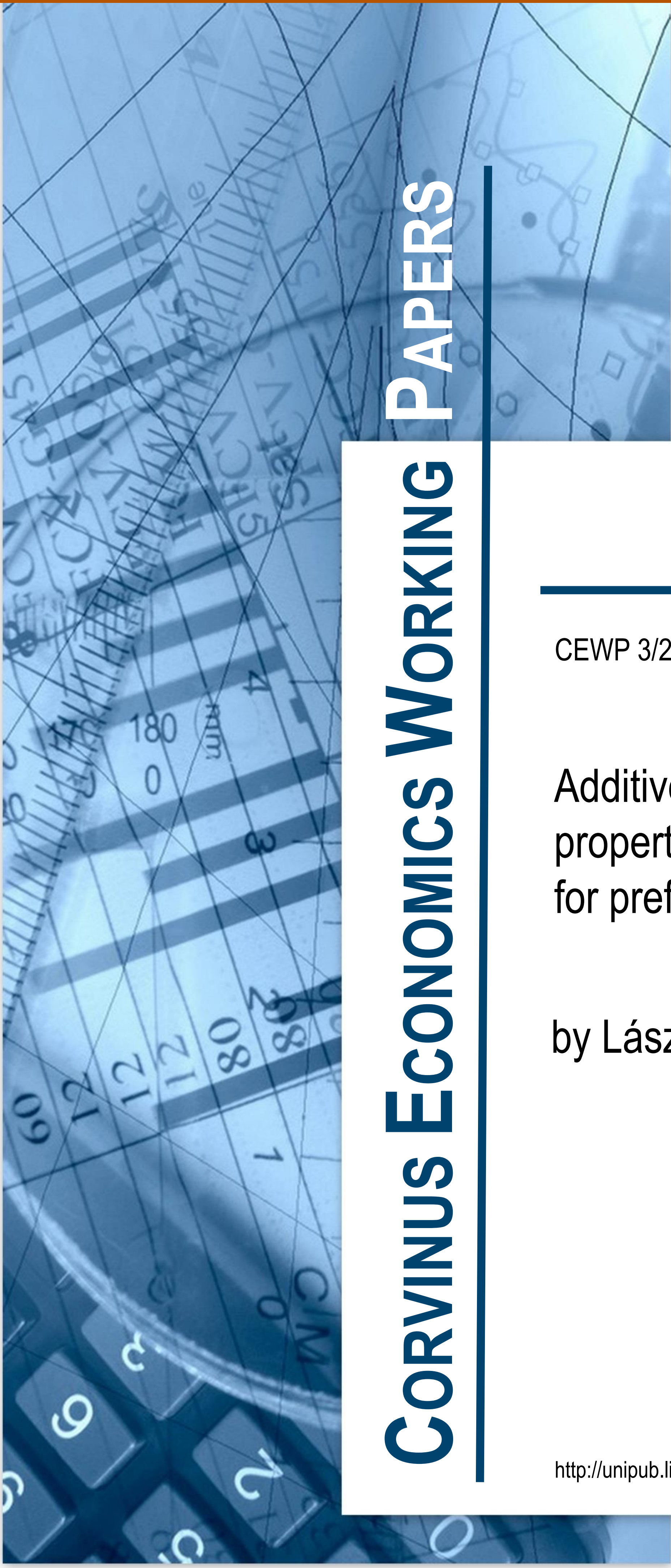


CORVINUS ECONOMICS WORKING PAPERS

CORVINUS UNIVERSITY of BUDAPEST
Faculty of Economics

CEWP 3/2014

Additive and multiplicative properties of scoring methods for preference aggregation

by László Csató

http://unipub.lib.uni-corvinus.hu/1562

Additive and multiplicative properties of scoring methods for preference aggregation*

László Csató†

Department of Operations Research and Actuarial Sciences
Corvinus University of Budapest
MTA-BCE "Lendület" Strategic Interactions Research Group
Budapest, Hungary

May 15, 2014

Abstract

The paper reviews some additive and multiplicative properties of ranking procedures used for generalized tournaments with missing values and multiple comparisons. The methods analysed are the score, generalised row sum and least squares as well as fair bets and its variants. It is argued that generalised row sum should be applied not with a fixed parameter, but a variable one proportional to the number of known comparisons. It is shown that a natural additive property has strong links to independence of irrelevant matches, an axiom judged unfavourable when players have different opponents.

JEL classification number: D71

Keywords: Preference aggregation, Tournament ranking, Paired comparison, Axiomatic approach

1 Introduction

Paired-comparison based ranking problems are given by a set of objects and a tournament matrix, which represents the performance of objects against each other. They arise in many different fields like social choice theory (Chebotarev and Shamis, 1998), sports (Landau, 1895, 1914; Zermelo, 1928) or psychology (Thurstone, 1927). The usual goal is to determine a winner (a set of winners) or a complete ranking for the objects. There were some attempts to link the two areas (i.e. Bouyssou (2004)), however, their results seem to be limited. We will deal only with the latter problem, allowing for different preference intensities (including ties), incomplete and multiple comparisons among the objects.

Ranking procedures are usually given as functions associating a score for each object and a higher score corresponds to a better ranking. The literature on these methods has expanded significantly (for reviews, see Laslier (1997) and Chebotarev and Shamis

*We are grateful to Julio González-Díaz for his remark about homogeneity.
†e-mail: laszlo.csato@uni-corvinus.hu

(1998)), so there is a real need for some guidelines about the choice of the appropriate procedure. It may be achieved by studying their axiomatic properties. Characterization results on this general domain are limited, we know them only for fair bets (Slutzki and Volij, 2005) and invariant methods (Slutzki and Volij, 2006). Our goal is to investigate some scoring procedures with respect to a set of properties, naturally arising from the setting. Our results supplement González-Díaz et al. (2014)'s findings by analysing new methods and axioms.

Because of the large amount of ranking methods discussed in the different fields, some selection is needed. We will mainly concentrate on the following procedures.

- **Score**: a natural method for binary tournaments (for characterizations on restricted domains, see Young (1974); Hansson and Sahlquist (1976); Rubinstein (1980); Nitzan and Rubinstein (1981); Bouyssou (1992)).

- **Least squares**: a well-known procedure in statistics and psychology (see Thurstone (1927); Gulliksen (1956); Kaiser and Serlin (1978)).

- **Generalised row sum**: a parametric family of ranking methods resulting in the score and least squares as limits (see Chebotarev (1989, 1994)).

- **Fair bets**: an extensively studied method in social choice theory as well as for ranking the nodes of directed graphs (see Daniels (1969); Moon and Pullman (1970); Slutzki and Volij (2005, 2006); Slikker et al. (2012)).

- **Dual fair bets**: a scoring procedure obtained from fair bets by 'reversing' an axiom in its characterization (see Slutzki and Volij (2005)).

- **Copeland fair bets**: a new method introduced in the current paper by applying the idea of Herings et al. (2005) for fair bets.

The main contribution of this paper is to study the ranking methods above in the view of a set of axioms. It helps in understanding the different procedures and reveals the connections of the properties investigated. For instance, Copeland fair bets is proposed because González-Díaz et al. (2014) considers that its major weakness is the violation of inversion, which imposes the requirement that if all the results are reversed, then the corresponding ranking should be obtained by reversing the original ranking as well. The significance of certain axioms for applications will also be emphasized. Thorough analysis of these properties may support the work towards the characterization of some methods, too.

The paper is structured as follows. In Section 2 our setting and definitions are presented. Section 3 exhibits three main properties with a significance for later discussion. In Section 4 we deal with two possible multiplicative axioms. Section 5 reviews four axioms linked to adding of ranking problems. In Section 6, we argue that the strongest additive property has unfavourable implications on the general domain used. Finally, Section 7 concludes the results, summarized visually in a table and a graph.

2 Notations and rating methods

Let $N = \{X_1, X_2, \ldots, X_n\}$, $n \in \mathbb{N}$ be a *set of objects* and $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a *tournament* matrix such that $a_{ij} + a_{ji} \in \mathbb{N}$. a_{ij} represents the aggregate score of object

X_i against X_j, $a_{ij}/(a_{ij}+a_{ji})$ may be interpreted as the likelihood that object X_i is better than object X_j. $a_{ii} = 0$ is assumed for all $i = 1, 2, \ldots, n$, diagonal entries will have no significance in the discussion. A possible derivation of the tournament matrix can be found in González-Díaz et al. (2014) and Csató (2014). This notation also follows Chebotarev and Shamis (1998).

The pair (N, A) is called a *ranking problem*.[1] The set of ranking problems is denoted by $\mathcal{R}$. A *scoring procedure* f is an $\mathcal{R} \to \mathbb{R}^n$ function, $f_i = f_i(N, A)$ is the rating of object X_i. It immediately determines a ranking (transitive and complete weak order on the set $N \times N$) $\succeq$, where $f_i \geq f_j$ means that X_i is ranked weakly above X_j, denoted by $X_i \succeq X_j$. Ratings provide cardinal and rankings provide ordinal information about the objects. Throughout the paper, the notions of rating and ranking methods will be used analogously since only scoring procedures are discussed.

To each ranking problem $(N, A) \in \mathcal{R}$, we associate a *results matrix* $R = A - A^\top = (r_{ij}) \in \mathbb{R}^{n \times n}$ and a *matches matrix* $M = A + A^\top = (m_{ij}) \in \mathbb{N}^{n \times n}$, where m_{ij} is the number of the comparisons between X_i and X_j, whose the outcome is given by r_{ij}. $d_i = \sum_{j=1}^{n} m_{ij}$ is the *total number of comparisons* of object X_i. $m = \max_{X_i, X_j \in N} m_{ij}$ is the *maximal number of comparisons* in the ranking problem. A ranking problem is called *round-robin* if $m_{ij} = m$ for all $X_i \neq X_j$, namely, every object has been compared with all the others exactly as many times. The set of round-robin ranking problems is denoted by $\mathcal{R}^R$. Note that ranking problem can be also defined by matrices R and M with the restriction $|r_{ij}| \leq m_{ij}$ for all $X_i, X_j \in N$.

Ranking of the objects involves three main challenges. The first one is the possible appearance of circular triads, when object X_i is better than X_j (that is, $a_{ij} > a_{ji}$), X_j is better than X_k, but X_k is better than X_i. If preference intensities also count as in the model above, other triplets (X_i, X_j, X_k) may produce problems, too. The second problem is that the performance of objects compared with X_i strongly influences the observable paired comparison outcomes a_{ij}. For example, if X_i was compared only with X_j, then its rating may depend on the results of X_j. The third difficulty is given by the different number of comparisons of the objects, $d_i \neq d_j$. According to David (1987, p. 1), it must be realized that there can be no entirely satisfactory way of ranking if the number of replications of each object varies appreciably. Despite this we do not deal with the question whether a given dataset may be globally ranked in a meaningful way or the data is inherently inconsistent, an issue investigated for example by Jiang et al. (2011). Since each problem occur if $n \geq 3$, the case of two objects becomes trivial.

Matrix M can be represented by an undirected multigraph $G := (V, E)$ where vertex set V corresponds to the object set N, and the number of edges between objects X_i and X_j is equal to m_{ij}. Therefore the degree of node X_i is d_i. Graph G is called the *comparison multigraph* associated with the ranking problem (N, R, M), however, it is independent of the results matrix R. The *Laplacian matrix* $L = (\ell_{ij})$, $i, j = 1, 2, \ldots, n$ of graph G is given by $\ell_{ij} = -m_{ij}$ for all $X_i \neq X_j$ and $\ell_{ii} = d_i$ for all $X_i \in N$.

A *path* from X_{k_1} to X_{k_t} is a sequence of objects $X_{k_1}, X_{k_2}, \ldots, X_{k_t}$ such that $m_{k_\ell k_{\ell+1}} > 0$ for all $\ell = 1, 2, \ldots, t-1$. Two objects are connected if there exists a path between them. Ranking problem $(N, A) \in \mathcal{R}$ is said to be *connected* if every pair of objects is connected. The set of connected ranking problems is denoted by $\mathcal{R}^C$.

A *directed path* from X_{k_1} to X_{k_t} is a sequence of objects $X_{k_1}, X_{k_2}, \ldots, X_{k_t}$ such that $a_{k_\ell k_{\ell+1}} > 0$ for all $\ell = 1, 2, \ldots, t-1$. Objects X_i and X_j are in the same *league* if there

[1] In certain cases we will denote it only by the tournament matrix A, whose rows already determine the set of objects N.

exists a directed path from X_i to X_j and from X_j to X_i. Ranking problem $(N, A) \in \mathcal{R}$ is called *irreducible* if every pair of objects is in the same league. The set of irreducible ranking problems is denoted by $\mathcal{R}^I$.

Let $\mathbf{e} \in \mathbb{R}^n$ be the unit column vector, that is, $e_i = 1$ for all $i = 1, 2, \ldots, n$. Let $I \in \mathbb{R}^{n \times n}$ be the identity matrix, and L be the Laplacian matrix of the comparison multigraph G associated with the ranking problem (N, A).

Now we define some rating methods. The first one does not take the comparison structure into account.

Definition 2.1. ***Score****:* $\mathbf{s}(N, R, M) = R\mathbf{e}$.

Score will also be referred to as *row sum*. The following parametric rating method was constructed axiomatically by Chebotarev (1989) and thoroughly analyzed in Chebotarev (1994).

Definition 2.2. ***Generalized row sum****: it is the unique solution* $\mathbf{x}(\varepsilon)(N, R, M)$ *of the system of linear equations* $(I + \varepsilon L)\mathbf{x}(\varepsilon)(N, R, M) = (1 + \varepsilon mn)\mathbf{s}$*, where* $\varepsilon > 0$ *is a parameter.*

It adjusts the standard score s_i by accounting for the performance of objects compared with X_i, and adds an infinite depth to this argument since scores of all objects available on a path appear in the calculation. ε indicates the importance attributed to this correction. An alternative solution would be to count only the scores of direct opponents as in David (1987).

Lemma 2.1. *Generalised row sum results in row sum if* $\varepsilon \to 0$*:* $\lim_{\varepsilon \to 0} \mathbf{x}(\varepsilon)(N, R, M) = \mathbf{s}(N, R, M)$.

There are few information about the choice of parameter ε. In our case, the value of r_{ij} is limited by $-m_{ij}$ and m_{ij}, thus some conditions can be made on ε.

Definition 2.3. Reasonable choice of ε *(Chebotarev, 1994, Proposition 5.1): Let* $(N, R, M) \in \mathcal{R}$ *be a ranking problem. Reasonableness for the choice of* ε *amounts to satisfying the constraint*

$$0 < \varepsilon \leq \frac{1}{m(n-2)}.$$

Reasonable upper bound *of* ε *is* $1/\left[m(n-2)\right]$.

The reasonable choice is not well-defined in the trivial case of $n = 2$, thus $n \geq 3$ is implicitly assumed in the following.

Proposition 2.1. *For the generalised row sum method with a reasonable choice of* ε*,* $-m(n-1) \leq x_i(\varepsilon)(N, R, M) \leq m(n-1)$ *for all* $X_i \in N$.

Proof. See Chebotarev (1994, Property 13). □

It is favourable as in a round-robin ranking problem $-m(n-1) \leq s_i(N, R, M) \leq m(n-1)$ for all $X_i \in N$.

Both the score and generalized row sum ratings are well-defined and easily computable from a system of linear equations for all ranking problems $(N, R, M) \in \mathcal{R}$.

The least squares method was suggested by Thurstone (1927) and Horst (1932). Other references can be found in Csató (2014).

Definition 2.4. ***Least squares***: *it is the solution* $\mathbf{q}(N, R, M)$ *of the system of linear equations* $L\mathbf{q}(N, R, M) = \mathbf{s}(N, R, M)$ *and* $\mathbf{e}^\top \mathbf{q}(N, R, M) = 0$.

Lemma 2.2. *Generalised row sum results in least squares if* $\varepsilon \to \infty$*:*

$$\lim_{\varepsilon \to \infty} \mathbf{x}(\varepsilon)(N, R, M) = mn\mathbf{q}(N, R, M).$$

Proposition 2.2. *The least squares rating is unique if and only if comparison multigraph* G *is connected.*

Proof. See Bozóki et al. (2014). Chebotarev and Shamis (1999, p. 220) mention this fact without further discussion. □

An extensive analysis and a graph interpretation of the least squares method can be found in Csató (2014).

Several scoring procedures build upon the idea of rewarding wins without punishing losses. Two early contributions in this field are Wei (1952) and Kendall (1955). They have been studied in social choice and game theory by Borm et al. (2002); Herings et al. (2005); Slikker et al. (2012); Slutzki and Volij (2005, 2006). One of the most widely used method within this framework is the fair bets method, originally suggested by Daniels (1969) and Moon and Pullman (1970), and axiomatically characterized by Slutzki and Volij (2005) and Slutzki and Volij (2006). Its properties have been investigated by González-Díaz et al. (2014).

Let $F = \mathrm{diag}(A^\top \mathbf{e})$, an $n \times n$ diagonal matrix with the number of losses for each object.

Definition 2.5. ***Fair bets***: *it is the solution* $\mathbf{fb}(N, A)$ *of the system of linear equations* $F^{-1}A\mathbf{fb}(N, A) = \mathbf{fb}(N, A)$ *and* $\mathbf{e}^\top \mathbf{fb}(N, A) = 1$.

Proposition 2.3. *The fair bets rating is unique if the ranking problem is irreducible.*

Proof. See Moon and Pullman (1970). □

For reducible ranking problems, Perron-Frobenius theorem does not guarantee that the eigenvector corresponding to the dominant eigenvalue is strictly positive.

Therefore we restrict our analysis to the class of connected ranking problems $\mathcal{R}^C$, and to the set of irreducible ranking problems $\mathcal{R}^I$ in the case of fair bets. However, for ranking problems without a connected comparison multigraph, rating of all objects on a common scale seems to be arbitrary.

The idea behind the fair bets method is to give more weight to wins against better objects than to losses against worse objects. It includes a subjective judgement in it: analogously, one may argue that the latter is more favourable. This approach is taken by the dual fair bets method (Slutzki and Volij, 2005) using the transposed tournament matrix $A^\top$, however, in this case a lower value is better.

Definition 2.6. ***Dual fair bets***: *it is the opposite of the solution* $\mathbf{dfb}^*(N, A)$ *of system of linear equations* $[diag(A\mathbf{e})]^{-1} A^\top \mathbf{dfb}^*(N, A) = \mathbf{dfb}^*(N, A)$ *and* $\mathbf{e}^\top \mathbf{dfb}^*(N, A) = 1$.

The transformation $\mathbf{dfb}(N, A) = -\mathbf{dfb}^*(N, A)$ is necessary in order to ensure that $X_i \succeq X_j \Leftrightarrow dfb_i(N, A) \geq dfb_j(N, A)$ for all $X_i, X_j \in N$.

In fact, the axiomatization of fair bets also characterizes the dual fair bets by changing only one property, negative responsiveness to losses with positive responsiveness to wins (Slutzki and Volij, 2005, Remark 1). This differentiation can be seen in the case of positional power, too, by the definition of positional power and positional weakness (Herings et al., 2005). Similarly to the latter paper's Copeland positional value, we introduce Copeland fair bets method.

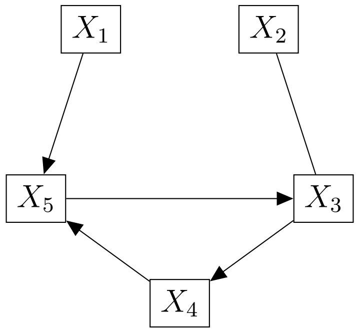
Definition 2.7. ***Copeland fair bets:*** $\mathbf{Cfb}(N, A)$ *it is the sum of the fair bets and dual fair bets ratings,* $\mathbf{Cfb}(N, A) = \mathbf{Cfb}(N, A) + \mathbf{dfb}(N, A)$.

Now $X_i \succeq X_j \Leftrightarrow Cfb_i(N, A) \geq Cfb_j(N, A)$ as earlier. According to our knowledge, we are the first to define this scoring procedure.

These are the six scoring procedures (or, in the case of generalised row sum, a family of them) discussed in the article. González-Díaz et al. (2014) have analysed the least squares and fair bets methods, as well as generalised row sum with the parameter $\varepsilon = 1/\left[m(n-2)\right]$. They use a different version of the score, s_i/d_i for all $X_i \in N$.

Ranking problem $(N, R, M) \in \mathcal{R}$ can be represented by a graph such that the nodes are the objects, k times $(X_i, X_j) \in N \times N$ undirected edge means $r_{ij} (= r_{ji}) = 0$, $m_{ij} = k$, and k times $(X_i, X_j) \in N \times N$ directed edge means $r_{ij} = k$ $(r_{ji} = -k)$, $m_{ij} = k$. We think it helps a lot in understanding the examples.

Figure 1: Ranking problem of Example 1



Example 1. *(Chebotarev, 1994, Example 2) Let* $(N, R, M) \in \mathcal{R}$ *be the ranking problem in Figure 1 with the set of objects* $N = \{X_1, X_2, X_3, X_4, X_5\}$.

The corresponding tournament, results and matches matrices are as follows

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The solutions with generalised row sum for various values of ε *are given in Table 1.*

Here $m = 1$ *and* $n = 5$*, thus* $\varepsilon = 1/3$ *is the reasonable upper bound. For all parameter greater than* 0*, the ranking is* $X_1 \succ X_2 \succ X_3 \succ X_4 \succ X_5$ *since* X_1 *dominates* X_5*, which effects* X_3 *and* X_4 *through the circular triad* (X_3, X_4, X_5)*. However,* X_3 *has a draw against* X_2*. Note that* $X_2 \sim X_3 \sim X_4$ *for the score and least squares methods, referring to a kind of neglect of the comparison* (X_2, X_3)*.*

It is an irreducible ranking problem, so fair bets rating is not unique. Nevertheless, a ranking can be obtained by the application of its extension according to Slutzki and Volij (2005): X_1 *is the best object as no other has a chance to defeat it, and the remaining four*

Table 1: $\mathbf{x}(\varepsilon)$ rating vectors of Example 1

ε	0	1/100	1/4	1/3	1	5	$\to \infty$
X_1	1.0000	1.0296	1.7165	2.2649	2.4242	3.4369	4.0000
X_2	0.0000	−0.0001	−0.0613	−0.1917	−0.2424	−0.6819	−1.0000
X_3	0.0000	−0.0099	−0.2452	−0.4314	−0.4848	−0.8183	−1.0000
X_4	0.0000	−0.0100	−0.2759	−0.4878	−0.5455	−0.8609	−1.0000
X_5	−1.0000	−1.0096	−1.1341	−1.1540	−1.1515	−1.0757	−1.0000

form an irreducible component. It gives the fair bets ranking as $X_1 \succ (X_2 \sim X_3 \sim X_4 \sim X_5)$, which coincides with the one from least squares. Similarly, both dual fair bets and Copeland fair bets result in $X_1 \succ (X_2 \sim X_3 \sim X_4 \sim X_5)$.

3 Structural invariance properties

Our axiomatic discussion begins with some basic properties already known from the literature.

Definition 3.1. Neutrality *(NEU) (Young, 1974): Let $(N, R, M) \in \mathcal{R}$ be a ranking problem and $\sigma : N \to N$ be a permutation on the set of objects. Let $\sigma(N, R, M) \in \mathcal{R}$ be the ranking problem obtained from (N, R, M) by this permutation. Scoring procedure $f : \mathcal{R} \to \mathbb{R}^n$ is* neutral *if for all $X_i, X_j \in N$: $f_i(N, R, M) \geq f_j(N, R, M) \Leftrightarrow f_{\sigma i}[\sigma(N, R, M)] \geq f_{\sigma j}[\sigma(N, R, M)]$.*

In some articles, it is called *anonymity* (Bouyssou, 1992; Slutzki and Volij, 2005; González-Díaz et al., 2014). It is equivalent with requiring that the permutation of two objects do not affect the ranking as in González-Díaz et al. (2014).

Remark 1. *Let $f : \mathcal{R} \to \mathbb{R}^n$ be a neutral scoring procedure. If for the objects $X_i, X_j \in N$, $m_{ij} = 0$, and $r_{ik} = r_{jk}$, $m_{ik} = m_{jk}$ for all $X_k \in N \setminus \{X_i, X_j\}$, then $f_i(N, R, M) = f_j(N, R, M)$ (Bouyssou, 1992, p. 62).*

Lemma 3.1. *All methods presented above satisfy NEU.*

Definition 3.2. Symmetry *(SYM) (González-Díaz et al., 2014): Let $(N, R, M) \in \mathcal{R}$ be a ranking problem such that $R = \mathbf{0}$. Scoring procedure $f : \mathcal{R} \to \mathbb{R}^n$ is* symmetric *if $f_i(N, R, M) = f_j(N, R, M)$ for all $X_i, X_j \in N$.*

Symmetry does not require $d_i = d_j$ for the pair (X_i, X_j). Young (1974) and Nitzan and Rubinstein (1981, Axiom 4) use the axiom *cancellation* for round-robin ranking problems, which coincides with symmetry on this set.

Lemma 3.2. *All methods presented above satisfy SYM.*

Definition 3.3. Inversion *(INV) (Chebotarev and Shamis, 1998): Let $(N, R, M) \in \mathcal{R}$ be a ranking problem. Scoring procedure $f : \mathcal{R} \to \mathbb{R}^n$ is* invertible *if $f_i(N, R, M) \geq f_j(N, R, M) \Leftrightarrow f_i(N, -R, M) \leq f_j(N, -R, M)$ for all $X_i, X_j \in N$.*

Inversion means that taking the opposite of each result changes the ranking accordingly. It establishes a uniform treatment of victories and defeats. Chebotarev (1994, Property 7) defines *transposability* such that the ratings change their sign and keep the same absolute value.

Remark 2. *Let $f : \mathcal{R} \to \mathbb{R}^n$ be a scoring procedure satisfying INV. Then $f_i(N, R, M) > f_j(N, R, M) \Leftrightarrow f_i(N, -R, M) < f_j(N, -R, M)$ for all $X_i, X_j \in N$.*

The following result is mentioned by González-Díaz et al. (2014, p. 150).

Corollary 1. *INV implies SYM.*

Lemma 3.3. *The score, generalised row sum and least squares methods satisfy INV.*

Proof. It is the immediate consequence of $\mathbf{s}(N, -R, M) = -\mathbf{s}(N, R, M)$. □

Lemma 3.4. *Fair bets and dual fair bets methods do not satisfy INV on the set $\mathcal{R}^R$.*

Proof. For fair bets, see González-Díaz et al. (2014, Example 4.4). The same counterexample with a transposed tournament matrix proves the statement for dual fair bets. □

So axiom INV is not satisfied by the two methods still on the restricted domain of round-robin problems.

Lemma 3.5. *Copeland fair bets satisfies INV.*

Proof. Take ranking problems (N, A) and $(N, A^\top)$. $\mathbf{Cfb}(N, A) = \mathbf{fb}(N, A) + \mathbf{dfb}(N, A) = -\mathbf{dfb}(N, A^\top) - \mathbf{fb}(N, A^\top) = -\mathbf{Cfb}(N, A^\top)$. □

Hence Copeland fair bets eliminates the major weakness of fair bets according to González-Díaz et al. (2014, p. 164), while retains its favourable properties.

4 Multiplicative properties

The following axiom refers to proportional modification of the ranking problem.

Definition 4.1. Homogeneity *(HOM) (González-Díaz et al., 2014): Let $(N, R, M) \in \mathcal{R}$ be a ranking problem and $k > 0$ such that $(N, kR, kM) \in \mathcal{R}$. Scoring procedure $f : \mathcal{R} \to \mathbb{R}^n$ is* homogeneous, *if $f_i(N, R, M) \geq f_j(N, R, M) \Leftrightarrow f_i(N, kR, kM) \geq f_j(N, kR, kM)$ for all $X_i, X_j \in N$.*[2]

In our setting the elements of kM should be integers, which is not required by González-Díaz et al. (2014).

Lemma 4.1. *The score and least squares methods satisfy HOM.*

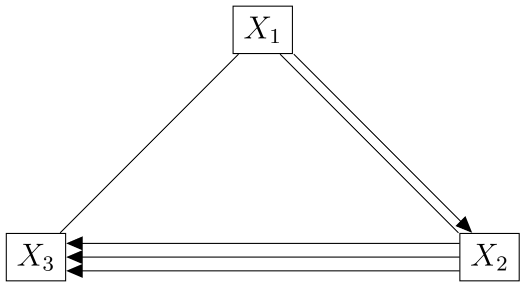
Generalised row sum should be examined with some caution since it is a whole family of scoring procedures. First, we regard it with a constant parameter ε.

Proposition 4.1. *The generalised row sum method with a fixed ε violates HOM.*

Proof.

[2] Since $k > 0$, positive homogeneity may be a better name for this axiom, but we wanted to retain the original definition.

Figure 2: Ranking problem of Example 2



Example 2. *Let $(N, R, M) \in \mathcal{R}$ be the ranking problem in Figure 2 with the set of objects $N = \{X_1, X_2, X_3\}$ and tournament matrix*

$$A = \begin{pmatrix} 0 & 1.5 & 1 \\ 0.5 & 0 & 3 \\ 0.5 & 0 & 0 \end{pmatrix}.$$

Let $k = 2$.

Here $m = 3$ and $n = 3$, the reasonable upper bound of ε is $1/3$. Let choose it as a fixed parameter:

$$\mathbf{x}(1/5)(N, R, M) = [2.0000;\ 2.0000;\ -4.0000]^\top, \text{ and}$$

$$\mathbf{x}(1/5)(N, 2R, 2M) = [4.5352;\ 3.9437;\ -8.4789]^\top,$$

implying $X_1 \sim^{\mathbf{x}(1/3)}_{(N,R,M)} X_2$ but $X_1 \succ^{\mathbf{x}(1/3)}_{(N,2R,2M)} X_2$. □

Now allow ε to depend on the matches matrix M.

Proposition 4.2. *The generalised row sum method with a variable ε satisfies HOM if $\varepsilon(k)$ is inversely proportional to k, that is, $\varepsilon(k) = \varepsilon(1)/k = \varepsilon/k$.*

Proof. It yields from some basic calculations:

$$\mathbf{x}\left(\varepsilon(k)\right)(N, kR, kM) = (1 + \varepsilon mn)(I + \varepsilon L)^{-1}\mathbf{s}(N, kR, kM) = k\mathbf{x}(\varepsilon)(N, R, M).$$

□

Remark 3. *The reasonable upper bound $\varepsilon = 1\left[m(n-2)\right]$ is inversely proportional to k.*

Conjecture 1. *The proof of Proposition 4.2 suggests that generalised row sum violates HOM if ε is not inversely proportional to k.*

Lemma 4.2. *Fair bets, dual fair bets and Copeland fair bets methods satisfy HOM.*

Proof. Regarding fair bets, $\mathbf{fb}(N, kA) = \mathbf{fb}(N, A)$ since $(kF)^{-1} = (1/k)F^{-1}$. It shows the homogeneity of dual fair bets as well, which proves *HOM* for Copeland fair bets. □

It makes sense to deal only with the multiplication of results.

Definition 4.2. Admissible transformation of the results*: Let $(N, R, M) \in \mathcal{R}$ be a ranking problem. An* admissible transformation of the results *provides a ranking problem $(N, kR, M) \in \mathcal{R}$ such that $k > 0$, $k \in \mathbb{R}$ and $kr_{ij} \in [-m_{ij}, m_{ij}]$ for all $X_i, X_j \in N$.*

k cannot be arbitrarily large in order to retain the condition $|r_{ij}| \leq m_{ij}$. For instance, in Example 2 $\max_{X_i,X_j \in N} r_{ij}/m_{ij} = \max\{1/3; 3/5\} = 3/5$, therefore $0 < k \leq 5/3$ makes $(N, kR, M) \in \mathcal{R}$ a ranking problem provided through an admissible transformation of the results. On the other hand, $0 < k \leq 1$ is always possible.

Definition 4.3. Scale invariance *(SI): Let $(N, R, M), (N, kR, M) \in \mathcal{R}$ be two ranking problems such that (N, kR, M) is obtained from (N, R, M) through an admissible transformation of the results. Scoring procedure $f : \mathcal{R} \to \mathbb{R}^n$ is* scale invariant *if $f_i(N, R, M) \geq f_j(N, R, M) \Leftrightarrow f_i(N, kR, M) \geq f_j(N, kR, M)$ for all $X_i, X_j \in N$.*

Scale invariance implies that the ranking is invariant to a proportional modification of wins ($r_{ij} > 0$) and losses ($r_{ij} < 0$). It seems to be important for applications. If the paired comparison outcomes cannot be measured on a continuous scale, it is not trivial how to transform them into r_{ij} values. SI provides that it is not a problem in several cases. For example, if only three outcomes are possible, the coding ($r_{ij} = \kappa$ for wins; $r_{ij} = 0$ for draws; $r_{ij} = -\kappa$ for losses) makes the ranking independent from κ. It may be advantageous, too, when relative intensities, such as a regular win is two times better than an overtime triumph, are known.

Remark 4. *Take the tournament matrix A, where $a_{ij} = (r_{ij} + m_{ij})/2$. Through an admissible transformation of the results, every reducible A can be made irreducible (by all $k < 1$) if the ranking problem is connected.*

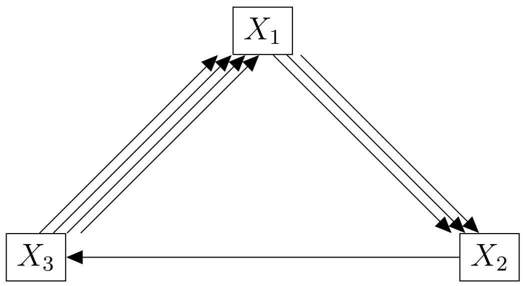
Remark 4 offers a way to extend scale invariant scoring procedures unique on the domain $\mathcal{R}^I$ to $\mathcal{R}^C$.

Lemma 4.3. *The score, generalised row sum and least squares methods satisfy SI.*

Proof. It is the immediate consequence of $\mathbf{s}(N, kR, M) = k\mathbf{s}(N, R, M)$. □

Proposition 4.3. *Fair bets, dual fair bets and Copeland fair bets methods violate SI.*

Figure 3: Ranking problem of Example 3



Proof.

Example 3. *Let $(N, A) \in \mathcal{R}$ be the ranking problem in Figure 3 with the set of objects $N = \{X_1, X_2, X_3\}$ and tournament matrix*

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix}.$$

Table 2: Fair bets and associated rating vectors of Example 3

	$\mathbf{fb}(N,A)$	$\mathbf{dfb}(N,A)$	$\mathbf{Cfb}(N,A)$	$\mathbf{fb}(N,A')$	$\mathbf{dfb}(N,A')$	$\mathbf{Cfb}(N,A')$
X_1	$1/3$	$-1/3$	0	$67/247$	$-75/247$	$-8/247$
X_2	$1/3$	$-1/3$	0	$3/13$	$-121/247$	$-64/247$
X_3	$1/3$	$-1/3$	0	$123/247$	$-51/247$	$72/247$

Let $k = 0.5$, resulting in the tournament matrix

$$A' = \begin{pmatrix} 0 & 2.25 & 1 \\ 0.75 & 0 & 0.75 \\ 3 & 0.25 & 0 \end{pmatrix}.$$

The rating vectors are given in Table 2. Thus $fb_1(N,A) \leq fb_2(N,A)$, $dfb_1(N,A) \leq dfb_2(N,A)$ and $Cfb_1(N,A) \leq Cfb_2(N,A)$, but $fb_1(N,A') > fb_2(N,A')$, $dfb_1(N,A') > dfb_2(N,A')$ and $Cfb_1(N,A') > Cfb_2(N,A')$, which is a contradiction. □

Similarly to Lemma 3.3, it is worth to examine the two multiplicative properties on the domain of round-robin ranking problems. Note that the set $\mathcal{R}^R$ is closed under modifications allowed by HOM and SI (admissible transformation of the results).

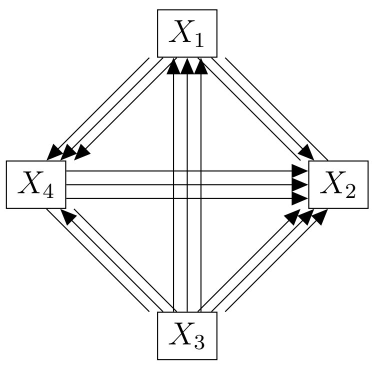
Lemma 4.4. *The generalised row sum satisfies HOM on the set $\mathcal{R}^R$.*

Proof. Due to the axiom *agreement* (Chebotarev, 1994, Property 3), generalised row sum coincides with the score on this set of problems, so Lemma 4.1 holds. □

Example 3 is unbalanced, the degree of the three nodes varies from 4 to 7. As it was mentioned in Section 2, in this case the ranking may be not meaningful. It also justifies the investigation of round-robin problems.

Proposition 4.4. *Fair bets, dual fair bets and Copeland fair bets methods violate SI on the set $\mathcal{R}^R$.*

Figure 4: Ranking problem of Example 4



Proof.

Example 4. *Let $(N, A) \in \mathcal{R}^R$ be the round-robin ranking problem in Figure 4 with the set of objects $N = \{X_1, X_2, X_3, X_4\}$ and tournament matrix*

$$A = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 2 \\ 0 & 3 & 1 & 0 \end{pmatrix}.$$

Let $k = 0.5$, resulting in the tournament matrix

$$A' = \begin{pmatrix} 0 & 1.75 & 0.75 & 2.25 \\ 1.25 & 0 & 0.75 & 0.75 \\ 2.25 & 2.25 & 0 & 1.75 \\ 1.75 & 2.25 & 1.25 & 0 \end{pmatrix}.$$

Table 3: Fair bets and associated rating vectors of Example 4

	$\mathbf{fb}(N, A)$	$\mathbf{dfb}(N, A)$	$\mathbf{Cfb}(N, A)$	$\mathbf{fb}(N, A')$	$\mathbf{dfb}(N, A')$	$\mathbf{Cfb}(N, A')$
X_1	8/66	−10/66	−1/33	67/282	−59/282	4/141
X_2	1/66	−47/66	−23/33	35/282	−121/282	−43/141
X_3	47/66	−1/66	23/33	121/282	−35/282	43/141
X_4	10/66	−8/66	1/33	59/282	−67/282	−4/141

The rating vectors are given in Table 3. Hence $fb_1(N, A) < fb_4(N, A)$, $dfb_1(N, A) < dfb_4(N, A)$ and $Cfb_1(N, A) < Cfb_4(N, A)$, but $fb_1(N, A') > fb_4(N, A')$, $dfb_1(N, A') > dfb_4(N, A')$ and $Cfb_1(N, A') > Cfb_4(N, A')$, which proves the statement for fair bets and Copeland fair bets. □

The other partial version of positive homogeneity, when matrix M is multiplied by $k > 0$, has no relevance. Moreover, HOM and SI already imply the respective property.

5 Additive properties

Definition 5.1. Consistency *(CS) (Young, 1974): Let $(N, R, M), (N, R', M') \in \mathcal{R}$ be two ranking problems and $X_i, X_j \in N$ be two objects. Let $f : \mathcal{R} \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, R, M) \geq f_j(N, R, M)$ and $f_i(N, R', M') \geq f_j(N, R', M')$. f is called* consistent *if $f_i(N, R+R', M+M') \geq f_j(N, R+R', M+M')$, moreover, $f_i(N, R+R', M+M') > f_j(N, R+R', M+M')$ if $f_i(N, R, M) > f_j(N, R, M)$ or $f_i(N, R', M') > f_j(N, R', M')$.*

CS is the most intuitive version of additivity.

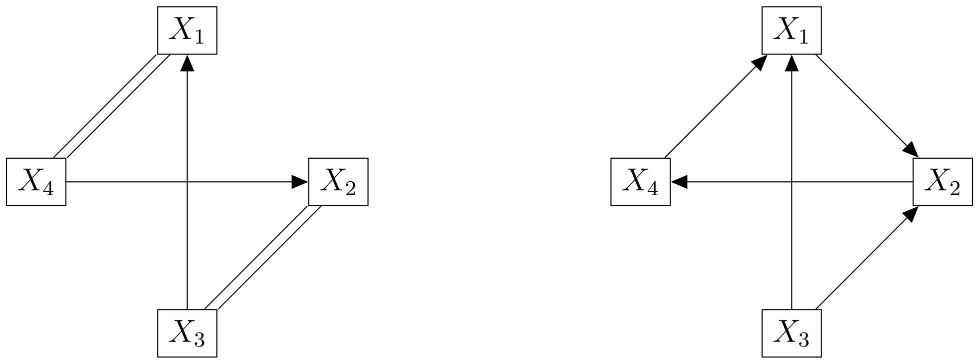
Lemma 5.1. *The score method satisfies CS.*

Proposition 5.1. *The generalised row sum and least squares methods violate CS.*

Proof.

Figure 5: Ranking problems of Example 5

(a) Ranking problem (N, R, M) (b) Ranking problem (N, R', M')



Example 5. *Let $(N, R, M) \in \mathcal{R}$ and $(N, R', M') \in \mathcal{R}$ be the ranking problems in Figure 5 with the set of objects $N = \{X_1, X_2, X_3, X_4\}$ and tournament matrices*

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $(N, R'', M'') = (N, R + R', M + M') \in \mathcal{R}$ be the sum of these two ranking problems.

Let $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{x}(\varepsilon)$, $\mathbf{x}(\varepsilon)(N, R', M') = \mathbf{x}(\varepsilon)'$, $\mathbf{x}(\varepsilon)(N, R'', M'') = \mathbf{x}(\varepsilon)''$ and $\mathbf{q}(N, R, M) = \mathbf{q}$, $\mathbf{q}(N, R', M') = \mathbf{q}'$, $\mathbf{q}(N, R'', M'') = \mathbf{q}''$. Now $n = 4$, $m = m' = 1$, thus $m'' = 2$ and

$$x_1(\varepsilon) = x_2(\varepsilon) = -\frac{1 + 10\varepsilon + 32\varepsilon^2 + 32\varepsilon^3}{1 + 12\varepsilon + 44\varepsilon^2 + 48\varepsilon^3}, \text{ and}$$

$$x_1(\varepsilon)' = x_2(\varepsilon)' = -1, \text{ but}$$

$$x_1(\varepsilon)'' - x_2(\varepsilon)'' = -\frac{2\varepsilon + 36\varepsilon^2 + 160\varepsilon^3}{1 + 22\varepsilon + 154\varepsilon^2 + 340\varepsilon^3} < 0.$$

It implies that $X_1 \sim^{\mathbf{x}(\varepsilon)}_{(N,R,M)} X_2$ and $X_1 \sim^{\mathbf{x}(\varepsilon)}_{(N,R',M)} X_2$, however, $X_1 \prec^{\mathbf{x}(\varepsilon)}_{(N,R'',M'')} X_2$. Generalised row sum is not consistent for any ε.

Regarding the least squares method, on the basis of Lemma 2.2:

$$q_1 = \frac{\lim_{\varepsilon \to \infty} x_1(\varepsilon)}{mn} = \frac{\lim_{\varepsilon \to \infty} x_2(\varepsilon)}{mn} = q_2, \text{ and}$$

$$q_1' = \frac{\lim_{\varepsilon \to \infty} x_1(\varepsilon)'}{m'n} = \frac{\lim_{\varepsilon \to \infty} x_2(\varepsilon)'}{m'n} = q_2', \text{ but}$$

$$q_1'' - q_2'' = \frac{\lim_{\varepsilon \to \infty} [x_1(\varepsilon)'' - x_2(\varepsilon)'']}{m''n} = -\frac{1}{17} < 0.$$

Hence $X_1 \sim^{\mathbf{q}}_{(N,R,M)} X_2$ and $X_1 \sim^{\mathbf{q}}_{(N,R',M')} X_2$ but $X_1 \prec^{\mathbf{q}}_{(N,R'',M'')} X_2$. □

González-Díaz et al. (2014, Example 4.2) have shown the violation of a somewhat weaker property called *order preservation* for the least squares and generalised row sum with $\varepsilon = 1/[m(n-2)]$, since there $d_i/d_j = d_i'/d_j'$ for all $X_i, X_j \in N$.

We will return later to the examination of fair bets and connected methods.

González-Díaz et al. (2014) discusses the following restricted version of additivity.

Definition 5.2. Flatness preservation *(FP) (Slutzki and Volij, 2005): Let* (N, R, M), $(N, R', M') \in \mathcal{R}$ *be two ranking problems. Let* $f : \mathcal{R} \to \mathbb{R}^n$ *be a scoring procedure such that* $f_i(N, R, M) = f_j(N, R, M)$ *and* $f_i(N, R', M') = f_j(N, R', M')$ *for all* $X_i, X_j \in N$. f preserves flatness *if* $f_i(N, R+R', M+M') = f_j(N, R+R', M+M')$ *for all* $X_i, X_j \in N$.

Flatness preservation demands additivity only for problems with all objects ranked equally.

Corollary 2. *CS implies FP.*

Lemma 5.2. *The score, generalised row sum and least squares methods satisfy FP.*

Proof. It has been shown in González-Díaz et al. (2014, Corollary 4.3) for the least squares, and in González-Díaz et al. (2014, Proposition 4.2) for generalised row sum with $\varepsilon = 1/[m(n-2)]$.

The score method preserves flatness due to Lemma 5.1 and Corollary 2.

If $x_i(\varepsilon)(N, R, M) = x_j(\varepsilon)(N, R, M)$ for all $X_i, X_j \in N$, then $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{0}$. We prove that $\mathbf{s}(N, R, M) = \mathbf{0} \Leftrightarrow \mathbf{x}(\varepsilon)(N, R, M) = \mathbf{0}$. $s_i(N, R, M) = s_j(N, R, M)$ for all $X_i, X_j \in N$ implies $\mathbf{s}(N, R, M) = \mathbf{0}$, therefore $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{0}$. If $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{0}$, then $(1 + \varepsilon mn)\mathbf{s} = \mathbf{0}$, so $\mathbf{s} = \mathbf{0}$.

The same argument can be applied in the case of least squares. □

Lemma 5.3. *Fair bets, dual fair bets and Copeland fair bets methods satisfy FP.*

Proof. Regarding the fair bets see Slutzki and Volij (2005, Theorem 1). According to Slutzki and Volij (2005, Remark 1), it is true for dual fair bets. It implies that Copeland fair bets preserves flatness. □

All objects ranked equally seems to be a strong condition, so it makes sense to require additivity on a larger set. A natural choice can be that only the objects examined are ranked equally.

Definition 5.3. Equality preservation *(EP): Let* $(N, R, M), (N, R', M') \in \mathcal{R}$ *be two ranking problems and* $X_i, X_j \in N$ *be two objects. Let* $f : \mathcal{R} \to \mathbb{R}^n$ *be a scoring procedure such that* $f_i(N, R, M) = f_j(N, R, M)$ *and* $f_i(N, R', M') = f_j(N, R', M')$. f preserves equality *if* $f_i(N, R+R', M+M') = f_j(N, R+R', M+M')$.

Corollary 3. *CS implies EP.*
EP implies FP.

Lemma 5.4. *The score method satisfies EP.*

Proof. It comes from Lemma 5.1 and Corollary 3. □

Lemma 5.5. *The generalised row sum and least squares methods violate EP.*

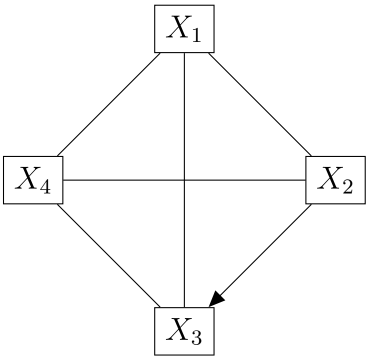
Proof. $x_1(\varepsilon) = x_2(\varepsilon)$ and $q_1 = q_2$ as well as $x_1(\varepsilon)' = x_2(\varepsilon)'$ and $q_1' = q_2'$, but $x_1(\varepsilon)'' = x_2(\varepsilon)''$ and $q_1'' = q_2''$ in Example 5. □

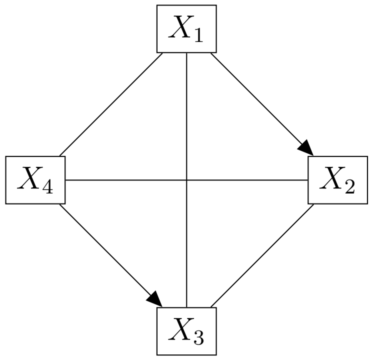
Proposition 5.2. *Fair bets, dual fair bets and Copeland fair bets methods violate EP.*

Proof.

Figure 6: Ranking problems of Example 6

(a) Ranking problem (N, A) (b) Ranking problem (N, A')





Example 6. *Let $(N, A) \in \mathcal{R}$ and $(N, A') \in \mathcal{R}$ be the ranking problems in Figure 6 with the set of objects $N = \{X_1, X_2, X_3, X_4\}$ and tournament matrices*

$$A = \begin{pmatrix} 0 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 1 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 0 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \end{pmatrix}.$$

Let $(N, A'') = (N, A + A') \in \mathcal{R}$ be the sum of these two ranking problems.

Table 4: Fair bets and associated rating vectors of Example 6

	fb(A)	**dfb**(A)	**Cfb**(A)	**fb**(A')	**dfb**(A')	**Cfb**(A')	**fb**(A'')	**dfb**(A'')	**Cfb**(A'')
X_1	1/4	−1/4	0	3/8	−1/8	1/4	163/512	−101/512	31/256
X_2	3/8	−1/8	1/4	1/8	−3/8	−1/4	117/512	−115/512	1/256
X_3	1/8	−3/8	−1/4	1/8	−3/8	−1/4	75/512	−205/512	−65/256
X_4	1/4	−1/4	0	3/8	−1/8	1/4	157/512	−91/512	33/256

The rating vectors are given in Table 4, where $fb_1(N, A) = fb_4(N, A)$, $dfb_1(N, A) = dfb_4(N, A)$, $Cfb_1(N, A) = Cfb_4(N, A)$, similarly, $fb_1(N, A') = fb_4(N, A')$, $dfb_1(N, A') = dfb_4(N, A')$, $Cfb_1(N, A') = Cfb_4(N, A')$, but $fb_1(N, A'') > fb_4(N, A'')$, $dfb_1(N, A'') < dfb_4(N, A'')$, $Cfb_1(N, A'') < Cfb_4(N, A'')$, which is a contradiction. □

Lemma 5.6. *Fair bets, dual fair bets and Copeland fair bets methods violate CS.*

Proof. It comes from Proposition 5.2 with Corollary 3. □

In the weakening of axiom CS, another natural restriction can be to allow only for the combination of ranking problems with the same matches matrix, when the effects of different comparison multigraphs are eliminated.

Definition 5.4. Result consistency *(RCS): Let $(N, R, M), (N, R', M) \in \mathcal{R}$ be two ranking problems and $X_i, X_j \in N$ be two objects. Let $f : \mathcal{R} \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, R, M) \geq f_j(N, R, M)$ and $f_i(N, R', M) \geq f_j(N, R', M)$. f is called* result consistent *if $f_i(N, R + R', 2M) \geq f_j(N, R + R', 2M)$, moreover, $f_i(N, R + R', 2M) > f_j(N, R + R', 2M)$ if $f_i(N, R, M) > f_j(N, R, M)$ or $f_i(N, R', M) > f_j(N, R', M)$.*

Corollary 4. *CS implies RCS.*

Corollary 5. *RCS (hence CS) implies HOM for all positive integer k.*

Because of Corollary 5, generalised row sum with a constant value of $\varepsilon = 1/3$ violates of consistency according to Example 2 as $k = 2$.

Proposition 5.3. *RCS and SYM implies INV.*

Proof. Take a ranking problem $(N, R, M) \in \mathcal{R}$, assume that $f_i(N, R, M) \geq f_j(N, R, M)$ for objects $X_i, X_j \in N$. If $f_i(N, -R, M) > f_j(N, -R, M)$, then $f_i(N, \mathbf{0}, M) > f_j(N, \mathbf{0}, M)$ due to RCS, which contradicts to SYM. Therefore $f_i(N, -R, M) \leq f_j(N, -R, M)$. □

Corollary 6. *CS and SYM implies INV.*

Corollary 6 was proved by Nitzan and Rubinstein (1981, Lemma 1) in the case of round-robin ranking problems.

Lemma 5.7. *The score method satisfies RCS.*

Proof. It comes from Lemma 5.1 and Corollary 4. □

Proposition 5.4. *The least squares method satisfies RCS.*

Proof. Let $\mathbf{q}(N, R, M) = \mathbf{q}$, $\mathbf{q}(N, R', M) = \mathbf{q}'$ and $\mathbf{q}(N, R + R', M + M) = \mathbf{q}''$. It is shown that $2\mathbf{q}'' = \mathbf{q} + \mathbf{q}'$. The Laplacian matrix of the comparison multigraph associated with matches matrix $M + M$ is $2L$, so

$$2L\mathbf{q}'' = L\left(\mathbf{q} + \mathbf{q}'\right) = \mathbf{s}(N, R, M) + \mathbf{s}(N, R', M) = \mathbf{s}(N, R + R', M + M)$$

as well as $\mathbf{e}^\top \mathbf{q}'' = \mathbf{e}^\top \left[(1/2)\mathbf{q} + (1/2)\mathbf{q}'\right] = 0$. □

Regarding the generalised row sum, we repeatedly distinguish two cases.

Lemma 5.8. *The generalised row sum method with a fixed ε violates RCS.*

Proof. Corollary 5 can be applied because of $k = 2$ in Example 2. □

Proposition 5.5. *The generalised row sum method with a variable ε satisfies RCS if ε is inversely proportional to the number of added ranking problems.*

Proof. Let $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{x}(\varepsilon)$, $\mathbf{x}(\varepsilon)(N, R', M) = \mathbf{x}(\varepsilon)'$ and $\mathbf{x}(\varepsilon)(N, R + R', M + M) = \mathbf{x}(\varepsilon)''$. It yields from some basic calculations:

$$\begin{aligned} \mathbf{x}(\varepsilon/2)'' &= (1 + \varepsilon mn)(I + \varepsilon L)^{-1}\mathbf{s}(N, R + R', M + M) = \\ &= (1 + \varepsilon mn)(I + \varepsilon L)^{-1}\left[\mathbf{s}(N, R, M) + \mathbf{s}(N, R', M)\right] = \mathbf{x} + \mathbf{x}(\varepsilon)'. \end{aligned}$$

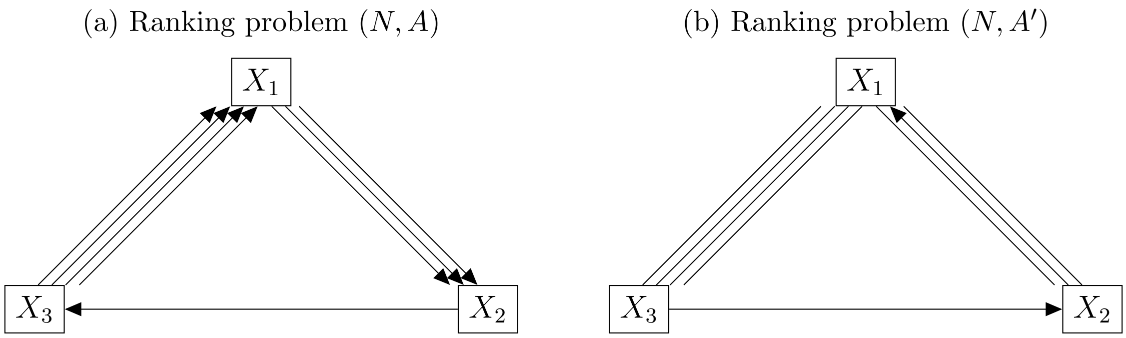
□

Remark 5. *The reasonable upper bound of $\varepsilon = 1\left[m(n-2)\right]$ is inversely proportional to the number of added ranking problems.*

Conjecture 2. *The proof of Proposition 5.5 suggests that generalised row sum violates HOM if ε is not inversely proportional to the number of added ranking problems.*

Lemma 5.9. *Fair bets and dual fair bets methods violate RCS.*

Figure 7: Ranking problems of Example 7



Proof. It is a consequence of Lemmata 3.2 and 3.4 with Proposition 5.3. □

Proposition 5.6. *Copeland fair bets method violates RCS.*

Proof.

Example 7. *Let $(N, A) \in \mathcal{R}$ and $(N, A') \in \mathcal{R}$ be the ranking problems in Figure 7 with the set of objects $N = \{X_1, X_2, X_3\}$ and tournament matrices*

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

with the same matches matrix

$$M = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Note that ranking problem (N, A) is the same as in Example 3. Let $(N, A'') = (N, A+A') \in \mathcal{R}$ be the sum of these two ranking problems.

Table 5: Fair bets and associated rating vectors of Example 7

	fb(A)	**dfb**(A)	**Cfb**(A)	**fb**(A')	**dfb**(A')	**Cfb**(A')	**fb**(A'')	**dfb**(A'')	**Cfb**(A'')
X_1	3/19	−1/3	−10/57	2/7	−6/15	−12/105	7/29	−2/6	−16/174
X_2	4/19	−1/3	−7/57	2/7	−5/15	−5/105	6/29	−3/6	−51/174
X_3	12/19	−1/3	17/57	3/7	−4/15	17/105	16/29	−1/6	67/174

The rating vectors are given in Table 5: $Cfb_1(N, A) < Cfb_2(N, A)$ and $Cfb_1(N, A') < Cfb_2(N, A')$, but $Cfb_1(N, A'') > Cfb_2(N, A'')$. □

Now we analyse the special case of round-robin ranking problems. Note that the set $\mathcal{R}^R$ is closed under summation.

Lemma 5.10. *The generalised row sum and least squares methods satisfy CS on the set $\mathcal{R}^R$.*

Proof. Due to the axioms *agreement* (Chebotarev, 1994, Property 3) and *score consistency* (González-Díaz et al., 2014), both the generalised row sum and least squares methods coincide with the score on this set of problems, so Lemma 5.1 holds. □

Lemma 5.10 shows that lack of additivity in Example 5 is due to the different structure of the comparison multigraphs.

Lemma 5.11. *The generalised row sum satisfies RCS on the set* $\mathcal{R}^R$.

Proof. It comes from Lemma 5.10 and Corollary 4. □

Lemma 5.12. *Fair bets, dual fair bets and Copeland fair bets methods violate EP on the set* $\mathcal{R}^R$.

Proof. Both $(N, A) \in \mathcal{R}^R$ and $(N, A') \in \mathcal{R}^R$ are round-robin ranking problems in Example 6. □

Lemma 5.13. *Fair bets, dual fair bets and Copeland fair bets methods violate CS on the set* $\mathcal{R}^R$.

Proof. It comes from Proposition 5.7 and Corollary 4. □

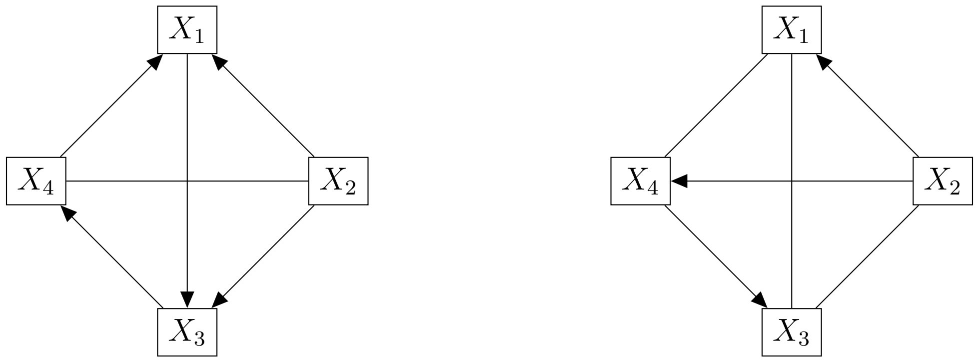
Lemma 5.14. *Fair bets and dual fair bets methods violate RCS.*

Proof. It is a consequence of Lemmata 3.2 and 3.4 with Proposition 5.3. □

Proposition 5.7. *Copeland fair bets methods violate RCS on the set* $\mathcal{R}^R$.

Figure 8: Ranking problems of Example 8

(a) Ranking problem (N, A) (b) Ranking problem (N, A')



Proof.

Example 8. *Let* $(N, A) \in \mathcal{R}$ *and* $(N, A') \in \mathcal{R}$ *be the ranking problems in Figure 8 with the set of objects* $N = \{X_1, X_2, X_3, X_4\}$ *and tournament matrices*

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \\ 1 & 0.5 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0.5 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \end{pmatrix}$$

with the same matches matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $(N, A'') = (N, A + A') \in \mathcal{R}^R$ *be the sum of these two ranking problems.*

The rating vectors are given in Table 6: $Cfb_1(N, A) < Cfb_3(N, A)$ and $Cfb_1(N, A') < Cfb_3(N, A')$, but $Cfb_1(N, A'') > Cfb_3(N, A'')$. □

Table 6: Fair bets and associated rating vectors of Example 8

	$\mathbf{fb}(A)$	$\mathbf{dfb}(A)$	$\mathbf{Cfb}(A)$	$\mathbf{fb}(A')$	$\mathbf{dfb}(A')$	$\mathbf{Cfb}(A')$	$\mathbf{fb}(A'')$	$\mathbf{dfb}(A'')$	$\mathbf{Cfb}(A'')$
X_1	1/17	−6/19	−83/323	5/64	−23/64	−9/32	17/236	−79/244	−906/3599
X_2	10/17	−1/19	173/323	39/64	−5/64	17/32	145/236	−15/244	1990/3599
X_3	2/17	−7/19	−81/323	11/64	−25/64	−7/32	31/236	−97/244	−958/3599
X_4	4/17	−5/19	−9/323	9/64	−11/64	−1/32	43/236	−53/244	−126/3599

6 Additivity and irrelevant comparisons

Definition 6.1. Independence of irrelevant matches *(IIM): Let $(N, R, M) \in \mathcal{R}$ be a ranking problem and $X_i, X_j, X_k, X_\ell \in N$ be four different objects. Let $f : \mathcal{R} \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, R, M) \geq f_j(N, R, M)$ and $(N, R', M) \in \mathcal{R}$ be a ranking problem identical to (N, R, M) except for the result $r'_{k\ell} \neq r_{k\ell}$. f is called* independent of irrelevant matches *if $f_i(N, R', M) \geq f_j(N, R', M)$.*

IIM means that all comparisons not involving the chosen objects are irrelevant from the perspective of their relative ranking. It appears as *independence* in Rubinstein (1980, Axiom III) and Nitzan and Rubinstein (1981, Axiom 5) for round-robin ranking problems. The name independence of irrelevant matches was introduced by González-Díaz et al. (2014), however, they also allowed for a change in the number of matches between two objects (i.e. $a'_{k\ell} \neq a_{k\ell}$ implies that possibly both $r'_{k\ell} \neq r_{k\ell}$ and $m'_{k\ell} \neq m_{k\ell}$), which seems to be too general for us. Altman and Tennenholtz (2008, Definition 8.4) introduces a still stronger axiom called *Arrow's independence of irrelevant alternatives* by permitting modifications of comparisons involving X_i and X_j if $r_{ih} - r'_{ih} = r_{jh} - r'_{jh}$ holds for all $X_h \in N \setminus \{X_i, X_j\}$.

Remark 6. *Property IIM has a meaning if $n \geq 4$.*

Sequential application of independence of irrelevant matches can result in a ranking problem $(N, R', M) \in \mathcal{R}$, for which $r'_{gh} = r_{gh}$ if $\{X_g, X_h\} \cap \{X_i, X_j\} \neq \emptyset$, but all other paired comparisons are arbitrary.

Lemma 6.1. *The score method satisfies IIM.*

Proposition 6.1. *The generalised row sum, least squares, fair bets, dual fair bets and Copeland fair bets methods violate IIM.*

Proof.

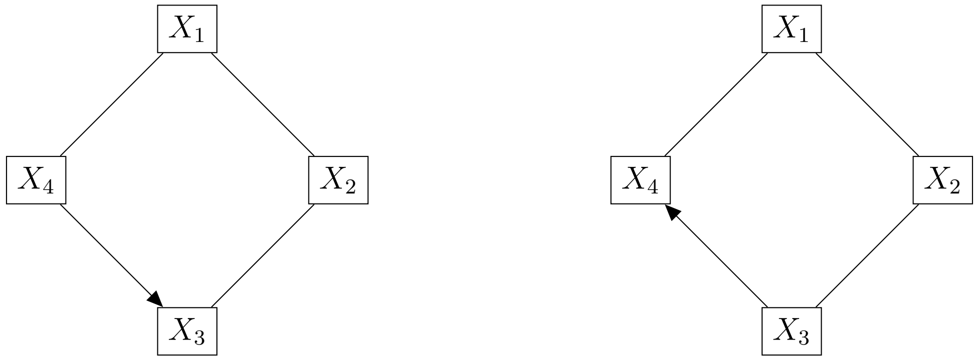
Example 9. *Let $(N, R, M) \in \mathcal{R}$ and $(N, R', M) \in \mathcal{R}$ be the ranking problems in Figure 9 with set of objects $N = \{X_1, X_2, X_3, X_4\}$ and tournament matrices*

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 1 \\ 0.5 & 0 & 0 & 0 \end{pmatrix},$$

where $a'_{34} \neq a_{34}$.

Figure 9: Ranking problems of Example 9

(a) Ranking problem (N, R, M) (b) Ranking problem (N, R', M)



IIM requires that $f_1(N, R, M) \geq f_2(N, R, M) \Leftrightarrow f_1(N, R', M) \geq f_2(N, R', M)$. Let $\mathbf{x}(\varepsilon)(N, R, M) = \mathbf{x}(\varepsilon)$, $\mathbf{x}(\varepsilon)(N, R', M') = \mathbf{x}(\varepsilon)'$ and $\mathbf{q}(N, R, M) = \mathbf{q}$, $\mathbf{q}(N, R', M') = \mathbf{q}'$. Here

$$x_1(\varepsilon) = x_2(\varepsilon)' = (1 + \varepsilon mn)\frac{\varepsilon}{(1+2\varepsilon)(1+4\varepsilon)} = \frac{\varepsilon}{1+2\varepsilon} \text{ and}$$

$$x_1(\varepsilon)' = x_2(\varepsilon) = (1 + \varepsilon mn)\frac{-\varepsilon}{(1+2\varepsilon)(1+4\varepsilon)} = \frac{-\varepsilon}{1+2\varepsilon},$$

that is, $X_1 \succ^{\mathbf{x}(\varepsilon)}_{(N,R,M)} X_2$ but $X_1 \prec^{\mathbf{x}(\varepsilon)}_{(N,R',M)} X_2$.

Regarding the least squares method, on the basis of Lemma 2.2:

$$q_1 = \frac{\lim_{\varepsilon\to\infty} x_1(\varepsilon)}{mn} = q_2' = \frac{\lim_{\varepsilon\to\infty} x_2(\varepsilon)'}{mn} = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} \text{ and}$$

$$q_1' = \frac{\lim_{\varepsilon\to\infty} x_1(\varepsilon)'}{mn} = q_2 = \frac{\lim_{\varepsilon\to\infty} x_2(\varepsilon)}{mn} = -\frac{1}{2}\cdot\frac{1}{4} = -\frac{1}{8}.$$

Hence $X_1 \succ^{\mathbf{q}}_{(N,R,M)} X_2$ but $X_1 \prec^{\mathbf{q}}_{(N,R',M)} X_2$.

Table 7: Fair bets and associated rating vectors of Example 9

	$\mathbf{fb}(N, A)$	$\mathbf{dfb}(N, A)$	$\mathbf{Cfb}(N, A)$	$\mathbf{fb}(N, A')$	$\mathbf{dfb}(N, A')$	$\mathbf{Cfb}(N, A')$
X_1	5/16	−3/16	1/8	3/16	−5/16	−1/8
X_2	3/16	−5/16	−1/8	5/16	−3/16	1/8
X_3	1/16	−7/16	−3/8	7/16	−1/16	3/8
X_4	7/16	−1/16	3/8	1/16	−7/16	−3/8

The other three rating vectors are given in Table 7. Thus $fb_1(N, A) > fb_2(N, A)$, $dfb_1(N, A) > dfb_2(N, A)$ and $Cfb_1(N, A) > Cfb_2(N, A)$, but $fb_1(N, A') < fb_2(N, A')$, $dfb_1(N, A') < dfb_2(N, A')$ and $Cfb_1(N, A') < Cfb_2(N, A')$, which is a contradiction. □

Remark 7. *In Example 9, the two ranking problems coincide with the permutation $\sigma(X_1) = X_2$ and $\sigma(X_3) = X_4$. Therefore independence of irrelevant matches demands that $f_1(N, R, M) = f_2(N, R, M)$, which is violated by all ranking methods discussed except for the score.*

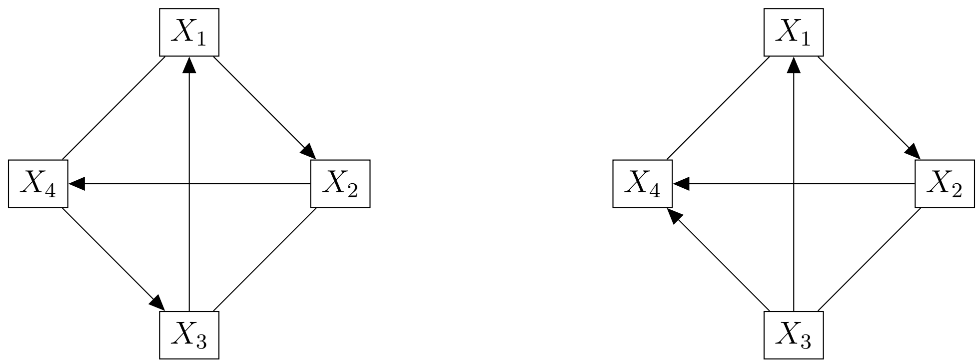
Lemma 6.2. *The generalised row sum and least squares methods satisfy IIM on the set $\mathcal{R}^R$.*

Proof. Due to the axioms *agreement* (Chebotarev, 1994, Property 3) and *score consistency* (González-Díaz et al., 2014), both the generalised row sum and least squares methods coincide with the score on this set of problems, so Lemma 6.1 holds. □

Proposition 6.2. *Fair bets, dual fair bets and Copeland fair bets methods violate IIM on the set $\mathcal{R}^R$.*

Figure 10: Ranking problems of Example 10

(a) Ranking problem (N, A) (b) Ranking problem (N, A')



Proof.

Example 10. *Let $(N, A) \in \mathcal{R}$ and $(N, A') \in \mathcal{R}$ be the ranking problems in Figure 10 with the set of objects $N = \{X_1, X_2, X_3, X_4\}$ and tournament matrices*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0.5 & 1 \\ 1 & 0.5 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0.5 & 1 \\ 1 & 0.5 & 0 & 1 \\ 0.5 & 0 & 0 & 0 \end{pmatrix},$$

where $a'_{34} \neq a_{34}$.

Table 8: Fair bets and associated rating vectors of Example 10

	$\mathbf{fb}(N, A)$	$\mathbf{dfb}(N, A)$	$\mathbf{Cfb}(N, A)$	$\mathbf{fb}(N, A')$	$\mathbf{dfb}(N, A')$	$\mathbf{Cfb}(N, A')$
X_1	1/4	−1/4	0	5/32	−7/32	−1/16
X_2	1/4	−1/4	0	7/32	−5/32	1/16
X_3	1/4	−1/4	0	19/32	−1/32	9/16
X_4	1/4	−1/4	0	1/32	−19/32	−9/16

IIM requires that $f_1(N, R, M) \geq f_2(N, R, M) \Leftrightarrow f_1(N, R', M) \geq f_2(N, R', M)$. The rating vectors are given in Table 8. Thus $fb_1(N, A) \geq fb_2(N, A)$, $dfb_1(N, A) \geq dfb_2(N, A)$ and $Cfb_1(N, A) \geq Cfb_2(N, A)$, but $fb_1(N, A') < fb_2(N, A')$, $dfb_1(N, A') < dfb_2(N, A')$ and $Cfb_1(N, A') < Cfb_2(N, A')$, which is a contradiction. □

González-Díaz et al. (2014, p. 165) consider independence of irrelevant matches a drawback of the score method because outside the subdomain of round-robin ranking problems, it makes sense for the scoring procedure to be responsive to the strength of the opponents.

Our discussion is finalized by linking IIM to additivity.

Theorem 6.1. *NEU, CS and SYM imply IIM.*

Proof. For the round-robin case, see Nitzan and Rubinstein (1981, Lemma 3).

Assume to the contrary, and let $(N, R, M) \in \mathcal{R}$ be a ranking problem, $X_i, X_j, X_k, X_\ell \in N$ be four different objects such that $f_i(N, R, M) \geq f_j(N, R, M)$, $(N, R', M) \in \mathcal{R}$ is identical to (N, R, M) except for the result $a'_{k\ell} \neq a_{k\ell}$, but $f_i(N, R', M) < f_j(N, R', M)$.

Corollary 6 implies that a consistent and symmetric scoring procedure satisfies INV, hence $f_i(N, -R, M) \leq f_j(N, -R, M)$. Denote $\sigma : N \to N$ the permutation $\sigma(X_i) = X_j$, $\sigma(X_j) = X_i$, and $\sigma(X_k) = X_k$ for all $X_k \in N \setminus \{X_i, X_j\}$. By neutrality, $f_i[\sigma(N, R, M)] \leq f_j[\sigma(N, R, M)]$, and $f_i[\sigma(N, -R', M)] < f_j[\sigma(N, -R', M)]$ due to inversion and Remark 2. With the definition $R'' = \sigma(R) - \sigma(R') - R + R = \mathbf{0}$,

$$(N, R'', M'') = \sigma(N, R, M) + \sigma(N, -R', M) - (N, R, M) + (N, R', M).$$

Symmetry implies $f_i(N, R'', M) = f_j(N, R'', M)$, consistency results in $f_i(N, R'', M) < f_j(N, R'', M)$, a contradiction. □

Since NEU and SYM are difficult to debate, CS is an axiom one would rather not have in the general case. It highlights the significance of Section 5 as weakening of consistency seems to be desirable in axiomatizations on the whole set $\mathcal{R}$.

7 Conclusions

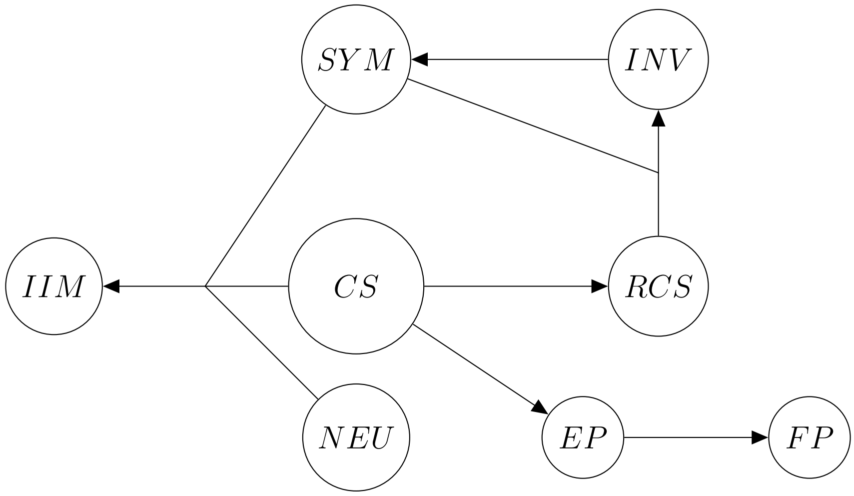
Table 9: Axiomatic properties of ranking methods

Property	Score	Generalised row sum (fixed ε)	Generalised row sum (variable ε)	Least squares	Fair bets / Dual fair bets	Copeland fair bets
NEU	✔	✔	✔	✔	✔	✔
SYM	✔	✔	✔	✔	✔	✔
INV	✔	✔	✔	✔	✗	✔
HOM	✔	✗	✔	✔	✔	✔
SI	✔	✔	✔	✔	✗	✗
CS	✔	✗	✗	✗	✗	✗
FP	✔	✔	✔	✔	✔	✔
EP	✔	✗	✗	✗	✗	✗
RCS	✔	✗	✔	✔	✗	✗
IIM	✔	✗	✗	✗	✗	✗

Our results are summarized in Table 9. Score satisfies all properties, however, we have seen that IIM is not favourable in the presence of missing and multiple comparisons. The findings recommend the use of generalised row sum with a variable parameter, somewhat proportional to the number of matches like the upper bound of reasonable choice as $\varepsilon = 1/[m(n-2)]$. It is not surprising given the statistical background of the method (Chebotarev, 1994). With this definition, generalised row sum and least squares cannot be distinguished with respect to the axioms examined.[3] A drawback of fair bets (and its

[3] Their main differences are highlighted by González-Díaz et al. (2014).

Figure 11: Connections among the axioms (arrows sign implications)



dual) was eliminated by the introduction of Copeland fair bets, but it does not have much effect for other properties. Chebotarev and Shamis (1999)'s results about self-consistent monotonicity also confirm that 'manipulation' with win-loss combining scoring procedures is not able to correct some inherent features of this class.

Copeland fair bets has two significant weakness compared to generalised row sum and least squares, the breaking of scale invariance and result consistency. SI is essential for practical applications and offers a way to convert all ranking problems into an irreducible one. Therefore it is worth to investigate scale invariance for other methods defined on the domain $\mathcal{R}^I$, some of them are presented in Chebotarev and Shamis (1999, Table 1). Result consistency means a problem because of its immediate interpretation. For instance, Proposition 5.7 suggests that a player worse in both half of a round-robin tournament than another can overtake it on the basis of aggregated results. It may have strange consequences.

We have aspired to give simple counterexamples, minimal with respect to the number of objects and matches. It shows that the violation of these properties remains an issue for small problems. All axioms have also been analysed on the restricted domain of round-robin ranking problems. Then the generalised row sum and least squares coincides with the score, while fair bets and its peers violate all properties on this set, too (in some cases with a marginal increase in complexity). According to our opinion, it makes their use debatable for ranking purposes, despite González-Díaz et al. (2014) does not mention as a drawback that fair bets deviates from the score on the set $\mathcal{R}^R$.

Figure 11 gives a comprehensive picture about our axioms. HOM and SI does not fit into it, however, Corollary 5 refers to some link between homogeneity and result consistency. HOM certainly does not imply RCS (see the properties of fair bets), but it remains an open question whether there exists a scoring procedure satisfying only the latter or not. These results shed light on some discoveries of Table 9. The strong connection of IIM and CS justifies the violation of both properties. EP was constructed as a weak form of additivity by an idea from FP, but it does not proved to be successful. Result consistency yields some positive outcomes and, besides SI, it motivates our setting with the differentiation of result and matches matrices. We think that further investigation of additivity may help the understanding of scoring procedures.

We see two main directions for future research. The first is to extend the scope of the analysis with other scoring procedures. For example, Slikker et al. (2012) defines a general framework for ranking the nodes of directed graphs, resulting in fair bets as a limit. Positional power (Herings et al., 2005) is also worth to analyse since it is the pair of least squares from a graph-theoretic point of view (Csató, 2014). The second course is the introduction of new axioms with the final aim of characterization results, an intended end goal of our analysis. Nevertheless, it may be a difficult road.

References

A. Altman and M. Tennenholtz. Axiomatic foundations for ranking systems. *Journal of Artificial Intelligence Research*, 31(1):473–495, 2008.

P. Borm, R. van den Brink, and M. Slikker. An iterative procedure for evaluating digraph competitions. *Annals of Operations Research*, 109(1-4):61–75, 2002.

D. Bouyssou. Ranking methods based on valued preference relations: a characterization of the net flow method. *European Journal of Operational Research*, 60(1):61–67, 1992.

D. Bouyssou. Monotonicity of 'ranking by choosing': a progress report. *Social Choice and Welfare*, 23(2):249–273, 2004.

S. Bozóki, L. Csató, L. Rónyai, and J. Tapolcai. Robust peer review decision process. Manuscript, 2014.

P. Chebotarev. Generalization of the row sum method for incomplete paired comparisons. *Automation and Remote Control*, 50(3):1103–1113, 1989.

P. Chebotarev. Aggregation of preferences by the generalized row sum method. *Mathematical Social Sciences*, 27(3):293–320, 1994.

P. Chebotarev and E. Shamis. Characterizations of scoring methods for preference aggregation. *Annals of Operations Research*, 80:299–332, 1998.

P. Chebotarev and E. Shamis. Preference fusion when the number of alternatives exceeds two: indirect scoring procedures. *Journal of the Franklin Institute*, 336(2):205–226, 1999.

L. Csató. A graph interpretation of the least squares ranking method. *Social Choice and Welfare*, 2014. URL `http://link.springer.com/article/10.1007%2Fs00355-014-0820-0`. Forthcoming.

H. E. Daniels. Round-robin tournament scores. *Biometrika*, 56(2):295–299, 1969.

H. A. David. Ranking from unbalanced paired-comparison data. *Biometrika*, 74(2):432–436, 1987.

J. González-Díaz, R. Hendrickx, and E. Lohmann. Paired comparisons analysis: an axiomatic approach to ranking methods. *Social Choice and Welfare*, 42(1):139–169, 2014.

H. Gulliksen. A least squares solution for paired comparisons with incomplete data. *Psychometrika*, 21(2):125–134, 1956.

B. Hansson and H. Sahlquist. A proof technique for social choice with variable electorate. *Journal of Economic Theory*, 13(2):193–200, 1976.

P. J.-J. Herings, G. van der Laan, and D. Talman. The positional power of nodes in digraphs. *Social Choice and Welfare*, 24(3):439–454, 2005.

P. Horst. A method for determining the absolute affective value of a series of stimulus situations. *Journal of Educational Psychology*, 23(6):418–440, 1932.

X. Jiang, L.-H. Lim, Y. Yao, and Y. Ye. Statistical ranking and combinatorial Hodge theory. *Mathematical Programming*, 127(1):203–244, 2011.

H. F. Kaiser and R. C. Serlin. Contributions to the method of paired comparisons. *Applied Psychological Measurement*, 2(3):423–432, 1978.

M. G. Kendall. Further contributions to the theory of paired comparisons. *Biometrics*, 11:43–62, 1955.

E. Landau. Zur relativen Wertbemessung der Turnierresultate. *Deusches Wochenschach*, 11:366–369, 1895.

E. Landau. Über Preisverteilung bei Spielturnieren. *Zeitschrift für Mathematik und Physik*, 63:192–202, 1914.

J.-F. Laslier. *Tournament solutions and majority voting*. Springer, Berlin, 1997.

J. W. Moon and N. J. Pullman. On generalized tournament matrices. *SIAM Review*, 12 (3):384–399, 1970.

S. Nitzan and A. Rubinstein. A further characterization of Borda ranking method. *Public Choice*, 36(1):153–158, 1981.

A. Rubinstein. Ranking the participants in a tournament. *SIAM Journal on Applied Mathematics*, 38(1):108–111, 1980.

M. Slikker, P. Borm, and R. van den Brink. Internal slackening scoring methods. *Theory and Decision*, 72(4):445–462, 2012.

G. Slutzki and O. Volij. Ranking participants in generalized tournaments. *International Journal of Game Theory*, 33(2):255–270, 2005.

G. Slutzki and O. Volij. Scoring of web pages and tournaments – axiomatizations. *Social Choice and Welfare*, 26(1):75–92, 2006.

L. L. Thurstone. A law of comparative judgment. *Psychological Review*, 34(4):273–286, 1927.

T. H. Wei. *The Algebraic Foundations of Ranking Theory*. PhD thesis, University of Cambridge, 1952.

H. P. Young. An axiomatization of Borda's rule. *Journal of Economic Theory*, 9(1):43–52, 1974.

E. Zermelo. Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, 29:436–460, 1928.