h^{b,\Delta}_{1,2}(\cdot),\ \mu^y(\cdot),\ h^y(\cdot),\ f(\cdot),\ g(\cdot),\ f_1(\cdot),\ g_1(\cdot)\right\}$$

satisfies, uniformly in the control variables, the Lipschitz continuity

$$|\psi(t_1,\boldsymbol{y}_1,\boldsymbol{v})-\psi(t_2,\boldsymbol{y}_2,\boldsymbol{v})|\le K(|t_1-t_2|+||\boldsymbol{y}_1-\boldsymbol{y}_2||),$$

and the linear growth conditions

$$|\psi(t_1,\boldsymbol{y}_1,\boldsymbol{v})|\le K(1+|t_1|+||\boldsymbol{y}_1||),$$

where $\boldsymbol{y}_{1,2}\in\mathcal{O}$ is the vector of state variables and $\boldsymbol{v}\in\mathcal{Z}$ is the vector of controls associated to every function (the latter may also be empty). Standard results—see, e.g., Ikeda and Watanabe [26]—ensure that there exists a strong and path-wise unique solution of the price, the spread and the wealth models defined by Equations (2.2), (2.3) and (3.3).

6.3 Moments estimates

We now provide some moments estimates of S^b_u. We note that both, the following proposition and its proof, are analogous for Δ_u, X_u and Y_u. In the remainder of the paper, we write $\boldsymbol{X}_{t,x}(u)$ for the state variables with initial values $(t,\boldsymbol{x})$ since we need the initial conditions explicitly.

Proposition 6.2. *Fix $p=1,2$ and let $S^b_{t,s^b}(u)$ be the random variable at a fixed time $u\in[t,T]$ with initial values $(t,s^b)\in[0,T]\times\mathbb{R}_+$. Then, for any $v\in\mathcal{V}$ and for any stopping time $\tau_0\le h\in[0,T]$ there exists a constant $K=K(p,C,M,T)>0$ such that*

$$\begin{aligned}\mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)\big|^p\Big]&\le K\left(1+\left|s^b\right|^p\right),\\ \mathbb{E}\Big[\,\big|S^b_{t,s^b_1}(\tau_0)-S^b_{t,s^b_2}(\tau_0)\big|^p\Big]&\le K\left(\left|s^b_1-s^b_2\right|^p\right),\\ \mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)-s^b\big|^p\Big]&\le K\left(1+\left|s^b\right|^p\right)(h-t)^{\frac{p}{2}},\\ \mathbb{E}\Big[\sup_{0\le u\le h}\big|S^b_{t,s^b}(u)-s^b\big|^p\Big]&\le K\left(1+\left|s^b\right|^p\right)(h-t)^{\frac{p}{2}}.\end{aligned}\tag{6.9}$$

Proof. We adapt the proof in Pham [35] to the present work and indeed we shall consider the proof only for $p = 2$ as it suffices to ensure the relation for $p = 1$, according to Hölder's inequality. In order to reduce notation, here K is a generic positive constant which may take different values in different places. Define $\mathscr{T}_h$ as the set of all stopping times smaller than $h \in [0, T]$. By the optional sampling theorem and the Lévy-Itô isometry, we have

$$\mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)\big|^2\Big] \le$$
$$K\mathbb{E}\left[\,\big|s^b\big|^2 + \int_t^{\tau_0}\Big|\mu^b\Big(u, S^b_{t,s^b}(u), \nu_u\Big)\Big|^2 \mathrm{d}u + \sum_{i=1}^{2}\lambda^b_i \mathbb{E}\Big[\big|z^{b,i}_1\big|^2\Big]\int_{t_1}^{\tau_0}\Big|h^b_i\Big(u, S^b_{t,s^b}(u)\Big)\Big|^2 \mathrm{d}u\right],$$

for $\tau_0 \in \mathscr{T}_h$. By the linear growth conditions on μ^b, h^b_1, h^b_2, we have

$$\mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)\big|^2\Big] \le K\left[1 + \big|s^b\big|^2 + \mathbb{E}\int_t^{\tau_0}\big|S^b_{t,s^b}(u)\big|^2 \mathrm{d}u\right]. \tag{6.10}$$

As noted in Pham (1998), if τ_0 were a deterministic time, (6.10) would yield

$$\mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)\big|^2\Big] \le K\Big[1 + \big|s^b\big|^2\Big].$$

By definition of $\mathscr{T}_h$, we note that

$$\mathbb{E}\left[\int_t^{\tau_0}\big|S^b_{t,s^b}(u)\big|^2 \mathrm{d}u\right] \le \mathbb{E}\left[\int_t^{h}\big|S^b_{t,s^b}(u)\big|^2 \mathrm{d}u\right].$$

Thus, by applying Fubini's theorem to exchange the order of integration and by Gronwall's lemma,

$$\mathbb{E}\Big[\,\big|S^b_{t,s^b}(\tau_0)\big|^2\Big] \le K\left[1 + \big|s^b\big|^2 + \int_t^{h}\mathbb{E}\Big[\,\big|S^b_{t,s^b}(u)\big|^2\Big]\mathrm{d}u\right] \le K\Big[1 + \big|s^b\big|^2\Big], \tag{6.11}$$

for a suitable constant $K = K(p, C, M, T)$. Define the process Z_u by

$$Z_u = S^b_{t,s^b_1}(u) - S^b_{t,s^b_2}(u).$$

Then by an application of Itô's formula to $|Z_u|^2$, we have

$$\mathbb{E}\Big[\big|Z_{\tau_0}\big|^2\Big] = \mathbb{E}\Bigg[\big|s^b_1 - s^b_2\big|^2 + \int_t^{\tau_0} 2Z_u\Big(\mu^b\Big(u, S^b_{t,s^b_1}(u), \nu_u\Big) - \mu^b\Big(u, S^b_{t,s^b_2}(u), \nu_u\Big)\Big)\mathrm{d}u$$
$$+ \sum_{i=1}^{2}\lambda^b_i \mathbb{E}\Big[\big|z^{b,i}_1\big|^2\Big]\int_t^{\tau_0}\Big|h^b_i\Big(u, S^b_{t,s^b_1}(u)\Big) - h^b_i\Big(u, S^b_{t,s^b_2}(u)\Big)\Big|^2 \mathrm{d}u\Bigg].$$

From the Lipschitz condition on μ^b, h_1^b and h_2^b, it follows that

$$\mathbb{E}\Big[\,\big|Z_{\tau_0}\big|^2\Big] \leq K\mathbb{E}^{(z^{b,i})}\bigg[\,\big|s_1^b - s_2^b\big|^2 + \int_t^{\tau_0}\big|S^b_{t,s_1^b}(u) - S^b_{t,s_2^b}(u)\big|^2 \mathrm{d}u\bigg].$$

By making use of Fubini's Theorem and Gronwall's Lemma we get

$$\mathbb{E}\Big[\,\big|S^b_{t,s_1^b}(\tau_0) - S^b_{t,s_2^b}(\tau_0)\big|^2\Big] \leq K\mathbb{E}\bigg[\,\big|s_1^b - s_2^b\big|^2 + \int_t^h \big|Z_u\big|^2 \mathrm{d}u\bigg] \leq K\,\big|s_1^b - s_2^b\big|^2,$$

for a suitable constant $K = K(p, C, M, T)$. For the third moment estimate, we make use of the first moment estimate in (6.9) to obtain

$$\mathbb{E}\Big[\,\big|S^b_{t,s^b}(t) - s^b\big|^2\Big] \leq K\bigg[\int_t^{\tau_0}\Big(1 + \mathbb{E}\Big[\,\big|S^b_{t,s^b}(u)\big|^2\Big]\Big)\mathrm{d}u\bigg] \leq K\Big(1 + \big|s^b\big|^2\Big)(h-t).$$

The fourth moment estimate in (6.9) follows from the third moment estimate, Doob's maximal inequality, and the fact that the constant K does not depend on the control process. □

6.4 Viscosity solution

In what follows, we note that it suffices to show the viscosity property for the model presented in Section 3, since the model discussed in Section 2.1 is a special case.

Proposition 6.3. *The value function $V : [0, T] \times \mathcal{O} \to \mathbb{R}$ defined in (3.4) is continuous on $[0, T] \times \mathcal{O}$. Furthermore, for $K > 0$ and $\forall\ \boldsymbol{x} \in \mathcal{O}$, it satisfies*

$$V(t, \boldsymbol{x}) \leq K(1 + \|\boldsymbol{x}\|). \tag{6.12}$$

Proof. We proceed in two steps. We first show that the value function is Lipschitz continuous in $\boldsymbol{x}$, uniformly in t. Next we show that it is continuous in t. We take $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{O}$ and since $|\sup(A) - \sup(B)| \leq \sup|(A-B)|$, we have that

$$\begin{aligned}
|V(t,\boldsymbol{x}) - V(t,\boldsymbol{y})| = &\bigg|\sup_{\nu\in\mathscr{Z}}\mathbb{E}\bigg[\int_t^\tau \mathrm{e}^{-r(u-t)} f_1\big(u, \boldsymbol{X}_{t,\boldsymbol{x}}(u), \nu_u\big)\,\mathrm{d}u + \mathrm{e}^{-r(\tau-t)} g_1\big(\boldsymbol{X}_{t,\boldsymbol{x}}(\tau)\big)\bigg] \\
&- \sup_{\nu\in\mathscr{Z}}\mathbb{E}\bigg[\int_t^\tau \mathrm{e}^{-r(u-t)} f_1\big(u, \boldsymbol{X}_{t,\boldsymbol{y}}(u), \nu_u\big)\,\mathrm{d}u + \mathrm{e}^{-r(\tau-t)} g_1\big(\boldsymbol{X}_{t,\boldsymbol{y}}(\tau)\big)\bigg]\bigg| \\
\leq &\sup_{\nu\in\mathscr{Z}}\bigg|\mathbb{E}\bigg[\int_t^\tau \mathrm{e}^{-r(u-t)}\Big(f_1\big(u, \boldsymbol{X}_{t,\boldsymbol{x}}(u), \nu_u\big)\,\mathrm{d}u - f_1\big(u, \boldsymbol{X}_{t,\boldsymbol{y}}(u), \nu_u\big)\Big)\mathrm{d}u \\
&+ \mathrm{e}^{-r(\tau-t)}\Big(g_1\big(\boldsymbol{X}_{t,\boldsymbol{x}}(\tau)\big) - g_1\big(\boldsymbol{X}_{t,\boldsymbol{y}}(\tau)\big)\Big)\bigg]\bigg|.
\end{aligned}$$

The Lipschitz continuity of f_1 and g_1 gives

$$\begin{aligned}
&|V(t,\boldsymbol{x})-V(t,\boldsymbol{y})| \\
&\qquad \le \sup_{\nu\in\mathscr{Z}} \mathbb{E}\bigg[\int_t^\tau \mathrm{e}^{-r(u-t)} K|\boldsymbol{X}_{t,x}(u)-\boldsymbol{X}_{t,y}(u)|\mathrm{d}u + \mathrm{e}^{-r(\tau-t)}K|\boldsymbol{X}_{t,x}(\tau)-\boldsymbol{X}_{t,y}(\tau)|\bigg] \\
&\qquad \le K||\boldsymbol{x}-\boldsymbol{y}||,
\end{aligned}$$

where the last inequality is justified by the moments estimates in Proposition 6.2. We note that an analogous calculation will produce Equation (6.12). We now take $0 \le t_1 < t_2 < T$ and we apply the DP to obtain

$$\begin{aligned}
|V(t_1,\boldsymbol{x})-V(t_2,\boldsymbol{x})| = \bigg|\sup_{\nu\in\mathscr{Z}} \mathbb{E}\bigg[&\int_{t_1}^{t_2\wedge\tau} \mathrm{e}^{-r(u-t_1)} f_1\big(u,\boldsymbol{X}_{t_1,x}(u),\nu_u\big)\,\mathrm{d}u \\
&+ \mathrm{e}^{-r(t_2-t_1)}V\big(t_2,\boldsymbol{X}_{t_1,x}(t_2)\big)\,\mathbb{1}_{\{\tau\ge t_2\}} + \mathrm{e}^{-r(\tau-t_1)}g_1\big(\boldsymbol{X}_{t_1,x}(\tau)\big)\,\mathbb{1}_{\{\tau<t_2\}}\bigg] - V(t_2,\boldsymbol{x})\bigg|.
\end{aligned}$$

We can add and subtract the quantity

$$\mathbb{1}_{\{\tau<t_2\}}\mathrm{e}^{-r(\tau-t_1)}(g_1(\boldsymbol{x})+V(t_2,\boldsymbol{x})) + \mathbb{1}_{\{\tau\ge t_2\}}\mathrm{e}^{-r(t_2-t_1)}V(t_2,\boldsymbol{x}),$$

to obtain

$$\begin{aligned}
&|V(t_1,\boldsymbol{x})-V(t_2,\boldsymbol{x})| \le \\
&\sup_{\nu\in\mathscr{Z}} \mathbb{E}\bigg[\int_{t_1}^{t_2\wedge\tau} \mathrm{e}^{-r(u-t_1)}|f_1\big(u,\boldsymbol{X}_{t_1,x}(u),\nu_u\big)|\mathrm{d}u + \mathbb{1}_{\{\tau\ge t_2\}}\mathrm{e}^{-r(t_2-t_1)}|V\big(t_2,\boldsymbol{X}_{t_1,x}(t_2)\big)-V(t_2,\boldsymbol{x})| \\
&+ \mathbb{1}_{\{\tau<t_2\}}\mathrm{e}^{-r(\tau-t_1)}|g_1\big(\boldsymbol{X}_{t_1,x}(\tau)\big)-g_1(\boldsymbol{x})| + \mathbb{1}_{\{\tau<t_2\}}\mathrm{e}^{-r(\tau-t_1)}|g_1(\boldsymbol{x})-V(t_2,\boldsymbol{x})| \\
&+ \mathbb{1}_{\{\tau<t_2\}}|(\mathrm{e}^{-r(\tau-t_1)}-1)V(t_2,\boldsymbol{x})| + \mathbb{1}_{\{\tau\ge t_2\}}|(\mathrm{e}^{-r(t_2-t_1)}-1)V(t_2,\boldsymbol{x})|\bigg] \le K|t_2-t_1|(1+||\boldsymbol{x}||),
\end{aligned}$$

where the last inequality is justified by: (i) the linear growth of f_1, g_1 and V, the Lipschitz continuity of g_1 and of V in $\boldsymbol{x}$ uniformly in t, and the moment estimates in Proposition 6.2. Thus, we can conclude that

$$|V(t_1,\boldsymbol{x})-V(t_2,\boldsymbol{y})| \le K(|t_2-t_1|(1+||\boldsymbol{x}||)+||\boldsymbol{x}-\boldsymbol{y}||).$$

□

Proposition 6.4. *The value function V defined by Equation (3.4) is a viscosity solution of the HJB PIDE (6.8).*

Proof. We show that $V(t,\boldsymbol{x})$ is a continuous viscosity solution of (6.8) by proving that it is both a supersolution and a subsolution. We proceed along the same lines of Øksendal and Sulem

[32] and we first show the supersolution property. We define a test function $\phi:[0,T)\times\mathscr{O}\to\mathbb{R}$ such that $\phi\in\mathscr{C}^{1,1}([0,T)\times\mathscr{O})\cap PB$ and, without loss of generality, we assume that $V-\phi$ reaches its minimum at $(\bar{t},\bar{\boldsymbol{x}})$, such that

$$V(\bar{t},\bar{\boldsymbol{x}})-\phi(\bar{t},\bar{\boldsymbol{x}})=0. \tag{6.13}$$

We let τ_1 be a stopping time defined by $\tau_1=\inf\{u>\bar{t}\,|\,\boldsymbol{X}_{\bar{t},\bar{\boldsymbol{x}}}(u)\notin B_\epsilon(\bar{\boldsymbol{x}})\}$, where $B_\epsilon(\bar{\boldsymbol{x}})$ is the ball of radius ϵ centred in $\bar{\boldsymbol{x}}$. Then we define the stopping time $\tau^*=\tau_1\wedge(\bar{t}+h)$ for $0<h<T-\bar{t}$ and note that $\bar{\gamma}:=\mathbb{E}_{\bar{t}\,\bar{\boldsymbol{x}}}[\tau^*]>0$. From the first part of DP and the definition of ϕ, it follows that, for arbitrary $\boldsymbol{\nu}\in\mathscr{Z}$,

$$V(\bar{t},\bar{\boldsymbol{x}})\geq\mathbb{E}\left[\int_{\bar{t}}^{\tau^*}\mathrm{e}^{-r(u-\bar{t})}f_1\left(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),\nu_u\right)\mathrm{d}u+\mathrm{e}^{-r(\tau^*-\bar{t})}\phi\left(\tau^*,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(\tau^*)\right)\right].$$

By applying Dynkin's formula to $\mathrm{e}^{-r(\tau^*-\bar{t})}\phi\left(\tau^*,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(\tau^*)\right)$ at $(\bar{t},\bar{\boldsymbol{x}})$, we get

$$\begin{aligned}\mathbb{E}\Bigg[\int_{\bar{t}}^{\tau^*}\Bigg(&\mathrm{e}^{-r(u-\bar{t})}f_1\left(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),\nu_u\right)-\mathrm{e}^{-r(u-\bar{t})}\Bigg\{r\phi\left(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u)\right)-\frac{\partial\phi}{\partial t}\left(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u)\right)\\&-\bar{\mathscr{H}}_1\left(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),D_{\boldsymbol{x}}\phi,\nu_u\right)-\mathscr{B}_{b,\Delta}(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),\phi)-\mathscr{B}_{\Delta}(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),\phi)\\&-\bar{\mathscr{B}}_y(u,\boldsymbol{X}^{\boldsymbol{\nu}}_{\bar{t},\bar{\boldsymbol{x}}}(u),\phi,\eta_u)\Bigg\}\Bigg)\mathrm{d}u\Bigg]\leq 0,\end{aligned}$$

where $\bar{\mathscr{H}}_1$ and $\bar{\mathscr{B}}_y$ are defined respectively by

$$\bar{\mathscr{H}}_1(t,\boldsymbol{x},\boldsymbol{p},\nu)=-\nu p_1+\mu^b\left(t,s^b,\nu\right)p_2+\mu^y(t,s^b,\Delta,\nu)p_3+\mu^\Delta(t,\Delta,\nu)p_4,$$

and

$$\bar{\mathscr{B}}_y(t,\boldsymbol{x},\varphi,n)=\lambda^y\mathbb{E}^{(z^y)}\Big[\varphi(t,x-nz^y,s^b,\Delta,y+h^y(t,s^b,\Delta,n)z^y)-\varphi(t,\boldsymbol{x})\Big].$$

We divide both sides by $-\bar{\gamma}$ and let $h\to 0$, resulting in

$$\begin{aligned}r\phi(\bar{t},\bar{\boldsymbol{x}})-\frac{\partial\phi}{\partial t}(\bar{t},\bar{\boldsymbol{x}})-\bar{\mathscr{H}}_1(\bar{t},\bar{\boldsymbol{x}},D_{\boldsymbol{x}}\phi,\nu)-f_1(\bar{t},\bar{\boldsymbol{x}},\nu)-\mathscr{B}_{b,\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)-\mathscr{B}_{\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)&\\-\bar{\mathscr{B}}_y(\bar{t},\bar{\boldsymbol{x}},\phi,\eta)\geq 0.&\end{aligned}$$

Due to the arbitrariness of $\boldsymbol{\nu}$, we can rewrite the above as

$$r\phi(\bar{t},\bar{\boldsymbol{x}})-\frac{\partial\phi}{\partial t}(\bar{t},\bar{\boldsymbol{x}})-\mathscr{H}_1(\bar{t},\bar{\boldsymbol{x}},D_{\boldsymbol{x}}\phi)-\mathscr{B}_{b,\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)-\mathscr{B}_{\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi)-\mathscr{B}_y(\bar{t},\bar{\boldsymbol{x}},\phi)\geq 0,$$

which proves the supersolution inequality. We now prove the subsolution inequality. We let

ϕ be a smooth and polinomially-bounded test function such that $V - \phi$ has its maximum at $(\bar{t}, \bar{\boldsymbol{x}})$. Without loss of generality, we assume $V(\bar{t}, \bar{\boldsymbol{x}}) - \phi(\bar{t}, \bar{\boldsymbol{x}}) = 0$. We shall show that the following inequality

$$r\phi(\bar{t},\bar{\boldsymbol{x}}) - \frac{\partial \phi}{\partial t}(\bar{t},\bar{\boldsymbol{x}}) - \mathscr{H}_1(\bar{t},\bar{\boldsymbol{x}},D_{\boldsymbol{x}}\phi) - \mathscr{B}_{b,\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi) - \mathscr{B}_{\Delta}(\bar{t},\bar{\boldsymbol{x}},\phi) - \mathscr{B}_{y}(\bar{t},\bar{\boldsymbol{x}},\phi) \leq 0, \quad (6.14)$$

holds. We define $\tau_1 = \inf\{u > \bar{t} \,|\, (u, \boldsymbol{X}_{\bar{t},\bar{x}}(u)) \notin B_\epsilon(\bar{t},\bar{\boldsymbol{x}})\}$ and we define the stopping time $\tau^* = \tau_1 \wedge (\bar{t} + h)$. By the second part of the DP, there exist a control $\boldsymbol{\nu}^* \in \mathscr{Z}$ such that, for $\delta > 0$, we have

$$V(\bar{t},\bar{\boldsymbol{x}}) \leq \mathbb{E}\left[\int_{\bar{t}}^{\tau^*} \mathrm{e}^{-r(u-\bar{t})} f_1\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), \nu^*_u\right)\mathrm{d}u + \mathrm{e}^{-r(\tau^*-\bar{t})}\phi\left(\tau^*, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(\tau^*)\right)\right] + \delta\,\bar{h}.$$

We divide both sides by $-\bar{\gamma}$ and get

$$\begin{aligned}-\delta \geq \frac{1}{\bar{\gamma}}\mathbb{E}\Bigg[\int_{\bar{t}}^{\tau^*} -\Bigg(&\mathrm{e}^{-r(u-\bar{t})} f_1\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), \nu^*_u\right) - \mathrm{e}^{-r(u-\bar{t})}\Big\{ r\phi\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u)\right) \\ &- \frac{\partial\phi}{\partial t}\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u)\right) - \bar{\mathscr{H}}_1\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), D_{\boldsymbol{x}}\phi, \nu^*_u\right) - \mathscr{B}_{b,\Delta}\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), \phi\right) \\ &- \mathscr{B}_{\Delta}\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), \phi\right) - \bar{\mathscr{B}}_{y}\left(u, \boldsymbol{X}^{\boldsymbol{\nu}^*}_{\bar{t},\bar{x}}(u), \eta^*_u, \phi\right)\Big\}\Bigg)\mathrm{d}u\Bigg].\end{aligned}$$

For $h \to 0$ and by the arbitrariness of δ, (6.14) follows. □

Proposition 6.5. *Let U (resp. V) be a viscosity subsolution (resp. supersolution) of (6.8). If $U(T,\boldsymbol{x}) \leq V(T,\boldsymbol{x})$ on $\mathscr{O}$, then $U \leq V$ on $[0,T] \times \mathscr{O}$.*

Proof. Let U be a subsolution and V be a supersolution. Since U, $V \in PB$, then there exist a $p > 1$ such that

$$\frac{|U(t,\boldsymbol{x})| + |V(t,\boldsymbol{x})|}{\left(1 + \|\boldsymbol{x}\|_p^p\right)} < \infty, \quad (6.15)$$

where the operator $\|\cdot\|_p^p$ is the L_p-norm raised to the p-th power. Let $\tilde{V}^\epsilon(t,\boldsymbol{x}) := V(t,\boldsymbol{x}) + \epsilon\kappa(t,\boldsymbol{x})$, where $\epsilon > 0$ and $\kappa(t,\boldsymbol{x}) = \mathrm{e}^{-\zeta t}\left(1 + \|\boldsymbol{x}\|_{2p}^{2p}\right)$, for $\zeta > 0$. Then $\tilde{V}^\epsilon$ is a supersolution of (6.8). Indeed, let $\phi(t,\boldsymbol{x})$ be the test function for $\tilde{V}^\epsilon$, then the test function for V is $\phi(t,\boldsymbol{x}) - \epsilon\kappa(t,\boldsymbol{x})$. First note that we have

$$\begin{aligned}r\epsilon\kappa(t,\boldsymbol{x}) - \frac{\partial \epsilon\kappa}{\partial t}(t,\boldsymbol{x}) - \sup_{\boldsymbol{\nu}\in\mathscr{Z}} \bar{\mathscr{H}}_1(t,\boldsymbol{x},D_{\boldsymbol{x}}\epsilon\kappa,\boldsymbol{\nu}) - \mathscr{B}_{b,\Delta}(t,\boldsymbol{x},\epsilon\kappa)& \\ -\mathscr{B}_{\Delta}(t,\boldsymbol{x},\epsilon\kappa) - \mathscr{B}_{y}(t,\boldsymbol{x},\epsilon\kappa) &\geq 0,\end{aligned}$$

for ζ sufficiently large. By the supersolution property of V, we have

$$
\begin{aligned}
& r(\phi-\epsilon\kappa)(t,\boldsymbol{x}) - \frac{\partial(\phi-\epsilon\kappa)}{\partial t}(t,\boldsymbol{x}) - \mathcal{H}_1(t,\boldsymbol{x},D_{\boldsymbol{x}}(\phi-\epsilon\kappa)) - \mathcal{B}_{b,\Delta}(t,\boldsymbol{x},\phi-\epsilon\kappa) \\
& - \mathcal{B}_{\Delta}(t,\boldsymbol{x},\phi-\epsilon\kappa) - \mathcal{B}_{y}(t,\boldsymbol{x},\phi-\epsilon\kappa) \geq 0,
\end{aligned}
$$

and recalling that $\sup(A+B) \leq \sup A + \sup B$, we have

$$
\begin{aligned}
r\phi(t,\boldsymbol{x}) - \frac{\partial\phi}{\partial t}(t,\boldsymbol{x}) - \mathcal{H}_1(t,\boldsymbol{x},D_{\boldsymbol{x}}\phi) - \mathcal{B}_{b,\Delta}(t,\boldsymbol{x},\phi) - \mathcal{B}_{\Delta}(t,\boldsymbol{x},\phi) - \mathcal{B}_{y}(t,\boldsymbol{x},\phi) \geq \\
r\epsilon\kappa(t,\boldsymbol{x}) - \frac{\partial\epsilon\kappa}{\partial t}(t,\boldsymbol{x}) - \sup_{\boldsymbol{\nu}\in\mathcal{Z}} \bar{\mathcal{H}}_1(t,\boldsymbol{x},D_{\boldsymbol{x}}\epsilon\kappa,\boldsymbol{\nu}) - \mathcal{B}_{b,\Delta}(t,\boldsymbol{x},\epsilon\kappa) \\
-\mathcal{B}_{\Delta}(t,\boldsymbol{x},\epsilon\kappa) - \mathcal{B}_{y}(t,\boldsymbol{x},\epsilon\kappa) \geq 0.
\end{aligned}
$$

Since by (6.15) $\lim_{\boldsymbol{x}\to\infty}\sup_{[0,T]}(U-\tilde{V}^{\epsilon})(t,\boldsymbol{x}) = -\infty$, we can assume w.l.o.g. that

$$
\mathcal{M} := \max_{[0,T]\times\mathcal{O}} \left(U(t,\boldsymbol{x}) - V(t,\boldsymbol{x})\right),
$$

is attained at $(\bar{t},\bar{\boldsymbol{x}}) \in [0,T]\times\Sigma$, where $\Sigma \subset \mathcal{O}$ is a compact set. In order to prove Proposition 6.5, it suffices to show that $\mathcal{M} < 0$. Suppose by contradiction that there exists a $(\bar{t},\bar{\boldsymbol{x}}) \in [0,T)\times\Sigma$ such that $\mathcal{M} > 0$. For $\epsilon > 0$, we define the function Ψ^{ϵ} by

$$
\Psi^{\epsilon}(t_1,t_2,\boldsymbol{x}_1,\boldsymbol{x}_2) = U(t_1,\boldsymbol{x}_1) - V(t_2,\boldsymbol{x}_2) - \psi^{\epsilon}(t_1,t_2,\boldsymbol{x}_1,\boldsymbol{x}_2),
$$

where

$$
\psi^{\epsilon}(t_1,t_2,\boldsymbol{x}_1,\boldsymbol{x}_2) := \frac{1}{2\epsilon}\left(|t_1-t_2|^2 + \|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2^2\right).
$$

The function Ψ^{ϵ} is continuous and admits a maximum point $\mathcal{M}^{\epsilon}$, where $\mathcal{M} \leq \mathcal{M}^{\epsilon}$, at $m = \left(t_1^{\epsilon},t_2^{\epsilon},\boldsymbol{x}_1^{\epsilon},\boldsymbol{x}_2^{\epsilon}\right)$. That is, the function $U(t_1,\boldsymbol{x}_1) - \psi^{\epsilon}(t_1,t_2,\boldsymbol{x}_1,\boldsymbol{x}_2)$ has its maximum at m and $V(t_2,\boldsymbol{x}_1) - (-\psi^{\epsilon}(t_1,t_2,\boldsymbol{x}_1,\boldsymbol{x}_2))$ has its minimum at m. Also, formally $\lim_{\epsilon\to 0}\left(t_1^{\epsilon},t_2^{\epsilon},\boldsymbol{x}_1^{\epsilon},\boldsymbol{x}_2^{\epsilon}\right)$ converges, up to a subsequence, to $(\bar{t},\bar{t},\bar{\boldsymbol{x}},\bar{\boldsymbol{x}})$ (see Crandall et al. [14] for details). We let $o^{\epsilon} = (t_1-t_2)/\epsilon$, and define the vector $\boldsymbol{p}^{\epsilon}$ by

$$
\boldsymbol{p}^{\epsilon} = \left(p_1^{\epsilon},p_2^{\epsilon},p_3^{\epsilon},p_4^{\epsilon}\right) = \left(\frac{1}{\epsilon}\left(x_1^{\epsilon}-x_2^{\epsilon}\right), \frac{1}{\epsilon}\left(s_1^{b,\epsilon}-s_2^{b,\epsilon}\right), \frac{1}{\epsilon}\left(\Delta_1^{\epsilon}-\Delta_2^{\epsilon}\right), \frac{1}{\epsilon}\left(y_1^{\epsilon}-y_2^{\epsilon}\right)\right).
$$

We can apply the viscosity subsolution and supersolution properties at the point m such that $U-\psi^{\epsilon}$ $(V-(-\psi^{\epsilon}))$ has its maximum (minumum) at $t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon} \to \bar{t},\bar{\boldsymbol{x}}$ $\left(t_2^{\epsilon},\boldsymbol{x}_2^{\epsilon} \to \bar{t},\bar{\boldsymbol{x}}\right)$. We thus have

$$
rU\left(t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon}\right) - o^{\epsilon} - \mathcal{H}_1\left(t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon},\boldsymbol{p}^{\epsilon}\right) - \mathcal{B}_{b,\Delta}\left(t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon},U\right) - \mathcal{B}_{\Delta}\left(t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon},U\right) - \mathcal{B}_{y}\left(t_1^{\epsilon},\boldsymbol{x}_1^{\epsilon},U\right) \leq 0
$$

and

$$rV\left(t_2^\epsilon,\boldsymbol{x}_2^\epsilon\right)-o^\epsilon-\mathscr{H}_1\left(t_2^\epsilon,\boldsymbol{x}_2^\epsilon,\boldsymbol{p}^\epsilon\right)-\mathscr{B}_{b,\Delta}\left(t_2^\epsilon,\boldsymbol{x}_2^\epsilon,V\right)-\mathscr{B}_{\Delta}\left(t_2^\epsilon,\boldsymbol{x}_2^\epsilon,V\right)-\mathscr{B}_{y}\left(t_2^\epsilon,\boldsymbol{x}_2^\epsilon,V\right)\geq 0$$

We can subtract the two inequalities and take the limit for $\epsilon\to 0$ to get $r\left[U\left(\bar{t},\bar{\boldsymbol{x}}\right)-V\left(\bar{t},\bar{\boldsymbol{x}}\right)\right]\geq 0$, which concludes the proof. □

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