OTKA T068477: Kváziöröklődő és standardul rétegezett algebrák
Záróbeszámoló

A kváziöröklődő algebrák osztályát az 1980-as évek végén definiálta Cline, Parshall és Scott, és igen hamar megindult az intenzív kutatásuk. (Csak érdekességként jegyzem meg, hogy az eredeti cikkre, mely 1988-ban jelent meg, a Googlescholar 444 hivatkozást sorol föl; a témában írt cikkek száma ennél nyilván jóval nagyobb.) A 90-es évek óta pedig számos általánosításuk is ismert: a legtermészetesebb ilyen osztály a standardul rétegezett algebráké, melyekre a kváziöröklődő algebrák számos strukturális és homologikus tulajdonságát már sikerült általánosítani. A legáltalánosabb ezen általánosítások közül az ún. CPS-rétegezett algebrák osztálya, melyeknek a vizsgálata Auslander, Platzeck és Todorov, illetve Cline, Parshall és Scott 90-es években megjelent cikkeire vezethető vissza. Ennek az osztálynak a strukturális és homologikus vizsgálata sok tekintetben még várat magára.

Hogy a beszámoló érthetőbb legyen, definiáljuk a témához tartozó főbb fogalmakat. Valamennyi itt szereplő osztálynál a következő alaphelyzettel számolunk. Adva van egy A véges dimenziós algebra egy K test fölött, melyről föltesszük, hogy bázisalgebra, s rögzítjük primitív ortogonális idempotensek egy teljes rendszerének egy (az indexeikkel kifejezett) teljes rendezését: $1 = e_1 + e_2 + \cdots + e_n$. Ezt az alaphelyzetet $(A, \mathbf{e})$ jelöli, ahol $\mathbf{e} = (e_1, \ldots, e_n)$. Ekkor az $\varepsilon_i = e_i + \cdots + e_n$ idempotensek által generált $I_i = A\varepsilon_i A$ nyomideálok egy filtrálását adják az algebrának: $0 \subset I_n \subset I_{n-1} \subset \cdots \subset I_1 = A$. Az i-edik direkt fölbonthatatlan projektív modulusnak, e_iA-nak az $i+1$-edik nyomideál szerinti faktorát nevezzük i-edik *standard modulusnak*: $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$. Az i-edik *valódi standard modulus* $\bar{\Delta}(i) = \Delta(i)/(\mathrm{rad}\,\Delta(i))e_iA$. Azoknak a modulusoknak a részkategóriáját, melyeknek van Δ-kkal való filtrálása, $\mathcal{F}(\Delta)$ jelöli; hasonlóképpen beszélhetünk $\mathcal{F}(\bar{\Delta})$-ról is. Egy A algebrát akkor nevezünk *standardul rétegezettnek*, ha $A_A \in \mathcal{F}(\Delta)$ vagy $A_A \in \mathcal{F}(\bar{\Delta})$ (azaz A Δ-filtrált vagy $\bar{\Delta}$-filtrált); s egy algebra akkor *kváziöröklődő*, ha standardul rétegezett, és $\bar{\Delta}(i) = \Delta(i)$ minden i-re, azaz $\mathcal{F}(\Delta) = \mathcal{F}(\bar{\Delta})$. A bal oldali standard modulusok K-duálisait mondjuk *kostandard modulusoknak*, s a jelük ∇, illetve $\bar{\nabla}$. Egy I idempotens ideált *rétegező ideálnak* nevezünk, ha minden X, Y modulusra, melyet I annullál, $\mathrm{Ext}^k_{A/I}(X,Y) = \mathrm{Ext}^k_A(X,Y)$ minden $k \geq 0$-ra. Egy $(A,\mathbf{e})$ algebrát CPS-rétegezettnek nevezünk, ha mindegyik $A\varepsilon_iA$ nyomideál rétegező ideál. Könnyen bizonyítható, hogy minden standardul rétegezett algebra egyúttal CPS-rétegezett is.

[1] *Ágoston, I., Dlab, V., Lukács, E.*: Approximations of algebras by standardly stratified algebras, *J. Alg.* **319** (2008), 4177–4198.

A pályázati időszak elején egy korábban megkezdett munkánkat fejeztük be. Ebben a munkánkban Dlab és Ringel egy korábbi cikkéből indultunk ki. A cikkben Dlab és Ringel axiomatikus jellemzését adta azoknak a modulus-részkategóriáknak, amelyek ekvivalensek egy A kváziöröklődő algebrához tartozó $\mathcal{F}_A(\Delta)$ részkategóriával. Mi ugyanezt a kérdést tettük föl a standardul rétegezett algebrák megfelelő részkategóriáira, $\mathcal{F}(\Delta)$-ra és $\mathcal{F}(\bar{\Delta})$-ra.

Az axiomatikus jellemzésből kiderült, hogy ezek a kategóriák már meglehetősen általánosak, de ez az általánosság egyúttal új kapcsolatokra világított rá. Ugyanis bármely $(A, \mathbf{e})$ véges dimenziós algebra esetén értelmezhetők az $\mathcal{F}_A(\Delta)$ és az $\mathcal{F}_A(\bar{\Delta})$ kategóriák, s megmutattuk, hogy Morita-ekvivalencia erejéig egyértelműen létezik egy $\Sigma(A)$-val, illetve egy $\Omega(A)$-val jelölt algebra, melyekre az igaz, hogy $\Sigma(A)$ Δ-filtrált, $\Omega(A)$ $\bar{\Delta}$-filtrált, és $\mathcal{F}_A(\Delta)$ ekvivalens $\mathcal{F}_{\Sigma(A)}(\Delta)$-val, $\mathcal{F}_A(\bar{\Delta})$ pedig ekvivalens $\mathcal{F}_{\Omega(A)}(\bar{\Delta})$-sal. Ezzel algebráknak két ekvivalenciáját definiáltuk, s az ekvivalenciaosztályok mindegyikében pontosan egy Δ- illetve $\bar{\Delta}$-filtrált algebra van. Még meglepőbb volt, hogy ha a Σ és Ω idempotens operátorokat fölváltva alkalmazzuk egy algebrára, akkor az így kapott algebrasorozat véges sok lépés után megáll egy olyan algebránál, mely egyszerre Δ és $\bar{\Delta}$-filtrált. A stabilizálódás helyére sikerült pontos fölső becslést adnunk az egyszerű modulusok számának függvényében. Ez azt jelenti, hogy ha vesszük az összes standardul rétegezett algebra gráfját, melyben irányított és színezett éleket határoznak meg a Σ és Ω operátorok, akkor ennek a gráfnak a komponensei paraméterezhetők az egyszerre Δ- és $\bar{\Delta}$-filtrált algebrákkal. – A cikk eredményeiről Ágoston István konferenciaelőadást tartott 2007 végén Toruńban az ICRA XII Nemzetközi Reprezentációelméleti Konferencián, és ugyanezen évben Szegeden a Bolyai Intézet algebra szemináriumán. Mivel a cikk eredményeinek nagy része még a korábbi OTKA-pályázatunk idején született, ezért annak a száma van rajta föltüntetve, noha a végső publikáció előkészítése már a jelen pályázat idejére esett.

[2] *Ágoston, I., Dlab, V., Lukács, E.*: Constructions of stratified algebras *Comm. Algebra* **39** (2011), 2545–2553.

Ismét a kváziöröklődő algebrák esetéből indultunk ki. Minden kváziöröklődő algebrához hozzárendelhető kváziöröklődő algebráknak két sorozata: egyik a nyomideálok szerinti faktoralgebráké: $B_i = A/A\varepsilon_{i+1}A$; a másik pedig az $\varepsilon_i A$ projektívek endomorfizmusalgebráié, az ún. $C_i = \varepsilon_i A \varepsilon_i$ centralizátoralgebráké. A kváziöröklődő algebrák rekurzív definíciójához az első konstrukció áll közelebb, de egyes algebrai geometriai alkalmazásoknál a második konstrukció játszik szerepet. Dlab és Ringel mutatták meg, hogy hogyan lehet a centralizátoralgebrákat fölhasználni arra, hogy minden kváziöröklődő algebrát megkaphassunk egy konstrukció iterált alkalmazásával lokális algebrákból és alkalmasan filtrált bimodulusokból. A cikkünkben azt mutatjuk meg, hogy az eljárás általánosítható a standardul rétegezett algebrákra is, de az egyszerű modulusok bonyolultabb homologikus viselkedése miatt (itt már lehet az egyszerű modulusoknak önmagukkal vett bővítése) az egész eljárás is bonyolultabbá válik. Azon túlmenően, hogy alkalmas standardul rétegezett algebrából, egy lokális algebrából, két megfelelően filtrált bimodulusból, valamint egy bimodulushomorfizmusból kiindulva egy adott természetes eljárással standardul rétegezett algebrát kapunk, esetenként szükség lehet egy további lépésre is, hogy valamennyi standardul rétegezett algebrát megkapjuk. A hiányzó algebrákat úgy állíthatjuk elő az előbbi konstrukcióból, hogy egy jól meghatározott típusú ideállal faktorizálunk. Megjegyzendő, hogy a faktorizációs lépés megengedésével megszabadulhatunk attól a föltevéstől, melyet Dlab és Ringel tettek a kváziöröklődő esetben, nevezetesen hogy az alaptest perfekt. A cikk elején kiegészítő eredményeket találhatunk CPS-rétegezett algebrákra is.

[3] *Ágoston, I., Lukács, E.*: Stratifying pairs of subcategories for CPS-stratified algebras *(közlésre benyújtva)*, 11 oldal

Már a standardul rétegezett algebrák finitisztikus dimenziójával kapcsolatos vizsgálataink során (melyeknek eredményét még 2000-ben publikáltuk) fölmerült az a probléma, hogy megfelelő struktúraelmélet nélkül az ottani eredmények nem vihetők át CPS-rétegezett algebrákra. A standardul rétegezett algebrák struktúráját és a moduluskategória homologikus tulajdonságait is nagymértékben befolyásolja az a tény, hogy Δ-filtrált algebra esetén az $\mathcal{F}(\Delta)$ és $\mathcal{F}(\bar{\nabla})$ kategóriák egymásra merőlegesek (azaz nincs egymással bővítésük), sőt, $\mathcal{F}(\bar{\nabla}) = \mathcal{F}(\Delta)^{\perp}$ és $\mathcal{F}(\Delta) = {}^{\perp}\mathcal{F}(\bar{\nabla})$, azaz a mondott kategóriák egymás „merőlegesei". Ez pl. lehetővé tette a standardul rétegezett algebrákra vonatkozó finitisztikusdimenzió-sejtés bizonyításánál, hogy homologikus tulajdonságokból strukturális következtetéseket vonjunk le. Ezt a helyzetet próbáltuk meg átvinni a CPS-rétegezett algebrákra, ahol a homologikus definícióból eleve nem következik semmilyen szerkezeti (pl. filtráltsági) tulajdonság. A kiinduló fogalmat a rétegező ideálok egy jellemzése adta: egy $I = AeA$ idempotens ideál pontosan akkor rétegező ideál, ha $I_A \in {}^{\perp}\text{mod-}A/I$. Ezért azoknak a modulusok kategóriáit vizsgáltuk először, amelyeknek létezik egy olyan filtrálása, melyben a faktorok benne vannak a $\mathcal{P}_i(\mathbf{e}) = ({}^{\perp}\text{mod-}A/A\varepsilon_i A) \cap \text{Gen}(\mathbf{e}_i A) \cap \text{mod-}A/A\varepsilon_{i+1}A$ részkategóriákban. Bizonyos általános zártsági feltételek teljesülése esetén az ilyen típusú kategóriákat *rétegező részkategóriának* neveztük. Ezzel a megfogalmazással azt is mondhatjuk, hogy egy algebra pontosan akkor CPS-rétegezett, ha van rétegező részkategóriája. Δ-filtrált algebrák esetén pl. $\mathcal{F}(\Delta)$ ilyen részkategória; ha az algebra $\bar{\Delta}$-filtrált, akkor $\mathcal{F}(\bar{\Delta})$ lesz feltétlenül rétegező részkategória. A duális fogalom az ún. *korétegező részkategória* fogalma. Megmutatható, hogy egy rétegező részkategóriának az ortogonálisa korétegező, és azt is igazoltuk, hogy minden CPS-rétegezett algebra esetén létezik rétegező, ill. korétegező részkategóriáknak egy olyan párja, melyek egymás merőlegesei. Könnyen látható, hogy pl. kváziöröklődő algebrák esetén az egyetlen ilyen pár az $\big(\mathcal{F}(\Delta), \mathcal{F}(\nabla)\big)$ pár, de példát mutatunk olyan CPS-rétegezett algebrára is, melyhez végtelen sok ilyen részkategória-pár létezik. Mivel egy rétegező részkategória létezése egyúttal a projektív modulusok egyfajta filtrálását vonja maga után, úgy tekinthetünk ezekre a részkategóriákra mint egy struktúraelmélet építőköveire. – A cikk eredményeiről Lukács Erzsébet számolt be a Rényi Alfréd Matematikai Kutatóintézet algebra szemináriumán 2011 júniusában.

[4] *Ágoston, I., Lukács, E.*: Construction of CPS-stratified algebras *(közlésre benyújtva)*, 14 oldal

Az előző két cikkünkre alapozva ebben a munkánkban a CPS-rétegezett algebrák konstrukciójával foglalkozunk. Noha a [2]-es cikkben egyes eredmények a CPS-rétegezett algebrákra is alkalmazhatók voltak, az ottani konstrukció alapvetően épített a szereplő bimodulusok szeleteinek relatív projektivitására (azaz arra, hogy Δ-filtráltak). Ebben a munkánkban a [3]-as cikkben kidolgozott rétegező részkategóriák fogalmának segítségével építjük föl azt az eljárást, mellyel az összes CPS-rétegezett algebrát megkaphatjuk. Itt is a centralizátoralgebrák sorozatára koncentrálunk, s először azt mutatjuk meg, hogyan kapcsolódnak az eredeti $(A, \mathbf{e})$ algebra fölötti rétegező párok a C_i centralizátorok fölötti párokhoz. Ebben a folyamatban döntő szerepet játszik három funktorcsalád: a $\text{Hom}_A(\varepsilon_i A, -)$, a $- \underset{\varepsilon_i A \varepsilon_i}{\otimes} \varepsilon_i A$ és a $\text{Hom}_{C_i}(A\varepsilon_i, -)$ (ezek a kváziöröklődő algebrák elméletéből ismert ún. *recollement* funktorai). Ezekre az eredményekre támaszkodva megmutatjuk, hogyan jellemezhető egy CPS-rétegezett algebra az ε_i idempotensek Peirce-fölbontásának komponensei segítségével, s megadjuk az eljárást az összes CPS-rétegezett algebra megkonstruálására.

[5] *Volodymyr Mazorchuk*: Koszul duality for stratified algebras I. Balanced quasi-hereditary algebras, *Manuscripta Mathematica* **131** (2010), 1–10

[6] *Volodymyr Mazorchuk*: Koszul duality for stratified algebras II. Properly stratified algebras, *J. Australian Math. Soc.* **89** (2010), 23–49.

A fönti cikkekben azoknak az eredményeknek a továbbfejlesztése szerepel, melyeket Mazorchuk Ovsienkóval együttműködve kapott. A továbblépés egyik legfontosabb eszköze az ún. standard Koszul algebrák fogalma és eszköztára. Ezt a fogalmat Ágoston, Dlab és Lukács vezették be 2003-ban, majd 2005-ben általánosították standardul rétegezett algebrákra. A szerző a cikkben többek között megmutatja pl., hogy a standard Koszul kváziöröklődő algebrák osztályán a Ringel-, illetve a Koszul-duális képzésének a sorrendje fölcserélhető. A cikk második felében alkalmas feltételek mellett az első rész eredményeinek az általánosítása olvasható standardul rétegezett algebrákra. – A cikkek alapvetően Mazorchuk eredményeit tartalmazzák, melyek az OTKA-támogatással megvalósult budapesti tartózkodása alatt születtek az Ágoston Istvánnal és Lukács Erzsébettel folytatott beszélgetések nyomán. A cikkek egyszerzősek, s a támogatás ténye csak köszönetnyilvánítás formájában szerepel a cikkben, maga a pályázat sorszáma nem.

Végezetül pár szót szólnánk a „be nem fejezett” dolgokról is.

A) Az [1]-es cikk eredményeinek a továbbfejlesztéséhez, valamint algebrák reprezentációdimenziójának a kiszámolásához szükségünk volt egyes modulusok endomorfizmusalgebrájára. Az esetleges további vizsgálódások megkönnyítésére Lukács Erzsébet egy GAP-programot írt, melyet modulusok endomorfizmusalgebrájának kiszámítására, szerkezetének kiderítésére szeretnénk használni. A program működik, de a fejlesztése abbamaradt, amikor a kutatásunk – eredmények híján – a témáról más írányba terelődött. Ettől függetlenül a program hasznos segédeszköze lehet a további munkánknak.

B) Kutatási területünk másik témája a finitisztikus dimenzió témaköre. Ebben a témában váratlanul igen érdekes eredmények születtek világszerte. A 2000-es évek eleje óta nőtt meg ismét az érdeklődés az algebrák reprezentációdimenziója iránt. (Ezt a fogalmat még Auslander vezette be a múlt század 70-es éveiben.) Igusa és Todorov megmutatták, hogy ha egy algebra reprezentációdimenziója legföljebb 3, akkor a finitisztikus dimenziója véges. Iyama, szintén a 2000-es évek elején, bebizonyította (fölhasználva a kváziöröklődő algebrák fogalmát), hogy tetszőleges véges dimenziós algebra reprezentációdimenziója véges, és sokáig nem volt ismeretes olyan példa, melynek a reprezentációdimenziója nagyobb 3-nál. Sajnos, Rouquier 2004-ben mutatott példát olyan algebrára, amelynek a reprezentációdimenziója nagyobb, mint 3, s a konstrukciót Oppermann később általánosította. 2008 tavaszán jelent meg Zhang és Zhang cikke, melyben megmutatták, hogy amennyiben egy A algebra reprezentációdimenziója legföljebb 3, akkor minden olyan algebra finitisztikus dimenziója is véges, amely előáll mint egy A fölötti projektív modulus endomorfizmuslagebrája. Közismert, hogy minden véges dimenziós algebra beágyazható ilyen módon egy kváziöröklődő algebrába, tehát fölmerült kérdésként, hogy a kváziöröklődő algebrák reprezentációdimenziójáról mit lehet mondani. 2008 nyarán kanadai tartózkodásunk idején Dlabbal közösen észrevettük, hogy az az eljárás mellyel Opperman konstruált akármekkora reprezentációdimenziójú algebrákat, ilyen típusú kváziöröklődő algebrákat is szolgáltat. A finitisztikus dimenzióval kapcsolatos sejtés általános bizonyítása tehát nem oldható meg egyszerűen, a Zhang–Zhang-cikk közvetlen fölhasználásával. Viszont azt sejtjük, hogy azoknak a kváziöröklődő algebráknak, melyeknek a Δ- és ∇-típusa véges, a reprezentációdimenziója valóban 3, s ezáltal sok új algebráról lehet igazolni, hogy a finitisztikus dimenziója véges. Az állítás igazolására (részben a programunk segíségével) kiszámolt sok-sok példa alátámasztani látszik a sejtésünket. A kiegészítő lépéshez azt próbáltuk igazolni, hogy minden kváziöröklődő algebra beágyazható egy fenti típusúba; ez a próbálkozásunk egyelőre akadályokba ütközött.

C) Közvetlen és megfogható célunk a finitisztikus dimenzió végességének bizonyítása CPS-rétegezett algebrákra. A [3]-as cikkben szereplő rétegező részkategóriák segítségével sikerült a kérdést átfogalmaznunk és állítások hierarchiáját fölállítanunk a kérdéskörben. Az eszközök ilyen jellegű használata azonban meglehetősen új; döntő eredmény nem született a témakörben.

APPROXIMATIONS OF ALGEBRAS BY STANDARDLY STRATIFIED ALGEBRAS

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Abstract. The paper has its origin in an attempt to answer the following question: Given an arbitrary finite dimensional associative K-algebra A, does there exist a quasi-hereditary algebra B such that the subcategories of all A-modules and all B-modules, filtered by the corresponding standard modules are equivalent. Such an algebra will be called a *quasi-hereditary approximation* of A. The question is answered in the appropriate language of standardly stratified algebras: For any K-algebra A, there is a uniquely defined basic algebra $B = \Sigma(A)$ such that B_B is Δ-filtered and the subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ of all Δ-filtered modules are equivalent; similarly there is a uniquely defined basic algebra $C = \Omega(A)$ such that C_C is $\bar{\Delta}$-filtered and the subcategories $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ of all $\bar{\Delta}$-filtered modules are equivalent. These subcategories play a fundamental role in the theory of stratified algebras. Since, in general, it is difficult to localize these subcategories in the category of all A-modules, the construction of $\Sigma(A)$ and $\Omega(A)$ often helps to describe them explicitly. By applying consecutively the operators Σ and Ω for an algebra, we get a sequence of standardly stratified algebras which, after a finite number of steps, stabilizes in a properly stratified algebra. Thus, all standardly stratified algebras are partitioned into (generally infinite) trees, indexed by properly stratified algebras (as their roots).

1. Introduction

Let $(A, \mathbf{e})$ be a finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \ldots, e_n)$ of primitive orthogonal idempotents. Let $\Delta_A = (\Delta(1), \Delta(2), \ldots, \Delta(n))$ and $\bar{\Delta}_A = (\bar{\Delta}(1), \bar{\Delta}(2), \ldots, \bar{\Delta}(n))$ be the respective sequences of (right) standard and properly standard A-modules. Hence, we have the well-defined (full) subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of all Δ_A-filtered and $\bar{\Delta}_A$-filtered A-modules, of the category mod-A of all finite dimensional (right) A-modules, respectively.

The concept of standardly stratified algebra (i. e. of Δ- and of $\bar{\Delta}$-filtered algebra) has its origin in the concept of a quasi-hereditary algebra introduced by Cline–Parshall–Scott [CPS] in order to deal with highest weight categories as they arise in the representation theory of semisimple complex Lie algebras and algebraic

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groups. The subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of mod-A of all Δ- and $\bar{\Delta}$-filtered modules of such algebras play a fundamental role in the theory.

In [DR] Dlab and Ringel established a simple characterization of the category $\mathcal{F}(\Delta_A)$ of a quasi-hereditary algebra in terms of a "standardizable" set of an abelian K-category. Their method, consisting of presenting the quasi-hereditary algebra as the endomorphism algebra of the direct sum of the relevant indecomposable Ext-projective objects, has been reformulated and applied in a number of papers (e.g. [ES], [MMS1], [MMS2]).

Note that one of the corollaries of their result is the following statement: *Given an arbitrary algebra whose standard and proper standard modules coincide, there is a unique basic quasi-hereditary algebra A_q such that $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_{A_q})$ are equivalent via an exact functor.*

Here and throughout the paper we shall assume that the equivalence functors between $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ are exact, meaning that sequences of Delta-filtered modules which are short exact in mod-A or mod-B are mapped into short exact sequences in the other module category.

In the present paper we are going to use this method to extend this result to standardly stratified algebras (Theorem 2.2 and 2.3) and to investigate two equivalences $\overset{\Delta}{\approx}$ and $\overset{\bar{\Delta}}{\approx}$ in the class of all algebras $(A, \mathbf{e})$: we shall say that $(A, \mathbf{e}) \overset{\Delta}{\approx} (A', \mathbf{e}')$ if and only if $\mathcal{F}(\Delta_A) \approx \mathcal{F}(\Delta_{A'})$ and $(A, \mathbf{e}) \overset{\bar{\Delta}}{\approx} (A', \mathbf{e}')$ if and only if $\mathcal{F}(\bar{\Delta}_A) \approx \mathcal{F}(\bar{\Delta}_{A'})$, the equivalence in both cases being induced by an exact functor. The respective equivalence classes are, up to fully described exceptions, infinite (cf. Theorem 3.3. and 3.5). The main point is the fact that every $\overset{\Delta}{\approx}$-equivalence class is represented by a unique basic Δ-filtered algebra and every $\overset{\bar{\Delta}}{\approx}$-equivalence class by a unique basic $\bar{\Delta}$-filtered algebra (cf. Theorem 2.2 and Theorem 2.3).

This process allows us to define two operators Σ and Ω on the class of all algebras $(A, \mathbf{e})$ with a given ordering on the simple types. The range of these operators will be the union of the class $\mathcal{A}(\Delta)$ of all basic Δ-filtered algebras and the class $\mathcal{A}(\bar{\Delta})$ of all basic $\bar{\Delta}$-filtered algebras. Recall that for a basic algebra $(A, \mathbf{e}) \in \mathcal{A}(\Delta)$ means that the regular representation A_A belongs to $\mathcal{F}(\Delta_A)$ and $(A, \mathbf{e}) \in \mathcal{A}(\bar{\Delta})$ means that $A_A \in \mathcal{F}(\bar{\Delta})$. Thus the class $\mathcal{A}(\Delta) \cap \mathcal{A}(\bar{\Delta})$ consists of all properly stratified algebras in the sense of [D2].

We define $\Sigma(A)$ as the unique algebra such that $\Sigma(A) \in \mathcal{A}(\Delta)$ and $A \overset{\Delta}{\approx} \Sigma(A)$. Similarly we define $\Omega(A)$ by $\Omega(A) \in \mathcal{A}(\bar{\Delta})$ and $A \overset{\bar{\Delta}}{\approx} \Omega(A)$. Note that Σ acts as the identity operator on $\mathcal{A}(\Delta)$ while Ω acts as the identity operator on $\mathcal{A}(\bar{\Delta})$. We shall investigate the action of the operators Σ and Ω, mostly on $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$.

In particular, we shall show that for every algebra A with n (non-isomorphic) simple modules

$$(\Omega\Sigma)^{n-1}(A) = \Sigma(\Omega\Sigma)^{n-1}(A)$$

(see Theorem 4.1). Defining a partial order $\preceq$ on $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$ by taking $A' \preceq A$ if and only if A' can be obtained from A by successive applications of the operators Σ and Ω, the class $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$ becomes a (disjoint) union of rooted trees whose roots are in one-to-one correspondence with the properly stratified algebras. In other words, the orbits of the action of the semigroup generated by the operators Σ and Ω carry a natural tree structure and they are indexed by properly stratified algebras.

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2. Δ and $\bar{\Delta}$ equivalence of algebras

Throughout the paper we shall assume that A is a finite dimensional basic algebra over a field K. We shall fix in A a complete set of primitive orthogonal idempotents: $\mathbf{e} = (e_1, \dots, e_n)$ such that $1 = e_1 + \cdots + e_n$, together with its ordering inherited from the natural ordering of the index set. The indecomposable projective (right) modules will be denoted by $P(i) \simeq e_i A$, and the corresponding simple tops by $S(i) = P(i)/\operatorname{rad} P(i)$, while the *standard modules* (with respect to the given order) are $\Delta(i) = e_i A \big/ e_i A(e_{i+1} + \cdots + e_n)A$ and the *proper standard modules* are $\bar{\Delta}(i) = e_i A \big/ e_i \operatorname{rad} A(e_i + \cdots + e_n)A$ for $1 \le i \le n$. Thus the standard module $\Delta(i)$ is the largest quotient of $P(i)$ such that the composition multipicity $[\Delta(i) : S(j)]$ is 0 for $j > i$, while $\bar{\Delta}(i)$ is the largest quotient of $\Delta(i)$ such that $[\bar{\Delta}(i) : S(i)] = 1$.

Recall that in some of the earlier papers $(A, \mathbf{e})$ is said to be *standardly stratified* if the right regular module A_A belongs to $\mathcal{F}(\Delta_A)$ while in others it is said to be standardly stratified if $A_A \in \mathcal{F}(\bar{\Delta}_A)$. Let us reiterate that $\mathcal{F}(\Delta_A)$ (or $\mathcal{F}(\bar{\Delta}_A)$) is the full subcategory of mod-A consisting of modules X with a filtration $X = X_0 \supseteq X_1 \supseteq \dots \supseteq X_\ell \supseteq X_{\ell+1} = 0$ such that for every $0 \le j \le \ell$ the quotient $X_j/X_{j+1} \simeq \Delta(i)$ (or $X_j/X_{j+1} \simeq \bar{\Delta}(i)$) for some $1 \le i \le n$. By a result of [D1] $A_A \in \mathcal{F}(\bar{\Delta}_A)$ if and only if $A^{opp}_{A^{opp}} \in \mathcal{F}(\Delta_{A^{opp}})$. In this spirit, in order to streamline our formulations, we shall use throughout the paper the terminology of Δ-*filtered algebras* (i. e. when $A_A \in \mathcal{F}(\Delta_A)$) and $\bar{\Delta}$-*filtered algebras* (i. e. when $A_A \in \mathcal{F}(\bar{\Delta}_A)$). Those algebras that are either Δ-filtered or $\bar{\Delta}$-filtered will be then called *standardly stratified*. We believe that this terminology is more appropriate and hope that it will be generally accepted.

The algebra $(A, \mathbf{e})$ is *quasi-hereditary* if and only if it is Δ-filtered and $\Delta(i) = \bar{\Delta}(i)$ for all $1 \le i \le n$. Note that quasi-hereditary algebras are those Δ-filtered algebras which have finite global dimension. For elementary properties of standard modules, quasi-hereditary algebras and standardly stratified algebras we refer to [DR], [ADL] and [CPS].

Theorem 2 of [DR] provides a full characterization of the category $\mathcal{F}(\Delta_A)$ for a quasi-hereditary algebra A by listing some characterizing homological properties of the standard modules. This characterization also leads to an explicit construction: given a subcategory $\mathcal{C}$ of modules satisfying these requirements we can construct a unique quasi-hereditary algebra $(A, \mathbf{e})$ such that its $\mathcal{F}(\Delta_A)$ is equivalent to $\mathcal{C}$.

It turns out that by making several adjustments and by taking care of some technicalities, we can establish a similar characterization in the case of standardly stratified algebras (see Proposition 2.1). In fact, such a generalization can be found also in the paper [ES] (although with slightly different emphasis and not explicitly referring to the corresponding 'standardization theorem' of [DR]). As a consequence, given an algebra A, there is a uniquely defined representative in the class of all basic Δ-filtered algebras B whose categories $\mathcal{F}(\Delta_B)$ are equivalent to $\mathcal{F}(\Delta_A)$ (Theorem 2.2). In a similar spirit, we can establish the existence of a uniquely defined representative in the class of all basic $\bar{\Delta}$-filtered algebras C whose categories $\mathcal{F}(\bar{\Delta}_C)$ are equivalent to $\mathcal{F}(\bar{\Delta}_A)$ for a given algebra A (Theorem 2.3).

Let us recall here the above mentioned characterization of the category $\mathcal{F}(\Delta)$ over a quasi-hereditary algebra (cf. Theorem 2 of [DR]). Given a subcategory $\mathcal{C}$ of a module category mod-A, this subcategory $\mathcal{C}$ is equivalent to $\mathcal{F}(\Delta_B)$ for some quasi-hereditary algebra $(B, \mathbf{e})$ if and only if $\mathcal{C} = \mathcal{F}(\Theta)$, for a finite set of indecomposable objects $\Theta = \{\, \Theta(i) \in \mathcal{C} \mid 1 \leq i \leq n \,\}$ satisfying the following conditions:

(1) $\mathrm{Hom}_A(\Theta(i), \Theta(j)) = 0$ for $1 \leq j < i \leq n$;
(2) $\mathrm{Ext}^1_A(\Theta(i), \Theta(j)) = 0$ for $1 \leq j < i \leq n$;
(3) $\mathrm{Ext}^1_A(\Theta(i), \Theta(i)) = 0$ for $1 \leq i \leq n$;
(4) $\mathrm{Hom}_A(\Theta(i), \Theta(i))$ is a division algebra for $1 \leq i \leq n$.

Note that the indecomposability of the objects in Θ actually follows from condition (4). However we prefer assuming indecomposability in our formulation since for characterizing Δ-filtered modules of Δ-filtered algebras we just omit the condition (4). In [DR] the elements of Θ are called *standardizable objects* of $\mathcal{C}$. Let us note here that standardizable objects may be identified within the category as the only objects which do not admit a non-trivial filtration within this category.

It is a well-known fact that standard modules over a quasi-hereditary algebra satisfy these conitions. To prove the sufficiency of these conditions one can show first that there are enough Ext-projective objects in the category $\mathcal{C}$. In fact, there are precisely n indecomposable (non-isomorphic) Ext-projective modules. Denoting by M their direct sum, $B = \mathrm{End}_A(M)$ is basic quasi-hereditary algebra and $\mathrm{Hom}_A(M, -)$ defines a categorical equivalence between $\mathcal{C} = \mathcal{F}(\Theta)$ and $\mathcal{F}(\Delta_B)$. (Let us point out that the endomorphisms of right A-modules will be written from the left.) Since for a quasi-hereditary algebra $\mathcal{F}(\Delta)$ contains the projective modules (and they can be identified as the Ext-projective objects of the category), the algebra itself is uniquely determined by $\mathcal{F}(\Delta)$ as the endomorphism algebra of the direct sum of the indecomposable Ext-projective objects. (Note that in [ES], using a dual approach and dealing with Ext-injective objects instead of Ext-projectives such systems, consisting of standardizable objects and the indecomposable Ext-injectives were called *stratifying systems*.)

The differences between quasi-hereditary algebras and Δ-filtered algebras stem from the fact that standard modules of Δ-filtered algebras are not necessarily Schurian, i. e. condition (4) above is not, in general, satisifed. If we retain the remaining conditions, we get a characterization of $\mathcal{F}(\Delta_A)$ for Δ-filtered algebras.

Proposition 2.1. *Let $\mathcal{C}$ be a full subcategory of* mod-A *of an arbitrary finite dimensional algebra. Then $\mathcal{C}$ is equivalent to $\mathcal{F}(\Delta_B)$ of a Δ-filtered algebra $(B, \mathbf{e})$ via an exact functor if and only if $\mathcal{C} = \mathcal{F}(\Theta)$ for a finite set of indecomposable objects $\Theta = \{\, \Theta(i) \in \mathcal{C} \mid 1 \leq i \leq n \,\}$ satisfying the conditions* (1), (2) *and* (3) *above. Moreover, the algebra B is unique up to Morita equivalence.*

Proof. For the proof, we refer to Theorem 2 of [DR]. The only major difference is that in the recursive construction of the Ext-projective objects $P_\Theta(i)$, the resulting module does not have to be indecomposable, but it will have a unique indecomposable direct summand containing $\Theta(i)$ in its top. Note that, in general, the Ext-projective modules will not be local. (See Example 2.9 at the end of this section). □

It is easy to see that the set of standard modules of any algebra A satisfies the above conditions (1)-(3). Thus an immediate consequence of Proposition 2.1 is the following theorem.

THEOREM 2.2. *Let $(A, \mathbf{e})$ be a finite dimensional algebra. Then there exists a unique basic Δ-filtered algebra $(B, \mathbf{f})$ such that the categories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ are equivalent via an exact functor. In this case the number of isomorphism types of simple A-modules and simple B-modules is the same.*

Unlike standard modules, proper standard modules are Schurian. Thus, they satisfy the condition (4). On the other hand, in general, proper standard modules have self-extensions, i. e. they fail to satisfy (3). However, we can formulate a statement parallel to Theorem 2.2.

THEOREM 2.3. *Let $(A, \mathbf{e})$ be a finite dimensional algebra. Then there exists a unique basic $\bar{\Delta}$-filtered algebra $(C, \mathbf{g})$ such that the categories $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ are equivalent via an exact functor. In this case the number of isomorphism types of simple A-modules and simple C-modules is the same.*

Proof. Let us follow the line of proof of Theorem 2 in [DR], by constructing enough Ext-projective objects in $\mathcal{F}(\bar{\Delta}_A)$, namely n indecomposable modules $N(i)$, $1 \le i \le n$, such that:

- (i) $N(i) \in \mathcal{F}(\bar{\Delta}(i), \bar{\Delta}(i+1), \dots, \bar{\Delta}(n))$;
- (ii) there exists an epimorphism $N(i) \to \bar{\Delta}(i)$ and
- (iii) $N(i)$ is Ext-projective in $\mathcal{F}(\bar{\Delta}_A)$, i. e. $\mathrm{Ext}^1_A(N(i), \bar{\Delta}(\ell)) = 0$ for all $1 \le \ell \le n$.

The modules $N(i)$ will be defined recursively, step by step, constructing a sequence of A-modules $Q(i,j)$, $i \le j \le n$, such that each $Q(i,j)$ satisfies the following conditions:

- $(i)'$ $Q(i,j) \in \mathcal{F}(\bar{\Delta}(i), \bar{\Delta}(i+1), \dots \bar{\Delta}(j))$;
- $(ii)'$ there exists an epimorphism $Q(i,j) \to \bar{\Delta}(i)$;
- $(iii)'$ $\mathrm{Ext}^1_A(Q(i,j), \bar{\Delta}(\ell)) = 0$ for $1 \le \ell \le j$.

Obviously, $N(i) = Q(i,n)$ will then satisfy the conditions (i), (ii) and (iii).

Let us start the construction by defining $Q(i,i)$ to be the maximal quotient of $\Delta(i)$ belonging to $\mathcal{F}(\bar{\Delta}(i))$. Due to the fact that $\mathrm{Ext}^1_A(\bar{\Delta}(i), \bar{\Delta}(\ell)) = 0$ for all $\ell < i$, only the condition $\mathrm{Ext}^1_A(Q(i,i), \bar{\Delta}(i)) = 0$ requires a proof. Applying, for $1 \le \ell \le i-1$, the functor $\mathrm{Hom}_A(-, S(\ell))$ to the exact sequence

$$0 \to Z \to \Delta(i) \to Q(i,i) \to 0 \tag{2.3.1}$$

we see that $\mathrm{Hom}_A(Z, S(\ell))$=0 and thus, due to the maximality of $Q(i,i)$, we get that $\mathrm{Hom}(Z, \bar{\Delta}(i))) = 0$. Consequently, applying $\mathrm{Hom}_A(-, \bar{\Delta}(i))$ to (2.3.1), we conclude that $\mathrm{Ext}^1_A(Q(i,i), \bar{\Delta}(i)) = 0$, as required.

Proceeding by induction, assume that $Q(i, j-1)$for some $i < j \leq n$ has already been constructed. For convenience we write $Q(i, j-1) = Q$ and consider the universal extension U_1 of Q by $\bar{\Delta}(j)$:

$$0 \to X_1 = \bigoplus_{d_1} \bar{\Delta}(j) \to U_1 \to Q \to 0. \tag{2.3.2}$$

Here $d_1 = \dim_{D_j} \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$, where $D_j = \mathrm{End}_A(\bar{\Delta}(j))$. (The universality of the extension means that the pushout sequences along the projection maps $X_1 \to \bar{\Delta}(j)$ form a basis for $\mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$.) Clearly, in addition to the conditions $(i)'$ and $(ii)'$, U_1 satisfies, by recursion, $\mathrm{Ext}^1_A(U_1, \bar{\Delta}(\ell)) = 0$ for all $1 \leq \ell \leq j-1$. In general, however, $\mathrm{Ext}^1_A(U_1, \bar{\Delta}(j)) \neq 0$; denote its D_j-dimension by d_2 and construct the universal extension U_2 of $\bar{\Delta}(j)$ by U_1:

$$0 \to X_2 = \bigoplus_{d_2} \bar{\Delta}(j) \to U_2 \to U_1 \to 0.$$

This sequence yields the following derived exact sequence:

$$0 \to \overline{X}_2 \to U_2 \to Q \to 0,$$

where $\overline{X}_2 \in \mathcal{F}(\bar{\Delta}(j))$ is an extension of X_2 by X_1. If $\mathrm{Ext}^1_A(U_2, \bar{\Delta}(j)) \neq 0$, we continue this process. In t steps we get – again by means of constructing the universal extensions

$$0 \to X_t \to U_t \to U_{t-1} \to 0 \tag{2.3.3}$$

of $\bar{\Delta}(j)$ by U_{t-1} – the corrseponding sequence:

$$0 \to \overline{X}_t \to U_t \to Q \to 0.$$

Note that, in each step of this procedure, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X_t & \to & \overline{X}_t & \to & \overline{X}_{t-1} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & X_t & \to & U_t & \to & U_{t-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & Q & = & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.3.4}$$

Here, by recursion $(i)'$, $(ii)'$ and $\mathrm{Ext}^1_A(U_t, \bar{\Delta}(\ell)) = 0$ hold for $1 \leq \ell \leq j-1$.

We are going to show that after a finite number of steps, the process of constructing the universal extensions will stabilize, i.e. that $\mathrm{Ext}^1_A(U_{t_0}, \bar{\Delta}(j)) = 0$ for some t_0.

Indeed, we can show by induction that $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$. The statement clearly holds for $\overline{X}_1 = X_1$ by the universality of the extension (2.3.2). For arbitrary $t > 1$ we can apply the functor $\mathrm{Hom}_A(-, \bar{\Delta}(j))$ to the diagram in (2.3.4) to get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & & & \uparrow \\
0 & \to & \mathrm{Hom}(\overline{X}_{t-1}, \bar{\Delta}(j)) & \xrightarrow{\alpha} & \mathrm{Hom}(\overline{X}_t, \bar{\Delta}(j)) & \to & \mathrm{Hom}(X_t, \bar{\Delta}(j)) & \xrightarrow{\beta} & \mathrm{Ext}^1(\overline{X}_{t-1}, \bar{\Delta}(j)) \\
 & & \uparrow & & \uparrow & & \| & & \uparrow \delta \\
0 & \to & \mathrm{Hom}(U_{t-1}, \bar{\Delta}(j)) & \to & \mathrm{Hom}(U_t, \bar{\Delta}(j)) & \to & \mathrm{Hom}(X_t, \bar{\Delta}(j)) & \xrightarrow{\gamma} & \mathrm{Ext}^1(U_{t-1}, \bar{\Delta}(j)) \\
 & & & & & & & & \uparrow \\
 & & & & & & & & \mathrm{Ext}^1(Q, \bar{\Delta}(j)) \\
 & & & & & & & & \uparrow \varphi \\
 & & & & & & & & \mathrm{Hom}(\overline{X}_{t-1}, \bar{\Delta}(j)) \\
 & & & & & & & & \uparrow
\end{array}$$

Here γ is an isomorphism since (2.3.3) is a universal extension, furthermore φ is an isomorphism by induction. Thus we get that δ is injective and so is β. This implies that α is an isomorphism which, in view of the induction hypothesis, yields the statement.

Observe that the isomorphism $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$ implies that $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Hom}_A(X_1, \bar{\Delta}(j))$ for each t. Note also that $\overline{X}_t$ is an extension of a module in $\mathcal{F}(\bar{\Delta}(j))$ by $X_1 = \bigoplus_{d_1} \bar{\Delta}(j)$. Hence, the previous isomorphism implies that $\overline{X}_t$ is a homomorphic image of $\oplus_{d_1} \Delta(j)$, and thus its dimension is bounded. Since $\dim X_1 < \dim \overline{X}_2 < \ldots < \dim \overline{X}_t$ we get that the sequence of the universal extensions must, after a finite number of steps, stabilize, i. e. $\mathrm{Ext}^1_A(U_{t_0}, \bar{\Delta}(j)) = 0$ for some t_0. We set $Q(i,j) = U_{t_0}$.

Thus, using this recursion we have constructed the Ext-projective objects $N(i)$ in $\mathcal{F}(\bar{\Delta}_A)$. To show that the modules $N(i)$ are indecomposable, we need the following lemma.

Lemma 2.4. *$\mathcal{F}(\bar{\Delta}_A)$ is closed under taking direct summands.*

Proof. Let M be an element of $\mathcal{F}(\bar{\Delta}_A)$, and suppose that $M = U \oplus V$. Since $\mathrm{Ext}_A(\bar{\Delta}_A(n), \bar{\Delta}_A(i)) = 0$ for $i \neq n$, $Me_nA = Ue_nA \oplus Ve_nA \in \mathcal{F}(\bar{\Delta}_A(n))$ and $M/Me_nA \simeq U/Ue_nA \oplus V/Ve_nA \in \mathcal{F}(\bar{\Delta}_A(1), \ldots, \bar{\Delta}_A(n-1))$. So it suffices to prove the statement for $Me_nA \in \mathcal{F}(\bar{\Delta}_A(n))$, and apply induction on the factor module. For simplicity assume that $M = Me_nA$. Then, $M \in \mathcal{F}(\bar{\Delta}_A(n))$ implies that $0 \neq \mathrm{Hom}_A(M, \bar{\Delta}_A(n)) = \mathrm{Hom}_A(U, \bar{\Delta}_A(n)) \oplus \mathrm{Hom}_A(V, \bar{\Delta}_A(n))$ so one of the summands, say, $\mathrm{Hom}_A(U, \bar{\Delta}_A(n))$ is nontrivial. But the top of U, and thus the top of any nonzero homomorphic image of U is filtered by $S(n)$, so a nonzero homomorphism from U to $\bar{\Delta}_A(n)$ must be an epimorphism. This means that $M/(U_1 \oplus V)$ is isomorphic to $\bar{\Delta}_A(n)$ for some $U_1 \leq U$, and thus $U_1 \oplus V$ is $\bar{\Delta}_A(n)$-filtered because $\mathcal{F}(\bar{\Delta}_A(n))$ is closed under kernels of epimorphisms (cf. [ADL]). Recursively we can prove that both U and V are $\bar{\Delta}_A(n)$-filtered. □

Now we can prove the indecomposability of $N(i)$, by showing that in the recursive construction of $N(i)$, every module $Q(i,j)$ is indecomposable. The initial module $Q(i,i)$ is a quotient of the local module $\Delta(i)$, hence it is indecomposable. Suppose now that $Q(i,j-1)$ is indecomposable for some $i < j \leq n$. We constructed $Q(i,j)$ as an extension of a $\bar{\Delta}_A(j)$-filtered module X by $Q(i,j-1)$:

$$0 \to X \to Q(i,j) \to Q(i,j-1) \to 0,$$

and we also know that in the long exact sequence

$$\cdots \to \mathrm{Hom}_A(Q(i,j), \bar{\Delta}_A(j)) \xrightarrow{\beta} \mathrm{Hom}_A(X, \bar{\Delta}_A(j)) \xrightarrow{\alpha} \mathrm{Ext}^1_A(Q(i,j-1), \bar{\Delta}_A(j)) \to \cdots$$

the morphism α is an isomorphism. Thus $\beta = 0$.

Now suppose that $Q(i,j) = U \oplus V$ is a proper decomposition of $Q(i,j)$. Since $X = Q(i,j)e_jA = Ue_jA \oplus Ve_jA$, we have $Q(i,j-1) \simeq U/Ue_jA \oplus V/Ve_jA$. The indecomposability of $Q(i,j-1)$ implies that one of the components in the latter decomposition is 0. We may assume that $U \subseteq X$. The previous lemma implies that U is $\bar{\Delta}_A(j)$-filtered. But then an epimorphism from U to $\bar{\Delta}_A(j)$ gives a homomorphism in $\mathrm{Hom}_A(Q(i,j), \bar{\Delta}_A(j))$, which has a nonzero restriction to X. This is a contradiction, since $\beta = 0$.

This proves that each $Q(i,j)$ and thus each $N(i)$ must be indecomposable for $1 \leq i \leq n$.

Put $N = \overset{n}{\underset{i=1}{\oplus}} N(i)$ and $C = \mathrm{End}_A(N)$.

To show that C is a basic $\bar{\Delta}$-filtered algebra and that the functor $\mathrm{Hom}_A(N, -)$ induces an equivalence between $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ we can follow almost word by word the rest of the proof of Theorem 2 in [DR]. This task is left to the reader. □

In view of Theorems 2.2 and 2.3, we can introduce the following definitions.

Definition 2.5. (1) The algebras $(A, \mathbf{e})$ and $(B, \mathbf{f})$ are called Δ-equivalent if the respective full subcategories $\mathcal{F}(\Delta_A) \subseteq$ mod-A and $\mathcal{F}(\Delta_B) \subseteq$ mod-B are equivalent via an exact functor; in this case we write $(A, \mathbf{e}) \overset{\Delta}{\sim} (B, \mathbf{f})$ or simply $A \overset{\Delta}{\sim} B$.

(2) The algebras $(A, \mathbf{e})$ and $(B, \mathbf{f})$ are called $\bar{\Delta}$-equivalent if the respective full subcategories $\mathcal{F}(\bar{\Delta}_A) \subseteq$ mod-A and $\mathcal{F}(\bar{\Delta}_B) \subseteq$ mod-B are equivalent via an exact functor; in this case we write $(A, \mathbf{e}) \overset{\bar{\Delta}}{\sim} (B, \mathbf{f})$ or simply $A \overset{\bar{\Delta}}{\sim} B$.

In this way we get two equivalence relations on the class of all algebras (or rather, on Morita equivalence classes of algebras).

Definition 2.6. For an arbitrary algebra $(A, \mathbf{e})$ we define $\Sigma(A)$ to be the unique algebra satisfying:

(i) $(\Sigma(A), \mathbf{f})$ is Δ-filtered and basic;

(ii) $A \overset{\Delta}{\sim} \Sigma(A)$.

Similarly we define $\Omega(A)$ to be the unique algebra satisfying:

$(i)'$ $(\Omega(A), \mathbf{f})$ is $\bar{\Delta}$-filtered and basic;

$(ii)'$ $A \overset{\bar{\Delta}}{\sim} \Omega(A)$.

Thus, $A \overset{\Delta}{\sim} B$ if and only if $\Sigma(A) \simeq \Sigma(B)$ with the isomorphism preserving the corresponding orderings. In a similar fashion, $A \overset{\bar{\Delta}}{\sim} B$ if and only if $\Omega(A) \simeq \Omega(B)$.

The explicit construction in the proof of Proposition 2.1 and Theorem 2.3 gives us a bound on the dimension of these algebras:

PROPOSITION 2.7. *There exist functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that for any algebra A we have:*

$$\dim \Sigma(A) \leq f(\dim A) \qquad \textit{and} \qquad \dim \Omega(A) \leq g(\dim A).$$

Proof. We will not make any attempt to give an optimal bound: our estimate will be very rough and in most cases far from the best possible bound.

Since $\Sigma(A)$ and $\Omega(A)$ can be obtained as the endomorphism algebras of the direct sum of indecomposable Ext-projective modules in $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\Delta})$, respectively, it is enough to show that there is an upper bound on the dimension of these indecomposable Ext-projective modules, since their number is n, the number of isomorphism types of simple A-modules, and this is not greater than $\dim A = d$.

First we show that for modules of bounded dimension the dimension of their first extension group is also bounded. Let us take two A-modules, X and Y with their dimensions bounded by x and y, respectively. If $0 \to \Omega_1(X) \to P_0 \to X \to 0$ is a projective cover of X, then $\dim \Omega_1(X) \leq \dim P_0 \leq \dim A \cdot \dim X \leq dx$. The long exact sequence $\cdots \to \mathrm{Hom}_A(\Omega_1(X), Y) \to \mathrm{Ext}^1_A(X, Y) \to 0$ yields that $\dim \mathrm{Ext}^1_A(X, Y) \leq \dim \mathrm{Hom}_A(\Omega_1(X), Y) \leq dxy$.

Thus if Z is the universal extension of X with Y, i.e. we have $0 \to Y^k \to Z \to X \to 0$ with $k = \dim \mathrm{Ext}^1_A(X, Y)$, then $\dim Z \leq x + ky \leq x + dxy^2 = x(1 + dy^2)$.

We can apply this estimate to the recursive construction of the indecomposable Ext-projective modules $M_\Delta(i)$ in $\mathcal{F}(\Delta_A)$, to their direct sum M and to $\Sigma(A) = \mathrm{End}_A(M)$. We use the bound $\dim(\Delta(i)) \leq d$ to get:

$$\dim M_\Delta(i) \leq d(1 + d^3)^{n-i} \leq d(1 + d^3)^n$$
$$\dim M \leq nd(1 + d^3)^n$$
$$\dim \Sigma(A) \leq n^2 d^2 (1 + d^3)^{2n}$$

Since the number of simple module types n is cleary not more than d, we get the desired function f.

In the recursive construction of the indecomposble Ext-projective modules $N_{\bar{\Delta}}(i)$ in $\mathcal{F}(\bar{\Delta}_A)$ we have seen that when one of the intermediate modules X is extended by a module filtered by $\bar{\Delta}(j)$-s then the latter module is the homomorphic image of the direct sum of k copies of $\Delta(j)$-s where $k = \dim \mathrm{Ext}^1_A(X, \bar{\Delta}(j)$. Hence we get the earlier recursive estimate for the dimension of the indecomposable Ext-projective objects: $\dim N_{\bar{\Delta}}(i) \leq d(1 + d^3)^n$. Thus we also get the same estimate for $\dim \Omega(A)$ as for $\dim \Sigma(A)$, namely: $\dim \Omega(A) \leq n^2 d^2 (1 + d^3)^{2n}$. This gives the function g. □

At the end of this section, let us give some examples for these constructions.

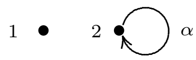
EXAMPLE 2.8. Let us consider the algebra $A = KQ_A/I_A$ whose quiver Q_A and right regular representation are as follows:

$$Q_A\colon \quad 1 \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet\ 2\ ; \qquad I_A = \langle \alpha\beta \rangle\ ; \qquad A_A = {\scriptstyle\begin{matrix}1\\2\end{matrix}} \oplus {\scriptstyle\begin{matrix}2\\1\\2\end{matrix}}\ .$$

Then the direct sum M of the indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is:

$$M_A = 1 \oplus {\scriptstyle\begin{matrix}2\\1\\2\end{matrix}}\ ;$$

hence $\Sigma(A) = \mathrm{End}_A(M) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ is given by:

$Q_{\Sigma(A)}$:  ; $\qquad I_{\Sigma(A)} = \langle \alpha^2 \rangle$;

$${}_{\Sigma(A)}\Sigma(A) = 1 \oplus {\scriptstyle\begin{matrix}2\\2\end{matrix}}\ ; \qquad \Sigma(A)_{\Sigma(A)} = 1 \oplus {\scriptstyle\begin{matrix}2\\2\end{matrix}}\ .$$

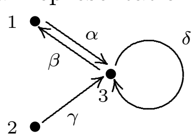
On the other hand, for the Ext-projective object N in $\mathcal{F}(\bar{\Delta}_A)$ we get

$$N_A = 1 \oplus {\scriptstyle\begin{matrix}2\\1\end{matrix}}\ ;$$

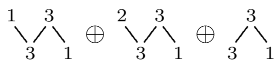
hence $\Omega(A) = \mathrm{End}_A(N) = KQ_{\Omega(A)}$ is given by:

$$Q_{\Omega(A)}\colon \quad 1 \bullet \longleftarrow \bullet\ 2\ ; \qquad {}_{\Omega(A)}\Omega(A) = {\scriptstyle\begin{matrix}1\\2\end{matrix}} \oplus 2\ ; \qquad \Omega(A)_{\Omega(A)} = 1 \oplus {\scriptstyle\begin{matrix}2\\1\end{matrix}}\ .$$

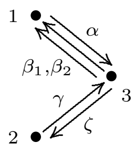
EXAMPLE 2.9. Let us take the algebra $A = KQ_A/I_A$ whose quiver Q_A and right regular representation are as follows:

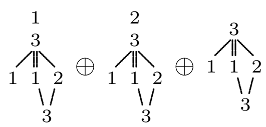
Q_A:  ; $\qquad I_A = \langle \delta^2, \delta\beta, \alpha\beta, \gamma\delta, \beta\alpha \rangle$; $\qquad A_A = {\scriptstyle\begin{matrix}1\\3\\3\end{matrix}} \oplus {\scriptstyle\begin{matrix}2\\3\\1\end{matrix}} \oplus {\scriptstyle\begin{matrix} & 3 & \\ 1 & & 3\end{matrix}}$.

Then the direct sum M of the indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is:

$M_A =$ 

and its endomorphism ring $\Sigma(A) = \mathrm{End}_A(M) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ is given by:

$Q_{\Sigma(A)}$:  ; $\qquad I_{\Sigma(A)} = \langle \beta_1\alpha, \beta_2\alpha - \zeta\gamma, \beta_2\alpha\beta_1, \beta_2\alpha\beta_2, \beta_2\alpha\zeta \rangle$;

$\Sigma(A)_{\Sigma(A)} =$  .

The Ext-projective object N in $\mathcal{F}(\bar{\Delta})$ is given by

$$N_A = 1 \oplus {\scriptstyle\begin{matrix}2\\3\\1\end{matrix}} \oplus {\scriptstyle\begin{matrix}3\\1\end{matrix}}$$

and its endomorphism algebra $\Omega(A) = KQ_{\Omega(A)}$ is as follows:

$$Q_{\Omega(A)}\colon \quad \underset{1}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{2}{\bullet}\ ; \qquad \Omega(A)_{\Omega(A)} = 1 \oplus {\scriptstyle\begin{matrix}2\\3\\1\end{matrix}} \oplus {\scriptstyle\begin{matrix}3\\1\end{matrix}}\ .$$

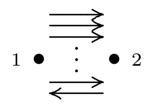
3. The size of equivalence classes

In this section we will look more closely at the equivalence classes with respect to the relations $\overset{\Delta}{\sim}$ and $\overset{\bar{\Delta}}{\sim}$.

It may happen that the categories $\mathcal{F}(\Delta)$ or $\mathcal{F}(\bar{\Delta})$ fully determine the algebra, more precisely the whole module category. For example, when all standard A modules are simple — note that this fact can be recognized within $\mathcal{F}(\Delta_A)$ since this means that the standardizable objects are Schurian and there are no non-trivial homomorphisms between different standardizable objects — then $\mathcal{F}(\Delta_A)$ is the full module category. Thus any algebra Δ-equivalent to A must be Morita equivalent to A. A similar situation arises when the proper standard modules are simple.

In the above situations the corresponding $\overset{\Delta}{\sim}$ or $\overset{\bar{\Delta}}{\sim}$ class has only one (basic) element. On the other hand the following two examples show that some equivalence classes are infinite.

Example 3.1. Let A_k for $k \geq 1$ be the algebras whose quiver and right regular representation are as follows:



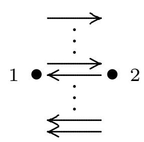
and $(A_k)_{A_k} = \begin{smallmatrix} & 1 & \\ 2 & 2 \cdots & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$

with k arrows heading from 1 to 2. Here the Ext-projective module M in $\mathcal{F}(\Delta_A)$ and its endomorphism algebra $\Sigma(A_k)$ are given by:

$$M_{A_k} = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \;; \qquad {}_{\Sigma(A_k)}\Sigma_{A_k} = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2 \;; \qquad \Sigma(A_k)_{\Sigma_{A_k}} = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \;;$$

thus, $\Sigma(A_k)$ is independent of k, i.e. it is isomorphic for every algebra A_k. Note that $\mathcal{F}(\Delta_{A_k}) = \mathcal{F}(\bar{\Delta}_{A_k})$ and $\mathcal{F}(\Delta_{\Sigma(A_k)}) = \mathcal{F}(\bar{\Delta}_{\Sigma(A_k)})$; hence, $\Omega(A_k) = \Sigma(A_k)$.

Example 3.2. Let us now consider the algebras B_k for $k \geq 1$ whose quivers and right regular representation are as follows:



and $(B_k)_{B_k} = \begin{smallmatrix} & 1 & \\ 2 & 2 \cdots & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & 1 \cdots & 1 \\ & 2 & \end{smallmatrix}$;

here, there are k arrows $\alpha_1, \ldots, \alpha_k$ from 1 to 2 and $k+1$ arrows $\beta_0, \ldots, \beta_k$ from 2 to 1 satisfying the following relations: $\alpha_j \beta_\ell = 0$ for any $1 \leq j \leq k$ and $0 \leq \ell \leq k$ and $\beta_i \alpha_j = 0$ for $i \neq j$ and $\beta_i \alpha_i = \beta_j \alpha_j$ for any $1 \leq i, j \leq k$. Then an easy calculation shows that $\Delta_{B_k}(1)$ and $\Delta_{B_k}(2)$ are Ext-projective in $\mathcal{F}(\Delta_{B_k})$. By taking for M their direct sum, the algebra $\Sigma(B_k) = \mathrm{End}_{B_k}(M)$ does not depend on k and it can be described by the regular representations

$${}_{\Sigma(B_k)}\Sigma(B_k) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \;; \qquad \Sigma(B_k)_{\Sigma(B_k)} = 1 \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix} \;.$$

It turns out that these two extreme cases exhaust all possibilities: apart from one element classes, the equivalence classes are always infinite.

THEOREM 3.3. *Let* $(A, \mathbf{e})$ *be an arbitrary algebra. Then the number of Morita equivalence classes of algebras which are* Δ*-equivalent to* A *is:*
(*i*) *one if all standard modules* $\Delta(i)$ *for* $2 \leq i \leq n$ *are simple;*
(*ii*) *infinite otherwise.*

Proof. The fact that a standard module $\Delta(i)$ is simple is clearly invariant under Δ-equivalence: it means that $\mathrm{Hom}_A(\Delta(j), \Delta(i)) = 0$ for $j \neq i$ and $\Delta(i)$ is Schurian. Furthermore, if all standard modules $\Delta_A(i)$ are simple for $2 \leq i \leq n$ then the algebra A must be Δ-filtered. Since every Δ-equivalence class contains a unique basic Δ-filtered algebra, we are done with case (i).

We have to show now that if at least one of the standard modules $\Delta(i)$ for $i \geq 2$ is not simple then there are infinitely many non-isomorphic basic algebras which are Δ-equivalent to A.

To this end let us first formulate a technical lemma, giving a general framework for the construction of these algebras.

LEMMA 3.4. *Let* U *be an* (A, A)*-bimodule,* $\Phi = \mathrm{Hom}_A(U_A, A_A)$ *and* $X = \mathrm{Tr}_A(U_A)$ *the trace of* U_A *in* A_A *(i. e.* $X = \sum \{ \mathrm{Im}\, \varphi \mid \varphi \in \Phi \}$*). Thus* Φ *and* X *also carry a natural* (A, A)*-bimodule structure. Assume that* $XU = UX = 0$*. Then the* (A, A)*-bimodule*

$$\tilde{A} = A \oplus U \oplus \Phi$$

can be given an associative algebra structure as follows: multiplication by elements of A *is given by the* (A, A)*-bimodule structure;* $U\Phi = UU = \Phi\Phi = 0$*; finally* $\varphi \cdot u = \varphi(u)$ *for* $\varphi \in \Phi$ *and* $u \in U$*. Furthermore* U *is a right ideal of* $\tilde{A}$ *such that*

$$\mathrm{End}(\tilde{A}/U)_{\tilde{A}} \simeq A.$$

Proof. First, the assumption $XU = UX = 0$ implies that $XX = 0$ and $\Phi X = X\Phi = 0$. Using these relations, it is easy to verify that the multiplication

$$(a, u, \varphi) \cdot (a', u', \varphi') = (aa' + \varphi(u'), au' + ua', a\varphi' + \varphi a')$$

is associative.

Clearly, since $(0, u, 0)(a', u', \varphi') = (0, ua', 0)$, U is a right ideal of $\tilde{A}$.

Moreover, every endomorphism of the (right) $\tilde{A}$-module $\tilde{A}/U$ is induced by left multiplication by an element (a_0, u_0, φ_0) of $\tilde{A}$ such that $(a_0, u_0, \varphi_0)U \subseteq U$. As a consequence, in view of $(a_0, u_0, \varphi_0)(0, u, 0) = (\varphi_0(u), a_0 u, 0)$ for all $u \in U$, we have $\varphi_0 = 0$. But then, modulo U,

$$(a_0, u_0, 0)(a, 0, \varphi) = (a_0 a, u_0 a, a_0 \varphi) \sim (a_0 a, 0, a_0 \varphi) = (a_0, 0, 0)(a, 0, \varphi).$$

Thus, $\mathrm{End}_{\tilde{A}}(\tilde{A}/U) \simeq A$.

□

Returning to the proof of Theorem 3.3, we define the (A, A)-bimodule $U = \oplus S^\circ(1) \otimes_K e_iA/e_i \operatorname{rad} A(e_1 + \ldots + e_i)A$ (here $\oplus S^\circ(1)$ is the direct sum of any finite number of copies of the left A-module $S^\circ(1)$). Define $\tilde{A}$ as in Lemma 3.4. We are going to prove that $\Sigma(\tilde{A}) = A$.

Note that the conditions that $\Delta(i)$ is not simple but $\Delta(j)$ is simple for all $j > i$ imply that $e_i \operatorname{rad} A(e_1 + \ldots + e_i) \neq 0$ and $e_j \operatorname{rad} Ae_k = 0$ for all $j > i$ and $k \leq j$.

First we verify that the relations in the construction of $\tilde{A}$ in Lemma 3.4 are satisfied, i.e. that $X := Tr_A(U) \subseteq \operatorname{rad} A$ and, as a consequence, $XU = UX = 0$. Indeed, the definition of U yields $U = Ue_iA$ and $U \operatorname{rad} A(e_1 + \ldots + e_i) = 0$, so $X = Xe_iA$ and $X \operatorname{rad} A(e_1 + \ldots + e_i) = 0$. Now for $j \neq i$, $e_jX = e_jXe_iA \subseteq e_i \operatorname{rad} A$. To prove that e_iX is also in $\operatorname{rad} A$, we first observe that $e_iX \operatorname{rad} A(e_1 + \ldots + e_i) = 0$; but $e_i \operatorname{rad} A(e_1 + \ldots + e_i) \neq 0$, so $e_i \notin e_iX$. Thus e_iX is a proper submodule of the local module e_iA, hence $e_iX \subseteq e_i \operatorname{rad} A$. This finishes the proof of the first statement. The rest follows from $XU \subseteq (\operatorname{rad} A)U = 0$ and $UX = UXe_iA \subseteq U(\operatorname{rad} A)e_iA = 0$.

Second, let us show that $\tilde{A}/U$ is Δ-filtered. Observe that the condition that $\Delta(j)$ are simple for $j > i$ means that $e_j \operatorname{rad} Ae_k = 0$ for $j > i$ and $k \leq j$, and that this property is inherited by the algebra $\tilde{A}$: $(e_{i+1} + \ldots + e_n)U = 0$ and $(e_{i+1} + \ldots + e_n)\Phi U = (e_{i+1} + \ldots + e_n)\Phi Ue_iA \subseteq (e_{i+1} + \ldots + e_n)Ae_iA = 0$ implies $(e_{i+1} + \ldots + e_n)\Phi = 0$, and thus $e_j \operatorname{rad} \tilde{A}e_k \subseteq e_j \operatorname{rad} Ae_k = 0$ for $j > i$ and $k \leq j$.

It is easy to check that $\tilde{A}/\tilde{A}(e_{i+1} + \ldots + e_n)\tilde{A}$ is isomorphic to the algebra that we obtain by the same construction from $A/A(e_{i+1} + \ldots + e_n)A$. So it is sufficient to prove that $\tilde{A}/U$ is Δ-filtered in the case when $i = n$.

In this case, since A is Δ-filtered, Ae_nA is $\Delta(n)$-filtered, i.e. $Ae_nA \simeq \oplus\, e_nA$. The ismorphism naturally induces an isomorphism from $Ae_n\Phi = \operatorname{Hom}(U_A, Ae_nA_A)$ to the direct sum of copies of $e_n\Phi = \operatorname{Hom}(U_A, e_nA_A)$ as right A-modules. On the other hand, $\tilde{A}e_n\tilde{A}/U = (Ae_nA + Ae_n\Phi + Ue_nA)/U = (Ae_nA + Ae_n\Phi)/U$, while $e_n\tilde{A} = e_nA + e_n\Phi$, so this proves that $\tilde{A}e_n\tilde{A}/U$ is Δ-filtered . To finish the proof we only need to observe that $\tilde{A}/U + \tilde{A}e_n\tilde{A} = \tilde{A}/\tilde{A}e_n\tilde{A} \cong A/Ae_nA$, since $U + \Phi \subseteq Ae_nA$, and this shows that the $\Delta(j)'s$ of $\tilde{A}$ for $j < n$ are the same as those of A and $\tilde{A}/U$ is Δ-filtered.

Finally, we show that $\tilde{A}/U$ is the direct sum of indecomposable Ext-projectives in the category of Δ-filtered right $\tilde{A}$-modules:

Since $\tilde{A}/U$ is the direct sum of local modules with tops $S(1), \ldots, S(n)$, the only thing left to prove is that $\operatorname{Ext}^1(\tilde{A}/U, \Delta_{\tilde{A}}(j)) = 0$ for all j. If we apply the $\operatorname{Hom}(-, \Delta_{\tilde{A}}(j))$ functor on the short exact sequence $0 \to U \to \tilde{A} \to \tilde{A}/U \to 0$, then we see that $\operatorname{Ext}^1(\tilde{A}/U, \Delta_{\tilde{A}}(j)) = 0$ if and only if the morphism $\operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(j)) \to \operatorname{Hom}(U, \Delta_{\tilde{A}}(j))$ is surjective. This condition is easily satisfied for $j \neq i$ because in that case $U = Ue_iA$ (and the simplicity of $\Delta_{\tilde{A}}(j)$ for $j > i$) implies that $\operatorname{Hom}(U, \Delta_{\tilde{A}}(j)) = 0$.

In the case when $j = i$, we can assume again that $i = n$. Under this condition $\Delta_{\tilde{A}}(n) = e_n\tilde{A} = e_nA + e_n\Phi$, and $\Phi = \Phi e_1$, while $Ue_1 = 0$, so $\operatorname{Hom}(U, \Delta_{\tilde{A}}(n)) = \operatorname{Hom}(U, e_nA) = e_n\Phi$. Let $\varphi \in \operatorname{Hom}(U, e_nA)$, and define $\alpha \in \operatorname{Hom}(\tilde{A}, e_n\tilde{A})$ with $\alpha(\tilde{a}) = \varphi\tilde{a}$. Since $\varphi\tilde{A} \subseteq e_n\Phi\tilde{A} \subseteq e_n\tilde{A}$, we get that $\alpha \in \operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(n))$, and $\alpha(u) = \varphi u = \varphi(u)$, so α is an extension of φ. This proves that the morphism $\operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(n)) \to \operatorname{Hom}(U, \Delta_{\tilde{A}}(n))$ is surjective, thus implying that $\tilde{A}/U$ is an Ext-projective module in the construction of $\Sigma(\tilde{A})$. Now, applying Lemma 3.4, this shows that $\Sigma(\tilde{A}) \cong A$.

□

Let us now formulate the parallel statement for $\bar{\Delta}$-equivalence.

THEOREM 3.5. *Let $(A, \mathbf{e})$ be an arbitrary algebra. Then the number of Morita equivalence classes of algebras which are $\bar{\Delta}$-equivalent to A is*
(i) one if all standard modules $\bar{\Delta}(i)$ for $2 \le i \le n$ are simple;
(ii) infinite otherwise.

Proof. The proof of case (i) is similar to that of the corresponding case of Theorem 3.3.

To prove case (ii), we could slightly modify the construction in the proof of Theorem 3.3. For later use, however, we shall give now a different construction showing that if at least one of the modules $\bar{\Delta}(i)$ for $i \ge 2$ is non-simple then there are infinitely many algebras in the $\bar{\Delta}$-equivalence class of A. (Recall that $\bar{\Delta}(1)$ is always a simple module.)

Thus, let i be such that $\bar{\Delta}(i)$ is not simple and $\bar{\Delta}(j)$ is simple for all $j > i$. Let us define the following (A, A)-bimodule: $L = Ae_i \otimes_K S(i)$. Finally let $\tilde{A}$ be defined as the trivial extension of A by L, i. e.

$$\tilde{A} = L \rtimes A = \left\{ \begin{pmatrix} a & \ell \\ 0 & a \end{pmatrix} \Big| \, a \in A, \ \ell \in L \right\}.$$

(Note that for path algebras this means adding one extra loop α at vertex i and an additional defining relation $\alpha^2 = 0$.) We want to show that A and $\tilde{A}$ are $\bar{\Delta}$-equivalent. Then, repeating the construction we can get infinitely many non-isomorphic basic algebras which are all $\bar{\Delta}$-equivalent to A.

Note that there is a natural action of $\tilde{A}$ on all A-modules and the modules $S(i)$ for $1 \le i \le n$ give a natural set of representatives of all simple $\tilde{A}$-modules. Furthermore, L is an ideal in A contained in $\operatorname{rad} A$, isomorphic as a right $\tilde{A}$-module to a direct sum of simple modules of type $S(i)$. This implies that for each indecomposable projective $\tilde{A}$-module $P_{\tilde{A}}(j)$, we get an exact sequence of $\tilde{A}$-modules

$$0 \to K(j) \to P_{\tilde{A}}(j) \to P_A(j) \to 0,$$

where $P_A(j)$ is the corresponding indecomposable projective A-module; moreover, $K(j) \simeq \oplus S(i)$.

Now, let us observe that the proper standard A-modules $\bar{\Delta}_A(j)$ are — as $\tilde{A}$-modules — isomorphic to the proper standard modules $\bar{\Delta}_{\tilde{A}}(j)$ for $1 \le j \le n$. This holds because the choice (the maximality) of i implies that L has a trivial intersection with the idecomposable projectives $e_j\tilde{A}$ for $j > i$, while for $j \le i$ the kernel of the epimorphism $e_j\tilde{A} \to \bar{\Delta}_{\tilde{A}}(j)$ contains $L \cap e_j\tilde{A}$ since L is a direct sum of $S(i)$-s, contained in the radical of A.

This also implies that modules in $\mathcal{F}(\Delta_A)$ also belong to $\mathcal{F}(\Delta_{\tilde{A}})$. In particular the direct sum M of indecomposable Ext-projective modules in $\mathcal{F}(\Delta_A)$ belongs to $\mathcal{F}(\Delta_{\tilde{A}})$. To show that $\bar{\Delta}(A)$ and $\bar{\Delta}(\tilde{A})$ are isomorphic, it is enough to show that M remains Ext-projective in $\mathcal{F}(\Delta_{\tilde{A}})$.

To this end let us take the projective cover $P_A(M)$ of M over A and the projective cover $P_{\tilde{A}}(M)$ of M over $\tilde{A}$. Then we get the following diagram of $\tilde{A}$-modules:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & K' & \to & K'' & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \tilde{K} & \to & P_{\tilde{A}}(M) & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K & \to & P_A(M) & \to & M & \to & 0 \end{array}$$

Here, as mentioned earlier, $K' \simeq K'' \simeq \oplus S(i)$, moreover the map $\tilde{K} \to K$ must be surjective. In view of our choice of i, there are no non-zero homomorphisms $S(i) \to \bar{\Delta}(j)$ for $1 \leq j \leq n$, and thus $\mathrm{Hom}_{\tilde{A}}(K, \bar{\Delta}(j)) \simeq \mathrm{Hom}_{\tilde{A}}(\tilde{K}, \bar{\Delta}(j))$ and $\mathrm{Hom}_{\tilde{A}}(P_A(M), \bar{\Delta}(j)) \simeq \mathrm{Hom}_{\tilde{A}}(P_{\tilde{A}}(M), \bar{\Delta}(j))$. Since $\mathrm{Ext}^1_A(M, \bar{\Delta}(j)) = 0$, the map $\mathrm{Hom}_A(P_A(M), \bar{\Delta}(j)) \to \mathrm{Hom}_A(K, \bar{\Delta}(j))$ is surjective. Using the previous isomorphisms we get that $\mathrm{Hom}_{\tilde{A}}(P_{\tilde{A}}(M), \bar{\Delta}(j)) \to \mathrm{Hom}_{\tilde{A}}(\tilde{K}, \bar{\Delta}(j))$ is also surjective. This means that $\mathrm{Ext}^1_{\tilde{A}}(M, \bar{\Delta}(j)) = 0$ and shows that M is Ext-projective in $\mathcal{F}(\bar{\Delta}_{\tilde{A}})$. The proof is completed. □

Let us observe that from the construction of $\tilde{A}$ it is easy to derive (say, by a dimension counting argument) that if the original algebra A is Δ-filtered then so is $\tilde{A}$. Thus, we get the following corollary.

Corollary 3.6. *If a $\bar{\Delta}$-equivalence class has more than one element and contains at least one Δ-filtered algebra then it contains infinitely many non-isomorphic basic Δ-filtered algebras.*

4. The orbit graph of the operators Σ and Ω

As before, all algebras in this section will be basic. Let us point out that the equivalence $(A, \mathbf{e}) \overset{\Delta}{\sim} (B, \mathbf{f})$ (or $(A, \mathbf{e}) \overset{\bar{\Delta}}{\sim} (B, \mathbf{f})$) implies the respective equivalence for the factor algebras $\mathrm{fact}_i(A) = A \big/ A(e_{i+1} + \cdots + e_n)A$ and $\mathrm{fact}_i(B) = B \big/ B(f_{i+1} + \cdots + f_n)B$ for all $1 \leq i \leq n$. This follows from the fact that in the equivalence between the categories of Δ-filtered (or $\bar{\Delta}$-filtered) modules over A and B, the modules filtered by $\Delta(j)$'s (or $\bar{\Delta}(j)$'s) with $j \leq i$ correspond to each other. Consequently,

$$\Sigma(\mathrm{fact}_i(A)) \simeq \mathrm{fact}_i(\Sigma(A))$$

and

$$\Omega(\mathrm{fact}_i(A)) \simeq \mathrm{fact}_i(\Omega(A)).$$

Theorem 4.1. *Denote the number of the (non-isomorphic) simple A-modules by n. Then the algebra $(\Omega\Sigma)^{n-1}(A)$ is properly stratified.*

For the proof of the theorem we shall need the following lemma:

Lemma 4.2. *Let A be a Δ-filtered algebra such that the factor algebra* $\mathrm{fact}_{n-1}(A)$ *is $\bar{\Delta}$-filtered. Then $\Omega(A)$ is properly stratified.*

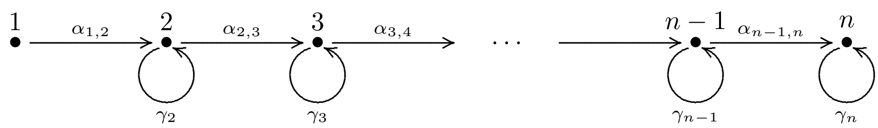
Proof. Let e_nA/e_nX be a maximal $\bar{\Delta}$-filtered factor of e_nA. Then $\mathrm{Hom}(e_nX, \bar{\Delta}(j)) = 0$ is true for all j: for $j < n$ it follows from $[\bar{\Delta}(j) : S(n)] = 0$, while for $j = n$, any nontrivial homomorphism from e_nX to $\bar{\Delta}(n)$ must be surjective, and thus bijective, so the existence of such a homomorphism would contradict the maximality of the factor e_nA/e_nX.

Now, consider the ideal $I = Ae_nX$ of A. Since Ae_nA is a direct sum of the modules $\Delta(n) = e_nA$, the ideal I is a direct sum of the submodules e_nX. Thus $\mathrm{Hom}(I, \bar{\Delta}(j)) = 0$ for all $1 \le j \le n$. Since $\mathrm{Ext}^1(A, \bar{\Delta}(j)) = 0$, also $\mathrm{Ext}^1(A/I, \bar{\Delta}(j)) = 0$ for all $1 \le j \le$. Consequently, in view of the fact that A/I is a direct sum of n $\bar{\Delta}$-filtered factors of the projective modules $P(j) = e_jA$, A/I is the Ext-projective module used in the construction of $\Omega(A)$, i. e. $\Omega(A) \simeq \mathrm{End}_A(A/I) \simeq A/I$.

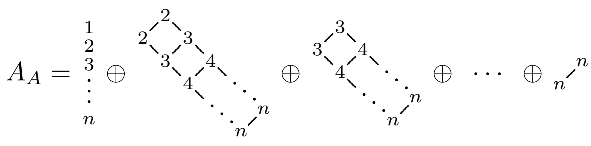
Since $\Omega(A)$ must be $\bar{\Delta}$-filtered, we only need to prove that A/I is Δ-filtered. The assumption of the lemma gives that A/Ae_nA is Δ-filtered, and we saw that $Ae_nA/I \simeq \oplus\, e_nA/e_nI$, so Ae_nA is $\Delta(n)$-filtered. This finishes the proof that $\Omega(A)$ is properly stratified. □

Proof of Theorem 4.1. Let us proceed by induction. The statement trivially holds for $n = 1$. Assume now that the statement holds for algebras with $n-1$ simple modules. Then $(\Omega\Sigma)^{n-2}(A\big/Ae_nA)$ is properly stratified. Thus, denoting $\Sigma(\Omega\Sigma)^{n-2}$ by Π, we have $\Pi(A\big/Ae_nA) \simeq \Pi(A)\big/\Pi(A)e_n\Pi(A)$ is $\bar{\Delta}$-filtered. Furthermore, $\Pi(A)$ is Δ-filtered by definition. Hence, applying the lemma to $\Pi(A)$, we get $\Omega\Pi(A) = (\Omega\Sigma)^{n-1}(A)$ is properly stratified. □

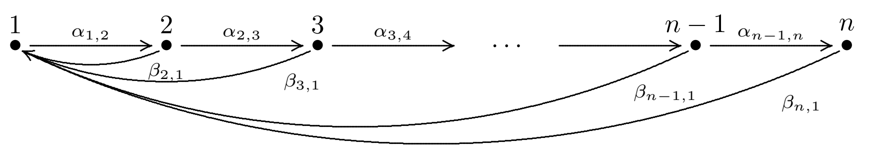
Example 4.3. The following example shows that the bound in Theorem 4.1 is optimal. Let A be the algebra given as $A = KQ_A/I_A$, where Q_A is given by:



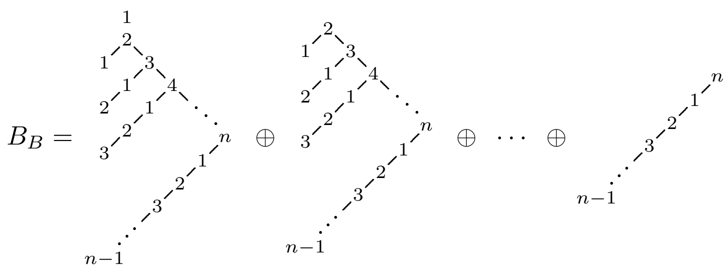
and $I_A = \langle \gamma_i^2, \gamma_i\alpha_{i,i+1} - \alpha_{i,i+1}\gamma_{i+1} \,|\, 2 \le i \le n-1\rangle$. Thus the right regular representation of A can be described as follows:



Then $(\Omega\Sigma)^{n-1}(A) = B = KQ_B/I_B$, where Q_B is given by:



and $I_B = \langle \beta_{i1}\alpha_{12}\alpha_{23}\cdots\alpha_{i-1,i} \,|\, 2 \le i \le n \rangle$ with regular decomposition:



Here $\Sigma(\Omega\Sigma)^{n-2}(A)$ is Δ-filtered but not $\bar{\Delta}$-filtered. The last projective (i.e. the last standard module) is a uniserial module with a composition series of length $n+1$ as follows: $S(n), S(1), S(2), \ldots, S(n-1), S(n)$.

Let us now take the Cayley-graph of this action of the operators Σ and Ω, restricted to the class of all standardly stratified algebras (A, e). Thus, we define an arrow of type Σ from A to $\Sigma(A)$ and an arrow of type Ω from A to $\Omega(A)$.

For this graph, as an immediate consequence of Theorem 4.1 we get the following corollary.

Corollary 4.4. *The family of all basic standardly stratified algebras with n non-isomorphic simple modules is a disjoint union of oriented trees of algebras, indexed by properly stratified algebras as their roots. The height of these trees is bounded by* $2(n-1)$.

Note that although Theorem 4.1 is valid for general algebras, we have restricted our formulation of Corollary 4.4. to standardly stratified algebras, where it seems to be possible to describe also the proper preimages $\Sigma^{-1}(A)$ and $\Omega^{-1}(A)$ (i. e. the preimages not including the algebra itself), of a given algebra. In the family of all algebras this may be an impossible task. A more detailed description of the structure of this graph will be presented in a separate paper. Here we conclude our discussion with two remarks only, illustrating the complexity of the question.

Corollary 3.6 immediately implies that if A is a standardly stratified algebra then the proper preimage $\Omega^{-1}(A)$ is either empty or it is infinite. (Note that we have excluded the algebra A from its proper preimage.) Namely, if $\Omega^{-1}(A)$ is non-empty then A is $\bar{\Delta}$-filtered and its $\bar{\Delta}$-equivalence class contains at least one Δ-filtered algebra, not isomorphic to A. Thus by Corollary 3.6 it contains infinitely many Δ-filtered elements, hence $\left|\Omega^{-1}(A)\right| = \infty$.

On the other hand the following example shows that the cardinality of $\Sigma^{-1}(A)$ can be equal to any natural number.

Example 4.5. The following examples of algebras show that the Δ-equivalence classes of algebras can contain an arbitrary finite number of $\bar{\Delta}$-filtered algebras.

Let $k \in \mathbb{N}$, $k \ge 1$ be given and consider the algebras $A_{i,k}$ defined for $1 \le i \le k$ as $A_{i,k} = KQ_{A_{i,k}}/I_{A_{i,k}}$ with $Q_{A_{i,k}}$ having two vertices, one arrow α from 1 to 2 and k loops at 2, denoted by $\beta_1, \ldots, \beta_k$, subject to the relations $I_{A_{i,k}} = \langle \beta_p\beta_q,\ \alpha\beta_r,\ \mid 1 \le$

$p, q \le k,\ i \le r \le n\rangle$. Thus the right regular decomposition of $A_{i,k}$ can be described as follows:

$$(A_{i,k})_{A_{i,k}} = \begin{smallmatrix} 1 \\ 2 \\ \underbrace{2 \cdots 2}_{i-1 \text{ copies}} \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ \underbrace{2\ 2 \cdots 2}_{k \text{ copies}} \end{smallmatrix}$$

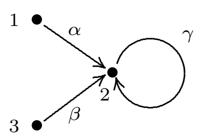
Clearly each algebra $A_{i,k}$ is $\bar{\Delta}$-filtered, moreover $A_{i,k}$ is a homomorphic image of $A_{j,k}$ for $i \le j$. In this way we can say that the standard modules for $A_{1,k}$ are also standard modules for each $A_{i,k}$ and $\dim \mathrm{Ext}^1_{A_{i,k}}(\Delta(1), \Delta(2)) = k$ for each $1 \le i \le k$. Hence the universal extension construction of $\Delta(2)$ by $\Delta(1)$ over $A_{1,k}$ gives the Ext-projective module for every algebra $A_{i,k}$, $1 \le i \le k$. This implies that $\Sigma(A_{i,k}) \simeq \Sigma(A_{j,k})$ for $1 \le i, j \le k$.

We want to show that there is no other $\bar{\Delta}$-filtered algebra A for which $\Sigma(A)$ is isomorphic to $\Sigma(A_{i,k})$. Suppose that A and $A_{1,k}$ are Δ-equivalent and A is $\bar{\Delta}$-filtered. Then it is easy to see that $\Delta_A(2)$ must not contain a simple module of type $S_A(1)$ in its socle, since this would give a nonzero homomorphism in $\mathrm{Hom}_A(\Delta_A(1), \Delta_A(2))$ although such a homomorphism does not exist in $\mathcal{F}(\Delta_{A_{i,k}})$. Since A is $\bar{\Delta}$-filtered, we get that $\Delta_A(2)$ is homogeneous, containing only simple factors of type $S(2)$. Now it is easy to see that the structure of $\Delta_A(2)$ is well described by its endomorphism ring $\mathrm{End}_A(\Delta_A(2))$ which is isomorphic to $\mathrm{End}_{A_{1,k}}(\Delta_{A_{i,k}}(2))$. Now, knowing the structure of $\mathrm{Ext}^1_A(\Delta_A(1), \Delta_A(2)$, we get that $\mathrm{rad}\, P_A(1)/\mathrm{rad}^2\, P_A(1)$ is isomorphic to $S_A(2)$, hence $\mathrm{rad}\, P_A(1)$ is a homomorphic image of $\Delta_A(2)$. This implies that, depending on the composition length of $P_A(1)$, the algebra A must be isomorphic to one of the algebras $A_{i,k}$.

5. An example of Δ-equivalence

Let us conclude the paper by exhibiting the subcategories of Δ-filtered modules in one particular case. Compare the inclusions of the subcateogories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_{\Sigma(A)})$ in the Auslander–Reiten quiver of A and $\Sigma(A)$.

Example 5.1. Let $A = KQ_A/I_A$ be the algebra given by the following quiver and right regular representation:

Q_A: 

; $\quad I_A = \langle \alpha\gamma, \gamma^2 \rangle$; $\quad A_A = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 2 \end{smallmatrix}$.

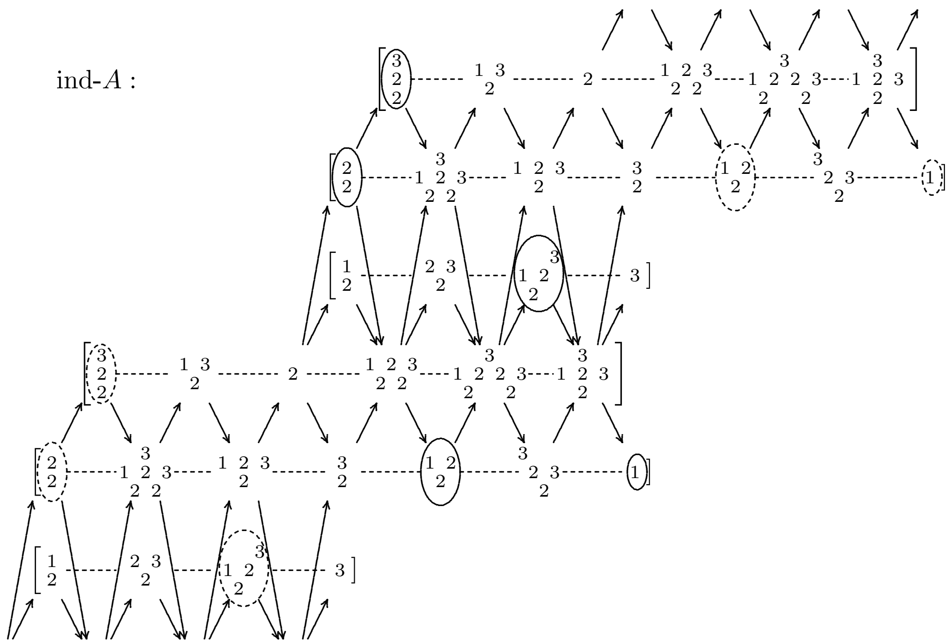
Thus A is a $\bar{\Delta}$-filtered algebra. The standard and proper standard modules are given by:

$$\Delta_A(1) = 1; \qquad \Delta_A(2) = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}; \qquad \Delta_A(3) = \begin{smallmatrix} 3 \\ 2 \\ 2 \end{smallmatrix};$$

$$\bar{\Delta}_A(1) = 1; \qquad \bar{\Delta}_A(2) = 2; \qquad \bar{\Delta}_A(3) = 3 .$$

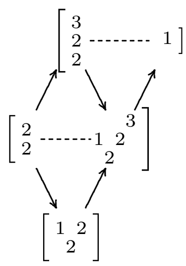
The Auslander–Reiten quiver of the indecomposable right A-modules is as follows (encircled are the elements of $\mathcal{F}(\Delta_A)$):

ind-A :



Thus there are 17 indecomposable A-modules in three τ-orbits, while there are five indecomposable modules in $\mathcal{F}(\Delta_A)$, forming the relative Auslander–Reiten quiver:

$\mathcal{F}_{ind}(\Delta_A)$:

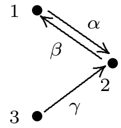


The direct sum of indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is given by

$$M_A = \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\2\end{smallmatrix}.$$

Here, $\Sigma(A) = \mathrm{End}_A(M)$ is given by $\Sigma(A) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ with the quiver and regular representation as follows:

$Q_{\Sigma(A)}$:



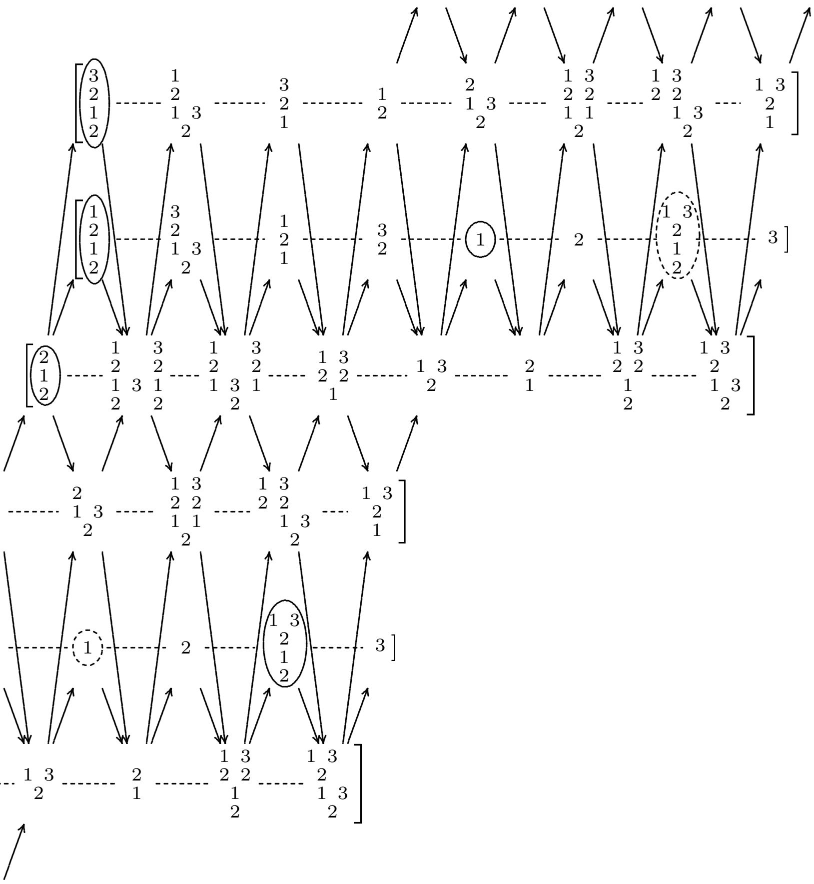
; $\quad I_{\Sigma(A)} = \langle \beta\alpha\beta \rangle$; $\quad \Sigma(A)_{\Sigma(A)} = \begin{smallmatrix}1\\2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\\2\end{smallmatrix}$.

Clearly, $\Sigma(A)$ is Δ-filtered. The standard modules are given by:

$$\Delta_{\Sigma(A)}(1) = 1; \qquad \Delta_{\Sigma(A)}(2) = \begin{smallmatrix}2\\1\\2\end{smallmatrix}; \qquad \Delta_{\Sigma(A)}(3) = \begin{smallmatrix}3\\2\\1\\2\end{smallmatrix}.$$

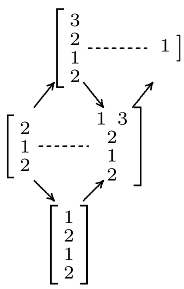
The Auslander–Reiten quiver of the indecomposable right A-modules is as follows (encircled are the elements of $\mathcal{F}(\Delta_{\Sigma(A)})$):

ind-$\Sigma(A)$:



Thus there are 24 indecomposable $\Sigma(A)$-modules in three τ-orbits, and five indecomposable modules in $\mathcal{F}(\Delta_{\Sigma(A)})$, forming the relative Auslander–Reiten quiver:

$\mathcal{F}_{ind}(\Delta_A)$:



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CONSTRUCTIONS OF STRATIFIED ALGEBRAS

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Abstract. In this paper a construction to build recursively all basic finite dimensional standardly stratified algebras is given. In comparison to the construction described by Dlab and Ringel for the quasi-hereditary case ([DR3]) some new features appear here.

1. Introduction

The concept of standardly stratified algebras (or Δ-filtered algebras) appears as a natural generalization of the concept of quasi-hereditary algebras. The class of quasi-hereditary algebras was introduced by Cline, Parshall and Scott (see [CPS1], [PS]) in connection with their study of highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups. The study of quasi-hereditary algebras grew into an extensive volume of contributions starting with the seminal papers [DR1], [R], [DR2]. The concept of standardly stratified algebras was introduced independently in [D1] and in the comprehensive study [CPS2] and further extended in [ADL1] and [ADL2]. It may be also pointed out that the concept of a stratifiying ideal of [CPS2] appeared already as a strongly idempotent ideal in [APT]. A particular type of standardly stratified algebras, namely properly stratified algebras of [D2] illustrates again a very close relationship to the representation theory of Lie algebras (see also [FM], [FKM]).

Ever since their introduction, standardly stratified algebras have drawn much attention; their structural and homological properties were investigated among others in [AHLU1], [AHLU2], [ADL3], [ChD], [ADL4], [M]. It is worth mentioning that the main body of results in this field is established for standardly stratified algebras and then easily generalized for particular types of these algebras such as quasi-hereditary and properly stratified algebras.

As in the case of quasi-hereditary algebras, the structure of standardly stratified algebras includes two recursive sequences of standardly stratified algebras. One sequence is obtained by taking consecutive quotients of the algebra modulo the respective idempotent trace ideals. The other sequence is obtained by taking centralizer algebras of the corresponding sequence of indecomposable projective

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modules. In most cases the first approach is used to study these algebras. On the other hand Dlab and Ringel ([DR3]) have shown that the sequence of centralizer algebras relates to the structure of the categories of perverse sheaves and provides a recursive construction of all finite dimensional basic quasi-hereditary algebras.

The goal of this paper is to extend this construction for the general situation. Much of the original results can be adopted for this case, however, the complexity of standard modules requires some extra precautions. Furthermore, to construct all standardly stratified algebras we need to introduce an extra step to make the procedure complete. Thus, the main result is the following theorem.

Theorem. *Let L be a local algebra, and C a basic algebra such that C_C is filtered by standard C-modules with respect to some order $(e_2, \ldots, e_n)$ of a complete set of primitive orthogonal idempotents in C. Furthermore, let ${}_LE_C$ and ${}_CF_L$ be bimodules such that E_C is filtered by standard C-modules and ${}_CF$ is filtered by proper standard C^{opp}-modules. In addition, suppose that $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ is a C-C bimodule homomorphism. Then $\tilde{A} = L \oplus (E \underset{C}{\otimes} F) \oplus E \oplus F \oplus C$ has an algebra structure such that $\tilde{A}_{\tilde{A}}$ is filtered by standard $\tilde{A}$-modules with respect to the order $(1_L, e_2, \ldots, e_n)$. Moreover, one can get all basic standardly stratified algebras recursively, starting with a local algebra, by constructing and taking suitable quotients of algebras $\tilde{A}$ obtained this way.*

2. CPS-stratified algebras

Let $(A, \mathbf{e})$ be a basic finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \ldots, e_n)$ of primitive orthogonal idempotents. Here K denotes an arbitrary field. We also use the notation $\varepsilon_i = e_i + \ldots + e_n$ throughout the paper.

Let us recall some of the characterizations of the so-called CPS-stratifying ideals.

Definition 2.1. (Cf. [CPS2], [APT], [ADL2]) An idempotent ideal AeA of the algebra A will be called *CPS-stratifying* (or *stratifying* for short) if it satisfies any of the equivalent conditions (S1), (S1′), (S2), (S3):

(S1) (i) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and
(ii) $\operatorname{Tor}_t^{eAe}(Ae, eA) = 0$ for all $t > 0$;

(S1′) (i) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and
(ii) $\operatorname{Ext}^t_{eAe}(Ae, D(eA)) = 0$ for all $t > 0$;

(S2) $\operatorname{Ext}^t_{A/AeA}(X, Y) = \operatorname{Ext}^t_A(X, Y)$ for all $t \geq 0$ and A/AeA-modules X and Y;

(S3) Each term in the minimal projective resolution of AeA_A is generated by eA.

Definition 2.2. $(A, \mathbf{e})$ is said to be *CPS-stratified* if either $n = 1$ (i. e. the algebra is local) or in case $n > 1$, the ideal Ae_nA is stratifying and $(A/Ae_nA, (e_1, \ldots, e_{n-1}))$ is CPS-stratified.

In the later sections of the paper we shall use some simple facts about CPS-stratified algebras (cf. also [CPS2]). In particular, we will need the following lemma.

LEMMA 2.3. *Let $(A, \mathbf{e})$ be CPS-stratified. Then $A\varepsilon_i A$ is a stratifying ideal and $(\varepsilon_i A\varepsilon_i, (e_i, \dots, e_n))$ is CPS-stratified for all $i = 1, \dots, n$.*

Proof. We prove both statements by induction on $n - i$, where the $i = n$ case immediately follows from the definition of CPS-stratified algebras. Now suppose that $n - i > 0$. Then the algebra $\bar{A} = A/Ae_nA$ is CPS-stratified with respect to the idempotents $\bar{\mathbf{e}} = (e_1, \dots, e_{n-1})$, so the lemma holds for $\bar{A}$ (for the same i) by the induction hypothesis.

To prove the first statement, we show that condition (S2) holds for $A\varepsilon_i A$. Let us take $X, Y \in$ mod-$A/A\varepsilon_i A$. Then we have $\mathrm{Ext}^t_{A/A\varepsilon_i A}(X, Y) = \mathrm{Ext}^t_{\bar{A}/\bar{A}\varepsilon_i\bar{A}}(X, Y) = \mathrm{Ext}^t_{\bar{A}}(X, Y) = \mathrm{Ext}^t_A(X, Y)$ for all $t > 0$, thus $A\varepsilon_i A$ is stratifying in A.

Next, the fact that Ae_nA is stratifying implies by (S3) that $\varepsilon_i Ae_nA\varepsilon_i$ is stratifying in $\varepsilon_i A\varepsilon_i$. On the other hand $\varepsilon_i A\varepsilon_i/\varepsilon_i Ae_nA\varepsilon_i \simeq \varepsilon_i\bar{A}\varepsilon_i$ is CPS-stratified by induction, hence $\varepsilon_i A\varepsilon_i$ is CPS-stratified. □

LEMMA 2.4. *Suppose that for some i the algebra $(\varepsilon_i A\varepsilon_i, (e_i, \dots, e_n))$ is CPS-stratified, and $A\varepsilon_i A$ is a stratifying ideal in A. Then the multiplication map $Ae_n \underset{e_nAe_n}{\otimes} e_nA \to Ae_nA$ is bijective.*

Proof. Let us denote $\varepsilon_j A\varepsilon_j$ by C_j. Since the multiplication map is clearly surjective, it is enough to show that $Ae_n \underset{C_n}{\otimes} e_nA$ and Ae_nA are isomorphic. So the following succession of isomorphisms provides a proof.

$$\begin{aligned}
Ae_n \underset{C_n}{\otimes} e_nA &= \\
A\varepsilon_i Ae_n \underset{C_n}{\otimes} e_nA\varepsilon_i A &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A)e_n \underset{C_n}{\otimes} e_n(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A) &\simeq \\
A\varepsilon_i \underset{C_i}{\otimes} (\varepsilon_i Ae_n \underset{C_n}{\otimes} e_nA\varepsilon_i) \underset{C_i}{\otimes} \varepsilon_i A &\simeq && \text{(since } \varepsilon_i A\varepsilon_i \text{ is CPS-stratified)}\\
A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i Ae_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &= \\
(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A)e_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
A\varepsilon_i Ae_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &= \\
A\varepsilon_i Ae_n(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A) &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
A\varepsilon_i Ae_nA\varepsilon_i A &= \\
Ae_nA.
\end{aligned}$$

□

3. Δ-filtered algebras

For the reader's convenience let us recall some basic definitions and results.

For a given algebra $(A, \mathbf{e})$ the *standard modules* are defined by $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$ and the *proper standard modules* by $\bar{\Delta}(i) = e_iA/e_i \operatorname{rad} A\varepsilon_iA$ for $1 \le i \le n$. Similarly one can define the left standard and left proper standard modules $\Delta^\circ(i)$ and $\bar{\Delta}^\circ(i)$. The (full) subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of the category mod-A of all finite dimensional right A-modules consist of those A-modules which have a filtration by standard modules (or proper standard modules, respectively).

It is well known (cf. for example [ADL1]) that a module $M \in$ mod-A belongs to $\mathcal{F}(\Delta_A)$ (in this case we will say that M is Δ-*filtered*) if and only if the trace Me_nA is projective (i. e. it is $\Delta(n)$-filtered) and M/Me_nA is Δ-filtered as an $\bar{A} = A/Ae_nA$-module. Similarly, $M \in \mathcal{F}(\bar{\Delta}_A)$ if and only if Me_nA has a filtration by $\bar{\Delta}(n)$ and M/Me_nA belongs to $\mathcal{F}(\bar{\Delta}_{\bar{A}})$.

Definition 3.1. An algebra $(A, \mathbf{e})$ is said to be Δ-*filtered* if the regular module A_A belongs to $\mathcal{F}(\Delta_A)$. Similarly, $(A, \mathbf{e})$ is said to be $\bar{\Delta}$-*filtered* if $A_A \in \mathcal{F}(\bar{\Delta}_A)$. An algebra is called *standardly stratified* if it is either Δ or $\bar{\Delta}$-filtered.

By a result of Dlab (cf. [D1]) $(A, \mathbf{e})$ is Δ-filtered if and only if $(A^{opp}, \mathbf{e})$ is $\bar{\Delta}^\circ$-filtered. Furthermore it is straightforward that Δ-filtered algebras are also CPS-stratified (cf. condition (S3) or [CPS2]). Hence the above result of Dlab implies that $\bar{\Delta}$-filtered algebras are also CPS-stratified algebras (since condition (S1) is obviously left-right symmetric).

In the following we want to describe the property that $(A, \mathbf{e})$ is Δ-filtered in terms of its centralizer algebra $\varepsilon_2A\varepsilon_2$, and the corresponding subalgebra and bimodules e_1Ae_1, $e_1A\varepsilon_2$ and ε_2Ae_1.

Theorem 3.2. *Given an algebra $(A, \mathbf{e})$ let us consider the local algebra $L = e_1Ae_1$, the centralizer algebra $C = \varepsilon_2A\varepsilon_2$ together with the order $\mathbf{e}' = (e_2, \dots, e_n)$ and the bimodules $E = e_1A\varepsilon_2$ and $F = \varepsilon_2Ae_1$. Then $(A, \mathbf{e})$ is Δ-filtered if and only if the following conditions hold:*
(1) $C_C \in \mathcal{F}(\Delta_C)$;
(2) $E_C \in \mathcal{F}(\Delta_C)$;
(3) ${}_CF \in \mathcal{F}({}_C\bar{\Delta}^\circ)$;
(4) *the multiplication map $E \underset{C}{\otimes} F \to L$ is injective.*

Proof. Let us note that the condition (4) is equivalent to the condition (4′) stating that the multiplication map $A\varepsilon_2 \underset{C}{\otimes} \varepsilon_2A \to A\varepsilon_2A$ is injective (in fact, bijective), since $A\varepsilon_2 = E \oplus C$, $\varepsilon_2A = F \oplus C$, and the injectivity of the multiplication map on the other three components is obvious.

First assume that A is Δ-filtered. Then it is also CPS-stratified and thus by Lemma 2.3, (4′) and hence (4) of the theorem holds. The conditions (1) and (2) follow from the fact that $A\varepsilon_2A$ is Δ-filtered. Hence $e_1A\varepsilon_2A$ and ε_2A are also Δ-filtered, and therefore $e_1A\varepsilon_2$ and $\varepsilon_2A\varepsilon_2$ are Δ-filtered over $\varepsilon_2A\varepsilon_2$. Similarly, (3) holds because $A\varepsilon_2A$ is $\bar{\Delta}^\circ$-filtered.

The opposite statement will be proved by induction on n. Thus, assume that the conditions (1)–(4) hold for A. We will show that Ae_nA_A is projective and that the conditions (1)–(4) hold for the factor algebra $\bar{A} = A/Ae_nA$.

First, let us prove that $A\varepsilon_2 A$ is a stratifying ideal. By the condition (4′) the map $A\varepsilon_2 \underset{C}{\otimes} \varepsilon_2 A \to A\varepsilon_2 A$ is injective. On the other hand, using the condition (1) and the dual of Theorem 3.1 of [ADL1], we conclude that $\mathrm{Ext}^t_C(\Delta_C, D({}_C\bar{\Delta}^{\circ})) = 0$ for all $t > 0$. Thus (2) and (3) imply that $\mathrm{Ext}^t_C(A\varepsilon_2, D(\varepsilon_2 A)) = \mathrm{Ext}^t_C(E\oplus C, D(F\oplus C)) = 0$ for all $t > 0$. Hence the condition (S1′) implies that $A\varepsilon_2 A$ is a stratifying ideal in A.

By the conditions (1) and (2), $A\varepsilon_2 = E\oplus C \in \mathcal{F}(\Delta_C)$, so the trace of $\Delta_C(n)$ on $A\varepsilon_2$ is projective: $Ae_nA\varepsilon_2 \simeq \oplus\, e_nA\varepsilon_2$. Thus $Ae_n \simeq \oplus\, e_nAe_n$ as e_nAe_n-modules. Hence we get that $Ae_n \underset{e_nAe_n}{\otimes} e_nA \simeq (\oplus\, e_nAe_n) \underset{e_nAe_n}{\otimes} e_nA \simeq \oplus\, e_nA$ is a projective A-module. Finally, by Lemma 2.4 (using that C is Δ-filtered, thus CPS-stratified as well), $Ae_n \underset{e_nAe_n}{\otimes} e_nA \simeq Ae_nA$, and so Ae_nA is a projective right A-module.

Now, take the factor algebra $\bar{A} = A/Ae_nA$. The corresponding objects to consider are $\bar{C} \simeq \varepsilon_2A\varepsilon_2/\varepsilon_2Ae_nA\varepsilon_2$, $\bar{E} \simeq e_1A\varepsilon_2/e_1Ae_nA\varepsilon_2$ and $\bar{F} \simeq \varepsilon_2Ae_1/\varepsilon_2Ae_nAe_1$. The remarks preceding Definiton 3.1 show that the conditions (1), (2) and (3) also hold for $\bar{A}$. Finally, since $A\varepsilon_2A$ and Ae_nA are stratifying, (S2) implies that $\bar{A}\bar{\varepsilon}_2\bar{A}$ is stratifying in $\bar{A}$: for any $X, Y \in \text{mod-}\bar{A}/\bar{A}\bar{\varepsilon}_2\bar{A}$ we have $\mathrm{Ext}^t_{\bar{A}/\bar{A}\bar{\varepsilon}_2\bar{A}}(X,Y) = \mathrm{Ext}^t_{A/A\varepsilon_2A}(X,Y) = \mathrm{Ext}^t_A(X,Y) = \mathrm{Ext}^t_{\bar{A}}(X,Y)$. Thus by (S1), the condition (4) also holds for $\bar{A}$. By induction we get that $\bar{A}$ is Δ-filtered, so A is also Δ-filtered.

□

Note that the data above correspond to the Peirce decomposition of the algebra $A \simeq \begin{pmatrix} L & E \\ F & C \end{pmatrix}$.

4. Construction of Δ-filtered algebras

In this chapter we proceed in the opposite direction and construct all Δ-filtered algebras from "smaller" algebras, using a recursive process.

Suppose L and C are algebras together with an L-C-bimodule E and a C-L-bimodule F and a C-C-bimodule homomorphism $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}(C)$. Then it is easy to see that the map

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \overset{\mathrm{id}_E \otimes \mu \otimes \mathrm{id}_F}{\longrightarrow} E \underset{C}{\otimes} C \underset{C}{\otimes} F \simeq E \underset{C}{\otimes} F$$

defines an algebra multiplication on the L-L-bimodule $E \underset{C}{\otimes} F$. Thus:

$$(e\otimes f)(e'\otimes f') = e\mu(f\otimes e')\otimes f' = e\otimes \mu(f\otimes e')f'.$$

The split extension $\tilde{L} = L \ltimes (E \underset{C}{\otimes} F)$ of the algebra L by $E \underset{C}{\otimes} F$ is defined, in the usual way, on the cartesian product with multiplication:

$$(l,u)(l',u') = (ll', lu' + ul' + uu').$$

Now we can extend the L-C and C-L bimodule structure of E and F respectively to $\tilde{L}$-C and C-$\tilde{L}$ structure, using the maps

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} E \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \xrightarrow{\mathrm{id}_E \otimes \mu} E \underset{C}{\otimes} C \simeq E$$

and

$$F \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \xrightarrow{\mu \otimes \mathrm{id}_F} C \underset{C}{\otimes} F \simeq F.$$

Finally, the C-C-bimodule map $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}(C)$ induces naturally a C-C-bimodule map $\tilde{\mu} : F \underset{\tilde{L}}{\otimes} E \to \mathrm{rad}\, C$, since $\mu(f(e' \underset{C}{\otimes} f') \underset{L}{\otimes} e) = \mu(\mu(f \underset{L}{\otimes} e')f' \underset{L}{\otimes} e) = \mu(f \underset{L}{\otimes} e')\mu(f' \underset{L}{\otimes} e) = \mu(f \underset{L}{\otimes} e' \mu(f' \underset{L}{\otimes} e)) = \mu(f \underset{L}{\otimes} (e' \underset{C}{\otimes} f')e)$.

It is easy to show that if L is a local algebra then $\tilde{L}$ is also local. Indeed, $E \underset{C}{\otimes} F$ is a nilpotent ideal in $\tilde{L}$, since $(E \underset{C}{\otimes} F)^k = E \underset{C}{\otimes} (\mu(F \underset{L}{\otimes} E))^{k-1} F \subseteq E \underset{C}{\otimes} (\mathrm{rad}\, C)^{k-1} F$. Thus, $\mathrm{rad}\, L + (E \underset{C}{\otimes} F)$ is a nilpotent ideal of $\tilde{L}$, and furthermore, $\tilde{L}/(\mathrm{rad}\, L + (E \underset{C}{\otimes} F)) \simeq L/\mathrm{rad}\, L$ is a simple $\tilde{L}$ module.

Now we can consider the matrix algebra $\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$ with the natural multiplication structure:

$$\begin{pmatrix} x & e \\ f & c \end{pmatrix} \begin{pmatrix} x' & e' \\ f' & c' \end{pmatrix} = \begin{pmatrix} xx' + e \underset{C}{\otimes} f' & xe' + ec' \\ fx' + cf' & \tilde{\mu}(f \underset{\tilde{L}}{\otimes} e') + cc' \end{pmatrix}$$

for arbitrary $x, x' \in \tilde{L}$, $e, e' \in E$, $f, f' \in F$ and $c, c' \in C$. The associativity of the multiplication follows directly from the definition of the bimodule structures ${}_{\tilde{L}}E_C$ and ${}_C F_{\tilde{L}}$. (Note that the algebra $\tilde{A}$ is usually called the *Morita ring* corresponding to the Morita context $(\tilde{L}, C, E, F, \iota, \tilde{\mu})$ where $\iota : E \underset{C}{\otimes} F \to \tilde{L}$ is the natural embedding.)

THEOREM 4.1. *Let L be a local algebra with identity element denoted by e_1 and let $(C, (e_2, \ldots, e_n))$ be a (basic) Δ-filtered algebra. Let ${}_L E_C$ and ${}_C F_L$ be two bimodules such that $E_C \in \mathcal{F}(\Delta_C)$ and ${}_C F \in \mathcal{F}({}_C\bar{\Delta}^{\circ})$, together with a bimodule map $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}\, C$. Then the algebra $\tilde{A}$ constructed above is a Δ-filtered algebra with respect to the sequence of idempotents $\mathbf{e} = (e_1, e_2, \ldots, e_n)$.*

Proof. We have seen that $\tilde{L} = e_1 \tilde{A} e_1$ is local, so $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is a complete sequence of primitive orthogonal idempotents. It is also clear that $\varepsilon_2 \tilde{A} \varepsilon_2 = C$ and that $e_1 \tilde{A} \varepsilon_2 = E$ and $\varepsilon_2 \tilde{A} e_1 = F$ satisfy the filtration conditons of Theorem 3.2 by the assumptions on C, E and F. Moreover the multiplication map $\iota : E \underset{C}{\otimes} F \to \tilde{L}$ is injective by definition, so $(\tilde{A}, \mathbf{e})$ is Δ-filtered. □

To construct all Δ-filtered algebras, we need the following concept.

DEFINITION 4.2. Let $(A, \mathbf{e})$ be a Δ-filtered algebra. An ideal $H \triangleleft A$ will be called *auxiliary* if $H \subseteq e_1(\mathrm{rad}\, A)e_1$ and $H \cap e_1 A \varepsilon_2 A e_1 = 0$.

LEMMA 4.3. *Let* $(A, \mathbf{e})$ *be a* Δ*-filtered algebra and* $H \triangleleft A$ *an auxiliary ideal. Then* $(A/H, \mathbf{e})$ *is also* Δ*-filtered.*

Proof. The conditions imply that $A\varepsilon_2 A \cap H = 0$, hence the trace ideal $A\varepsilon_2 A$ maps injectively into $\bar{A} = A/H$. Thus the Δ_A-filtration of $A\varepsilon_2 A_A$ gives a $\Delta_{\bar{A}}$-filtration of $\bar{A}\varepsilon_2\bar{A}_{\bar{A}}$. Since $\bar{A}/\bar{A}\varepsilon_2\bar{A} \simeq \Delta_{\bar{A}}(1)$, the algebra $(\bar{A}, \mathbf{e})$ is also Δ-filtered. □

Finally, we show that all Δ-filtered algebras can be obtained using the construction of Theorem 4.1, followed by factoring out an auxiliary ideal.

THEOREM 4.4. *Let* $(A, \mathbf{e})$ *be a basic* Δ*-filtered algebra. Take* $L = e_1 A e_1$, $C = \varepsilon_2 A \varepsilon_2$, $E = e_1 A \varepsilon_2$, $F = \varepsilon_2 A e_1$ *and let* $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ *and* $\nu : E \underset{C}{\otimes} F \to L$ *be the multiplication maps in* A*. Construct the algebras* $\tilde{L} = L \ltimes (E \underset{C}{\otimes} F)$ *and* $\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$ *as in Theorem 4.1. Then* $H = \left\{ \nu(u) - u \,|\, u \in E \underset{C}{\otimes} F \right\} \subseteq \tilde{L}$ *is an auxiliary ideal of* $\tilde{A}$ *and the algebra* $\tilde{A}/H$ *is isomorphic to* A*.*

Proof. First, let us observe that Theorem 3.2 implies that L, C, E and F satisify the conditions of Theorem 4.1, hence the algebra $\tilde{A}$ is Δ-filtered.

In order to show that H is an ideal in $\tilde{A}$, note first that for any $u, u' \in E \underset{C}{\otimes} F \subseteq \tilde{L}$ we have $u'u = u'\nu(u)$. Indeed, for $e, e' \in E$ and $f, f' \in F$ we get $(e' \otimes f')(e \otimes f) = e' \otimes \mu(f' \otimes e) f = e' \otimes (f'e) f = e' \otimes f'(ef) = (e' \otimes f')(ef) = (e' \otimes f')\nu(e \otimes f)$. Similarly, for any $f' \in F$ and $u \in E \underset{C}{\otimes} F$, we have $f'u = f'\nu(u)$, since $f'(e \otimes f) = \mu(f' \otimes e) f = (f'e) f = f'(ef) = f'\nu(e \otimes f)$. Thus, for $\tilde{a} \in \tilde{A}$ and $u \in E \underset{C}{\otimes} F$:

$$
\begin{aligned}
\tilde{a}(u - \nu(u)) &= \begin{pmatrix} l' + u' & e' \\ f' & c' \end{pmatrix} \begin{pmatrix} u - \nu(u) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} l'(u - \nu(u)) + u'(u - \nu(u)) & 0 \\ f'(u - \nu(u)) & 0 \end{pmatrix} \\
&= \begin{pmatrix} l'u - \nu(l'u) & 0 \\ 0 & 0 \end{pmatrix} \in H.
\end{aligned}
$$

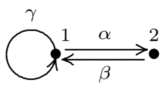
Similarly, one can show that $(u - \nu(u))\tilde{a} \in H$. Since H is clearly closed under addition, $H \triangleleft \tilde{A}$.

Also, $H \subseteq E \underset{C}{\otimes} F + \operatorname{rad} L = \operatorname{rad} \tilde{L} = e_1(\operatorname{rad} \tilde{A})e_1$. Since the map ν is injective, $u - \nu(u) \neq 0$ implies that $u \neq 0$ and $\nu(u) \neq 0$. Then $L \cap E \underset{C}{\otimes} F = 0$ yields that $u - \nu(u) \notin E \underset{C}{\otimes} F$ and $\notin L$, and consequently $H \cap E \underset{C}{\otimes} F = 0$ and $H \cap L = 0$. It follows that the condition $H \cap e_1 \tilde{A} \varepsilon_2 \tilde{A} e_1 = H \cap (E \underset{C}{\otimes} F) = 0$ holds for the ideal H, hence the ideal H is auxiliary. Furthermore, $H \cap A = H \cap \tilde{L} \cap A = H \cap L = 0$. Also, it is straightforward that $A + H = \tilde{A}$, so $\tilde{A}/H \simeq A$. □

COROLLARY 4.5. *Let* L, C, E, F *and* μ *be given as in Theorem 4.1 and let* I *be an auxiliary ideal of the algebra* $\tilde{A}$*. Then* $\tilde{A}/I$ *is a* Δ*-filtered algebra, and every basic* Δ*-filtered algebra can be obtained in this way.*

While all (basic) quasi-hereditary algebras over a perfect field can be recursively obtained by applying the construction described in Theorem 4.1, the next example illustrates that for standardly stratified algebras in some cases one cannot avoid factorization modulo an auxiliary ideal.

Example 4.6. Consider the algebra $A = KQ/I$, where the quiver Q is



and the admissible ideal $I = \langle \gamma\alpha, \gamma^2 - \alpha\beta, \beta\alpha, \beta\gamma \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & 1 & \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}.$$

Then the construction described in Theorem 4.4 results in $\tilde{A}$ with regular representation as follows:

$$\tilde{A}_{\tilde{A}} = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ 1 & & 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}.$$

Observe that the quiver of A and $\tilde{A}$ coincide, however the products γ^2 and $\alpha\beta$ are not yet indentified in $\tilde{A}$. This is done when we take the quotient modulo the auxiliary ideal $\langle \gamma^2 - \alpha\beta \rangle$. Note that $e_1 A \varepsilon_2 A e_1$ has no subalgebra complement in $e_1 A e_1$ hence A cannot be obtained directly in the form $\tilde{A}$ for a suitable local algebra L.

Example 4.7. Consider the algebra $A = KQ/I'$, where the quiver Q is the same as in Example 4.6 and $I' = \langle \gamma\alpha, \gamma^2, \beta\alpha, \beta\gamma \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & & 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}.$$

Here, the algebra $\tilde{A}$ constructed according to Theorem 4.4 has the following regular representation:

$$\tilde{A}_{\tilde{A}} = \begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & & 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}.$$

Hence in this example the quiver of $\tilde{A}$ differs from the quiver of A, since we get a new arrow corresponding to $\delta = \alpha\beta \in L$. This element is different from the product of arrows α and β, taken in $\tilde{A}$. We get A as a quotient of $\tilde{A}$ modulo the auxiliary ideal $\langle \delta - \alpha\beta \rangle$. Unlike the previous example, in this case we could obtain A directly as an algebra $\tilde{A}$: we would have to start with the local subalgebra $L = \langle e_1, \gamma \rangle$ instead of $e_1 A e_1$.

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STRATIFYING PAIRS OF SUBCATEGORIES FOR CPS-STRATIFIED ALGEBRAS

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Abstract. Two special types of module subcategories are defined over stratified algebras of Cline, Parshall and Scott. We show that for every stratified algebra there exists a (not necessarily unique) pair of subcategories which are the perpendicular categories of each other and which describe to a large extent the stratification structure of the algebra. These subcategories generalize the notion of modules with standard and costandard filtration for standardly stratified and quasi-hereditary algebras.

1. Introduction

In the theory of quasi-hereditary and standardly stratified (Δ or $\bar{\Delta}$-filtered) algebras the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\nabla})$ of modules with standard and proper costandard filtration play a crucial role (see for example [DR], [ADL1], [AHLU]). One of the key homological features of these subcategories is that they are perpendicular to each other. Much of the structure theory and a (limited) left-right symmetry for these algebras stems from this fact. On the other hand so far no such pairing is known for the more general case of so called strictly stratified algebras and CPS-stratified algebras (cf. [ADL2] and [CPS]) and they also lack a reasonable structure theory.

In this article we will present a setting in which to every CPS-stratified algebra A we will associate a perpendicular pair of subcategories so that the corresponding subcategories of modules with appropriate filtration will describe the structure of projective and injective A-modules, in particular, the structure of the regular module itself.

Thus in Section 2 we will define the concept of stratifying and costratifying subcategories and describe their basic properties. We will relate these subcategories to subcategories of modules with standard or costandard filtration over standardly stratified algebras. In Section 3 we will show that by taking the perpendicular category of a stratifying subcategory we get a costratifying subcategory (and vice

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versa). Moreover we show that for each CPS-stratified algebra we can find a pair of stratifying and costratifying subcategories so that each of these subcategories is the perpendicular category of the other (this will be called a stratifying pair). For quasi-hereditary algebras this pair is given by the category of modules with standard and costandard filtration, respectively. Finally we give an example of a CPS-stratified algebra for which an infinite number of stratifying pairs exist.

2. Stratifying and costratifying subcategories

Let K be an arbitrary field and $(A, \mathbf{e})$ a basic finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \ldots, e_n)$ of primitive orthogonal idempotents. Throughout the paper we shall be dealing with right A-modules. In particular, $P(i) = e_i A$ will stand for the ith indecomposable projective module, $Q(i) = \mathrm{Hom}_K(Ae_i, K)$ the ith indecomposable injective module and $S(i) \simeq P(i)/\operatorname{Rad} P(i) \simeq \operatorname{Soc} Q(i)$ the corresponding simple module. The category of all finitely generated right A-modules will be denoted by mod-A. If $\mathcal{C}$ is an arbitrary class of modules in mod-A then $\mathcal{F}(\mathcal{C})$ is the full subcategory of mod-A consisting of modules M with a $\mathcal{C}$-filtration, i. e. a chain of submdoules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ such that the factor modules M_i/M_{i-1} all belong to $\mathcal{C}$.

For $1 \le i \le n$ let us define the subclasses of modules $\mathcal{P}_i(\mathbf{e})$ as

$$\mathcal{P}_i(\mathbf{e}) = \{\, X \in \text{mod-}A \mid X \in \mathcal{F}(S(1), \ldots, S(i)),\ \mathrm{Ext}^t(X, S(j)) = 0\ \forall t \ge 0,\ j < i\,\},$$

and $\mathcal{P}(\mathbf{e}) = \mathcal{F}(\mathcal{P}_1(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e}))$. We call the algebra $(A, \mathbf{e})$ *CPS-stratified* if $A_A \in \mathcal{P}(\mathbf{e})$, i. e. all projective modules are in $\mathcal{P}(\mathbf{e})$ (cf. [CPS], [ADL2], [ADL3]).

Dually, we define the subclasses $\mathcal{Q}_i(\mathbf{e})$ as

$$\mathcal{Q}_i(\mathbf{e}) = \{\, Y \in \text{mod-}A \mid Y \in \mathcal{F}(S(1), \ldots, S(i)),\ \mathrm{Ext}^t(S(j), Y) = 0\ \forall t \ge 0,\ j < i\,\},$$

and $\mathcal{Q}(\mathbf{e}) = \mathcal{F}(\mathcal{Q}_1(\mathbf{e}), \ldots, \mathcal{Q}_n(\mathbf{e}))$. Since $(A, \mathbf{e})$ is CPS-stratified if and only if $(A^{opp}, \mathbf{e})$ is CPS-stratified (see for example [CPS]), all injective modules over a CPS-stratified algebra $(A, \mathbf{e})$ are in $\mathcal{Q}(\mathbf{e})$. Note that the definition implies that $\mathcal{P}_i(\mathbf{e})$ and $\mathcal{Q}_i(\mathbf{e})$ are closed under extensions, direct summands, kernels of epimorphisms, and cokernels of monomorphisms.

For an A-module X, we denote by $T_i(X)$ the trace of the projective module $P(i) \oplus \cdots \oplus P(n)$ in X; thus if $\varepsilon_i = e_i + \ldots + e_n$, then $T_i(X) = X\varepsilon_i A$. In other terms, $T_i(X)$ is the unique submodule of X such that $\mathrm{Hom}(T_i(X), S(j)) = 0$ for all $j < i$, and $X/T_i(X) \in \mathcal{F}(S(1), \ldots, S(i-1))$. Dually, let $R_i(X)$ be the reject of the injective module $Q(i) \oplus \cdots \oplus Q(n)$ in X, i. e. the largest submodule of X such that $R_i(X) \in \mathcal{F}(S(1), \ldots, S(i-1))$. Then $R_i(X)$ is the unique submodule of X for which $R_i(X) \in \mathcal{F}(S(1), \ldots, S(i-1))$ and $\mathrm{Hom}_A(S(j), X/R_i(X)) = 0$ for all $j < i$.

In the sequel we shall frequently make use of the following equivalence (cf. [CPS], [APT] or [ADL3]): the algebra $(A, \mathbf{e})$ is CPS-stratified if and only if for every $X, Y \in \text{mod-}(A/T_i(A))$ $(1 \le i \le n)$ and every $t \ge 0$ we have $\mathrm{Ext}^t_{A/T_i(A)}(X, Y) = \mathrm{Ext}^t_A(X, Y)$.

Lemma 2.1. *Let $X \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$, where $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for all i and take a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$ of X with factors $Y_r = X_r/X_{r-1}$ from $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_n$. Then X has a filtration with the same factors (up to isomorphism) but possibly in different order such that for the factors $Y_1', \ldots, Y_k'$ we have $Y_r' \in \mathcal{P}_{i_r}$ with $i_1 \geq i_2 \geq \ldots \geq i_k$.*

Proof. We use induction on k and the fact that $\operatorname{Ext}^1(\mathcal{P}_i(\mathbf{e}), \mathcal{P}_j(\mathbf{e})) = 0$ for $i > j$. □

Lemma 2.2. *Let $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$. Then $X \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ if and only if the trace factors $T_i(X)/T_{i+1}(X)$ are in $\mathcal{F}(\mathcal{P}_i)$ for $1 \leq i \leq n$.*

Proof. If the factors $T_i(X)/T_{i+1}(X)$ are in $\mathcal{F}(\mathcal{P}_i)$ for each i then clearly $X \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$. For the converse, let us observe first that by Lemma 2.1, we may take a filtration $0 = X_0 \subset \ldots \subset X_k = X$ with factors $Y_r = X_r/X_{r-1} \in \mathcal{P}_{i_r}$ such that $i_1 \geq \cdots \geq i_k$. Let s be the last index such that $i_s \geq i$. Since $\operatorname{Hom}(Y_r, S(j)) = 0$ for all $r \leq s$ and $j < i$, we have $\operatorname{Hom}(X_s, S(j)) = 0$ for $j < i$, and also $X/X_s \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{i-1}) \subseteq \mathcal{F}(S(1), \ldots, S(i-1))$, thus $X_s = T_i(X)$, and the statement follows □

Proposition 2.3. *Suppose that $(A, \mathbf{e})$ is CPS-stratified and let $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ be such that $\mathcal{F}(\mathcal{P}_i)$ are closed under kernels of epimorphisms. Then $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ is also closed under kernels of epimorphisms.*

Proof. We use induction on n, the number of simple modules. $\bar{A}$ will stand for the algebra $A/T_n(A)$ and in general, for $X \in$ mod-A we shall have $\bar{X} = X/T_n(X) \in$ mod-A.

Since for a CPS-stratified algebra $\operatorname{Ext}^t_{\bar{A}}(X, Y) = \operatorname{Ext}^t_A(X, Y)$ for all $X, Y \in$ mod-$\bar{A}$, the subclasses $\mathcal{P}_i(\mathbf{e}) \subseteq$ mod-A and $\mathcal{P}_i(\mathbf{e}') \subseteq$ mod-$\bar{A}$ will be equal for $1 \leq i \leq n-1$, where $\mathbf{e}' = (e_1, \ldots, e_{n-1})$. Thus we get by induction that $\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$ as a subcategory of mod-$\bar{A}$ is closed under kernels of epimorphisms, hence the same holds for $\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$ as a subcategory of mod-A.

Suppose now that $0 \to X \to Y \xrightarrow{g} Z \to 0$ is exact, and $Y, Z \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$. It is easy to see that g_1, the restriction of g maps surjectively $T_n(Y)$ to $T_n(Z)$ and we also have an induced surjection $\bar{Y} \xrightarrow{\bar{g}} \bar{Z}$. Thus by the Snake Lemma we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Ker} g_1 & \to & T_n(Y) & \xrightarrow{g_1} & T_n(Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & Y & \xrightarrow{g} & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Ker} \bar{g} & \to & \bar{Y} & \xrightarrow{\bar{g}} & \bar{Z} & \to & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.2, $T_n(Y), T_n(Z) \in \mathcal{F}(\mathcal{P}_n)$, and since $\mathcal{F}(\mathcal{P}_n)$ is closed under kernels of epimorphisms by assumption, $\operatorname{Ker} g_1 \in \mathcal{F}(\mathcal{P}_n)$. Similarly, $\bar{Y}$ and $\bar{Z}$ are in $\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$, hence by induction we get that $\operatorname{Ker} \bar{g} \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$. This implies that $X \in \mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$, as required. □

Let us recall that a subcategory of mod-A is called *resolving* if it is closed under extensions, direct summands and kernels of epimorphisms, and it contains all projective modules. Thus we have the following statement.

PROPOSITION 2.4. *If the algebra $(A, \mathbf{e})$ is CPS-stratified, then $\mathcal{P}(\mathbf{e})$ is a resolving subcategory of* mod-A.

Proof. From the definition of $\mathcal{P}(\mathbf{e})$ it is clear that it is closed under extensions. Observe that $\mathcal{P}_i(\mathbf{e}) = \mathcal{F}(\mathcal{P}_i(\mathbf{e}))$ is closed under direct summands, hence the fact that $\mathcal{P}(\mathbf{e})$ is closed under direct summands easily follows from Lemma 2.2, using that $T_i(X \oplus Y) = T_i(X) \oplus T_i(Y)$. Next, since $\mathcal{P}_i(\mathbf{e})$ is closed under kernels of epimorphisms, Lemma 2.3 implies that the same holds for $\mathcal{P}(\mathbf{e})$. Finally $A_A \in \mathcal{P}(\mathbf{e})$ holds for a CPS-stratified algebra, so all projective modules are in $\mathcal{P}(\mathbf{e})$. □

DEFINITION. Let $\mathcal{P}$ be a resolving subcategory of mod-A. We say that $\mathcal{P}$ is a *stratifying subcategory* if there are $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$ such that $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$.

LEMMA 2.5. *If $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ is a stratifying subcategory with $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ then $\mathcal{F}(\mathcal{P}_i) = \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$.*

Proof. We only need to prove that $\mathcal{P} \cap \mathcal{P}_i(\mathbf{e}) \subseteq \mathcal{F}(\mathcal{P}_i)$. Suppose $X \in \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$. Then $X \in \mathcal{F}(S(1), \ldots, S(i))$ implies that $T_{i+1}(X) = 0$. Hence $X \in \mathcal{P}_i(\mathbf{e})$ gives $X = T_i(X) = T_i(X)/T_{i+1}(X)$, so $X \in \mathcal{F}(\mathcal{P}_i)$ by Lemma 2.2. □

PROPOSITION 2.6. *A subcategory $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ with $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ is stratifying if and only if each $\mathcal{F}(\mathcal{P}_i)$ is closed under direct summands and kernels of epimorphisms, and $T_i(A_A)/T_{i+1}(A_A) \in \mathcal{F}(\mathcal{P}_i)$.*

Proof. Suppose $\mathcal{P}$ is stratifying. Then $\mathcal{F}(\mathcal{P}_i) = \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$ by Lemma 2.5, hence it is closed under the given operations, since both $\mathcal{P}$ and $\mathcal{P}_i(\mathbf{e})$ are closed. Furthermore, Lemma 2.2 and $A_A \in \mathcal{P}$ implies that $T_i(A)/T_{i+1}(A) \in \mathcal{F}(\mathcal{P}_i)$. In the opposite direction, the last condition implies that $(A, \mathbf{e})$ is CPS-stratified, i. e. all projective A-modules are in $\mathcal{P}$. Clearly, $\mathcal{P}$ is closed under extensions and by Proposition 2.3 $\mathcal{P}$ is closed under kernels of epimorphisms. Finally to prove that $\mathcal{P}$ is closed under direct summands we can follow a similar argument as in the proof of Proposition 2.4. □

We shall also need the duals of the previous statements. The proofs follow by straightforward dualization.

LEMMA 2.7. *Let $X \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$, where $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for all i and take a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$ of X with factors $Y_r = X_r/X_{r-1}$ from $\mathcal{Q}_1 \cup \ldots \cup \mathcal{Q}_n$. Then X has a filtration with the same factors (up to isomorphism) but possibly in different order such that for the factors $Y_1', \ldots, Y_k'$ we have $Y_r' \in \mathcal{Q}_{i_r}$ with $i_1 \leq i_2 \leq \ldots \leq i_k$.*

LEMMA 2.8. *Let $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for $1 \leq i \leq n$. Then $X \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ if and only if the factors $R_{i+1}(X)/R_i(X)$ are in $\mathcal{F}(\mathcal{Q}_i)$ for $1 \leq i \leq n$.*

PROPOSITION 2.9. *Suppose that* $(A, \mathbf{e})$ *is CPS-stratified and let* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *be such that* $\mathcal{F}(\mathcal{Q}_i)$ *are closed under cokernels of monomorphisms. Then* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *is also closed under cokernels of monomorphisms.*

A subcategory of mod-A is called *coresolving* if it is closed under extensions, direct summands and cokernels of monomorphisms, and it contains all injective modules. Thus we have the following statement.

PROPOSITION 2.10. *If the algebra* $(A, \mathbf{e})$ *is CPS-stratified, then* $\mathcal{Q}(\mathbf{e})$ *is a coresolving subcategory of* mod-A.

DEFINITION. Let $\mathcal{Q}$ be a coresolving subcategory of mod-A. We say that $\mathcal{Q}$ is a *costratifying subcategory* if there are $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for $1 \leq i \leq n$ such that $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$.

LEMMA 2.11. *If* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *is a costratifying subcategory with* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *then* $\mathcal{F}(\mathcal{Q}_i) = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$.

PROPOSITION 2.12. *A subcategory* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *with* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *is costratifying if and only if each* $\mathcal{F}(\mathcal{Q}_i)$ *is closed under direct summands and cokernels of monomorphisms, and* $R_{i+1}(D({}_AA))/R_i(D({}_AA)) \in \mathcal{F}(\mathcal{Q}_i)$.

Note that $(A, \mathbf{e})$ is a CPS-stratified algebra if and only if there exists a stratifying subcategory in mod-A, or equivalently, if there exists a costratifying subcategory in mod-A. In fact, for a CPS-stratified algebra $\mathcal{P}(\mathbf{e})$ is the largest stratifying and $\mathcal{Q}(\mathbf{e})$ is the largest costratifying subcategory. Examples of minimal stratifying and costratifying subcategories will be provided by subcategories of modules with standard and costandard filtration.

Let us first recall the definition of standard and costandard modules. For a given algebra $(A, \mathbf{e})$ the *standard module* $\Delta(i)$ is defined as $\Delta(i) = P(i)/T_{i+1}(P(i))$ for $1 \leq i \leq n$. Dually, the costandard module $\nabla(i)$ is defined as $\nabla(i) = R_{i+1}(Q(i))$. The *proper standard module* $\bar{\Delta}(i)$ is the largest quotient of $\Delta(i)$ such that the composition multiplicity $[\bar{\Delta}(i) : S(i)] = 1$. Similarly, the *proper costandard module* $\bar{\nabla}(i)$ is the largest submodule of $\nabla(i)$ such that $[\bar{\nabla}(i) : S(i)] = 1$. Then with the notation $\Delta = \{ \Delta(1), \ldots, \Delta(n) \}$, $\bar{\Delta} = \{ \bar{\Delta}(1), \ldots, \bar{\Delta}(n) \}$, $\nabla = \{ \nabla(1), \ldots, \nabla(n) \}$ and $\bar{\nabla} = \{ \bar{\nabla}(1), \ldots, \bar{\nabla}(n) \}$ we get the subcategories $\mathcal{F}(\Delta)$, $\mathcal{F}(\bar{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\bar{\nabla})$. An algebra is called Δ-*filtered* if $A_A \in \mathcal{F}(\Delta)$ and $\bar{\Delta}$-*filtered* if $A_A \in \mathcal{F}(\bar{\Delta})$. The algebra is *standardly stratified* if either A or A^{opp} is Δ-filtered. It is easy to see that A^{opp} is Δ-filtered if and only if $D({}_AA) \in \mathcal{F}(\nabla)$ and it is well-known (cf. [D]) that these conditions are equivalent to $A_A \in \mathcal{F}(\bar{\Delta})$, i. e. that A is $\bar{\Delta}$-filtered.

PROPOSITION 2.13. *Let* $(A, \mathbf{e})$ *be CPS-stratified, and* $\mathcal{P}$ *a stratifying subcategory. Then* $\mathcal{F}(\Delta) \subseteq \mathcal{P} \subseteq \mathcal{P}(\mathbf{e})$. *Furthermore,* $\mathcal{F}(\Delta)$ *is a stratifying subcategory if and only if* $A_A \in \mathcal{F}(\Delta)$. *Dually, for every costratifying subcategory* $\mathcal{Q}$ *we have* $\mathcal{F}(\nabla) \subseteq \mathcal{Q} \subseteq \mathcal{Q}(\mathbf{e})$, *moreover* $\mathcal{F}(\nabla)$ *is a costratifying subcategory if and only if* $D({}_AA) \in \mathcal{F}(\nabla)$ *(i. e.* A *is* $\bar{\Delta}$*-filtered).*

Proof. Since $P(i) \in \mathcal{P}$, we get $\Delta(i) = P(i)/T_{i+1}(P(i)) = T_i(P(i))/T_{i+1}(P(i)) \in \mathcal{P}_i$ by Lemma 2.2, so $\mathcal{F}(\Delta) \subseteq \mathcal{P}$.

It is clear that $\Delta(i)$ is the ith projective indecomposable module over $A/T_{i+1}(A)$. Since $T_{i+1}(A)$ is an idempotent ideal, the category $\mathcal{F}(\Delta(i))$ is the same over A as over the factor algebra $A/T_{i+1}(A)$, so it consists of the direct sums of copies of $\Delta(i)$. Consequently $\mathcal{F}(\Delta(i))$ is closed under direct summands and kernels of epimorphisms. If in addition, $A_A \in \mathcal{F}(\Delta)$, then by Lemma 2.2, $T_i(A)/T_{i+1}(A) \in \mathcal{F}(\Delta(i))$ for all i, so by Proposition 2.6, $\mathcal{F}(\Delta)$ is a stratifying subcategory.

The proof of the dual statement is omitted. □

Later we shall see that $\mathcal{F}(\bar{\Delta})$ is also a stratifying subcategory if A is $\bar{\Delta}$-filtered and $\mathcal{F}(\bar{\nabla})$ is a costratifying subcategory if A is Δ-filtered.

3. Stratifying pairs of subcategories

For a subcategory $\mathcal{C}$ of mod-A we use the notation

$$\mathcal{C}^{\perp} = \mathcal{C}^{\perp}_A = \{\, Y \in \text{mod-}A \,|\, \operatorname{Ext}^t(X,Y) = 0 \ \forall\ t > 0 \text{ and } X \in \mathcal{C}\,\}.$$

and

$$^{\perp}\mathcal{C} = {}^{\perp}\mathcal{C}_A = \{\, X \in \text{mod-}A \,|\, \operatorname{Ext}^t(X,Y) = 0 \ \forall\ t > 0 \text{ and } Y \in \mathcal{C}\,\}.$$

It is clear that if $\mathcal{C}$ is resolving (coresolving, respectively) then in the above definitions it is enough to require that $\operatorname{Ext}^1(X,Y) = 0$.

LEMMA 3.1. *Let $\mathcal{P}$ be a stratifying subcategory. Then*

(1) $\bar{\mathcal{P}} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}) = \mathcal{P} \cap \mathcal{F}(S(1), \ldots, S(n-1))$ *and $\bar{\mathcal{P}}$ is a stratifying subcategory over $\bar{A} = A/T_n(A)$;*

(2) $\bar{\mathcal{P}}^{\perp}_{\bar{A}} = \mathcal{P}^{\perp}_A \cap \mathcal{F}(S(1), \ldots, S(n-1))$.

Proof. (1) The equation $\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}) = \mathcal{P} \cap \mathcal{F}(S(1), \ldots, S(n-1))$ immediately follows from Lemma 2.2. Now we prove that $\bar{\mathcal{P}}$ is a stratifying subcategory over the factor algebra $\bar{A}$. As in the proof of Proposition 2.3 we get that the subclasses $\mathcal{P}_i(\mathbf{e}) \subseteq \text{mod-}A$ and $\mathcal{P}_i(\mathbf{e}') \subseteq \text{mod-}\bar{A}$ are the same for $1 \le i \le n-1$ and $\mathbf{e}' = (e_1, \ldots, e_{n-1})$, so $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e}')$. Furthermore, the subcategory $\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$ is the same over the two algebras, hence the required closure properties also hold. Finally, the projective modules of $\bar{A}$ are the factors $\bar{P} = P/T_n(P)$ of projective modules P over A, and Lemma 2.2 implies that $\bar{P} \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$.

(2) $\bar{\mathcal{P}}^{\perp}_{\bar{A}} \supseteq \mathcal{P}^{\perp}_A \cap \mathcal{F}(S(1), \ldots, S(n-1))$ is clear from the fact that A is CPS-stratified. We only need to show that $\bar{\mathcal{P}}^{\perp}_{\bar{A}} \subseteq \mathcal{P}^{\perp}_A$. But for any $Y \in \bar{\mathcal{P}}^{\perp}_{\bar{A}}$ we have $Y \in \mathcal{P}^{\perp}_n$, since $Y \in \mathcal{F}(S(1), \ldots, S(n-1))$, so $Y \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}, \mathcal{P}_n)^{\perp} = \mathcal{P}^{\perp}$. □

THEOREM 3.2. *If $\mathcal{P}$ is a stratifying subcategory over $(A, \mathbf{e})$ then $\mathcal{P}^{\perp}$ is costratifying. Dually, if $\mathcal{Q}$ is a costratifying subcategory over $(A, \mathbf{e})$ then ${}^{\perp}\mathcal{Q}$ is stratifying.*

For the proof we need two preparatory lemmas.

LEMMA 3.3. *Let $\mathcal{P}$ be a stratifying subcategory. If $Y \in \mathcal{P}^{\perp}$ such that $R_n(Y) = 0$, then $Y \in \mathcal{Q}_n(\mathbf{e})$.*

Proof. The condition $R_n(Y) = 0$ yields that $\mathrm{Hom}(S(i), Y) = 0$ for all $i < n$. By Proposition 2.13, $\Delta(i) \in \mathcal{P}_i$. Let us consider for some fixed $i < n$ the exact sequence

$$O \to U \to \Delta(i) \to S(i) \to 0.$$

This yields the long exact sequence

$$\cdots \to \mathrm{Hom}(U, Y) \to \mathrm{Ext}^1(S(i), Y) \to \mathrm{Ext}^1(\Delta(i), Y) \to \cdots$$

Here $\mathrm{Hom}(U, Y) = 0$ since $U \in \mathcal{F}(S(1), \ldots, S(i))$ and $\mathrm{Hom}(S(j), Y) = 0$ for $j < n$, furthermore $\mathrm{Ext}^1(\Delta(i), Y) = 0$, because $Y \in \mathcal{P}^{\perp}$. Thus $\mathrm{Ext}^1(S(i), Y) = 0$, and this is true for all $i < n$. Now we can prove by induction that $\mathrm{Ext}^t(S(i), Y) = 0$ for all $t > 0$ and $i < n$, using the following segment of the above sequence:

$$\cdots \to \mathrm{Ext}^t(U, Y) \to \mathrm{Ext}^{t+1}(S(i), Y) \to \mathrm{Ext}^{t+1}(\Delta(i), Y) \to \cdots$$

□

LEMMA 3.4. *Let $\mathcal{P}$ be a stratifying subcategory. If $Y \in \mathcal{P}^{\perp}$, then $R_n(Y) \in \mathcal{P}^{\perp}$ and $\widetilde{Y} = Y/R_n(Y) \in \mathcal{P}^{\perp}$.*

Proof. Consider the exact sequence

$$0 \to R_n(Y) \to Y \to \widetilde{Y} \to 0$$

and let us take a module $X \in \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{n-1})$. Then we get the induced long exact sequence

$$\cdots \to \mathrm{Hom}(X, \widetilde{Y}) \to \mathrm{Ext}^1(X, R_n(Y)) \to \mathrm{Ext}^1(X, Y) \to \cdots$$

Here $X \in \mathcal{F}(S(1), \ldots, S(n-1))$ implies that $\mathrm{Hom}(X, \widetilde{Y}) = 0$, furthermore by assumption $\mathrm{Ext}^1(X, Y) = 0$, thus $\mathrm{Ext}^1(X, R_n(Y)) = 0$. On the other hand, if $X \in \mathcal{P}_n$ then $\mathrm{Ext}^1(X, R_n(Y)) = 0$ since $R_n(Y) \in \mathcal{F}(S(1), \ldots, S(n-1))$. Thus $\mathrm{Ext}^1(X, R_n(Y)) = 0$ for any $X \in \mathcal{P}$. Using the fact that $\mathcal{P}$ is a resolving subcategory, we get that $R_n(Y) \in \mathcal{P}^{\perp}$. From the same long exact sequence we get now that $\mathrm{Ext}^t(X, \widetilde{Y}) = 0$ for all $t > 0$ and $X \in \mathcal{P}$, i. e. $\widetilde{Y} \in P^{\perp}$ as well. □

Proof of Theorem 3.2. Clearly, $\mathcal{Q} = \mathcal{P}^{\perp}$ is a coresolving subcategory. We only need to show that $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$, where $\mathcal{Q}_i = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$. By Lemma 3.4, every element of $\mathcal{Q}$ is filtered by $\mathcal{Q} \cap \mathcal{F}(S(1), \ldots, S(n-1)) \cup \{ Y \in \mathcal{Q} \mid R_n(Y) = 0 \}$. By Lemma 3.1 we know that $\mathcal{Q} \cap \mathcal{F}(S(1), \ldots, S(n-1)) = \bar{\mathcal{P}}^{\perp}_{\bar{A}}$ with $\bar{A} = A/T_n(A)$. Thus we may use induction on the number of simple types to get that $\mathcal{Q} \cap \mathcal{F}(S(1), \ldots, S(n-1)) = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})$. On the other hand, by Lemma 3.3 $\{ Y \in \mathcal{Q} \mid R_n(Y) = 0 \} = \mathcal{Q} \cap \mathcal{Q}_n(\mathbf{e})$, so $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$.

The dual statement can be proved along the same lines. □

Definition. A pair $(\mathcal{P}, \mathcal{Q})$ of subcategories in mod-A is called a *stratifying pair* if $\mathcal{P}$ is a stratifying subcategory and $\mathcal{Q}$ is a costratifying subcategory such that $\mathcal{P}^{\perp} = \mathcal{Q}$ and ${}^{\perp}\mathcal{Q} = \mathcal{P}$.

It is easy to find stratifying pairs over standardly strtaified algebras.

Theorem 3.5. *Let $(A, \mathbf{e})$ be a standardly stratified algebra.*

(1) *If A is $\bar{\Delta}$-filtered then $(\mathcal{F}(\bar{\Delta}), \mathcal{F}(\nabla))$ is a stratifying pair, and $\mathcal{F}(\bar{\Delta}) = \mathcal{P}(\mathbf{e})$.*
(2) *If A is Δ-filtered then $(\mathcal{F}(\Delta), \mathcal{F}(\bar{\nabla}))$ is a stratifying pair, and $\mathcal{F}(\bar{\nabla}) = \mathcal{Q}(\mathbf{e})$.*

Proof. We shall prove only (1); then (2) will follow by duality.

If A is $\bar{\Delta}$-filtered, then from Proposition 2.13 we get that $\mathcal{F}(\nabla)$ is a costratifying subcategory. Theorem 3.1 of [ADL1] implies that $\mathcal{F}(\bar{\Delta}) = {}^{\perp}\mathcal{F}(\nabla)$, hence by Theorem 3.2 we get that $\mathcal{F}(\bar{\Delta})$ is a stratifying subcategory. In order to prove that $(\mathcal{F}(\bar{\Delta}), \mathcal{F}(\nabla))$ is a stratifying pair we still have to show that $\mathcal{F}(\nabla) = \mathcal{F}(\bar{\Delta})^{\perp}$. We use induction on the number of simple types.

The statement clearly holds for $n = 1$, since in this case $\mathcal{F}(\bar{\Delta}) = \text{mod-}A$, and $\mathcal{F}(\nabla)$ is the category of injective modules. Now let $n > 1$ and assume that the statement is true for the $\bar{\Delta}$-filtered algebra $(\bar{A}, \mathbf{e}')$ with $\bar{A} = A/T_n(A)$ and $\mathbf{e}' = (e_1, \ldots, e_{n-1})$. Then $\mathcal{F}_A(\bar{\Delta})^{\perp} \cap \mathcal{Q}(\mathbf{e}') = \mathcal{F}_{\bar{A}}(\bar{\Delta})^{\perp}$ by Lemma 3.1, and the induction hypothesis implies that $\mathcal{F}_{\bar{A}}(\bar{\Delta})^{\perp} = \mathcal{F}_{\bar{A}}(\nabla) = \mathcal{F}_A(\nabla(1), \ldots, \nabla(n-1))$. On the other hand, we shall prove that every module $Y \in \mathcal{F}(\bar{\Delta})^{\perp} \cap \mathcal{Q}_n(\mathbf{e})$ is injective, i. e. $\text{Ext}^t(S(i), Y) = 0$ for all $i = 1, \ldots, n$ and $t > 0$. Indeed, the fact that $\text{Ext}^t(S(i), Y) = 0$ for $i < n$ and $t \geq 0$ follows from $Y \in \mathcal{Q}_n(\mathbf{e})$. Let us take the exact sequence

$$0 \to U \to \bar{\Delta}(n) \to S(n) \to 0$$

and the induced long exact sequence

$$\cdots \to \text{Ext}^t(U, Y) \to \text{Ext}^{t+1}(S(n), Y) \to \text{Ext}^{t+1}(\bar{\Delta}(n), Y) \to \cdots$$

Here $\text{Ext}^t(U, Y) = 0$ holds for $t \geq 0$ since $U \in \mathcal{F}(S(1), \ldots, S(n-1))$, while $\text{Ext}^{t+1}(\bar{\Delta}(n), Y) = 0$ follows from $Y \in \mathcal{F}(\bar{\Delta})^{\perp}$, so $\text{Ext}^{t+1}(S(n), Y) = 0$ for $t \geq 0$. This shows that $\text{Ext}^t(S(i), Y) = 0$ for $i = 1, \ldots, n$ and $t > 0$, i. e. Y is injective. Thus $\mathcal{F}(\bar{\Delta})^{\perp} \cap \mathcal{Q}_n(\mathbf{e}) = \mathcal{F}(\nabla(n))$. Since $\mathcal{F}(\bar{\Delta})^{\perp}$ is a costratifying subcategory, Lemma 2.11 implies that $\mathcal{F}(\bar{\Delta})^{\perp} = \mathcal{F}(\nabla)$.

Finally, $\mathcal{F}(\bar{\Delta}) \subseteq \mathcal{P}(\mathbf{e})$ implies $\mathcal{P}(\mathbf{e})^{\perp} \subseteq \mathcal{F}(\bar{\Delta})^{\perp} = \mathcal{F}(\nabla)$ and since $\mathcal{P}(\mathbf{e})^{\perp}$ is a costratifying subcategory, we get from Proposition 2.13 that $\mathcal{F}(\nabla) \subseteq \mathcal{P}(\mathbf{e})^{\perp}$. Thus $\mathcal{P}(\mathbf{e})^{\perp} = \mathcal{F}(\nabla)$. In this way we get $\mathcal{P}(\mathbf{e}) \subseteq {}^{\perp}(\mathcal{P}(\mathbf{e})^{\perp}) = {}^{\perp}\mathcal{F}(\nabla) = \mathcal{F}(\bar{\Delta}) \subseteq \mathcal{P}(\mathbf{e})$, hence $\mathcal{F}(\bar{\Delta}) = \mathcal{P}(\mathbf{e})$. □

Our next goal is to find stratifying pairs of subcategories for arbitrary CPS-stratified algebras. The first statement of the next proposition is an immediate consequence of Theorem 3.2.

THEOREM 3.6. *Suppose $(A, \mathbf{e})$ is a CPS-stratified algebra.*

(1) *If $\mathcal{P}$ is a stratifying subcategory in* mod-A *then $({}^\perp(\mathcal{P}^\perp), \mathcal{P}^\perp)$ is a stratifying pair. Dually, if $\mathcal{Q}$ is a costratifying subcategory in* mod-A *then $({}^\perp\mathcal{Q}, ({}^\perp\mathcal{Q})^\perp)$ is a stratifying pair. In particular, $(\mathcal{P}(\mathbf{e}), \mathcal{P}(\mathbf{e})^\perp)$ and $({}^\perp\mathcal{Q}(\mathbf{e}), \mathcal{Q}(\mathbf{e}))$ are stratifying pairs.*
(2) *Let $\mathcal{M} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n$, where $\mathcal{M}_i \subseteq \mathcal{P}_i(\mathbf{e})$. Then $\mathcal{Q} = \mathcal{M}^\perp \cap \mathcal{Q}(\mathbf{e})$ is a costratifying subcategory. Furthermore, $({}^\perp\mathcal{Q}, \mathcal{Q})$ form a stratifying pair in* mod-A.

Proof. Theorem 3.2 implies that the mappings $\mathcal{P} \mapsto \mathcal{P}^\perp$ and $\mathcal{Q} \mapsto {}^\perp\mathcal{Q}$ define an order reversing Galois connection between the set of stratifying and the set of costratifying subcategories. Hence the first statement of (1) follows immediately. Since $\mathcal{P}(\mathbf{e})$ is the largest stratifying subcategory, ${}^\perp(\mathcal{P}(\mathbf{e})^\perp) = \mathcal{P}(\mathbf{e})$, hence $(\mathcal{P}(\mathbf{e}), \mathcal{P}(\mathbf{e})^\perp)$ is a stratifying pair. The dual statements follow similarly.

Let us now prove the statements in (2). Since $\mathcal{M}^\perp$ and $\mathcal{Q}(\mathbf{e})$ are clearly coresolving, so is their intersection.

We shall use the notation $\mathcal{Q}_i = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$ and $\overline{\mathcal{M}} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{n-1}$. We still need to prove that $\mathcal{Q} = \mathcal{F}(Q_1, \ldots, \mathcal{Q}_n)$; we shall use induction on n. Thus we may assume that the statement holds for $\overline{\mathcal{M}} \subseteq \text{mod-}\bar{A}$ with $\bar{A} = A/T_n(A)$, i. e. $\overline{M}^\perp \cap \mathcal{Q}(\mathbf{e}') = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})$ with $\mathbf{e}' = (e_1, \ldots, e_{n-1})$.

For $Y \in \mathcal{Q}$ let us consider the short exact sequence

$$0 \to R_n(Y) \to Y \to \widetilde{Y} \to 0.$$

Lemma 2.8 implies that $R_n(Y) \in \mathcal{Q}(\mathbf{e}')$ and $\widetilde{Y} \in \mathcal{Q}_n(\mathbf{e})$. For arbitrary $X_1 \in \overline{\mathcal{M}}$ we get the long exact sequence

$$\cdots \to \operatorname{Ext}^t(X_1, \widetilde{Y}) \to \operatorname{Ext}^{t+1}(X_1, R_n(Y)) \to \operatorname{Ext}^{t+1}(X_1, Y) \to \cdots$$

Here $\operatorname{Ext}^t(X_1, \widetilde{Y}) = 0$ for $t \geq 0$, since $X_1 \in \mathcal{F}(S(1), \ldots, S(n-1))$ and $\widetilde{Y} \in \mathcal{Q}_n(\mathbf{e})$, while $\operatorname{Ext}^{t+1}(X_1, Y) = 0$ follows from $Y \in \mathcal{M}^\perp$. Thus the middle term is also 0, so $R_n(Y) \in \overline{M}^\perp \cap \mathcal{Q}(\mathbf{e}')$, i. e. $R_n(Y) \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})$ by the induction hypothesis. On the other hand, for any $X_2 \in \mathcal{M}_n \subseteq \mathcal{P}_n(\mathbf{e})$ we have $\operatorname{Ext}^t(X_2, R_n(Y)) = 0$ for all $t \geq 0$, since $R_n(Y) \in \mathcal{F}(S(1), \ldots, S(n-1))$. Thus, putting together the two cases, we get that $R_n(Y) \in \mathcal{M}^\perp$.

Let us take now an arbitrary $X \in \mathcal{M}$. Then in the long exact sequence

$$\cdots \to \operatorname{Ext}^t(X, Y) \to \operatorname{Ext}^t(X, \widetilde{Y}) \to \operatorname{Ext}^{t+1}(X, R_n(Y)) \to \cdots$$

$\operatorname{Ext}^t(X, Y) = \operatorname{Ext}^{t+1}(X, R_n(Y)) = 0$ for all $t > 0$. Thus $\widetilde{Y} \in \mathcal{M}^\perp \cap \mathcal{Q}_n(\mathbf{e}) = \mathcal{Q}_n$, so $Y \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$. This proves that $\mathcal{Q}$ is a costratifying subcategory.

It follows from Theorem 3.2 that ${}^\perp\mathcal{Q}$ is a stratifying subcategory. The inclusion $\mathcal{M} \subseteq {}^\perp(\mathcal{M}^\perp \cap \mathcal{Q}(\mathbf{e})) = {}^\perp\mathcal{Q}$ implies that $({}^\perp\mathcal{Q})^\perp \subseteq \mathcal{M}^\perp$. Since $({}^\perp\mathcal{Q})^\perp \subseteq \mathcal{Q}(\mathbf{e})$ clearly holds, we have $({}^\perp\mathcal{Q})^\perp \subseteq {}^\perp\mathcal{M} \cap \mathcal{Q}(\mathbf{e}) = \mathcal{Q}$. Finally, since the containment $({}^\perp\mathcal{Q})^\perp \supseteq \mathcal{Q}$ is obvious, we get from part (1) that $({}^\perp\mathcal{Q}, \mathcal{Q})$ is a stratifying pair. □

In general, not every perpendicular pair of modules $X \in \mathcal{P}(\mathbf{e})$ and $Y \in \mathcal{Q}(\mathbf{e})$ are contained in a stratifying pair of subcategories. However we have the following characterization.

Proposition 3.7. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra. For arbitrary modules $X \in \mathcal{P}(\mathbf{e})$ and $Y \in \mathcal{Q}(\mathbf{e})$ the following are equivalent.*

(i) *There is a stratifying pair $(\mathcal{P}, \mathcal{Q})$ in* mod-A *such that $X \in \mathcal{P}$ and $Y \in \mathcal{Q}$.*
(ii) $X_i = T_i(X)/T_{i+1}(X) \in {}^{\perp}\{ Y \}$ *for* $i = 1, \ldots, n$.
$(ii)'$ $Y_i = R_{i+1}(Y)/R_i(Y) \in \{ X \}^{\perp}$ *for* $i = 1, \ldots, n$.
(iii) $\mathrm{Ext}^t(X_i, Y_i) = 0$ *for all* $t > 0$ *and* $i = 1, \ldots, n$.

Proof. $(i) \Rightarrow (iii)$: By Lemma 2.2 and 2.8, $X_i \in \mathcal{P}_i \subseteq \mathcal{P}$ and $Y_i \in \mathcal{Q}_i \subseteq \mathcal{Q}$, so (iii) follows from the fact that $\mathcal{P}$ and $\mathcal{Q}$ are perpendicular to each other.

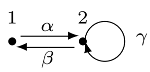
$(iii) \Rightarrow (ii)$: Since $X_i \in \mathcal{P}_i(\mathbf{e})$ and $Y_j \in \mathcal{Q}_j(\mathbf{e})$ by Lemma 2.2 and 2.8, we have $\mathrm{Ext}^t(X_i, Y_j) = 0$ for all $i \neq j$ and $t \geq 0$. Together with (iii) this implies that $\mathrm{Ext}^t(X_i, Y) = 0$ for all $t > 0$ and $i = 1, \ldots, n$.

$(ii) \Rightarrow (i)$: Take $\mathcal{M}_i = \{ X_i \}$ in part (2) of Theorem 3.6. Then $Y \in \mathcal{Q}$. Furthermore $X_i \in {}^{\perp}\mathcal{Q} = \mathcal{P}$ for $i = 1 \ldots, n$, implying that $X \in \mathcal{P}$.

Finally, $(iii) \Rightarrow (ii') \Rightarrow (i)$ follows by duality. □

A similar statement can be formulated giving a condition for arbitrary subclasses of $\mathcal{P}(\mathbf{e})$ and $\mathcal{Q}(\mathbf{e})$ to be included into a stratifying pair.

It is easy to see that if $(A, \mathbf{e})$ is *quasi-hereditary* (i. e. standardly stratified with $\Delta(i) = \bar{\Delta}(i)$ for $i = 1, \ldots, n$) then $(\mathcal{P}(\mathbf{e}), \mathcal{Q}(\mathbf{e}))$ is the only stratifying pair. This follows from the fact that by Proposition 2.13 and Theorem 3.5 $\mathcal{F}(\Delta) = \mathcal{F}(\bar{\Delta}) = \mathcal{P}(\mathbf{e})$ is the only stratifying subcategory. Non-quasi-hereditary examples with a unique stratifying pair can also be found. On the other hand the following example shows that for a general CPS-stratified algebra there may be even infinitely many different stratifying pairs of subcategories.

Example 3.8. Let $A = K\Gamma/I$, where Γ:  and $I = (\alpha\gamma, \beta\alpha\beta, \gamma^2)$.

Then

$$A_A = \begin{smallmatrix}1\\2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&2\\2&&1\\&2&\end{smallmatrix} \quad \text{and} \quad D({}_AA) = \begin{smallmatrix}1&2\\2&\\1&\end{smallmatrix} \oplus \begin{smallmatrix}1&2&\\2&&\\1&&2\\&2&\end{smallmatrix}.$$

Let $M = \begin{smallmatrix}2\\1\\2\end{smallmatrix}$, $N = \begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$, and $M_c = P(2)/(\beta\alpha - c\gamma)$ for any $0 \neq c \in K$. Then $\mathcal{M} = \{ M, M_c \,|\, 0 \neq c \in K \} \subseteq \mathcal{P}_2(\mathbf{e})$ and $\mathcal{N} = \{ N, M_c \,|\, 0 \neq c \in K \} \subseteq \mathcal{Q}_2(\mathbf{e})$. For any subset L of $K \setminus \{ 0 \}$ take $\mathcal{M}_L = \{ S(1), M, M_c \,|\, c \in L \}$. By Theorem 3.6, $\mathcal{Q}_L = \mathcal{M}_L^{\perp} \cap \mathcal{Q}(\mathbf{e})$ and $\mathcal{P}_L = {}^{\perp}\mathcal{Q}_L$ form a stratifying pair of subcategories. Easy calculation shows that $\mathcal{Q}_L \cap \mathcal{N} = \{ N, M_d \,|\, d \in K \setminus (\{ 0 \} \cup L) \}$, furthermore $\mathcal{P}_L \cap \mathcal{M} \subseteq {}^{\perp}(\mathcal{Q}_L \cap \mathcal{N}) \cap \mathcal{M} \subseteq \mathcal{M}_L$. Since the other inclusion is obvious, we have that $\mathcal{P}_L \cap \mathcal{M} = \mathcal{M}_L$, i. e. for each choice of $L \subseteq K \setminus \{ 0 \}$ we get a different stratifying pair.

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CONSTRUCTION OF CPS-STRATIFIED ALGEBRAS

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Abstract. The results of [DR] and [ADL2] gave a recursive construction for all quasi-hereditary and standardly stratified algebras starting with local algebras and suitable bimodules. Using the notion of stratifying pairs of subcategories, introduced in [AL], we generalize these earlier results to construct recursively all CPS-stratified algebras.

1. Introduction

Ever since their introduction by Cline, Parshall and Scott in the late 1980's quasi-hereditary algebras have drawn a lot of attention and they keep playing an important role. One of the key defining features of these algebras is the way how they are put together from simpler algebras (cf. the notions of *recollement* and *partial recollement*). Much of the homological properties and of the structure theory developed for quasi-hereditary algebras carry over to the class of so called *standardly stratified algebras* which is the most straightforward generalization of the original concept. On the other hand for so-called CPS-stratified algebras, which rely on the notion of *stratifying ideals*, defined by Cline, Parshall and Scott in [CPS] (but also investigated earlier by Auslander, Plateck and Todorov in [APT]) and which seem to be the most general class definable in terms of stratification, no such generalization is known. In particular the lack of adequate structure theory makes it more difficult to handle some general questions concerning these algebras.

In an attempt to provide a basis for such a structure theory, the notion of stratifying pairs of module subcategories was introduced in [AL]. This notion was modelled on the subcategories of modules with standard and costandard filtration over standardly stratified algebras. Their homological behaviour is to a large extent determined by the fact that modules in such a pair of subcategories are perpendicular to each other (actually, the individual strata of these modules also have this property), moreover the strata will also satisfy some further homological conditions.

In the present paper we use the language of stratifying subcategories and stratifying pairs to extend earlier results of [DR] and of [ADL2]. Namely, each of these classes, i. e. quasi-hereditary algebras, standardly stratified algebras and CPS-stratified algebras – when defined for an algebra together with a complete

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ordering on a complete set of orthogonal idempotents – come with two sequences of algebras: one is a sequence of quotient algebras and the second a sequence of centralizers (i. e. endomorphism algebras of projective modules). While it is more common to deal with these classes via the recursive approach which uses the sequence of consecutive factors, the papers mentioned earlier ([DR] and [ADL2]) deal with the sequence of centralizers and bimodules with appropriate filtration, and give an explicit construction for quasi-hereditary and standardly stratified algebras. In this way one can obtain each quasi-hereditary and standardly stratified algebra, starting with local algebras. We extend this result to the class of CPS-stratified algebras.

In section 2 of the paper we establish a few results about the functorial connection between stratifying subcategories for the original algebra and stratifying subcategories for the centralizer algebras. Then in section 3 first we give precise conditions in terms of the Peirce decompositions of the algebra to be CPS-stratified. Finally we show how these conditions can be applied to construct from suitable algebras and bimodules a CPS-stratified algebra. We also show that this construction is universal in the sense that every CPS-stratified algebra can be obtained this way, starting with local algebras. We conclude with examples.

For background and unexplained notions concerning quasi-hereditary and standardly stratified algebras we refer for example to [DR], [ADL2] and the literature quoted there, however we shall not need them in this paper.

2. Stratifying subcategories and centralizers

A will always denote a basic finite dimensional algebra over a field K. Modules – unless otherwise stated – will be right modules and mod-A (or A-mod) will stand for the category of finitely generated right A-modules (left A-modules, respectively).

Let us recall some of the basic characterizations of so-called stratifying ideals.

Definition. (Cf. [CPS], [APT], [ADL2]) An idempotent ideal AeA of the algebra A (with $e^2 = e \in A$) is called *stratifying* if it satisfies any of the following equivalent conditions (S1), (S1′), (S2), (S3):

(S1) (*i*) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and

(*ii*) $\mathrm{Tor}_t^{eAe}(Ae, eA) = 0$ for all $t > 0$;

(S1′) (*i*) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and

(*ii*) $\mathrm{Ext}^t_{eAe}(Ae, D(eA)) = 0$ for all $t > 0$, where D denotes K-duality;

(S2) $\mathrm{Ext}^t_{A/AeA}(X, Y) = \mathrm{Ext}^t_A(X, Y)$ for all $t \geq 0$ and $X, Y \in$ mod-A/AeA;

(S3) Each term in the minimal projective resolution of AeA_A is generated by eA.

The last condition is of particular interest to us.

Definition. Let $e \in A$ be an idempotent element. The subcategory $\mathcal{P}(e)$ consists of all those A-modules for which there is a projective resolution with all projective terms in $\mathrm{add}(eA)$. In particular, AeA is a stratifying ideal if and only if $AeA \in \mathcal{P}(e)$. Dually, $\mathcal{Q}(e)$ consists of all those A modules for wich there is an injective resolution with all injective terms in $\mathrm{add}(D(Ae))$.

It is easy to see that $M \in \mathcal{P}(e)$ if and only if $\operatorname{Ext}^t_A(M,N)=0$ for $t \geq 0$ and all modules N with $Ne=0$. This implies that $\mathcal{P}(e)$ is closed under extensions, direct summnads and kernels of epimorphisms.

For $\mathcal{C} \subseteq$ mod-A we use the notation $\mathcal{F}(\mathcal{C})$ for the class of modules filtered by elements of $\mathcal{C}$. Furthermore, take

$$\mathcal{C}^{\perp} = \mathcal{C}^{\perp}_A = \left\{ N \in \text{mod-}A \,|\, \operatorname{Ext}^t_A(M,N)=0 \ \forall\ t>0 \text{ and } M \in \mathcal{C} \right\},$$
$$^{\perp}\mathcal{C} = {}^{\perp}\mathcal{C}_A = \left\{ M \in \text{mod-}A \,|\, \operatorname{Ext}^t_A(M,N)=0 \ \forall\ t>0 \text{ and } N \in \mathcal{C} \right\}.$$

It is obvious that $^{\perp}\mathcal{C}$ is always a *resolving* subcategory in mod-A, i. e. it is closed under extensions, direct summands and kernels of epimorphisms, and it contains the projective modules, and similarly, $\mathcal{C}^{\perp}$ is necessarily a *coresolving* subcategory in mod-A, i. e. it is closed under extensions, direct summands and cokernels of monomorphisms, and it contains the injective modules.

Note that if $\mathcal{C}$ is resolving (coresolving, respectively) then in the above definitions it is enough to require that $\operatorname{Ext}^1(M,N)=0$.

We list here some further homological properties of modules in $\mathcal{P}(e)$. (A version of) the next statment can be found in [APT]. For the convenience of the readers we include a proof.

LEMMA 2.1. *Let AeA be a stratifying ideal and $X \in \mathcal{P}(e)$. Then:*
(*a*) $\operatorname{Tor}^{eAe}_t(Xe, eA) = 0$ *for all* $t>0$*;*
(*b*) $X \simeq Xe \underset{eAe}{\otimes} eA$.

Proof. Let us take the projective cover of X:

$$0 \to \Omega \to P \to X \to 0.$$

We can apply to this sequence the exact functor $\operatorname{Hom}_A(eA,-)$ to obtain

$$0 \to \Omega e \to Pe \to Xe \to 0 \tag{1}$$

and then the functor $- \underset{eAe}{\otimes} eA$ to get the exact sequence:

$$0 \to K \to \Omega e \underset{eAe}{\otimes} eA \to Pe \underset{eAe}{\otimes} eA \to Xe \underset{eAe}{\otimes} eA \to 0 \tag{2}$$

where $K = \operatorname{Tor}^{eAe}_1(Xe,eA)$. Note that the exactness follows from the fact that $\operatorname{Tor}^{eAe}_1(Pe,eA)=0$ by (S1)(*ii*). We can also observe that $Ke=0$ since by applying $\operatorname{Hom}_A(eA,-)$ to (2) we get back (1) with Ke standing in place of 0.

Let us factor out K from the first two non-zero terms of (2) and apply the natural multiplication maps β and γ to the last two terms to get the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & (\Omega e \underset{eAe}{\otimes} eA)/K & \to & Pe \underset{eAe}{\otimes} eA & \to & Xe \underset{eAe}{\otimes} eA & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & \Omega & \to & P & \to & X & \to & 0
\end{array}$$

$X \in \mathcal{P}(e)$ implies that $PeA = P$ and $XeA = X$, hence γ is surjective and by (S1)(i) the map β is an isomorphism. Thus, by the Snake lemma $\operatorname{Ker}\gamma \simeq \operatorname{Coker}\alpha$ and $\operatorname{Ker}\alpha = 0$. Since $(Xe \underset{eAe}{\otimes} eA)e \simeq Xe$, we get that $(\operatorname{Ker}\gamma)e = 0$. On the other hand, $X \in \mathcal{P}(e)$ clearly implies $\Omega \in \mathcal{P}(e)$ and this gives $\Omega = \Omega eA$. So $\operatorname{Coker}\alpha = (\operatorname{Coker}\alpha)eA \simeq (\operatorname{Ker}\gamma)eA = 0$. Thus γ and α are also isomorphisms. This proves (b).

$\Omega \in \mathcal{P}(e)$ and $Ke = 0$ gives that $\operatorname{Ext}^1(\Omega, K) = 0$. Thus the sequence $0 \to K \to \Omega e \underset{eAe}{\otimes} eA \to \Omega \to 0$ splits, giving that K is a direct summand of $\Omega e \underset{eAe}{\otimes} eA$. This implies that $K = KeA = 0$, i.e. $\operatorname{Tor}_1(Xe, eA) = 0$. Applying the result for the syzygies of X, we get the same statment for higher Tor's, hence we proved (a). □

For an arbitrary idempotent element $e \in A$ we denote by $C = eAe$ the corresponding centralizer algebra in A. Throughout the paper we shall also make use of the following functors:

$$\begin{array}{rcccl} \Phi & = & \operatorname{Hom}_A(eA, -) & : & \text{mod-}A \to \text{mod-}C \\ \Gamma & = & - \underset{C}{\otimes} eA & : & \text{mod-}C \to \text{mod-}A \\ \Theta & = & \operatorname{Hom}_C(Ae, -) & : & \text{mod-}C \to \text{mod-}A \end{array}$$

Observe that the functor Θ is naturally equivalent to $D\Gamma^\circ D$, where $D = \operatorname{Hom}_K(-, K)$ is the standard K-duality functor, and $\Gamma^\circ = Ae \underset{C}{\otimes} - : C\text{-mod} \to A\text{-mod}$. Indeed, we have $\operatorname{Hom}_K(Ae \underset{C}{\otimes} D(X), K) \simeq \operatorname{Hom}_C(Ae, \operatorname{Hom}_K(D(X), K)) = \Theta(DD(X)) \simeq \Theta(X)$. It is also easy to see that both $\Phi\Gamma$ and $\Phi\Theta$ are naturally equivalent to $\mathrm{id}_{\text{mod-}C}$, so Φ and Γ (or Φ and Θ) give an equivalence between mod-C and the image of Γ (or the image of Θ, respectively). We shall also use the following adjointness relations between these functors:

$$\operatorname{Hom}_A(\Gamma(X), Y) \simeq \operatorname{Hom}_C(X, \Phi(Y)) \quad \text{for } X \in \text{mod-}C,\ Y \in \text{mod-}A;$$

$$\operatorname{Hom}_C(\Phi(X), Y) \simeq \operatorname{Hom}_A(X, \Theta(Y)) \quad \text{for } X \in \text{mod-}A,\ Y \in \text{mod-}C.$$

Note that if AeA is a stratifying ideal, the functors Φ, Γ and Θ are the functors of the so called *recollement* on the module category level (cf. [CPS]).

We shall adopt the following convention: when $\mathcal{C}$ is an isomorphism-closed subcategory of mod-A or mod-C, respectively, then $\Phi(\mathcal{C})$, $\Gamma(\mathcal{C})$ and $\Theta(\mathcal{C})$ will stand for the isomorphism-closed subcategory of mod-C or mod-A, respectively, which is generated by modules of the form $\Phi(X_A)$, $\Gamma(Y_C)$ or $\Theta(Z_C)$.

Lemma 2.2. *Suppose that for an idempotent element $e \in A$ the ideal AeA is a stratifying ideal. Then $\mathcal{P}(e) \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{P}(e))$ and $\mathcal{Q}(e) \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{Q}(e))$ are equivalences between the given subcategories of* mod-A *and* mod-C*, with Φ, Γ and Θ being exact.*

Proof. We have already seen that $\Phi\Gamma \simeq \mathrm{id}_{\text{mod-}C}$, hence the same natural isomorphism applies to the restriction of Γ to $\Phi(\mathcal{P}(e))$. Next, Lemma 2.1(b) implies that

$\Gamma\Phi(X) \simeq X$ for every $X \in \mathcal{P}(e)$ and it is easy to see that the isomorphism is natural. Thus, Φ and Γ are inverse equivalences when restricted to $\mathcal{P}(e)$ and $\Phi(\mathcal{P}(e))$. The exactness of Φ is obvious, while the exactness of Γ when restricted to $\Phi(\mathcal{P}(e))$ follows from Lemma 2.1(a).

The statement about the equivalence of $\mathcal{Q}(e)$ and $\Phi(\mathcal{Q}(e))$ follows by K-duality from the previous part, since $D(\mathcal{Q}(e)) = \mathcal{P}^\circ(e)$, where $\mathcal{P}^\circ(e)$ consists of all left A-modules for which there is a projective resolution with all projective terms in add Ae. □

Lemma 2.3. *Suppose AeA is a stratifying ideal. Then*
(a) *$\Phi(\mathcal{P}(e))$ is a resolving and $\Phi(\mathcal{Q}(e))$ is a coresolving subcategory of* mod-C*;*
(b) *If $\mathcal{P}' \subseteq$ mod-C is a resolving subcategory, and $D(eA) \in (\mathcal{P}')^\perp$ then $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(e)$. Dually, if $\mathcal{Q}' \subseteq$ mod-C is a coresolving subcategory, and $Ae \in {}^\perp(\mathcal{Q}')$ then $\Theta(\mathcal{Q}') \subseteq \mathcal{Q}(e)$.*

Proof. To prove (a), we shall use the fact that $\mathcal{P}(e)$ is closed under extensions, direct summands and kernels of epimorphisms.

Let us take an exact sequence in mod-C:

$$0 \to X \to Y \to Z \to 0. \tag{3}$$

We can apply the functor $\Gamma = - \underset{C}{\otimes} eA$ to get

$$0 \to \Gamma(X) \to \Gamma(Y) \to \Gamma(Z) \to 0 \tag{4}$$

By Lemma 2.1 (a) the sequence in (4) is exact if $Z \in \Phi(\mathcal{P}(e))$. Furthermore, by Lemma 2.1 (b), if $M_A \in \mathcal{P}(e)$, then $\Gamma\Phi(M) \simeq M$, hence for any $N \in \Phi(\mathcal{P}(e))$ we get that $\Gamma(N) \in \mathcal{P}(e)$.

Thus if $X, Z \in \Phi(\mathcal{P}(e))$, then (4) is exact, and $\Gamma(X), \Gamma(Z) \in \mathcal{P}(e)$, giving that $\Gamma(Y) \in \mathcal{P}(e)$ and $Y \simeq \Phi\Gamma(Y) \in \Phi(\mathcal{P}(e))$. Similarly, if $Y, Z \in \Phi(\mathcal{P}(e))$ then $X \in \Phi(\mathcal{P}(e))$. Thus $\Phi(\mathcal{P}(e))$ is closed under extensions and kernels of epimorphisms.

When (3) is a split sequence then (4) is also split exact. In this case, if $Y \in \Phi(\mathcal{P}(e))$ then $\Gamma(Y) \in \mathcal{P}(e)$, so we get that $\Gamma(X), \Gamma(Z) \in \mathcal{P}(e)$ and $X \simeq \Phi\Gamma(X)$ and $Z \simeq \Phi\Gamma(Z)$ belong to $\Phi(\mathcal{P}(e))$, i. e. $\Phi(\mathcal{P}(e))$ is closed under taking direct summands.

Finally, since $eA \in \mathcal{P}(e)$ and $\Phi(eA) = eAe = C \in \Phi(\mathcal{P}(e))$, we get that $\Phi(\mathcal{P}(e))$ contains the projectives in mod-C, hence $\Phi(\mathcal{P}(e))$ is a resolving subcategory.

To prove (b), let X be a module in $\mathcal{P}'$ and let us take a minimal projective resolution of X in mod-C. We know that $\mathcal{P}'$ is resolving, hence each syzygy is in $\mathcal{P}'$. Since $D(eA) \in (\mathcal{P}')^\perp$ implies that $\mathrm{Tor}_t^C(M, eA) \simeq D(\mathrm{Ext}_C^t(M, D(eA)) = 0$ for $M \in \mathcal{P}'$, we get that Γ maps the projective resolution of X into an exact sequence. Clearly, projective C-modules are mapped to projective A-modules from add (eA), hence $\Gamma(X) \in \mathcal{P}(e)$. This shows that $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(e)$.

The dual statements about $\mathcal{Q}(e)$ and $\mathcal{Q}'$ can be proved by applying the statements about $\mathcal{P}(e)$ and $\mathcal{P}'$ to left modules and taking K-duals, using the natural equivalence between Θ and $D\Gamma^\circ D$ □

From now on we shall fix an order $\mathbf{e} = (e_1, \ldots, e_n)$ of primitive orthogonal idempotents and define the idempotents $\varepsilon_i = e_i + \ldots + e_n$. Let $e = \varepsilon_i$ for a fixed $i \geq 2$; then the corresponding centralizer algebra is $C = \varepsilon_i A \varepsilon_i$, with a fixed order of primitive orthogonal idempotents $\mathbf{e}' = (e_i, \ldots, e_n)$. The functors defined earlier are $\Phi = \mathrm{Hom}_A(\varepsilon_i A, -)$, $\Gamma = - \underset{C}{\otimes} \varepsilon_i A$, $\Theta = \mathrm{Hom}_C(A\varepsilon_i, -)$. We shall also use the notation $B = A/A\varepsilon_i A$ and $\mathbf{e}'' = (e_1, \ldots, e_{i-1})$.

We need to recall a few concepts from [AL].

Definition. For $(A, \mathbf{e})$ we define the subcategories

$$\mathcal{P}_i(\mathbf{e}) = \left\{ M \in \text{mod-}A \,|\, M\varepsilon_{i+1} = 0 \text{ and } \mathrm{Ext}^t(M, S(j)) = 0 \ \forall j < i, \ \forall t \geq 0 \right\},$$

where $S(j)$ denotes the simple top of the projective module $e_j A$.

$$\mathcal{Q}_i(\mathbf{e}) = \left\{ N \in \text{mod-}A \,|\, N\varepsilon_{i+1} = 0 \text{ and } \mathrm{Ext}^t(S(j), N) = 0 \ \forall j < i, \ \forall t \geq 0 \right\}.$$

Let us note that $\mathcal{P}_j(\mathbf{e}) \subseteq \mathcal{P}(\varepsilon_i)$ and $\mathcal{Q}_j(\mathbf{e}) \subseteq \mathcal{Q}(\varepsilon_i)$ for $j \geq i$, and in the case when $A\varepsilon_{i+1}A$ is a stratifying ideal, $\mathcal{P}_i(\mathbf{e}) = \mathcal{P}_{A/A\varepsilon_{i+1}A}(e_i)$ and $\mathcal{Q}_i(\mathbf{e}) = \mathcal{Q}_{A/A\varepsilon_{i+1}A}(e_i)$. Finally, let

$$\mathcal{P}(\mathbf{e}) = \mathcal{F}(\mathcal{P}_1(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e})) \text{ and}$$

$$\mathcal{Q}(\mathbf{e}) = \mathcal{F}(\mathcal{Q}_1(\mathbf{e}), \ldots, \mathcal{Q}_n(\mathbf{e})).$$

Definition. The algebra $(A, \mathbf{e})$ is *CPS-stratified* if $A_A \in \mathcal{P}(\mathbf{e})$, or equivalently, if $D({}_A A) \in \mathcal{Q}(\mathbf{e})$. (Cf. also [CPS] and [ADL1].)

Note that $(A, \mathbf{e})$ is CPS-stratified if and only if the ideals $A\varepsilon_i A$ are stratifying ideals in A for $1 \leq i \leq n$.

By [AL], $\mathcal{P}(\mathbf{e})$ is a resolving and $\mathcal{Q}(\mathbf{e})$ is a coresolving subcategory for $(A, \mathbf{e})$ if $(A, \mathbf{e})$ is CPS-stratified.

Definition. $\mathcal{P} \subseteq \text{mod-}A$ is a *stratifying subcategory for* $(A, \mathbf{e})$ if it is resolving, and $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ for some $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$. Similarly, $\mathcal{Q} \subseteq \text{mod-}A$ is a *costratifying subcategory for* $(A, \mathbf{e})$ if it is coresolving, and $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ for some $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$. A stratifying subcategory $\mathcal{P}$ and a costratifying subcategory $\mathcal{Q}$ for $(A, \mathbf{e})$ form a *stratifying pair for* $(A, \mathbf{e})$ if $\mathcal{Q} = \mathcal{P}^{\perp}$ and $\mathcal{P} = {}^{\perp}\mathcal{Q}$.

It was shown in [AL] that if $\mathcal{P}$ is a stratifying subcategory for $(A, \mathbf{e})$ then $\mathcal{P}^{\perp}$ is a costratifying subcategory and similarly, if $\mathcal{Q}$ is costratifying then ${}^{\perp}\mathcal{Q}$ is stratifying. Every CPS-stratified algebra has (at least one) stratifying pair: in fact, if $(A, \mathbf{e})$ is CPS-stratified then $\mathcal{P}(\mathbf{e})$ and $\mathcal{P}(\mathbf{e})^{\perp}$ form such a pair.

Lemma 2.4. *Let $A\varepsilon_i A$ be a stratifying ideal in $(A, \mathbf{e})$, and $j \geq i$. Then the pairs of functors $\mathcal{P}_j(\mathbf{e}) \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$ and $\mathcal{Q}_j(\mathbf{e}) \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{Q}(\varepsilon_i)) \cap \mathcal{Q}_j(\mathbf{e}')$ define equivalences between the corresponding subcategories of* mod-A *and* mod-C.

Proof. Since $\mathcal{P}_j(\mathbf{e}) \subseteq \mathcal{P}(\varepsilon_i)$, we can apply Lemma 2.2. So it suffices to prove that $\Phi(\mathcal{P}_j(\mathbf{e})) = \Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$.

Suppose that $X \in \mathcal{P}_j(\mathbf{e})$. Then X has a projective resolution in $\mathcal{P}(\varepsilon_i)$, whose projective terms belong to $\mathrm{add}\,(\varepsilon_j A)$, so $\Phi(X)$ has a projective resolution with projective terms in $\mathrm{add}\,(\varepsilon_j A\varepsilon_i)$. Furthermore, if $X \in \text{mod-}A/A\varepsilon_{j+1}A$, i. e. $X\varepsilon_{j+1} = 0$, then $(X\varepsilon_i)\varepsilon_{j+1} = 0$. Thus for $X \in \mathcal{P}_j(\mathbf{e})$, we have $\Phi(X) \in \mathcal{P}_j(\mathbf{e}')$.

Conversely, let X be in $\Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$, and consider a minimal projective resolution of X. By Lemma 2.3 (a) the syzygies of this resolution are in $\Phi(\mathcal{P}(\varepsilon_i))$. Lemma 2.1 yields that by applying the functor Γ to this resolution we get a projective resolution of $\Gamma(X)$ with projective terms in $\Gamma(\mathrm{add}\,(\varepsilon_j A\varepsilon_i)) = \mathrm{add}\,(\varepsilon_j A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A) = \mathrm{add}\,(\varepsilon_j A)$. Furthermore, if $X\varepsilon_{j+1} = 0$, then $\Gamma(X)\varepsilon_{j+1} = X \underset{C}{\otimes} \varepsilon_i A\varepsilon_{j+1} = X\varepsilon_i A\varepsilon_{j+1} \underset{C}{\otimes} \varepsilon_{j+1} = X\varepsilon_{j+1} \underset{C}{\otimes} \varepsilon_{j+1} = 0$. So $\Gamma(X) \in \mathcal{P}_j(\mathbf{e})$, and $X \simeq \Phi\Gamma(X) \in \Phi(\mathcal{P}_j(\mathbf{e}))$.

The second statement follows from the first by K-duality. □

The following two propositions give a connection between stratifying subcategories of $(A, \mathbf{e})$ and those of $(C, \mathbf{e}')$ and $(B, \mathbf{e}'')$.

Proposition 2.5. *Let $\mathcal{P}$ be a stratifying subcategory for $(A, \mathbf{e})$ and let $\mathcal{P}'$ be the image of $\mathcal{P}$ under the functor Φ, i. e. $\mathcal{P}' = \Phi(\mathcal{P}) = P\varepsilon_i$. Then $\mathcal{P}'$ is a stratifying subcategory for $(C, \mathbf{e}')$ and $\mathcal{P}'' = \mathcal{P} \cap (\text{mod-}B)$ is a stratifying subcategory for $(B, \mathbf{e}'')$.*

Proof. Let us observe that $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)) = \Phi(\mathcal{F}(\mathcal{P}_i, \ldots, \mathcal{P}_n))$, where $\mathcal{F}(\mathcal{P}_i, \ldots, \mathcal{P}_n) = \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ since for any $X \in \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ we have $X = X\varepsilon_i A$ and $X\varepsilon_j A/X\varepsilon_{j+1}A \in \mathcal{F}(\mathcal{P}_j)$ by Lemma 2.2 of [AL]. Since $\mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ is closed under extensions, kernels of epimorphisms and direct summands, Lemma 2.2 and Lemma 2.3 (a) give that $\Phi(\mathcal{P}) = \Phi(\mathcal{P} \cap \mathcal{P}(\varepsilon_i))$ is also closed under these operations. Furthermore, $\varepsilon_i A\varepsilon_i = \Phi(\varepsilon_i A) \in \Phi(\mathcal{P})$ also holds, thus $\Phi(\mathcal{P})$ is a resolving subcategory. By Lemma 2.4, $\Phi(\mathcal{P}_j) \subseteq \mathcal{P}_j(\mathbf{e}')$ for all $j \geq i$, so $\mathcal{P}' = \Phi(\mathcal{P}) = \mathcal{F}(\Phi(\mathcal{P}_i) \cdots, \Phi(\mathcal{P}_n))$ is a stratifying subcategory for $(C, \mathbf{e}')$.

To show that $\mathcal{P}'' = \mathcal{P} \cap \text{mod-}B = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_{i-1})$ is a stratifying subcategory for $(B, \mathbf{e}'')$, observe first that it is clearly resolving. Thus we have to show only that $\mathcal{P}_j \subseteq \mathcal{P}_j(\mathbf{e}'')$ for $j < i$. But this follows from the fact that $A\varepsilon_i A$ is a stratifying ideal, hence by (S2) the extensions of B-modules over B and A are the same, thus the subcategories $\mathcal{P}_j(\mathbf{e}'')$ and $\mathcal{P}_j(\mathbf{e})$ are identical for $j < i$. □

Proposition 2.6. *For a given $(A, \mathbf{e})$ suppose that*

(i) $\mathcal{P}'$ is a stratifying subcategory for $(C, \mathbf{e}')$, and $\mathcal{P}''$ is a stratifying subcategory for $(B, \mathbf{e}'')$;

(ii) $A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A \simeq A\varepsilon_i A$;

(iii) $A\varepsilon_i \in \mathcal{P}'$, $D(\varepsilon_i A) \in (\mathcal{P}')^{\perp}$.

Then $\mathcal{P} = \mathcal{F}(\mathcal{P}'', \Gamma(\mathcal{P}'))$ is a stratifying subcategory for $(A, \mathbf{e})$ such that $\Phi(\mathcal{P}) = \mathcal{P}'$.

Proof. Note first that conditions (*ii*) and (*iii*) imply by (S1′) that $A\varepsilon_i A$ is a stratifying ideal. It is also clear that $A_A \in \mathcal{P}$, since $A\varepsilon_i A \simeq A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A \in \Gamma(\mathcal{P}')$ and $A/A\varepsilon_i A \in \mathcal{P}''$.

Lemma 2.3 (*b*) implies that $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(\varepsilon_i)$, and from the equivalence given by the (exact) functors $\Gamma(\mathcal{P}') \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \mathcal{P}' = \Phi\Gamma(\mathcal{P}')$ (see Lemma 2.2) it follows that $\Gamma(\mathcal{P}')$ is closed under extensions, kernels of epimorphisms and direct summands.

Lemma 2.4 proves that $\Gamma(\mathcal{P}'_j) \subseteq \mathcal{P}_j(\mathbf{e})$ for $j \geq i$, and the elements of $\mathcal{P}$ are filtered by $\mathcal{P}''_1, \dots, \mathcal{P}''_{i-1}, \Gamma(\mathcal{P}'_i), \dots, \Gamma(\mathcal{P}'_n)$. Since the latter satisfy the closure properties of Proposition 2.6 in [AL], $\mathcal{P}$ is a stratifying subcategory for $(A, \mathbf{e})$.

Finally, $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}'', \Gamma(\mathcal{P}'))) = \Phi\Gamma(\mathcal{P}') = \mathcal{P}'$. □

To establish a similar connection between stratifying pairs of subcategories for a CPS-stratified algebra $(A, \mathbf{e})$ and its centralizer algebra $(C, \mathbf{e}')$, we need first the following lemma.

Lemma 2.7. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra, $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e}))$ and $Y \in \mathcal{F}(\mathcal{Q}_i(\mathbf{e}), \dots, \mathcal{Q}_n(\mathbf{e}))$. Then for arbitrary $t > 0$*

$$\operatorname{Ext}^t_A(X, Y) = 0 \iff \operatorname{Ext}^t_C(\Phi(X), \Phi(Y)) = 0.$$

Proof. We prove the statement only for $t = 1$; then the general statement will follow by a usual dimension shifting argument, using the fact that the syzygies of $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e})) = \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ also belong to $\mathcal{F}(\mathcal{P}_i(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e}))$, and Φ maps a projective resolution into a projective resolution.

Let us assume first that $\operatorname{Ext}^1_A(X, Y) = 0$. This is equivalent to saying that for the projective cover of X in mod-A:

$$0 \longrightarrow \Omega \xrightarrow{\alpha} P \longrightarrow X \longrightarrow 0$$

the map $\operatorname{Hom}_A(P, Y) \to \operatorname{Hom}_A(\Omega, Y)$ is surjective. That is to say, for every $\beta \in \operatorname{Hom}_A(\Omega, Y)$ there is $\gamma \in \operatorname{Hom}_A(P, Y)$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega & \xrightarrow{\alpha} & P & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\gamma} & & & & & \\ & & Y & & & & & & \end{array}$$

By applying the functor Φ to this diagram we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(\Omega) & \xrightarrow{\Phi(\alpha)} & \Phi(P) & \longrightarrow & \Phi(X) & \longrightarrow & 0 \\ & & {\scriptstyle\Phi(\beta)}\downarrow & \swarrow{\scriptstyle\Phi(\gamma)} & & & & & \\ & & \Phi(Y) & & & & & & \end{array}$$

where $\Phi(P)$ is projective. Note that $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e})) \subseteq \mathcal{P}(\varepsilon_i)$ implies that $\Omega \in \mathcal{P}(\varepsilon_i)$ so Lemma 2.1 (*b*) and the adjointness of the functors Γ and Φ give

$$\operatorname{Hom}_A(\Omega, Y) \simeq \operatorname{Hom}_A(\Gamma\Phi(\Omega), Y) \simeq \operatorname{Hom}_C(\Phi(\Omega), \Phi(Y)).$$

This shows that $\mathrm{Hom}_C(\Phi(P),\Phi(Y)) \to \mathrm{Hom}_C(\Phi(\Omega),\Phi(Y))$ is also surjective, hence $\mathrm{Ext}^1_C(\Phi(X),\Phi(Y)) = 0$.

Conversely, let us now assume that $\mathrm{Ext}^1_C(\Phi(X),\Phi(Y)) = 0$. This means that if we take the projective cover of $\Phi(X)$ in mod-C:

$$0 \longrightarrow \Omega' \xrightarrow{\alpha} P' \longrightarrow \Phi(X) \longrightarrow 0,$$

the map $\mathrm{Hom}_C(\mathcal{P}',\Phi(Y)) \to \mathrm{Hom}_C(\Omega',\Phi(Y))$ is surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega' & \xrightarrow{\alpha} & P' & \longrightarrow & \Phi(X) & \longrightarrow & 0. \\ & & {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\gamma} & & & & & \\ & & \Phi(Y) & & & & & & \end{array}$$

This gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\Omega') & \xrightarrow{\Gamma(\alpha)} & \Gamma(P') & \longrightarrow & \Gamma\Phi(X) & \longrightarrow & 0. \\ & & {\scriptstyle\beta'}\downarrow & \swarrow{\scriptstyle\gamma'} & & & & & \\ & & \Theta\Phi(Y) & & & & & & \end{array}$$

Here the maps β' and γ' are obtained from β and γ using the natural isomorphisms, coming from the adjointness of Φ and Θ:

$$\mathrm{Hom}_C(M,N) \simeq \mathrm{Hom}_C(\Phi\Gamma(M),N) \simeq \mathrm{Hom}_A(\Gamma(M),\Theta(N)).$$

As in the previous part, we get that $\mathrm{Hom}_A(\Gamma(P'),\Theta\Phi(Y)) \to \mathrm{Hom}_A(\Gamma(\Omega'),\Theta\Phi(Y))$ is surjective. Since $\Gamma(P')$ is projective, this proves that $\mathrm{Ext}^1_A(\Gamma\Phi(X),\Theta\Phi(Y)) = 0$. But $X \in \mathcal{P}(\varepsilon_i)$ implies $\Gamma\Phi(X) \simeq X$ and $Y \in \mathcal{Q}(\varepsilon_i)$ implies $\Theta\Phi(Y) \simeq Y$ by Lemma 2.1 (*b*) and its dual. So $\mathrm{Ext}^1_A(X,Y) = 0$. □

Proposition 2.8. *Let $(A,\mathbf{e})$ be a CPS-stratified algebra. Then the following are equivalent for a pair $(\mathcal{P}',\mathcal{Q}')$ of subcategories of* mod-C.
(1) $(\mathcal{P}',\mathcal{Q}')$ is a stratifying pair over $(C,\mathbf{e}')$ with $A\varepsilon_i \in \mathcal{P}'$ and $D(\varepsilon_i A) \in \mathcal{Q}'$.
(2) There is a stratifying pair $(\mathcal{P},\mathcal{Q})$ over $(A,\mathbf{e})$ such that $(\mathcal{P}',\mathcal{Q}') = (\Phi(\mathcal{P}),\Phi(\mathcal{Q}))$.

Proof. Let us fix a stratifying pair $(\mathcal{P}'',\mathcal{Q}'')$ for $(B,\mathbf{e}'')$.

Let $\mathcal{H}$ denote the set of all those pairs $(\mathcal{P},\mathcal{Q})$ of stratifying and costratifying subcategories for $(A,\mathbf{e})$ for which that $\mathcal{P} \cap (\text{mod-}B) = \mathcal{P}''$, $\mathcal{Q} \cap \text{mod-}B = \mathcal{Q}''$ and $\mathcal{Q} \subseteq \mathcal{P}^{\perp}$.

Let $\mathcal{H}'$ denote the set of all those pairs $(\mathcal{P}',\mathcal{Q}')$ of stratifying and costratifying subcategories for $(C,\mathbf{e}')$ for which $A\varepsilon_i \in \mathcal{P}'$, $D(\varepsilon_i A) \in \mathcal{Q}'$, and $\mathcal{Q}' \subseteq \mathcal{P}'^{\perp}$.

Consider the following maps:

$$\begin{array}{lll} \mu & : & (\mathcal{P},\mathcal{Q}) \mapsto (\Phi(\mathcal{P}),\Phi(\mathcal{Q})) \qquad \text{for each } (\mathcal{P},\mathcal{Q}) \in \mathcal{H} \\ \nu & : & (\mathcal{P}',\mathcal{Q}') \mapsto (\mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}')),\mathcal{F}(\mathcal{Q}'',\Theta(\mathcal{Q}'))) \qquad \text{for each } (\mathcal{P}',\mathcal{Q}') \in \mathcal{H}' \end{array}$$

Then μ maps every pair $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$ to a pair $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$, since Proposition 2.5 and its dual imply that $\mathcal{P}' = \Phi(\mathcal{P})$ is a stratifying and $\mathcal{Q}' = \Phi(\mathcal{Q})$

is a costratifying subcategory for $(C,\mathbf{e}')$; $A_A \in \mathcal{P}$ gives that $A\varepsilon_i \in \mathcal{P}'$, and similarly, $D({}_AA) \in \mathcal{Q}$ gives $D(\varepsilon_i A) = D(A)\varepsilon_i \in \mathcal{Q}'$; finally, $\mathcal{Q}' \subseteq \mathcal{P}'^{\perp}$ follows from Lemma 2.7, since $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}_i,\dots,\mathcal{P}_n))$, $\Phi(\mathcal{Q}) = \Phi(\mathcal{F}(\mathcal{Q}_i,\dots,\mathcal{Q}_n))$, and $\mathcal{F}(\mathcal{Q}_i,\dots,\mathcal{Q}_n) \subseteq \mathcal{F}(\mathcal{P}_i,\dots,\mathcal{P}_n)^{\perp}$.

Next we show that the map ν maps every pair $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$ to a pair $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$. Proposition 2.6 and its dual imply that $\mathcal{P} = \mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}'))$ is a stratifying and $\mathcal{Q} = \mathcal{F}(\mathcal{Q}'',\Theta(\mathcal{Q}'))$ is a costratifying subcategory for $(A,\mathbf{e})$. Furthermore, $\mathcal{P} \cap (\text{mod-}B) = \mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}')) \cap (\text{mod-}B) = \mathcal{P}''$, similarly, $\mathcal{Q} \cap (\text{mod-}B) = \mathcal{Q}''$. We still have to prove that $\mathcal{Q} \subseteq \mathcal{P}^{\perp}$. First, $\mathrm{Ext}^t_A(\mathcal{P}'',\mathcal{Q}'') = \mathrm{Ext}^t_B(\mathcal{P}'',\mathcal{Q}'') = 0$ for $t > 0$, since $(\mathcal{P}'',\mathcal{Q}'')$ is a stratifying pair for $(B,\mathbf{e}'')$ and $A\varepsilon_i A$ is a stratifying ideal. Next, $\mathrm{Ext}^t_A(\mathcal{P}'',\Theta(\mathcal{Q}')) = \mathrm{Ext}^t_A(\Gamma(\mathcal{P}'),\mathcal{Q}'') = 0$ for $t > 0$, since $\Theta(\mathcal{Q}') \in \mathcal{Q}(\varepsilon_i)$ and $\Gamma(\mathcal{P}') \in \mathcal{P}(\varepsilon_i)$ by Lemma 2.3 (b). Finally, $\Theta(\mathcal{Q}') = \mathcal{F}(\mathcal{Q}_i,\dots,\mathcal{Q}_n)$, $\Gamma(\mathcal{P}') = \mathcal{F}(\mathcal{P}_i,\dots,\mathcal{P}_n)$ and $\mathrm{Ext}^t_C(\mathcal{Q}',\mathcal{P}') = 0$ for $t > 0$, so Lemma 2.7 gives that $\mathrm{Ext}^t_A(\Gamma(\mathcal{P}'),\Theta(\mathcal{Q}') = 0$.

It is clear that $\mu\nu = \mathrm{id}_{\mathcal{H}'}$. On the other hand, for any pair $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$, $\mathcal{F}(\mathcal{P}'',\Gamma\Phi(\mathcal{P})) = \mathcal{P}$, since $\Gamma\Phi(\mathcal{P}) = \Gamma\Phi(\mathcal{F}(\mathcal{P}_i,\dots,\mathcal{P}_n))$, which is equal to $\mathcal{F}(\mathcal{P}_i,\dots,\mathcal{P}_n)$ by Lemma 2.2, and dually, $\mathcal{F}(\mathcal{Q}'',\Theta\Phi(\mathcal{Q})) = \mathcal{Q}$. So $\nu\mu = \mathrm{id}_{\mathcal{H}}$.

Now let us assume that $(\mathcal{P},\mathcal{Q})$ is a stratifying pair over $(A,\mathbf{e})$, and let $\mathcal{P}'' = \mathcal{P} \cap (\text{mod-}B)$ and $\mathcal{Q}'' = \mathcal{Q} \cap (\text{mod-}B)$. Consider the classes $\mathcal{H}$ and $\mathcal{H}'$ and the maps μ and ν with this fixed pair $(\mathcal{P}'',\mathcal{Q}'')$. Since $\mathcal{Q} = \mathcal{P}^{\perp}$, $\mathcal{Q}$ is the largest costratifying subcategory such that $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$. Thus, for $(\mathcal{P}',\mathcal{Q}') = \mu(\mathcal{P},\mathcal{Q})$, the subcategory $\mathcal{Q}'$ is the largest costratifying subcategory for $(C,\mathbf{e}')$ such that $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$. But $(\mathcal{P}',(\mathcal{P}')^{\perp})$ is also in $\mathcal{H}'$, so $(\mathcal{P}')^{\perp} \subseteq \mathcal{Q}' \subseteq (\mathcal{P}')^{\perp}$, i. e. $(\mathcal{P}')^{\perp} = \mathcal{Q}'$, and similarly, ${}^{\perp}(\mathcal{Q}') = \mathcal{P}'$. So $(\mathcal{P}',\mathcal{Q}')$ is a stratifying pair with $A\varepsilon_i \in \mathcal{P}'$ and $D(\varepsilon_i A) \in \mathcal{Q}'$. This proves that (2) implies (1).

With an analogous argument we get that for any stratifying pair $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$ the pair $(\mathcal{P},\mathcal{Q}) = \nu(\mathcal{P}',\mathcal{Q}')$ is a stratifying pair for $(A,\mathbf{e})$ with $\mathcal{P}' = \Phi(\mathcal{P})$ and $\mathcal{Q}' = \Phi(\mathcal{Q})$. So (1) implies (2). □

3. Recursive construction of CPS-stratified algebras

THEOREM 3.1. *For $(A,\mathbf{e})$ let $\varphi_i = e_1 + \ldots + e_{i-1} = 1 - \varepsilon_i$, and $E = \varphi_i A \varepsilon_i$, $F = \varepsilon_i A \varphi_i$, $C = \varepsilon_i A \varepsilon_i$, $B = A/A\varepsilon_i A$. Then $(A,\mathbf{e})$ is CPS-stratified if and only if the following conditions hold:*

(i) $(C,\mathbf{e}')$ and $(B,\mathbf{e}'')$ are CPS-stratified;
(ii) the multiplication map $E \otimes_C F \to \varphi_i A \varphi_i$ is injective;
(iii) there is a stratifying pair $(\mathcal{P}',\mathcal{Q}')$ for $(C,\mathbf{e}')$ such that $E \in \mathcal{P}'$ and $D(F) \in \mathcal{Q}'$.

Proof. Let us assume first that $(A,\mathbf{e})$ is CPS-stratified. Then $\mathcal{P} = \mathcal{P}(\mathbf{e})$ is a stratifying subcategory. By Proposition 2.5, $(C,\mathbf{e}')$ and $(B,\mathbf{e}'')$ are CPS-stratified, as stated in condition (i). Since $A\varepsilon_i A$ is a stratifying ideal, the multiplication map $A\varepsilon_i \otimes_C \varepsilon_i A \to A\varepsilon_i A$ is injective by (S1), so $\varphi_i A\varepsilon_i \otimes_C \varepsilon_i A\varphi_i \to \varphi_i A\varepsilon_i A\varphi_i$ is also injective, proving condition (ii). Finally, by Lemma 2.5, $\mathcal{P}' = \Phi(\mathcal{P}(\mathbf{e})) \subseteq \Phi(\mathcal{P}(\varepsilon_i))$ is a stratifying subcategory for $(C,\mathbf{e}')$ with $E = \Phi(\varphi_i A) \in \mathcal{P}'$, and by Lemma 2.1 (a) the costratifying subcategory $\mathcal{Q}' = (\mathcal{P}')^{\perp}$ contains $D(\varepsilon_i A)$ and its direct summand

$D(F) = D(\varepsilon_i A\varphi_i)$ as well. Since ${}^{\perp}(\mathcal{Q}')$ is a stratifying subcategory for $(C, \mathbf{e}')$, by Proposition 2.6 $\Gamma({}^{\perp}(\mathcal{Q}'))$ can be included in a stratifying subcategory for $(A, \mathbf{e})$, so it is in $\mathcal{P}(\mathbf{e})$, hence ${}^{\perp}(\mathcal{Q}') = \Phi\Gamma({}^{\perp}(\mathcal{Q}')) \subseteq \Phi(\mathcal{P}(\mathbf{e})) = \mathcal{P}'$. But the latter is clearly contained in ${}^{\perp}(\mathcal{Q}')$, so ${}^{\perp}(\mathcal{Q}') = \mathcal{P}'$. Thus $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$, satisfying condition (iii).

Now suppose that conditions (i), (ii) and (iii) hold. Condition (ii), i.e. the injectivity of the map $E \underset{C}{\otimes} F \to \varphi_i A\varphi_i$ implies that the multiplication map $A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A = (E \oplus C) \underset{C}{\otimes} (F \oplus C) \to A$ is also injective, since the injectivity for the other three direct components is obvious. Thus condition (ii) of Proposition 2.6 holds. Condition (iii) of the theorem implies that $A\varepsilon_i = E \oplus C \in \mathcal{P}'$ and $D(\varepsilon_i A) = D(F) \oplus D(C) \in \mathcal{Q}'$, since C_C is projective and $D({}_CC)$ is injective, so condition (iii) of Proposition 2.6 is also satisfied. Finally, we can take any stratifying subcategory for $(B, \mathbf{e}'')$ as $\mathcal{P}''$ to satisfy condition (i) of Proposition 2.6, so there exists a stratifying subcategory for $(A, \mathbf{e})$, i.e. $(A, \mathbf{e})$ is CPS-stratified. □

If $i = n$ in the previous theorem then $(C, \mathbf{e}')$ is automatically CPS-stratified since C is local. Furthermore condition (iii) is equivalent to saying that $\mathrm{Ext}^t(E, D(F)) = 0$ for all $t > 0$ (see Proposition 3.7 of [AL]), so (ii) and (iii) together give, by (S1′), the condition that $A\varepsilon_i A$ is a stratifying ideal. This is the usual recursive definition of a CPS-stratified algebra.

Similarly, if $i = 2$, then B is local, hence the condition on the algebra B can be dropped. Actually, we can use the previous theorem for this situation to construct all CPS-stratified algebras. The construction follows closely the construction of Δ- and $\bar{\Delta}$ filtered (i.e. standardly stratified) algebras in [ADL2], so we only prove what is different in this case.

Let us take a local algebra L with unit element e_1, and a CPS-stratified algebra $(C, \mathbf{e}')$, where $\mathbf{e}' = (e_2, \ldots, e_n)$. Furthermore, let ${}_LE_C$ and ${}_CF_L$ be bimodules such that $E_C \in \mathcal{P}'$ and $D({}_CF) \in \mathcal{Q}'$, where $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$. Note that by [AL], Proposition 3.7 such a stratifying pair exists if and only if $E_C \in \mathcal{P}(\mathbf{e}')$, $D({}_CF) \in \mathcal{Q}(\mathbf{e}')$ and $\mathrm{Ext}^t_C(E\varepsilon_j C/E\varepsilon_{j+1}C, D(C\varepsilon_j F/C\varepsilon_{j+1}F)) = 0$ for all $t > 0$ and $j \geq 2$. We also fix a C-C bimodule homomorphism $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$.

We extend L to a larger local algebra $\widetilde{L}$ so that

$$\widetilde{L} = L \ltimes (E \underset{C}{\otimes} F)$$

is the split extension of L by the algebra $E \underset{C}{\otimes} F$, where the algebra multiplication on $E \underset{C}{\otimes} F$ is defined by

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \xrightarrow{\mathrm{id}_E \otimes \mu \otimes \mathrm{id}_F} E \underset{C}{\otimes} C \underset{C}{\otimes} F \simeq E \underset{C}{\otimes} F.$$

In a similar fashion we extend the L-module structure on E and F to an $\widetilde{L}$-module structure. Thus E and F become $\widetilde{L}$-C and C-$\widetilde{L}$ bimodules. Finally, we define the algebra $\tilde{A}$ as

$$\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$$

with the natural algebra structure. In an obvious way $\mathbf{e} = (e_1, \ldots, e_n)$ gives a complete oredered set of primitive orthogonal idempotents in $\widetilde{A}$.

Since $\varepsilon_2 \widetilde{A} \varepsilon_2 = C$, $\widetilde{A}/\widetilde{A}\varepsilon_2\widetilde{A} \simeq L$, $(e_1\widetilde{A}\varepsilon_2)_C = E_C$ and ${}_C(\varepsilon_2\widetilde{A}e_1) = {}_CF$, conditions (i) and (iii) of Theorem 3.1 are satisfied. The construction of $\widetilde{L} = e_1\widetilde{A}e_1$ ensures that $E \underset{C}{\otimes} F = e_1\widetilde{A}\varepsilon_2\widetilde{A}e_1$, so condition (ii) of Theorem 3.1 also holds. Thus we proved the following theorem.

Theorem 3.2. *Let L be a local algebra with unit element e_1, $(C, \mathbf{e}')$ a CPS-stratified algebra with $\mathbf{e}' = (e_2, \ldots, e_n)$, ${}_LE_C$ and ${}_CF_L$ bimodules such that $E_C \in \mathcal{P}'$ and $D({}_CF) \in \mathcal{Q}'$, where $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$, and finally, $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ a bimodule homomorphism. If $\widetilde{A}$ is the algebra constructed above and $\mathbf{e} = (e_1, \ldots, e_n)$, then $(\widetilde{A}, \mathbf{e})$ is CPS-stratified.*

As in [ADL2], we can take an ideal $H \triangleleft \widetilde{A}$ such that $H \subseteq \operatorname{rad} \widetilde{L}$ and $H \cap (E \underset{C}{\otimes} F) = 0$. In [ADL2] such ideals were called *auxiliary ideals*. Then $A = \widetilde{A}/H$ is also CPS-stratified since C, E, F remain the same and the map $E \underset{C}{\otimes} F \to \widetilde{L}/H$ remains injective.

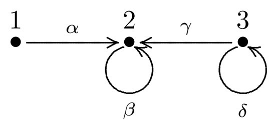
In this way we can construct all CPS-stratified algebras:

Theorem 3.3. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra and let us take $L = e_1Ae_1$, $C = \varepsilon_2 A \varepsilon_2$, $E = e_1 A \varepsilon_2$, $F = \varepsilon_2 A e_1$ and the multiplication map $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$. Then with the algebra $\widetilde{A}$ and an appropriate auxiliary ideal $H \subseteq e_1\widetilde{A}e_1$, we have $A \simeq \widetilde{A}/H$.*

Proof. Let us define $\nu : E \underset{C}{\otimes} F \to L$ to be the natural multiplication map, and $H = \left\{ u - \nu(u) \mid u \in E \underset{C}{\otimes} F \right\}$. Theorem 3.1 implies that conditions of Theorem 3.2 for C, E and F are satisfied, thus we can construct $\widetilde{A}$ in the prescribed way. Furthermore, the proof of Theorem 4.4 of [ADL2] can be applied to show that H is an auxiliary ideal, and $\widetilde{A}/H \simeq A$. □

Let us conclude with two examples.

Example 3.4. We give an example of a situation where a stratifying pair for $(C, \mathbf{e}')$ cannot be extended to a stratifying pair for $(A, \mathbf{e})$ (cf. Proposition 2.8). Let $A = KG/I$, where G is the graph



and $I = (\alpha\beta, \beta^2, \gamma\beta, \delta\gamma, \delta^2)$. So the right regular representation of A is given by

$$A_A = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 3 \end{smallmatrix}$$

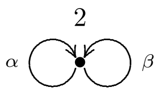
and for $C = \varepsilon_2 A \varepsilon_2$ we have

$$C_C = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 3 \end{smallmatrix}.$$

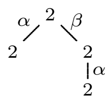
C is a Δ-filtered (hence standardly stratified) algebra with only two stratifying pairs: $(\operatorname{add} C_C = \mathcal{F}(\Delta), \text{mod-}C)$ and $(\mathcal{F}(\,2\,,\,{}_{2}{}^{3}{}_{3}\,), \mathcal{F}(\,{}^{2}_{2}\,,\,3\,))$. Here the first stratifying pair cannot be extended to a stratifying pair for $(A, \mathbf{e})$, since $e_1 A \varepsilon_2 \notin \operatorname{add} C_C$. On the other hand the second pair can be extended to the pair $(\mathcal{F}(\,1\,,\,2\,,\,{}_{2}{}^{3}{}_{3}\,), \mathcal{F}(\,1\,,\,{}^{2}_{2}\,,\,3\,))$.

Example 3.5. Here we show how Theorem 3.2 can be applied to construct CPS-stratified algebras, starting with two local algebras.

Let G be the graph with one vertex and two loops:



Let us take $C = KG/I$ with $I = (\alpha^2, \beta^2, \alpha\beta)$ and use the notation $e_2 = 1_C$. The regular representation of C can be described by

$C_C =$ 

For L we can take the base field K as a local K-algebra. Since C is local, to find suitable bimodules E and F we only have to satisfy the conditions $\operatorname{Ext}^t_C(E, D(F)) = 0$ for $t > 0$: by Propositon 3.7 of [AL] if these conditions are satisfied, we can always find a stratifying pair for (C, e_2), containing the given modules. To this end, let us consider the following modules: $X = C/\beta C$, $Y_\lambda = C/(\alpha - \lambda\beta)C$ for $0 \neq \lambda \in K$ and $Z = C/(\alpha C + \beta\alpha C)$. Thus we have:

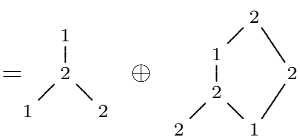
$$X_C = \begin{matrix} 2 \\ \mid \alpha \\ 2 \end{matrix} \qquad (Y_\lambda)_C = \begin{matrix} 2 \\ \alpha \mid\mid \lambda\beta \\ 2 \end{matrix} \qquad Z_C = \begin{matrix} 2 \\ \mid \beta \\ 2 \end{matrix}$$

One can check easily that the following extension modules are all zero:

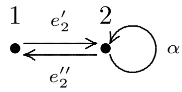
$$\operatorname{Ext}^t_C(X, Y_\lambda) = \operatorname{Ext}^t_C(X, Z) = \operatorname{Ext}^t_C(Y_\lambda, Y_\kappa) = \operatorname{Ext}^t_C(Y_\lambda, Z) = 0$$

for all $t > 0$ and $\lambda \neq \kappa$.

Thus we can start for example with $E_C = X$ and ${}_C F = D(Y_1)$. In order to define the map $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$, we fix a basis for C, L, E and F. Let $C = \langle e_2, \alpha, \beta, \beta\alpha \rangle$ and $L = \langle e_1 \rangle$; and similarly $E = \langle e_2', \alpha' \rangle$ with $e_2'\alpha = \alpha'$ and $e_2'\beta = \alpha'\alpha = \alpha'\beta = 0$ and $F = \langle e_2'', \alpha'' \rangle$ with $\alpha e_2'' = \beta e_2'' = \alpha''$ and $\alpha\alpha'' = \beta\alpha'' = 0$. Then we can define the C-C bimodule map μ as follows. Let us take $\mu(e_2'' \otimes e_2') = \beta$; then $\mu(e_2'' \otimes \alpha') = \beta\alpha$, furthermore $\mu(\alpha'' \otimes e_2') = \mu(\alpha'' \otimes \alpha') = 0$. Note that $E \underset{C}{\otimes} F$ is one dimensional and the multiplication on $E \underset{C}{\otimes} F$ becomes zero. Actually we can obtain the complete multiplication table of $\widetilde{A}$. We get that the regular module over $\widetilde{A}$ can be described by

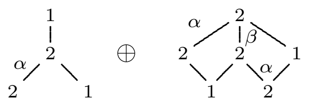
$\widetilde{A}_{\widetilde{A}} =$ 

and hence $\widetilde{A} = K\widetilde{G}/I$ where $\widetilde{G}$ is the graph



and $I = (\alpha^2,\ e_2'e_2''e_2',\ e_2''e_2'e_2'' - \alpha e_2'')$. In this way we get a CPS-stratified algebra, which is not standardly stratified. Since there are no non-zero auxiliary ideals, this input set (L, E, F, C, μ) gives only this algebra by our construction.

On the other hand by modifying the map μ we can obtain a completely different algebra. If $\mu(\mathbf{e}_2'' \otimes e_1') = \beta\alpha$ then we get an algebra $\widetilde{A}$ with regular decomposition

$\widetilde{A}_{\widetilde{A}} =$ 

Let us note also that by using different perpendicular pairs of bimodules (for example using the modules X, Y_λ and Z in a different setup) we can get infinitely many CPS-stratified algebras.

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KOSZUL DUALITY FOR STRATIFIED ALGEBRAS I. QUASI-HEREDITARY ALGEBRAS

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Abstract. We give a complete picture of the interaction between Koszul and Ringel dualities for quasi-hereditary algebras admitting linear tilting (co)resolutions of standard and costandard modules. We show that such algebras are Koszul, that the class of these algebras is closed with respect to both dualities and that on this class these two dualities commute. All arguments reduce to short computations in the bounded derived category of graded modules.

1. Introduction

Let A be a positively graded quasi-hereditary algebra. Then there exist two classical duals for A: the Ringel dual $R(A)$ ([Ri]), which is the endomorphism algebra of the characteristic tilting A-module, and the Koszul dual $E(A)$ ([ADL2]), which is the extension algebra of the direct sum of all simple A-modules. The algebra $R(A)$ is always quasi-hereditary, while the algebra $E(A)$ is quasi-hereditary only under some additional assumptions. For example, $E(A)$ is quasi-hereditary if both, projective resolutions of all standard A-modules and injective coresolutions of all costandard A-modules, are linear (see [ADL2]). Such algebras were called *standard Koszul* in [ADL2].

The natural question to ask is whether $R(E(A)) \cong E(R(A))$. This question was addressed in [MO], where it was shown that this is the case under some assumptions, which, roughly speaking, mean that the algebras A, $R(A)$, $E(A)$, $E(R(A))$ and $R(E(A))$ are standard Koszul with respect to the grading, induced from the grading on A. The main disadvantage of this result was that the condition was not formulated in terms of A-modules and hence was very difficult to check.

The main motivation for the present paper was to find an easier condition which would guarantee the isomorphism $R(E(A)) \cong E(R(A))$. For this we further develop the approach of [MO], based on the category of linear complexes of tilting A-modules. The main point of the paper is that we find an easy way to check Koszulity of A and quasi-heredity of $E(A)$ based on direct computations in the derived category. This looks much easier than, for example, the subtle analysis of the structure of projective resolutions, carried out in [ADL2].

A part of the condition, used in [MO], was formulated as follows: all standard and costandard A-modules have linear tilting (co)resolutions.

We call such algebras balanced. Using our computational approach we show that already this is enough to ensure that all algebras in the list A, $R(A)$, $E(A)$, $E(R(A))$ and $R(E(A))$ are standard Koszul with respect to the induced grading and derive as a corollary that Koszul and Ringel dualities on such A commute. Under our assumptions we reprove main results from [ADL2] and strengthen the main result from [MO]. Our main result is the following:

Theorem 1. *For every balanced quasi-hereditary algebra A we have:*

(i) The algebra A is Koszul and standard Koszul.
(ii) The algebras A, $R(A)$, $E(A)$, $E(R(A))$ and $R(E(A))$ are balanced.
(iii) Every simple A-module is represented by a linear complex of tilting modules.
(iv) $R(E(A)) \cong E(R(A))$ as graded quasi-hereditary algebras.

By [BGS, MOS] we also have equivalences of the corresponding bounded derived categories of graded modules for the algebras A, $E(A)$, $R(A)$ and $R(E(A)) \cong E(R(A))$. Another advantage of our approach is that it admits a straightforward generalization to stratified algebras, both in the sense of [ADL1] and [CPS]. There is, however, a technical complication in this generalization: In the case when a stratified algebra is not quasi-hereditary, it has infinite global dimension and hence the Koszul dual is infinite-dimensional. Thus to apply our approach one has first to develop a sensible tilting theory for infinite-dimensional stratified algebras. This is an extensive technical work, which will be carried out in the separate paper [Ma2]. In the present paper we avoid these technicalities to make our approach clearer. Another advantage of our approach is that it generalizes to infinite-dimensional quasi-hereditary algebras of finite homological dimension.

The paper is organized as follows: In Section 2 we collect all necessary preliminaries about graded quasi-hereditary algebras. In Section 3 we prove our main result. We complete the paper with some examples in Section 4.

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2. Graded quasi-hereditary algebras

By $\mathbb{N}$ we denote the set of all positive integers. By a module we always mean a *graded left* module, and by grading we always mean

$\mathbb{Z}$*-grading.* Let $\Bbbk$ be an algebraically closed field and A be a basic, finite-dimensional, positively graded and quasi-hereditary $\Bbbk$-algebra. Let $\Lambda = \{1, \dots, n\}$ and $\{e_\lambda : \lambda \in \Lambda\}$ be a complete set of pairwise orthogonal primitive idempotents for A such that the natural order on Λ is the one which defines the quasi-hereditary structure on A. Then $A = \oplus_{i\geq 0} A_i$, $A_0 \cong \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$ and $\mathrm{rad}(A) = \oplus_{i>0} A_i$.

Let $A-$gmod denote the category of all finite-dimensional graded A-modules. Morphisms in this category are homogeneous morphism of degree zero between graded A-modules. This is an abelian category with enough projectives and enough injectives. For $i \in \mathbb{Z}$ we denote by $\langle i \rangle$ the autoequivalence of $A-$gmod, which shifts the grading as follows: $(M\langle i \rangle)_j = M_{i+j}$, $j \in \mathbb{Z}$. We adopt the notation $\hom_A$ and ext^i_A to denote homomorphisms and extensions in $A-$gmod.

For $\lambda \in \Lambda$ we consider the graded indecomposable projective module $P(\lambda) = Ae_\lambda$, its graded simple quotient $L(\lambda) = P(\lambda)/\mathrm{rad}(A)P(\lambda)$ and the graded indecomposable injective envelop $I(\lambda)$ of $L(\lambda)$. Let $\Delta(\lambda)$ be the standard quotient of $P(\lambda)$ and $\nabla(\lambda)$ be the costandard submodule of $I(\lambda)$. By [MO, Corollary 5], there exists a graded lift $T(\lambda)$ of the indecomposable tilting module corresponding to λ such that $\Delta(\lambda)$ is a submodule of $T(\lambda)$ and $\nabla(\lambda)$ is a quotient of $T(\lambda)$.

For every $i \in \mathbb{Z}$ we will say that *centroids* of the modules $L(\lambda)\langle i \rangle$, $\Delta(\lambda)\langle i \rangle$, $\nabla(\lambda)\langle i \rangle$, $P(\lambda)\langle i \rangle$, $T(\lambda)\langle i \rangle$ and $T(\lambda)\langle i \rangle$ belong to $-i$. Simple, projective, injective, standard, costandard and tilting A-modules will be called *structural* modules. A complex $\mathcal{X}^\bullet$

$$(\mathcal{X}^\bullet, d_\bullet) : \ \cdots \xrightarrow{d_{i-2}} \mathcal{X}^{i-1} \xrightarrow{d_{i-1}} \mathcal{X}^{i} \xrightarrow{d_{i}} \mathcal{X}^{i+1} \xrightarrow{d_{i+1}} \cdots$$

of structural A-modules is called *linear* provided that for every $i \in \mathbb{Z}$ centroids of all indecomposable direct summands of $\mathcal{X}^i$ belong to $-i$.

The algebra A is called *standard Koszul* provided that all standard modules have linear projective resolutions and all costandard modules have linear injective coresolutions (see [ADL2]). The algebra A is called *balanced* provided that all standard modules have linear tilting coresolutions and all costandard modules have linear tilting resolutions (see [MO], where a stronger condition was imposed, however, we will show that both conditions are equivalent). The algebra A is called *Koszul* provided that projective resolutions of simple A-modules are linear (see [Pr, BGS, MOS]). Denote by $E(A)$ the opposite of the Yoneda extension algebra of the direct sum of of all simple A-modules. If A is Koszul, the algebra $E(A)$ is called the *Koszul dual* of A and we have that $E(A)$ is Koszul as well and $E(E(A)) \cong A$.

Let $\mathcal{D}^b(A)$ denote the bounded derived category of $A-$gmod. For $i \in \mathbb{Z}$ we denote by $[i]$ the autoequivalence of $\mathcal{D}^b(A)$, which shifts the position of the complex as follows: $\mathcal{X}[i]^j = \mathcal{X}^{i+j}$, $j \in \mathbb{Z}$ and $\mathcal{X}^\bullet \in \mathcal{D}^b(A)$. As usual, we identify A-modules with complexes concentrated in position 0. If A is Koszul, then the Koszul duality functor

$$\mathrm{K} = \mathcal{R}\mathrm{hom}_A(\oplus_{i\in\mathbb{Z}}\mathcal{P}\langle i\rangle[-i]^\bullet, \_),$$

where $\mathcal{P}^\bullet$ is the projective resolution of the direct sum of simple A-modules (see [BGS, MOS]), is well-defined and gives rise to an equivalence from $\mathcal{D}^b(A)$ to $\mathcal{D}^b(E(A))$.

Denote by $\mathfrak{LT}$ the full subcategory of $\mathcal{D}^b(A)$, which consists of all linear complexes of tilting A-modules. The category $\mathfrak{LT}$ is equivalent to $E(R(A))-$gmod and the simple objects of $\mathfrak{LT}$ have the form $T(\lambda)\langle -i\rangle[i]$, $\lambda\in\Lambda$, $i\in\mathbb{Z}$ ([MO]).

Let $R(A)$ denote the *Ringel dual* of A, which is the opposite of the (graded) endomorphism algebra of the characteristic tilting module $T = \oplus_{\lambda\in\Lambda}T(\lambda)$. The algebra $R(A)$ is quasi-hereditary with respect to the opposite order on Λ. The first Ringel duality functor

$$\mathrm{F} = \mathcal{R}\mathrm{hom}_A(\oplus_{i\in\mathbb{Z}}T\langle i\rangle, \_)$$

induces an equivalence from $\mathcal{D}^b(A)$ to $\mathcal{D}^b(R(A))$, which maps tilting modules to projectives, costandard modules to standard and injective modules to tilting. The second Ringel duality functor

$$\mathrm{G} = \mathcal{R}\mathrm{hom}_A(\_, \oplus_{i\in\mathbb{Z}}T\langle i\rangle)^*,$$

where $*$ denotes the usual duality, induces an equivalence from $\mathcal{D}^b(A)$ to $\mathcal{D}^b(R(A))$, which maps tilting modules to injectives, standard modules to costandard and projective modules to tilting.

3. The main result

The aim of this section is to prove Theorem 1. For this we fix a balanced algebra A throughout. For $\lambda\in\Lambda$ we denote by $\mathcal{S}_\lambda^\bullet$ and $\mathcal{C}_\lambda^\bullet$ the linear tilting coresolution of $\Delta(\lambda)$ and resolution of $\nabla(\lambda)$, respectively. We will need the following easy observation from [MO] and include the proof for the sake of completeness.

Lemma 2 ([MO])**.** *The natural grading on $R(A)$, induced from $A-$gmod, is positive.*

Proof. Let $\lambda,\mu\in\Lambda$. Then $T(\lambda)$ has a standard filtration and $T(\mu)$ has a costandard filtration ([Ri]). As standard modules are left orthogonal to costandard modules ([Ri]), every morphism from $T(\lambda)$ to $T(\mu)\langle j\rangle$, $j\in\mathbb{Z}$, in induced by a morphism from some standard module from a standard filtration of $T(\lambda)$ to some costandard module from a costandard filtration of $T(\mu)$. Hence to prove our claim it is enough to show that every standard module occurring in the standard filtration of $T(\lambda)$ and different from $\Delta(\lambda)$ has the form $\Delta(\nu)\langle j\rangle$ for some $j>0$; and that every costandard module occurring in the costandard filtration of $T(\mu)$ and different from $\nabla(\mu)$ has the form $\nabla(\nu)\langle j\rangle$ for some $j<0$.

We will prove the result for $T(\lambda)$ and for $T(\mu)$ the proof is similar. We use induction on λ. For $\lambda = 1$ the claim is trivial. For $\lambda > 1$ we consider the first two terms of the linear tilting coresolution of $\Delta(\lambda)$:

$$0 \to \Delta(\lambda) \to T(\lambda) \to X.$$

By linearity of our resolution, all direct summands of X have the form $T(\nu)\langle 1\rangle$ for some $\nu < \lambda$. All modules from the standard filtration of $T(\lambda)$, except for $\Delta(\lambda)$, occur in a standard filtration of X. Hence the necessary claim follows from the inductive assumption. □

From Lemma 2 we directly have the following:

Corollary 3. *We have* $\hom_A(T(\lambda)\langle i\rangle, T(\mu)) = 0$, $\lambda, \mu \in \Lambda$, $i \in \mathbb{N}$.

Corollary 3 allows us to formulate the following main technical tool of our analysis. Let $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ be two bounded complexes of tilting modules. We will say that $\mathcal{X}^\bullet$ *dominates* $\mathcal{Y}^\bullet$ provided that for every $i \in \mathbb{Z}$ the following holds: if the centroid of an indecomposable summand of $\mathcal{X}^i$ belongs to j and the centroid of an indecomposable summand of $\mathcal{Y}^i$ belongs to j', then $j < j'$.

Corollary 4. *Let $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ be two bounded complexes of tilting modules. Assume that $\mathcal{X}^\bullet$ dominates $\mathcal{Y}^\bullet$. Then* $\mathrm{Hom}_{\mathcal{D}^b(A)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet) = 0$.

Proof. Since tilting modules are self-orthogonal, by [Ha, Chapter III(2), Lemma 2.1] the necessary homomorphism space can be computed already in the homotopy category. Since $\mathcal{X}^\bullet$ dominates $\mathcal{Y}^\bullet$, from Corollary 3 we obtain $\mathrm{Hom}_A(\mathcal{X}^i, \mathcal{Y}^i) = 0$ for all i. The claim follows. □

Proposition 5. *For every $\lambda \in \Lambda$ the module $L(\lambda)$ is isomorphic in $\mathcal{D}^b(A)$ to a linear complex $\mathcal{L}^\bullet_\lambda$ of tilting modules.*

Proof. Consider a minimal projective resolution $\mathcal{P}^\bullet$ of $L(\lambda)$. Since A is positively graded, for every $i \in \mathbb{Z}$ centroids of all indecomposable projective modules in $\mathcal{P}^i$ belong to some j such that $j \geq -i$. Each projective has a standard filtration. Hence all centroids of standard subquotients in any standard filtration of an indecomposable projective module in $\mathcal{P}^i$ also belong to some j such that $j \geq -i$.

Resolving each standard subquotient $\Delta(\lambda)\langle j\rangle$ in every $\mathcal{P}^i$ using $\mathcal{S}_\lambda\langle j\rangle[i]^\bullet$, we obtain a complex $\overline{\mathcal{P}}^\bullet$ of tilting modules, which is isomorphic to $L(\lambda)$ in $\mathcal{D}^b(A)$. By construction and the previous paragraph, for each i all centroids of indecomposable summands in $\overline{\mathcal{P}}^i$ belong to some j such that $j \geq -i$.

Similarly, we consider a minimal injective coresolution $\mathcal{Q}^\bullet$ of $L(\lambda)$. Since A is positively graded, for every $i \in \mathbb{Z}$ centroids of all indecomposable injective modules in $\mathcal{Q}^i$ belong to some j such that $j \leq -i$. Resolving each standard subquotient $\nabla(\lambda)\langle j\rangle$ in every $\mathcal{Q}^i$ using $\mathcal{C}_\lambda\langle j\rangle[-i]^\bullet$, we obtain another complex, $\overline{\mathcal{Q}}^{\bullet}$, of tilting modules, which is isomorphic to $L(\lambda)$ in $\mathcal{D}^b(A)$. By construction, for each i all centroids of indecomposable summands in $\overline{\mathcal{Q}}^{i}$ belong to some j such that $j \leq -i$.

Because of the uniqueness of the minimal tilting complex $\mathcal{L}_{\lambda}^{\bullet}$, representing $L(\lambda)$ in $\mathcal{D}^b(A)$, we thus conclude that for all $i \in \mathbb{Z}$ centroids of all indecomposable summands in $\mathcal{L}_{\lambda}^{i}$ belong to $-i$. This means that $\mathcal{L}_{\lambda}^{\bullet}$ is linear and completes the proof. □

Corollary 6. *The algebra A is Koszul.*

Proof. Assume that $\mathrm{ext}_A^i(L(\lambda), L(\mu)\langle j\rangle) \neq 0$ for some $\lambda, \mu \in \Lambda$ and $j \in \mathbb{Z}$. Then $j \leq -i$ as A is positively graded. By Proposition 5, such a nonzero extension corresponds to a non-zero homomorphism from $\mathcal{L}_{\lambda}^{\bullet}$ to $\mathcal{L}_{\mu}\langle j\rangle[i]^{\bullet}$. Since both $\mathcal{L}_{\lambda}^{\bullet}$ and $\mathcal{L}_{\mu}\langle j\rangle[i]^{\bullet}$ are linear, the complex $\mathcal{L}_{\lambda}^{\bullet}$ dominates $\mathcal{L}_{\mu}\langle j\rangle[i]^{\bullet}$ for $j < -i$ and the homomorphism space vanish by Corollary 4. Therefore $j = -i$ and the claim follows. □

Corollary 7. *The algebra A is standard Koszul.*

Proof. That the minimal projective resolution of $\Delta(\lambda)$ is linear, is proved similarly to Corollary 6. To prove that the minimal injective coresolution of $\nabla(\mu)$ is linear we assume that $\mathrm{ext}_A^i(L(\lambda)\langle j\rangle, \nabla(\mu)) \neq 0$ for some $\lambda, \mu \in \Lambda$ and $j \in \mathbb{Z}$. Then $j \geq i$ as A is positively graded. As both $L(\lambda)$ and $\nabla(\mu)$ are represented in $\mathcal{D}^b(A)$ by linear complexes of tilting modules, one obtains that for $j > i$ the complex $\mathcal{L}_{\lambda}\langle j\rangle[-i]^{\bullet}$ dominates $\mathcal{C}_{\mu}^{\bullet}$, and thus the extension must vanish by Corollary 4. Therefore $j = i$ and the claim follows. □

Corollary 8. *The algebra $R(A)$ is balanced.*

Proof. By Lemma 2, the algebra $R(A)$ is positively graded with respect to the grading, induced from $A-$gmod. The functor F maps linear injective coresolutions of costandard A-modules to linear tilting coresolutions of standard $R(A)$-modules. The functor G maps linear projective resolutions of standard A-modules to linear tilting resolutions of costandard $R(A)$-modules. The claim follows. □

Remark 9. A standard Koszul quasi-hereditary algebra A is balanced if and only if $R(A)$ is positively graded with respect to the grading induced from $A-$gmod, see [MO, Theorem 7].

Corollary 10. *The algebra $R(A)$ is Koszul.*

Proof. This follows from Corollaries 6 and Corollaries 8. □

Proposition 11. *(i) The objects $\mathcal{S}_{\lambda}^{\bullet}$, $\lambda \in \Lambda$, are standard objects in $\mathfrak{LT}$ with respect to the natural order on Λ.*

(ii) The objects $\mathcal{C}_{\lambda}^{\bullet}$, $\lambda \in \Lambda$, are costandard objects in $\mathfrak{LT}$ with respect to the natural order on Λ.

Proof. We prove the claim (i), the claim (ii) is proved similarly. Let $\lambda, \mu \in \Lambda$ be such that $\lambda > \mu$. Every first extension ξ from $\mathcal{S}^\bullet_\lambda$ to $T(\mu)\langle -i\rangle[i]$, $i \in \mathbb{Z}$, is a complex and hence is obtained as the cone of some morphism φ from $\mathcal{S}[-1]^\bullet_\lambda$ to $T(\mu)\langle -i\rangle[i]$. The homology of the former complex is $\Delta(\lambda)$ and the homology of the latter is $T(\mu)$, which has a costandard filtration, where $\nabla(\lambda)$ does not occur (since $\mu < \lambda$). Since standard modules are left orthogonal to costandard modules, we get that all homomorphisms and extensions from $\Delta(\lambda)$ to $T(\mu)$ vanish. Therefore φ is homotopic to zero, which splits ξ. The claim follows. □

Proposition 12. *For all $\lambda, \mu \in \Lambda$ and $i, j \in \mathbb{Z}$ we have*

$$(1) \qquad \mathrm{Hom}_{\mathcal{D}^b(\mathfrak{LT})}(\mathcal{S}^\bullet_\lambda, \mathcal{C}_\mu\langle j\rangle[-i]^\bullet) = \begin{cases} \Bbbk, & \lambda = \mu, i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Via the equivalence K∘F, the equality (1) reduces to the equality

$$\mathrm{Hom}_{\mathcal{D}^b(A)}(\Delta(\lambda)^\bullet, \nabla(\mu)\langle j\rangle[-i]^\bullet) = \begin{cases} \Bbbk, & \lambda = \mu, i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The latter equality is true as standard modules are left orthogonal to costandard modules (see [Ri]). □

Corollary 13. *The algebra $E(R(A))$ is quasi-hereditary with respect to the natural order on Λ.*

Proof. By Propositions 11 and 12, standard $E(R(A))$-modules are left orthogonal to costandard. Now the claim follows from [DR, Theorem 1] (or [ADL1, Theorem 3.1]). □

Corollary 14. *The complexes $\mathcal{L}^\bullet_\lambda$, $\lambda \in \Lambda$, are tilting objects in $\mathfrak{LT}$.*

Proof. Because of [ADL1, Theorem 3.1] (or [DR, Ri]), we just need to show that any first extension from a standard object to $\mathcal{L}^\bullet_\lambda$ splits, and that any first extension from $\mathcal{L}^\bullet_\lambda$ to a costandard object splits. We prove the first claim and the second one is proved similarly.

Any first extension ξ from $\mathcal{S}_\mu\langle -i\rangle[i]^\bullet$, $\mu \in \Lambda$, $i \in \mathbb{Z}$, to $\mathcal{L}^\bullet_\lambda$ is a cone of some homomorphism φ from $\mathcal{S}_\mu\langle -i\rangle[i-1]^\bullet$ to $\mathcal{L}^\bullet_\lambda$. Thus φ corresponds to a (nonlinear) extension of degree $1-i$ from $\Delta(\mu)\langle -i\rangle$ to $L(\lambda)$. As A is standard Koszul by Corollary 7, we get that φ is homotopic to zero, and thus the extension ξ splits. The claim follows. □

Corollary 15. *There is an isomorphism $E(A) \cong R(E(R(A)))$ of graded algebras, both considered with respect to the natural grading induced from $\mathcal{D}^b(A)$. In particular, we have $R(E(A)) \cong E(R(A))$.*

Proof. By Corollary 14, the algebra $R(E(R(A)))$ is the opposite of the endomorphism algebra of $\oplus_{\lambda\in\Lambda}\mathcal{L}^\bullet_\lambda$. Since $\mathcal{L}^\bullet_\lambda$ is isomorphic to $L(\lambda)$ in $\mathcal{D}^b(A)$, from [Ha, Chapter III(2), Lemma 2.1] it follows that the same algebra is isomorphic to $E(A)$. The claim follows. □

Corollary 16. *Both $E(A)$ and $R(E(A))$ are positively graded with respect to the natural grading induced from $\mathcal{D}^b(A)$.*

Proof. For $E(A)$ the claim is obvious. By Corollary 15, we have $R(E(A)) \cong E(R(A))$. As $R(A)$ is positively graded with respect to the grading induces from $\mathcal{D}^b(A)$ (Lemma 2), the algebra $E(R(A))$ is positively graded with respect to the induces grading as well. □

Proposition 17. *The positively graded algebras $E(A)$ and $R(E(A))$ are balanced.*

Proof. Because of Corollary 8, it is enough to prove the claim for the algebra $E(A)$. Consider the algebra $E(R(A))$, whose module category is realized via $\mathfrak{LT}$.

Lemma 18. *The algebra $E(R(A))$ is standard Koszul.*

Proof. We already know that $E(R(A))$ is positively graded with respect to the grading, induced from $\mathcal{D}^b(A)$. Let us show that projective resolutions of standard $E(R(A))$-modules are linear. For injective resolutions of costandard modules the argument is similar.

We have to compute

$$\hom_{\mathcal{D}^b(\mathfrak{LT})}(\mathcal{S}^{\bullet}_{\lambda}, T(\mu)\langle j\rangle[i]) \tag{2}$$

for all $\lambda, \mu \in \Lambda$ and $i, j \in \mathbb{Z}$. Via the equivalence $\mathrm{K} \circ \mathrm{F}$, the space (2) is isomorphic to the space $\hom_{\mathcal{D}^b(A)}(\Delta(\lambda), T(\mu)\langle j\rangle[i])$. As $T(\mu)$ has a costandard filtration and standard modules are left orthogonal to costandard, we get that the later space is non-zero only if $i = 0$. As $R(A)$ is positively graded, we also get that $j < 0$. Applying [MOS, Theorem 22] we obtain that the standard $E(R(A))$-module $\mathcal{S}^{\bullet}_{\lambda}$ has only linear extensions with simple $E(R(A))$-modules. This completes the proof. □

Using Lemma 18, the proof of Proposition 17 is completed similarly to the proof of Corollary 8. □

Proof of Theorem 1. The claim (i) follows from Corollaries 6 and 7. The claim (ii) follows from Corollary 8 and Proposition 17. The claim (iii) follows from Proposition 5. Finally, the claim (iv) follows from Corollary 15. □

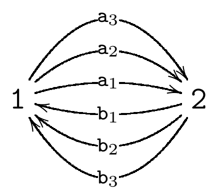
4. Examples

Example 19. Graded quasi-hereditary algebras, associated with blocks of the usual BGG category $\mathcal{O}$ and the parabolic category $\mathcal{O}$ for a semi-simple complex finite-dimensional Lie algebra, are balanced by [Ma1].

Example 20. The algebra A is called directed if either all standard or all costandard A-modules are simple (this is equivalent to the requirement that the quiver of A is directed with respect to the natural order on Λ). For a directed algebra A tilting modules are either injective (if standard modules are simple) or projective (if costandard modules are simple). Hence any directed Koszul algebra is balanced.

Example 21. Finite truncations $V_{\mathcal{T}}$ of Cubist algebras from [CT, Section 6] are balanced. Indeed, $V_{\mathcal{T}}$ is standard Koszul by [CT, Proposition 46], and that the Ringel dual of $V_{\mathcal{T}}$ is positively graded with respect to the induced grading follows from [CT, Corollary 71]. So, the fact that $V_{\mathcal{T}}$ is balanced follows from Remark 9.

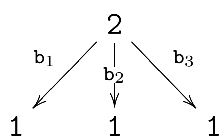
Example 22. One explicit example. Consider the path algebra A of the following quiver with relations:



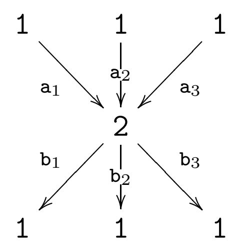
, $\mathtt{a}_i\mathtt{b}_j = 0,\ i,j = 1,2,3,$

We have $\Delta(1) \cong T(1) \cong L(1) \cong \nabla(1)$ and for $\lambda = 2$ we have the following standard and tilting modules:

$\Delta(2)$:



$T(2)$:

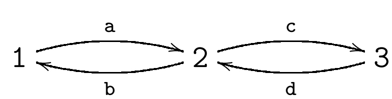


Hence we have the following linear tilting coresolution of $\Delta(2)$:

$$0 \to \Delta(2) \to T(2) \to T(1)\langle 1\rangle \oplus T(1)\langle 1\rangle \oplus T(1)\langle 1\rangle \to 0.$$

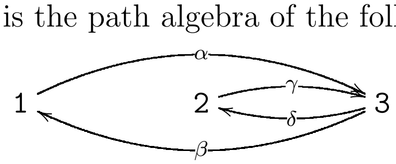
Swapping $\mathtt{a}_i$ and $\mathtt{b}_i$, $i = 1,2,3$, defines an antiinvolution on A, which preserves the primitive idempotents. Hence there is a duality on $A-$gmod, which preserves isomorphism classes of simple modules. Applying this duality to the above resolution gives a linear tilting resolution of $\nabla(2)$. Thus A is balanced. In this example one can also arbitrarily increase or decrease the number of arrows.

Example 23. One computes that the path algebra of the following quiver with relations



, $\mathtt{ab} = \mathtt{cd} = 0,$

is standard Koszul but not balanced. In fact, the Ringel dual of this algebra is the path algebra of the following quiver with relations



, $\beta\alpha = \delta\gamma = \beta\gamma\delta\alpha = 0,$

which is not Koszul (not even quadratic). So, our results can not be extended to all standard Koszul algebras.

Remark 24. Directly from the definition it follows that if the algebra A is balanced, then the algebra A/Ae_nA is balanced as well. It is also easy to see that if A and B are balanced, then both $A \oplus B$ and $A \otimes_{\Bbbk} B$ are balanced.

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KOSZUL DUALITY FOR STRATIFIED ALGEBRAS II. STANDARDLY STRATIFIED ALGEBRAS

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Abstract. We give a complete picture of the interaction between the Koszul and Ringel dualities for graded standardly stratified algebras (in the sense of Cline, Parshall and Scott) admitting linear tilting (co)resolutions of standard and proper costandard modules. We single out a certain class of graded standardly stratified algebras, imposing the condition that standard filtrations of projective modules are finite, and develop a tilting theory for such algebras. Under the assumption on existence of linear tilting (co)resolutions we show that algebras from this class are Koszul, that both the Ringel and Koszul duals belong to the same class, and that these two dualities on this class commute.

1. Introduction

In the theory of quasi-hereditary algebras there are two classical dualities: the Ringel duality, associated with the characteristic tilting module (see [Ri]), and the Koszul duality, associated with the category of linear complexes of projective modules (see [CPS2, ADL1, MO]). In [MO, Ma] it is shown that a certain class of Koszul quasi-hereditary algebras is stable with respect to taking both the Koszul and Ringel duals and that on this class of algebras the Koszul and Ringel dualities commute.

The approach of [Ma] is ultimately based on the possibility to realize the derived category of our algebra as the homotopy category of complexes of tilting modules. This also suggested that the arguments of [Ma] should work in a much more general setup, whenewer an appropriate stratification of the algebra and a sensible tilting theory with respect to this stratification exist. The aim of the present paper is to define a setup for the study of Koszulity for stratified algebras and to extend to this setup the main result of [Ma]. We note that Koszul standardly stratified algebras, which are not quasi-hereditary, appear naturally in [ADL2, Fr3, KKM].

The most general setup for stratified algebras seems to be the notion of standardly stratified algebras as introduced by Cline, Parshall and Scott in [CPS1]. The main problem which one faces, trying to generalize [Ma] to such stratified algebras, is that standardly stratified algebras have infinite global dimension in general. In particular, this means that the Koszul dual of such an algebra (in the case when the original algebra is Koszul) is always infinite dimensional. Therefore any reasonable extension of [Ma] to stratified algebras must deal with infinite dimensional stratified algebras, for which many of the classical results are not proved and lots of classical techniques are not developed.

In the present paper we study the class of positively graded standardly stratified algebras with finite dimensional homogeneous components satisfying the additional assumption that all projective modules have finite standard filtrations. For such algebras we develop an analogue of the classical tilting theory and Ringel duality. This follows closely the classcal theory, however, at some places one has to be careful as we work with infinite dimensional algebras, so some extension spaces might be infinite dimensional. We use the grading to split these infinite dimensional spaces into an (infinite) direct sum of finite dimensional ones. We also give some examples which justify our choice of algebras and show that outside the class we define the classical approach to tilting theory fails. The Ringel duality functor turns out to be an antiequivalence between three different kinds of derived categories.

Using the standard grading of a characteristic tilting module, we restrict our attention to those standardly stratified algebras, for which all tilting coresolutions of standard modules and all tilting resolutions of proper costandard modules are linear. For an algebra A let $R(A)$ and $E(A)$ denote the Ringel and Koszul duals of A, respectively. Generalizing the arguments of [Ma] we prove the following (see Section 2 for the definitions):

Theorem 1. *Let A be a positively graded standardly stratified algebra with finite dimensional homogeneous components. Assume that*

(a) Every indecomposable projective A-module has a finite standard filtration.
(b) Every standard A-module has a linear tilting coresolution.
(c) Every costandard A-module has a linear tilting resolution.

Then the following holds:

(i) The algebra A is Koszul.
(ii) The algebras A, $R(A)$, $E(A)$, $E(R(A))$ and $R(E(A))$ have properties (a), (b) *and* (c).
(iii) Every simple A-module is represented (in the derived category) by a linear complex of tilting modules.
(iv) $R(E(A)) \cong E(R(A))$ as graded standardly stratified algebras.

Theorem 1 extends and generalizes results from [ADL1, ADL2, MO, Ma].

The paper is organized as follows: in Section 2 we collected all necessary definitions and preliminaries. In Sections 3 and 4 we develop the tilting theory for graded standardly stratified algebras. This theory is used in Section 5 to prove Theorem 1. We complete the paper with several examples in Section 6.

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2. Graded standardly stratified algebras

By $\mathbb{N}$ we denote the set of all positive integers. By a grading we always mean a $\mathbb{Z}$-*grading* and by a module we always mean a *graded left* module.

Let $\Bbbk$ be an algebraically closed field and $A = \bigoplus_{i\geq 0} A_i$ be a graded $\Bbbk$-algebra. We assume that A is *locally finite*, that is $\dim_{\Bbbk} A_i < \infty$. Set $\mathrm{r}(A) := \bigoplus_{i>0} A_i$. We further assume that $A_0 \cong \bigoplus_{\lambda\in\Lambda} \Bbbk e_\lambda$ for some set $\{e_\lambda : \lambda \in \Lambda\}$ of pairwise orthogonal nonzero idempotents in A_0, where Λ is a nonempty finite set (using the classical Morita theory one extends all our results to the case when A_0 is a semi-simple algebra). Under these assumptions the algebra A is positively graded in the sense of [MOS]. In what follows we call A *positively graded* if it satisfies all assumptions of this paragraph. A typical example of a positively graded algebra is $\Bbbk[x]$, where 1 has degree zero and x has degree one.

Let A-gmod denote the category of all locally finite dimensional graded A-modules. Morphisms in this category are homogeneous morphism of degree zero between graded A-modules. Consider the full subcategories $A^{\uparrow}$-gmod and $A^{\downarrow}$-gmod of A-gmod, which consist of all graded modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$ for which there exists $n \in \mathbb{Z}$ such that $M_i = 0$ for all $i > n$ or $i < n$, respectively. All these categories are abelian, the category $A^{\downarrow}$-gmod has enough projectives and the category $A^{\uparrow}$-gmod has enough injectives. For $M \in A^{\downarrow}$-gmod we set

$$\mathfrak{b}(M) = \begin{cases} +\infty, & M = 0; \\ \min_{n\in\mathbb{Z}}\{M_n \neq 0\}, & M \neq 0. \end{cases}$$

For $i \in \mathbb{Z}$ we denote by $\langle i\rangle$ the autoequivalence of A-gmod, which shifts the grading as follows: $(M\langle i\rangle)_j = M_{i+j}$, where $j \in \mathbb{Z}$. This autoequivalence preserves both $A^{\uparrow}$-gmod and $A^{\downarrow}$-gmod. Denote by $\circledast$ the usual *graded duality* on A-gmod (it swaps $A^{\uparrow}$-gmod and $A^{\downarrow}$-gmod). We adopt the notation $\hom_A$ and ext^i_A to denote homomorphisms and extensions in A-gmod. Unless stated otherwise, all morphisms are considered in the category A-gmod.

For $\lambda \in \Lambda$ we consider the graded indecomposable projective module $P(\lambda) = Ae_\lambda$, its graded simple quotient $L(\lambda) = P(\lambda)/\mathrm{r}(A)P(\lambda)$ and the graded indecomposable injective envelop $I(\lambda)$ of $L(\lambda)$. Note that we always have the following: $P(\lambda) \in A^{\downarrow}$-gmod, $I(\lambda) \in A^{\uparrow}$-gmod and $L(\lambda) \in A^{\downarrow}\text{-gmod} \cap A^{\uparrow}\text{-gmod}$.

Let $\preceq$ be a partial preorder on Λ. For $\lambda, \mu \in \Lambda$ we write $\lambda \prec \mu$ provided that $\lambda \preceq \mu$ and $\mu \not\preceq \lambda$. We also write $\lambda \sim \mu$ provided that $\lambda \preceq \mu$ and $\mu \preceq \lambda$. Then $\sim$ is an equivalence relation. Let $\overline{\Lambda}$ denote the set of equivalence classes of $\sim$. Then the preorder $\preceq$ induces a partial order on $\overline{\Lambda}$, which we will denote by the same symbol, abusing notation. For $\lambda \in \Lambda$ we denote by $\overline{\lambda}$ the equivalence class from $\overline{\Lambda}$, containing λ. We also denote by $\preceq^{\mathrm{op}}$ the partial preorder on Λ, opposite to $\preceq$.

For $\lambda \in \Lambda$ we define the *standard module* $\Delta(\lambda)$ as the quotient of $P(\lambda)$ modulo the submodule, generated by the images of all possible morphisms $P(\mu)\langle i\rangle \to P(\lambda)$, where $\lambda \prec \mu$ and $i \in \mathbb{Z}$. We also define the *proper standard module* $\overline{\Delta}(\lambda)$ as the quotient of $P(\lambda)$ modulo the submodule, generated by the images of all possible morphisms $P(\mu)\langle i\rangle \to P(\lambda)$, where $\lambda \preceq \mu$ and $i \in \mathbb{Z}$ satisfies $i < 0$. By definition, the modules $\Delta(\lambda)$ and $\overline{\Delta}(\lambda)$ belong to $A^{\downarrow}$-gmod. Dually we define the *costandard module* $\nabla(\lambda)$ and the *proper costandard module* $\overline{\nabla}(\lambda)$ (which always belong to $A^{\uparrow}$-gmod).

The algebra A will be called *standardly stratified* (with respect to the preorder $\preceq$ on Λ) provided that for every $\lambda \in \Lambda$ the kernel $K(\lambda)$ of the canonical projection $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ has a *finite* filtration, whose subquotients are isomorphic (up to shift) to standard modules. This is a natural generalization of the original definition from [CPS1] to our setup. For example, the algebra A is always standardly stratified (with projective standard modules) in the case $|\Lambda| = 1$ and, more generally, in the case when the relation $\preceq$ is the full relation.

3. Tilting theory for graded standardly stratified algebras

Tilting theory for (finite dimensional) quasi-hereditary algebras was developed in [Ri]. It was extended in [AHLU] to (finite dimensional) strongly standardly stratified algebras and in [Fr2] to all (finite dimensional) standardly stratified algebras. For infinite dimensional algebras some versions of tilting theory appear in [CT, DM, MT]. This section is a further generalization of all these results, especially of those from [Fr2], to the case infinite dimensional positively graded algebras. In this section A is a positively graded standardly stratified algebra.

Let $\mathcal{C}(\Delta)$ denote the full subcategory of the category $A^{\downarrow}$-gmod, which consists of all modules M admitting a (possibly infinite) filtration

$$M = M^{(0)} \supseteq M^{(1)} \supseteq M^{(2)} \supseteq \dots, \qquad (1)$$

such that for every $i = 0, 1, \dots$ the subquotient $M^{(i)}/M^{(i+1)}$ is isomorphic (up to shift) to some standard module and $\lim\limits_{i\to+\infty} \mathfrak{b}(M^{(i)}) = +\infty$. Note that for $M \in A^{\downarrow}$-gmod with such a filtration we automatically get $\bigcap\limits_{i\geq 0} M^{(i)} = 0$. Denote by $\mathcal{F}^{\downarrow}(\Delta)$ the full subcategory of $A^{\downarrow}$-gmod, which consists of all modules M admitting a finite filtration with subquotients from $\mathcal{C}(\Delta)$. The category $\mathcal{F}^{\downarrow}(\Delta)$ is obviously closed with respect to finite extensions. Similarly we define $\mathcal{F}^{\downarrow}(\overline{\nabla})$. Let $\mathcal{F}^{b}(\Delta)$ and $\mathcal{F}^{b}(\overline{\nabla})$ be the corresponding full subcategories of modules with finite filtrations of the form (1). We start with the following result, which generalizes the corresponding results from [AB, AR, Ri, Fr2].

Theorem 2. *Let A be a positively graded standardly stratified algebra.*

(i) We have

$$\begin{aligned}\mathcal{F}^{\downarrow}(\Delta) &= \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^i_A(M, \overline{\nabla}(\lambda)\langle j\rangle) = 0, \forall j \in \mathbb{Z}, i > 0, \lambda \in \Lambda\}\\ &= \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^1_A(M, \overline{\nabla}(\lambda)\langle j\rangle) = 0, \forall j \in \mathbb{Z}, \lambda \in \Lambda\}.\end{aligned}$$

(ii) We have

$$\begin{aligned}\mathcal{F}^{\downarrow}(\overline{\nabla}) &= \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^i_A(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, i > 0, \lambda \in \Lambda\}\\ &= \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, \lambda \in \Lambda\}.\end{aligned}$$

To prove Theorem 2 we will need several auxiliary lemmata. We will often use the usual induction for stratified algebras. To define this let $\lambda \in \Lambda$ be maximal with respect to $\preceq$. Set $e_{\overline{\lambda}} = \sum_{\mu\in\overline{\lambda}} e_\mu$, $I_{\overline{\lambda}} = Ae_{\overline{\lambda}}A$ and $B_{\overline{\lambda}} = A/I_{\overline{\lambda}}$. The algebra $B_{\overline{\lambda}}$ inherits from A a positive grading and hence is a positively graded locally finite algebra. Further, just like in the case of usual stratified algebras, the algebra $B_{\overline{\lambda}}$ is stratified with respect to the restriction of the preorder $\preceq$ to $\Lambda \setminus \{\overline{\lambda}\}$. Any module M over $B_{\overline{\lambda}}$ can be considered as an A-module in the usual way. Set $P(\overline{\lambda}) = \bigoplus_{\mu\in\overline{\lambda}} P(\mu)$.

Lemma 3. *For all $M, N \in B^{\downarrow}_{\overline{\lambda}}$-gmod and all $i \geq 0$ we have*

$$\mathrm{ext}^i_{B_{\overline{\lambda}}}(M, N) = \mathrm{ext}^i_A(M, N).$$

Proof. Let $\mathcal{P}^\bullet$ denote the minimal projective resolution of M in $A^{\downarrow}$-gmod. As $M \in B^{\downarrow}_{\overline{\lambda}}$-gmod, there exists $k \in \mathbb{Z}$ such that $M_j = 0$ for all $j < k$. Since A is positively graded, we get $\mathcal{P}^i_j = 0$ for all $j < k$ and all i.

Consider the projective module $P = \bigoplus_{j\leq -k} P(\overline{\lambda})\langle j\rangle$. As A is standardly stratified, for every i the sum T^i of images of all homomorphisms from P to $\mathcal{P}^i$ has the form $\bigoplus_{j\leq -k} P_{i,j}$, where $P_{i,j} \in \mathrm{add} P(\overline{\lambda})\langle j\rangle$.

The differential of $\mathcal{P}^\bullet$ obviously maps T^i to T^{i-1}, which means that the sum of all T^i is a subcomplex of $\mathcal{P}^\bullet$, call it $\mathcal{T}^\bullet$. Since $M \in B^{\downarrow}_{\overline{\lambda}}$-gmod, the quotient $\overline{\mathcal{P}}^\bullet$ of $\mathcal{P}^\bullet$ modulo $\mathcal{T}^\bullet$ gives a minimal projective resolution of M over $B_{\overline{\lambda}}$.

Since $N \in B^{\downarrow}_{\overline{\lambda}}$-gmod, any homomorphism from $\mathcal{P}^i$ to N annihilates $\mathcal{T}^i$ and hence factors through $\overline{\mathcal{P}}^i$. The claim of the lemma follows. □

Lemma 4. *For all $\mu \in \Lambda$ we have $\overline{\nabla}(\mu) \in A^{\downarrow}$-gmod, in particular, $\overline{\nabla}(\mu)$ is finite dimensional.*

Proof. We proceed by induction on the cardinality of $\overline{\Lambda}$. If $|\overline{\Lambda}| = 1$, then all $\Delta(\lambda)$ are projective and all $\overline{\nabla}(\mu)$ are simple, so the claim is trivial.

Assume now that $|\overline{\Lambda}| > 1$. Let $\lambda \in \Lambda$ be maximal. Then for all $\mu \notin \overline{\lambda}$, the claim follows from the inductive assumption applied to the stratified algebra $B_{\overline{\lambda}}$.

Assume, finally, that $\mu \in \overline{\lambda}$ is such that $\overline{\nabla}(\mu) \notin A^{\downarrow}$-gmod. Then there exists $\nu \in \Lambda$ and an infinite sequence $0 < j_1 < j_2 < \dots$ of positive integers such that for any $l \in \mathbb{N}$ there exists a nonzero homomorphism from $P(\nu)\langle j_l \rangle$ to $\overline{\nabla}(\mu)$. Let M_l denote the image of this homomorphism. Then M_l has simple top $L(\nu)\langle j_l \rangle$ and simple socle $L(\mu)$ and all other composition subquotients of the form $L(\nu')\langle j \rangle$, where $\nu' \prec \mu$ and $1 \leq j \leq j_l - 1$.

The module $M_l\langle -j_l \rangle$ is thus a quotient of $P(\nu)$. Then the socle $L(\mu)\langle -j_l \rangle$ of $M_l\langle -j_l \rangle$ gives rise to a nonzero homomorphism from $P(\mu)\langle -j_l \rangle$ to $P(\nu)$. Since μ is maximal and all other composition subquotients of $M_l\langle -j_l \rangle$ are of the form $L(\nu')\langle j \rangle$ for some $\nu' \prec \mu$, the above homomorphism gives rise to an occurrence of the standard module $\Delta(\mu)\langle -j_l \rangle$ in the standard filtration of $P(\nu)$. However, we have infinitely many j_l's and, at the same time, the standard filtration of $P(\nu)$ is finite. This is a contradiction, which yields the claim of the lemma. □

Lemma 5. *For all $i, j \in \mathbb{Z}$ such that $i \geq 0$, and all $\lambda, \mu \in \Lambda$ we have*

$$\mathrm{ext}^i_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j \rangle) = \begin{cases} \Bbbk, & i = j = 0, \lambda = \mu; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We proceed by induction on the cardinality of $\overline{\Lambda}$. If $|\overline{\Lambda}| = 1$, then all $\Delta(\lambda)$ are projective and all $\overline{\nabla}(\mu)$ are simple, so the claim is trivial.

Assume now that $|\overline{\Lambda}| > 1$. Let $\lambda' \in \Lambda$ be maximal. Then, by definitions, the module $\Delta(\lambda)$ is projective for all $\lambda \in \overline{\lambda'}$. Hence for such λ the claim of the lemma follows from the definition of $\overline{\nabla}(\mu)$. If $\lambda, \mu \notin \overline{\lambda'}$, the claim follows from the inductive assumption applied to the standardly stratified algebra $B_{\overline{\lambda'}}$ and Lemma 3.

Consider now the case when $\mu \in \overline{\lambda'}$ and $\lambda \notin \overline{\lambda'}$. Then $\Delta(\lambda)$ does not have any composition subquotient of the form $L(\mu)\langle j \rangle$ and hence

$$\mathrm{hom}_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j \rangle) = 0.$$

Let us check that

$$\mathrm{ext}^1_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j\rangle) = 0 \tag{2}$$

for all j. Applying $\hom_A(\Delta(\lambda), \_)$ to the short exact sequence

$$\overline{\nabla}(\mu)\langle j\rangle \hookrightarrow I(\mu)\langle j\rangle \twoheadrightarrow \mathrm{Coker},$$

we obtain the exact sequence

$$\hom_A(\Delta(\lambda), \mathrm{Coker}) \to \mathrm{ext}^1_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j\rangle) \to \mathrm{ext}^1_A(\Delta(\lambda), I(\mu)\langle j\rangle).$$

Here the right term equals zero by the injectivity of $I(\mu)$. By the definition of $\overline{\nabla}(\mu)$, the socle of Coker has (up to shift) only simple modules of the form $L(\nu)$, where $\nu \in \overline{\lambda'}$, which implies that the left term equals zero as well. The equality (2) follows.

Now we prove our claim by induction on λ with respect to $\preceq$ (as mentioned above, the claim is true for λ maximal). Apply $\hom_A(\_, \overline{\nabla}(\mu)\langle j\rangle)$ to the short exact sequence

$$\mathrm{Ker} \hookrightarrow P(\lambda) \twoheadrightarrow \Delta(\lambda) \tag{3}$$

and, using the projectivity of $P(\lambda)$, for each $i > 1$ obtain the following exact sequence:

$$0 \to \mathrm{ext}^{i-1}_A(\mathrm{Ker}, \overline{\nabla}(\mu)\langle j\rangle) \to \mathrm{ext}^i_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j\rangle) \to 0.$$

Since A is standardly stratified, Ker has a finite filtration by standard modules of the form $\Delta(\nu)$, where $\lambda \prec \nu$, (up to shift). Hence, from the inductive assumption we get $\mathrm{ext}^{i-1}_A(\mathrm{Ker}, \overline{\nabla}(\mu)\langle j\rangle) = 0$. This yields $\mathrm{ext}^i_A(\Delta(\lambda), \overline{\nabla}(\mu)\langle j\rangle) = 0$ and completes the proof. □

Corollary 6. *Let A be a positively graded standardly stratified algebra.*

(i) *For any $M \in \mathcal{F}^{\downarrow}(\Delta)$, $\lambda \in \Lambda$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have* $\mathrm{ext}^i(M, \overline{\nabla}(\lambda)\langle j\rangle) = 0$.

(ii) *For any $M \in \mathcal{F}^{\downarrow}(\overline{\nabla})$, $\lambda \in \Lambda$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have* $\mathrm{ext}^i(\Delta(\lambda)\langle j\rangle, M) = 0$.

Proof. It is certainly enough to prove statement (i) in the case when M has a filtration of the form (1). As $\lim_{i\to+\infty} \mathfrak{b}(M^{(i)}) = +\infty$ and $\overline{\nabla}(\lambda)$ is finite dimensional (Lemma 4), there exists $n \in \mathbb{Z}$ such that for any $i \in \mathbb{Z}$ with $\overline{\nabla}(\lambda)\langle j\rangle_i \neq 0$ we have $i < \mathfrak{b}(M^{(n)})$. Since A is positively graded, there are no homomorphisms from any component of the projective resolution of $M^{(n)}$ to $\overline{\nabla}(\lambda)\langle j\rangle$. This means that all extentions from $M^{(n)}$ to $\overline{\nabla}(\lambda)\langle j\rangle$ vanish. At the same time, the quotient $M/M^{(n)}$ has a finite filtration by standard modules and hence all extensions from $M/M^{(n)}$ to $\overline{\nabla}(\lambda)\langle j\rangle$ vanish by Lemma 5. Statement (i) follows.

It is certainly enough to prove statement (ii) in the case when M has a filtration of the form (1) (with subquotients being proper costandard modules). Let $\mathcal{P}^{\bullet}$ be the minimal projective resolution of $\Delta(\lambda)\langle j\rangle$. As every indecomposable projective has a finite standard filtration, it follows that $\mathcal{P}^{\bullet}$ has only finitely many nonzero components, moreover, each of them is a finite direct sum of projective modules. As $\lim_{i\to+\infty}\mathfrak{b}(M^{(i)})=+\infty$, there exists $n\in\mathbb{N}$ such that there are no maps from any $\mathcal{P}^i$ to $M^{(n)}$, in particular, all extensions from $\Delta(\lambda)\langle j\rangle$ to $M^{(n)}$ vanish. At the same time, the quotient $M/M^{(n)}$ has a finite filtration by proper costandard modules and hence all extensions from $\Delta(\lambda)\langle j\rangle$ to $M/M^{(n)}$ vanish by Lemma 5. Statement (ii) follows and the proof is complete. □

The following lemma is just an observation that the category $\mathcal{F}^{\downarrow}(\overline{\nabla})$ can, in fact, be defined in a somewhat easier way than the one we used. For the category $\mathcal{F}^{\downarrow}(\Delta)$ this is not possible in the general case, see Example 43.

Lemma 7. *Any module from $\mathcal{F}^{\downarrow}(\overline{\nabla})$ has a filtration of the form* (1).

Proof. Let $X, Z\in\mathcal{C}$ and

$$X=X^{(0)}\supseteq X^{(1)}\supseteq X^{(2)}\supseteq\dots,$$

and

$$Z=Z^{(0)}\supseteq Z^{(1)}\supseteq Z^{(2)}\supseteq\dots,$$

be filtrations of the form (1). Assume that $Y\in A^{\downarrow}$-gmod is such that there is a short exact sequence

$$0\to X\to Y\to Z\to 0.$$

To prove the claim of the lemma it is enough to show that Y has a filtration of the form (1).

Since all costandard modules are finite dimensional (Lemma 4) and $\lim\limits_{i\to+\infty}\mathfrak{b}(Z^{(i)})=+\infty$, there exists $k\in\{0,1,2,\dots\}$ such that for any $i\in\mathbb{Z}$ with $(X^{(0)}/X^{(1)})_i\neq 0$ we have $i<\mathbf{b}(Z^{(k)})$.

Now for $i=0,1,\dots,k$, we let $Y^{(i)}$ be the full preimage of $Z^{(i)}$ in Y under the projection $Y\twoheadrightarrow Z$. In this way we get the first part of the filtration of Y with proper costandard subquotients. On the next step we let $Y^{(k+1)}$ denote the submodule of $Y^{(k)}$ generated by $X^{(1)}$ and $Y_i^{(k)}$, where $i\geq\mathbf{b}(Z^{(k)})$. Then $Y^{(k+1)}+X^{(0)}=Y^{(k)}$ by construction. At the same time, from our choice of k in the previous paragraph it follows that $Y^{(k+1)}\cap X^{(0)}=X^{(1)}$ and hence

$$Y^{(k)}/Y^{(k+1)}\cong X^{(0)}/X^{(1)},$$

which is a proper costandard module.

Now we proceed in the same way constructing a proper costandard filtration for $Y^{(k+1)}$. The condition $\lim\limits_{i\to+\infty}\mathfrak{b}(Y^{(i)})=+\infty$ follows from the construction. This completes the proof. □

Lemma 8. *Let $M\in A^{\downarrow}$-gmod be such that $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, M)=0$ for all λ and j. Then $M\in\mathcal{F}^{\downarrow}(\overline{\nabla})$.*

Proof. First let us show that the conditions of the lemma imply

$$\mathrm{ext}^i_A(\Delta(\lambda)\langle j\rangle, M) = 0 \tag{4}$$

for all j, all λ and all $i > 0$. If λ is maximal, then the corresponding $\Delta(\lambda)$ is projective and the claim is clear. Otherwise, we proceed by induction with respect to the preorder $\preceq$. We apply $\mathrm{hom}_A({}_-, M)$ to the short exact sequence (3) and the equality (4) follows from the inductive assumption by the dimension shift in the obtained long exact sequence.

We proceed by induction on the cardinality of $\overline{\Lambda}$. If $|\overline{\Lambda}| = 1$, then $\mathcal{F}^{\downarrow}(\overline{\nabla}) = A^{\downarrow}$-gmod and the claim is trivial.

Assume now that $|\overline{\Lambda}| > 1$ and let $\lambda' \in \Lambda$ be maximal. Let N denote the maximal submodule of M, which does not contain any composition factors of the form $L(\mu)$, where $\mu \in \overline{\lambda'}$ (up to shift). Let $\nu \notin \overline{\lambda'}$. Applying $\mathrm{hom}_A(\Delta(\nu)\langle j\rangle, {}_-)$ to the short exact sequence

$$N \hookrightarrow M \twoheadrightarrow \mathrm{Coker}, \tag{5}$$

we obtain the exact sequence

$$\mathrm{hom}_A(\Delta(\nu)\langle j\rangle, \mathrm{Coker}) \to \mathrm{ext}^1_A(\Delta(\nu)\langle j\rangle, N) \to \mathrm{ext}^1_A(\Delta(\nu)\langle j\rangle, M).$$

Here the right term is zero by our assumptions and the left term is zero by the definition of N. This implies that the middle term is zero, which yields $\mathrm{ext}^1_{B_{\overline{\lambda'}}}(\Delta(\nu)\langle j\rangle, N) = 0$ by Lemma 3. Applying the inductive assumption to the standardly stratified algebra $B_{\overline{\lambda'}}$, we obtain that $N \in \mathcal{F}^{\downarrow}(\overline{\nabla})$.

Since $\mathcal{F}^{\downarrow}(\overline{\nabla})$ is extension closed, to complete the proof we are left to show that $\mathrm{Coker} \in \mathcal{F}^{\downarrow}(\overline{\nabla})$. Applying $\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, {}_-)$ to (5) and using (4), the previous paragraph and Lemma 5, we obtain that

$$\mathrm{ext}^i_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}) = 0 \tag{6}$$

for all j, λ and $i > 0$.

If $\mathrm{Coker} = 0$, we are done. Otherwise, there exists some $\mu \in \overline{\lambda'}$ and a maximal possible $j' \in \mathbb{Z}$ such that there is a nonzero homomorphism from Coker to $I(\mu)\langle j'\rangle$. Let K denote the image of this homomorphism. Applying $\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, {}_-)$ to the short exact sequence

$$\mathrm{Ker} \hookrightarrow \mathrm{Coker} \twoheadrightarrow K, \tag{7}$$

and using the definition of K, we obtain that

$$\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, \mathrm{Ker}) = 0 \tag{8}$$

for all λ and j. The equality (8), the corresponding equalities (4) (for $M = \mathrm{Ker}$) and the dimension shift with respect to (7) then imply

$$\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, K) = 0 \tag{9}$$

for all λ and j.

By the definition of K we have a short exact sequence

$$K \hookrightarrow \overline{\nabla}(\mu)\langle j'\rangle \twoheadrightarrow C' \tag{10}$$

for some cokernel C'. By the definition of $\overline{\nabla}(\mu)$, all composition subquotients of C' have the form $L(\nu)$, where $\nu \prec \mu$ (up to shift). Let $\lambda \in \Lambda$ be such that $\lambda \prec \mu$. Applying $\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, {}_-)$ to (10) we get the exact sequence

(11)

$$\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, \overline{\nabla}(\mu)\langle j'\rangle) \to \mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, C') \to \mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, K).$$

Here the left term is zero by the definition of $\overline{\nabla}(\mu)$ and the right hand term is zero by (9). This yields that the middle term is zero as well and thus $C' = 0$, that is K is a proper costandard module.

We can now apply the same arguments as above to the module Ker in place of Coker and get the short exact sequence

$$\mathrm{Ker}' \hookrightarrow \mathrm{Ker} \twoheadrightarrow K',$$

where K' is proper costandard and $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, \mathrm{Ker}') = 0$ for all λ and j. Proceeding inductively we obtain a (possibly infinite) decreasing filtration

$$\mathrm{Coker} \supseteq \mathrm{Ker} \supseteq \mathrm{Ker}' \supseteq \dots$$

with proper costandard subquotients. That $\lim\limits_{i\to+\infty} \mathfrak{b}(\mathrm{Coker}^{(i)}) = +\infty$ follows from the construction since all our modules are from $A^{\downarrow}$-gmod, all proper costandard modules (subquotients of the filtration of Coker) are finite-dimensional by Lemma 4, and there are only finitely many proper costandard modules up to isomorphism and shift (which implies that dimensions of proper costandard modules are uniformly bounded). Therefore we get $\mathrm{Coker} \in \mathcal{F}^{\downarrow}(\overline{\nabla})$. The claim of the lemma follows. $\square$

Lemma 9. *Let $M \in A^{\downarrow}$-gmod be such that $\mathrm{ext}^1_A(M, \overline{\nabla}(\mu)\langle j\rangle) = 0$ for all μ and j. Then $M \in \mathcal{F}^{\downarrow}(\Delta)$.*

Proof. Let $M \in A^{\downarrow}$-gmod be such that $\mathrm{ext}^1_A(M, \overline{\nabla}(\mu)\langle j\rangle) = 0$ for all μ and j. We again proceed by induction on $|\overline{\Lambda}|$. If $|\overline{\Lambda}| = 1$, then proper costandard modules are simple and hence M is projective. All indecomposable projective modules belong to $\mathcal{F}^{\downarrow}(\Delta)$ as A is standardly stratified. Using this it is easy to check that all projective modules in $A^{\downarrow}$-gmod belong to $\mathcal{F}^{\downarrow}(\Delta)$. So, in the case $|\overline{\Lambda}| = 1$ the claim of the lemma is true.

If $|\overline{\Lambda}| > 1$, we take some maximal $\nu \in \Lambda$ and denote by N the sum of all images of all possible homomorphisms from $\Delta(\lambda)\langle j\rangle$, where $\lambda \in \overline{\nu}$ and $j \in \mathbb{Z}$, to M. Then we have a short exact sequence

$$N \hookrightarrow M \twoheadrightarrow \mathrm{Coker}. \tag{12}$$

Compare with (5) in the proof of Lemma 8. Using arguments similar to those in the latter proof, one shows that $\mathrm{ext}^1_A(\mathrm{Coker}, \overline{\nabla}(\mu)\langle j\rangle) = 0$ for all $\mu \in \Lambda \setminus \overline{\nu}$ and all j. By construction we have that Coker is in fact a

$B_{\overline{\nu}}$-module. Therefore, using Lemma 3 and the inductive assumption we get Coker $\in \mathcal{F}^{\downarrow}(\Delta)$. From Corollary 6(i) we thus get

$$\mathrm{ext}^i_A(\mathrm{Coker}, \overline{\nabla}(\mu)\langle j\rangle) = 0 \tag{13}$$

for all $\mu \in \Lambda$, $j \in \mathbb{Z}$ and $i \in \mathbb{N}$.

Furthermore, for any μ and j we also have the following part of the long exact sequence associated with (12):

$$\mathrm{ext}^1_A(M, \overline{\nabla}(\mu)\langle j\rangle) \to \mathrm{ext}^1_A(N, \overline{\nabla}(\mu)\langle j\rangle) \to \mathrm{ext}^2_A(\mathrm{Coker}, \overline{\nabla}(\mu)\langle j\rangle).$$

The left term is zero by our assumptions and the right term is zero by (13). Therefore for all μ and j we have

$$\mathrm{ext}^1_A(N, \overline{\nabla}(\mu)\langle j\rangle) = 0. \tag{14}$$

Fix now $\mu \in \Lambda$ and $j \in \mathbb{Z}$ and denote by C the cokernel of the natural inclusion $L(\mu)\langle j\rangle \hookrightarrow \overline{\nabla}(\mu)\langle j\rangle$. Applying $\mathrm{hom}_A(N, {}_-)$ to the short exact sequence

$$L(\mu)\langle j\rangle \hookrightarrow \overline{\nabla}(\mu)\langle j\rangle \twoheadrightarrow C,$$

and using (14) and the fact that $\mathrm{hom}_A(N, C) = 0$ by construction, we obtain that $\mathrm{ext}^1_A(N, L(\mu)\langle j\rangle) = 0$ for any μ and j. This yields that N is projective and thus belongs to $\mathcal{F}^{\downarrow}(\Delta)$. Since $\mathcal{F}^{\downarrow}(\Delta)$ is closed under extensions, the claim of the lemma follows. □

Proof of Theorem 2. Let

$$\begin{array}{rcl}\mathcal{X} & = & \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^i_A(M, \overline{\nabla}(\lambda)\langle j\rangle) = 0, \forall j \in \mathbb{Z}, i > 0, \lambda \in \Lambda\};\\ \mathcal{Y} & = & \{M \in A^{\downarrow}\text{-gmod} : \mathrm{ext}^1_A(M, \overline{\nabla}(\lambda)\langle j\rangle) = 0, \forall j \in \mathbb{Z}, \lambda \in \Lambda\}.\end{array}$$

The inclusion $\mathcal{X} \subseteq \mathcal{Y}$ is obvious. The inclusion $\mathcal{Y} \subseteq \mathcal{F}^{\downarrow}(\Delta)$ follows from Lemma 9. The inclusion $\mathcal{F}^{\downarrow}(\Delta) \subseteq \mathcal{X}$ follows from Corollary 6(i). This proves Theorem 2(i). Theorem 2(ii) is proved similarly using Lemma 8 instead of Lemma 9 and Corollary 6(ii) instead of Corollary 6(i). □

Corollary 10. *Let A be a positively graded standardly stratified algebra.*

(i) For every $M \in \mathcal{F}^{\downarrow}(\Delta)$, $\lambda \in \Lambda$ and $j \in \mathbb{Z}$ the multiplicity of $\Delta(\lambda)\langle j\rangle$ in any standard filtration of M is well-defined, finite and equals $\dim \mathrm{hom}_A(M, \overline{\nabla}(\lambda)\langle j\rangle)$.

(ii) For every $M \in \mathcal{F}^{\downarrow}(\overline{\nabla})$, $\lambda \in \Lambda$ and $j \in \mathbb{Z}$ the multiplicity of $\overline{\nabla}(\lambda)\langle j\rangle$ in any proper costandard filtration of M is well-defined, finite and equals $\dim \mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, M)$.

Proof. Follows from Lemma 5 by standard arguments (see e.g. [Ri]). □

Remark 11. Note that the ungraded multiplicity of $\Delta(\lambda)$ (or $\overline{\nabla}(\lambda)$) in M might be infinite.

Let $\mathcal{F}^{\uparrow}(\overline{\nabla})$ denote the full subcategory of the category $A^{\uparrow}$-gmod, which consists of all modules M admitting a (possibly infinite) filtration

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \dots \tag{15}$$

such that $M = \bigcup_{i\geq 0} M^{(i)}$ and for every $i = 0, 1, \dots$ the subquotient $M^{(i+1)}/M^{(i)}$ is isomorphic (up to shift) to some proper costandard module. Since all proper costandard modules are finite dimensional (Lemma 4) from the dual version of Lemma 7 one obtains that $\mathcal{F}^{\uparrow}(\overline{\nabla})$ is closed under finite extensions.

Theorem 12. *We have*

$$\begin{aligned} \mathcal{F}^{\uparrow}(\overline{\nabla}) &= \{M \in A^{\uparrow}\text{-gmod} : \mathrm{ext}_A^i(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, i > 0, \lambda \in \Lambda\} \\ &= \{M \in A^{\uparrow}\text{-gmod} : \mathrm{ext}_A^1(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, \lambda \in \Lambda\}. \end{aligned}$$

Proof. Set

$$\mathcal{X} = \{M \in A^{\uparrow}\text{-gmod} : \mathrm{ext}_A^1(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, \lambda \in \Lambda\},$$

$$\mathcal{Y} = \{M \in A^{\uparrow}\text{-gmod} : \mathrm{ext}_A^i(\Delta(\lambda)\langle j\rangle, M) = 0, \forall j \in \mathbb{Z}, i > 0, \lambda \in \Lambda\}.$$

Obviously, $\mathcal{Y} \subseteq \mathcal{X}$.

Let $M \in \mathcal{F}^{\uparrow}(\overline{\nabla})$, $\lambda \in \Lambda$ and $j \in \mathbb{Z}$. Assume that (15) gives a proper costandard filtration of M. As $M \in A^{\uparrow}$-gmod and $\Delta(\lambda) \in A^{\downarrow}$-gmod, it follows that there exists $k \in \mathbb{N}$ such that

$$\mathrm{ext}_A^i(\Delta(\lambda)\langle j\rangle, M/M^{(k)}) = 0$$

for all $i \geq 0$. At the same time we have

$$\mathrm{ext}_A^i(\Delta(\lambda)\langle j\rangle, M^{(k)}) = 0$$

for all $i > 0$ by Lemma 5. Hence

$$\mathrm{ext}_A^i(\Delta(\lambda)\langle j\rangle, M) = 0$$

for all $i > 0$ and thus $\mathcal{F}^{\uparrow}(\overline{\nabla}) \subseteq \mathcal{Y}$.

It is left to show that $\mathcal{X} \subseteq \mathcal{F}^{\uparrow}(\overline{\nabla})$. We will do this by induction on $|\overline{\Lambda}|$. If $|\overline{\Lambda}| = 1$, then all proper standard modules are simple, which yields $\mathcal{F}^{\uparrow}(\overline{\nabla}) = A^{\uparrow}$-gmod. In this case the inclusion $\mathcal{X} \subseteq \mathcal{F}^{\uparrow}(\overline{\nabla})$ is obvious.

If $|\overline{\Lambda}| > 1$ we fix some maximal $\mu \in \Lambda$. Let $M \in \mathcal{X}$. Denote by N the maximal submodule of M satisfying $[N : L(\nu)\langle j\rangle] = 0$ for all $\nu \in \overline{\mu}$ and $j \in \mathbb{Z}$. For $\lambda \in \Lambda$ and $j \in \mathbb{Z}$, applying the functor $\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, {}_-)$ to the short exact sequence

$$N \hookrightarrow M \twoheadrightarrow \mathrm{Coker},$$

and using $M \in \mathcal{X}$, gives the following exact sequences:

$$\mathrm{hom}_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}) \to \mathrm{ext}_A^1(\Delta(\lambda)\langle j\rangle, N) \to 0 \tag{16}$$

and

$$0 \to \mathrm{ext}_A^1(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}) \to \mathrm{ext}_A^2(\Delta(\lambda)\langle j\rangle, N). \tag{17}$$

By construction, any simple subquotient in the socle of Coker has the form $L(\nu)\langle j\rangle$ for some $\nu \in \overline{\mu}$ and $j \in \mathbb{Z}$. Therefore, since μ is maximal, in the case $\lambda \notin \overline{\mu}$ we have $\hom_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}) = 0$ and hence $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, N) = 0$ from (16). For $\lambda \in \overline{\mu}$ the module $\Delta(\lambda)\langle j\rangle$ is projective and hence $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, N) = 0$ as well. This implies $N \in \mathcal{X}$. As, by construction, $N \in B_{\overline{\mu}}$-mod, using Lemma 3 and the inductive assumption we obtain $N \in \mathcal{F}^{\uparrow}(\overline{\nabla})$. As the inclusion $\mathcal{F}^{\uparrow}(\overline{\nabla}) \subseteq \mathcal{Y}$ is already proved, we have $N \in \mathcal{Y}$ and from (17) it follows that $\mathrm{Coker} \in \mathcal{X}$.

Since $\mathcal{F}^{\uparrow}(\overline{\nabla})$ is closed under finite extensions, it is left to show that $\mathrm{Coker} \in \mathcal{F}^{\uparrow}(\overline{\nabla})$. If $\mathrm{Coker} = 0$, we have nothing to do. If $\mathrm{Coker} \neq 0$, we choose maximal $k \in \mathbb{Z}$ such that $\mathrm{Coker}_k \neq 0$. Denote by V the intersection of the kernels of all possible maps from Coker to $I(\nu)\langle j\rangle$, where $\nu \in \overline{\mu}$ and $-j < k$, and consider the corresponding short exact sequence

$$V \hookrightarrow \mathrm{Coker} \twoheadrightarrow \mathrm{Coker}'. \tag{18}$$

From the construction it follows that the socle of V is V_k and that for any $j < k$ every composition subquotient of V_j has the form $L(\nu)\langle -j\rangle$ for some $\nu \notin \overline{\mu}$. Therefore, taking the injective envelope of V and using the definition of proper standard modules, we obtain that V is a submodule of a finite direct sum of proper standard modules (such that the socles of V and of this direct sum agree). In particular, V is finite dimensional as both V_k and all proper standard modules are (Lemma 4). Hence $V \in A^{\downarrow}$-gmod.

For $\lambda \in \Lambda$ and $j \in \mathbb{Z}$, applying the functor $\hom_A(\Delta(\lambda)\langle j\rangle, {}_-)$ to (18) and using $\mathrm{Coker} \in \mathcal{X}$ gives the following exact sequences:

$$\hom_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}') \to \mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, V) \to 0 \tag{19}$$

and

$$0 \to \mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}') \to \mathrm{ext}^2_A(\Delta(\lambda)\langle j\rangle, V). \tag{20}$$

If $\lambda \notin \overline{\mu}$, then, by the definition of the module Coker', we have $\hom_A(\Delta(\lambda)\langle j\rangle, \mathrm{Coker}') = 0$ and hence $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, V) = 0$ from (19). If $\lambda \in \overline{\mu}$, then $\Delta(\lambda)\langle j\rangle$ is projective by the maximality of μ and $\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, V) = 0$ automatically. Hence $V \in \mathcal{X}$. Since $V \in A^{\downarrow}$-gmod as shown above, from Theorem 2(ii) we deduce that V has a (finite) proper standard filtration and thus $V \in \mathcal{F}^{\uparrow}(\overline{\nabla})$. Using the already proved inclusion $\mathcal{F}^{\uparrow}(\overline{\nabla}) \subseteq \mathcal{Y}$ and (20) we also get $\mathrm{Coker}' \in \mathcal{X}$. Note that $\mathrm{Coker}'_k = 0$ by construction.

Applying now the same arguments to Coker' and proceeding inductively (decreasing k) we construct a (possibly infinite) proper costandard filtration of Coker' of the form (15). This claim of the theorem follows. □

The following claim is a weak version of [Dl, Lemma 2.1] and [Fr2, Theorem 1]. The original statement also contains the converse assertion that the fact that indecomposable injective A-modules belong to $\mathcal{F}^{\uparrow}(\overline{\nabla})$ guarantees that A is standardly stratified.

Corollary 13 (Weak Dlab's theorem)**.** *All indecomposable injective A-modules belong to $\mathcal{F}^{\uparrow}(\overline{\nabla})$.*

Proof. If I in an indecomposable injective A-module, then we obviously have $\mathrm{ext}^i_A(\Delta(\lambda)\langle j\rangle, I) = 0$ for all $j \in \mathbb{Z}$, $i > 0$ and $\lambda \in \Lambda$, so the claim follows from Theorem 12. □

The following statement generalizes the corresponding results of [Ri, AHLU, Fr2]:

Theorem 14 (Construction of tilting modules)**.** *Let A be a positively graded standardly stratified algebra.*

(i) The category $\mathcal{F}^{\downarrow}(\Delta)\cap\mathcal{F}^{\downarrow}(\overline{\nabla})$ is closed with respect to taking direct sums and direct summands.

(ii) For every $\lambda \in \Lambda$ there is a unique indecomposable object $T(\lambda) \in \mathcal{F}^{\downarrow}(\Delta) \cap \mathcal{F}^{\downarrow}(\overline{\nabla})$ such that there is a short exact sequence

$$\Delta(\lambda) \hookrightarrow T(\lambda) \twoheadrightarrow \mathrm{Coker},$$

with Coker $\in \mathcal{F}^{\downarrow}(\Delta)$.

(iii) Every indecomposable object in $\mathcal{F}^{\downarrow}(\Delta) \cap \mathcal{F}^{\downarrow}(\overline{\nabla})$ has the form $T(\lambda)\langle j\rangle$ for some $\lambda \in \Lambda$ and $j \in \mathbb{Z}$.

We would need the following lemmata:

Lemma 15. *For all $\lambda, \mu \in \Lambda$, $i \geq 0$ and all $j \gg 0$ we have*

$$\mathrm{ext}^i_A(\Delta(\lambda)\langle j\rangle, \Delta(\mu)) = 0.$$

Proof. We proceed by induction with respect to $\preceq$. If λ is maximal, the module $\Delta(\lambda)$ is projective and the claim is trivial for $i > 0$. For $i = 0$ the claim follows from the fact that A is positively graded. Now, if λ is not maximal, we consider the short exact sequence (3). In this sequence Ker has a *finite* filtration by (shifted) standard modules, whose indexes are strictly greater than λ with respect to $\preceq$. Hence the claim follows by the usual dimension shift (note that it is enough to consider only finitely many values of i, namely $i \leq |\Lambda|$). □

Lemma 16. *For all $\lambda, \mu \in \Lambda$ and $j \in \mathbb{Z}$ the inequality*

$$\mathrm{ext}^1_A(\Delta(\lambda)\langle j\rangle, \Delta(\mu)) \neq 0.$$

implies $\lambda \prec \mu$.

Proof. If $\lambda \not\prec \mu$, then, using Lemma 3, we may assume that λ is maximal. In this case $\Delta(\lambda)$ is projective and the claim becomes trivial. □

Lemma 17. *For all $M \in \mathcal{F}^{\downarrow}(\Delta)$, $N \in \mathcal{F}^{\downarrow}(\overline{\nabla})$ and $i \in \mathbb{N}$ we have $\mathrm{ext}^i_A(M, N) = 0$.*

Proof. It is enough to prove the claim in the case when M has a filtration of the form (1). Let λ be a maximal index occurring in standard subquotients of M. Then from Lemma 16 we have that all corresponding standard subquotients do not extend any other standard subquotients of M. Therefore M has a submodule isomorphic to a direct sum of shifted $\Delta(\lambda)$ such that the cokernel has a standard filtration in which no subquotient of the form $\Delta(\lambda)$ (up to shift) occur. Since Λ is finite, proceeding inductively we construct a finite filtration of M whose subquotients are direct sums of standard modules. This means that it is enough to prove the claim in the case when M is a direct sum of standard modules. In this case the claim follows from Corollary 6(ii). □

Proof of Theorem 14. Statement (i) follows from the additivity of the conditions, which appear on the right hand side in the formulae of Theorem 2.

The existence part of statement (ii) is proved using the usual approach of universal extensions (see [Ri]). We start with $\Delta(\lambda)$ and go down with respect to the preorder $\preceq$. If all first extensions from all (shifted) standard modules to $\Delta(\lambda)$ vanish, we get $\Delta(\lambda) \in \mathcal{F}^{\downarrow}(\overline{\nabla})$ by Theorem 2(ii). Otherwise there exist $\mu \in \Lambda$ and $j' \in \mathbb{Z}$ such that

$$\mathrm{ext}^1_A(\Delta(\mu)\langle j'\rangle, \Delta(\lambda)) \neq 0.$$

We assume that μ is maximal with such property (we have $\mu \prec \lambda$ by Lemma 16) and use Lemma 15 to choose j' such that

$$\mathrm{ext}^1_A(\Delta(\nu)\langle j\rangle, \Delta(\lambda)) \neq 0$$

implies $j \leq j'$ for all $\nu \in \overline{\mu}$.

For every $\nu \in \overline{\mu}$ and $j \leq j'$ the space $\mathrm{ext}^1_A(\Delta(\nu)\langle j\rangle, \Delta(\lambda))$ is finite dimensional, say of dimension $l_{\nu,j}$. Consider the universal extension

$$X \hookrightarrow Y \twoheadrightarrow Z, \tag{21}$$

where $X = \Delta(\lambda)$ and

$$Z = \bigoplus_{\nu\in\overline{\mu}} \bigoplus_{j\leq j'} \Delta(\nu)\langle j\rangle^{l_{\nu,j}} \in \mathcal{F}^{\downarrow}(\Delta)$$

(note that $\mathrm{ext}^1_A(Z, Z) = 0$ by Lemma 16). We have $Y \in \mathcal{F}^{\downarrow}(\Delta)$ by construction. We further claim that Y is indecomposable. Indeed, Let $e \in \mathrm{end}_A(Y)$ be a nonzero idempotent (note that e is homogeneous of degree zero). As $\nu \prec \lambda$, we have $\hom_A(\Delta(\lambda), \Delta(\nu)\langle j\rangle) = 0$ for any ν and j as above. Therefore e maps X (which is indecomposable) to X. If $e|_X = 0$, then e provides a splitting for a nontrivial direct summand of Z in (21); if $e|_X = \mathrm{id}_X$ and $e \neq \mathrm{id}_Y$, then $\mathrm{id}_Y - e \neq 0$ annihilates X and hence provides a splitting for a nontrivial direct summand of Z in (21). This contradicts our construction of Y as the universal extension. Therefore $e = \mathrm{id}_Y$, which proves that the module Y is indecomposable.

By Lemma 16, there are no extensions between the summands of Z. From $\mathrm{ext}^1_A(Z,Z)=0$ and the universality of our extension, we get

$$\mathrm{ext}^1_A(\Delta(\nu)\langle j\rangle, Y)=0$$

for all $\nu\in\overline{\mu}$ and all j.

Now take the indecomposable module constructed in the previous paragraph as X, take a maximal μ' such that for some j we have $\mathrm{ext}^1_A(\Delta(\mu')\langle j\rangle, X)\neq 0$ and do the same thing as in the previous paragraph. Proceed inductively. In a finite number of steps we end up with an indecomposable module $T(\lambda)$ such that $\Delta(\lambda)\hookrightarrow T(\lambda)$, the cokernel is in $\mathcal{F}^{\downarrow}(\Delta)$, and

$$\mathrm{ext}^1_A(\Delta(\mu)\langle j\rangle, T(\lambda))=0$$

for all μ and j. By Theorem 2(ii), we have $T(\lambda)\in\mathcal{F}^{\downarrow}(\overline{\nabla})$. This proves the existence part of statement (ii). The uniqueness part will follow from statement (iii).

Let $M\in\mathcal{F}^{\downarrow}(\Delta)\cap\mathcal{F}^{\downarrow}(\overline{\nabla})$ be indecomposable and $\Delta(\lambda)\hookrightarrow M$ be such that the cokernel Coker has a standard filtration. Applying $\mathrm{hom}_A({}_-,T(\lambda))$ to the short exact sequence

$$\Delta(\lambda)\hookrightarrow M\twoheadrightarrow \mathrm{Coker}$$

we obtain the exact sequence

$$\mathrm{hom}_A(M,T(\lambda))\to\mathrm{hom}_A(\Delta(\lambda),T(\lambda))\to\mathrm{ext}^1_A(\mathrm{Coker},T(\lambda)).$$

Here the right term is zero by Lemma 17 and the definition of $T(\lambda)$. As the middle term is obviously nonzero, we obtain that the left term is nonzero as well. This gives us a nonzero map α from M to $T(\lambda)$. Similarly one constructs a nonzero map β from $T(\lambda)$ to M such that the composition $\alpha\circ\beta$ is the identity on $\Delta(\lambda)$. We claim the following:

Lemma 18. *Let $T(\lambda)$ be as above.*

(i) For any $n\in\mathbb{Z}$ there exists a submodule $N^{(n)}$ of $T(\lambda)$ with the following properties:
  - *(a) $N^{(n)}$ is indecomposable;*
  - *(b) $N^{(n)}$ has finite standard filtration starting with $\Delta(\lambda)$;*
  - *(c) $N^{(n)}_i=T(\lambda)_i$ for all $i\leq n$;*
  - *(d) every endomorphism of $T(\lambda)$ restricts to an endomorphism of $N^{(n)}$.*

(ii) The composition $\alpha\circ\beta$ is an automorphism of $T(\lambda)$.

Proof. Consider the multiset $\mathcal{M}$ of all standard subquotients of $T(\lambda)$. It might be infinite. However, for every $m\in\mathbb{Z}$ the multiset $\mathcal{M}_m$ of those subquotients X of $T(\lambda)$, for which $X_i\neq 0$ for some $i\leq m$ is finite since $T(\lambda)\in A^{\downarrow}$-mod. Construct the submultiset $\mathcal{N}$ of $\mathcal{M}$ in the following way: start with $\mathcal{M}_n\cup\{\Delta(\lambda)\}$, which is finite. From Lemma 15 it follows that every subquotient from $\mathcal{M}_n$ has a nonzero first extension with finitely many other subquotients from $\mathcal{M}$. Add to

$\mathcal{N}$ all such subquotients (counted with multiplicities), moreover, if we add some $\Delta(\mu)\langle j\rangle$, add as well all $\Delta(\nu)\langle i\rangle$, where $i \geq j$ and $\mu \preceq \nu$, occurring in $\mathcal{M}$. Obviously, the result will be a finite set. Repeat now the same procedure for all newly added subquotients and continue. By Lemma 16, on every next step we will add only $\Delta(\nu)\langle i\rangle$ such that $\mu \prec \nu$ (strict inequality!) for some minimal μ in the set indexing subquotients added on the previous step.

As Λ is finite, after finitely many steps we will get a finite submultiset $\mathcal{N}$ of $\mathcal{M}$ with the following properties: any subquotient from $\mathcal{N}$ does not extend any subquotient from $\mathcal{M}\setminus\mathcal{N}$; there are no homomorphisms from any subquotient from $\mathcal{N}$ to any subquotient from $\mathcal{M}\setminus\mathcal{N}$. Using the vanishing of the first extension one shows that there is a submodule $N^{(n)}$ of $T(\Lambda)$, which has a standard filtration with the multiset of subquotients being precisely $\mathcal{N}$, in particular, $N^{(n)}$ satisfies (ib). By construction, $N^{(n)}$ also satisfies (ic). The vanishing of homomorphisms from subquotients from $\mathcal{N}$ to subquotients from $\mathcal{M}\setminus\mathcal{N}$ implies that $N^{(n)}$ satisfies (id). That $N^{(n)}$ satisfies (ia) is proved similarly to the proof of the indecomposability of $T(\lambda)$. This proves statement (i).

To prove that $\alpha\circ\beta$ is an automorphism (statement (ii)) it is enough to show that for any $n\in\mathbb{Z}$ the restriction of $\alpha\circ\beta$ to $T(\lambda)_n$ is a linear automorphism. The restriction of $\alpha\circ\beta$ to $N^{(n)}$ (which is well defined by (id)) is not nilpotent as it is the identity on $\Delta(\lambda)$. As A is positively graded, the space $\hom_A(\Delta(\mu),\Delta(\nu)\langle j\rangle)$ is finite dimensional for all μ,ν and j. From this observation and (ib) it follows that the endomorphism algebra of $N^{(n)}$ is finite dimensional. This algebra is local by (ia). Therefore the restriction of $\alpha\circ\beta$ to $N^{(n)}$, being a non-nilpotent element of a local finite dimensional algebra, is an automorphism. Therefore the restriction of $\alpha\circ\beta$ to all $N_i^{(n)}$, in particular, to $N_n^{(n)} = T(\lambda)_n$ (see (ic)), is a linear automorphism. This completes the proof. $\square$

After Lemma 18, substituting α by $(\alpha\circ\beta)^{-1}\circ\alpha$, we may assume that $\alpha\circ\beta = \mathrm{id}_{T(\lambda)}$. We also have that β is injective and α is surjective. The gives us splittings for the following two short exact sequences:

$$0 \longrightarrow \mathrm{Ker}(\alpha) \hookrightarrow M \underset{\alpha}{\overset{\beta}{\rightleftarrows}} T(\lambda) \longrightarrow 0$$

$$0 \longrightarrow T(\lambda) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} M \twoheadrightarrow \mathrm{Coker}(\beta) \longrightarrow 0$$

As M is assumed to be indecomposable, we obtain $\mathrm{Ker}(\alpha) = \mathrm{Coker}(\beta) = 0$, which implies that α and β are isomorphisms. Therefore $M \cong T(\lambda)$, which completes the proof of the theorem. $\square$

The objects of the category $\mathcal{F}^{\downarrow}(\Delta)\cap\mathcal{F}^{\downarrow}(\overline{\nabla})$ are called *tilting modules*.

Remark 19. Note that a tilting module may be an infinite direct sum of indecomposable tilting modules. Note also that the direct sum of all indecomposable tilting modules (with all shifts) does not belong to $A^{\downarrow}$-gmod. It might happen that it does not belong to A-gmod either, since local finiteness is an issue.

Corollary 20. *Let A be a positively graded standardly stratified algebra.*

(i) *Every $M \in \mathcal{F}^{\downarrow}(\Delta)$ has a coresolution by tilting modules of length at most $|\Lambda| - 1$.*
(ii) *Every $M \in \mathcal{F}^{\downarrow}(\overline{\nabla})$ has a (possibly infinite) resolution by tilting modules.*

Proof. This follows from Theorem 14 and the definitions by standard arguments. □

Remark 21. Note that the standard filtration of $T(\lambda)$ may be infinite, see Example 43.

Unfortunately, Remark 21 says that one can not hope for a reasonable analogue of Ringel duality on the class of algebras we consider. We can of course consider the endomorphism algebra of the direct sum of all tilting modules, but from Remark 21 it follows that projective modules over such algebras might have infinite standard filtrations and hence we will not be able to construct tilting modules for them. Another obstruction is that we actually can not guarantee that the induced grading on this endomorphism algebra will be positive (see examples in [MO, Ma]). To deal with these problems we have to introduce some additional restrictions.

4. Ringel duality for graded standardly stratified algebras

Consider the $\Bbbk$-linear category $\mathfrak{T}$, which is the full subcategory of $A^{\downarrow}$-gmod, whose objects are $T(\lambda)\langle j\rangle$, where $\lambda \in \Lambda$ and $j \in \mathbb{Z}$. The group $\mathbb{Z}$ acts freely on $\mathfrak{T}$ via $\langle j\rangle$ and the quotient of $\mathfrak{T}$ modulo this free action is a $\mathbb{Z}$-graded $\Bbbk$-linear category $\overline{\mathfrak{T}}$, whose objects can be identified with $T(\lambda)$, where $\lambda \in \Lambda$ (see [DM, MOS] for more details). Thus the ungraded endomorphism algebra $R(A) = \mathrm{End}_A(T)$, where $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$ becomes a $\mathbb{Z}$-graded $\Bbbk$-algebra in the natural way. The algebra $R(A)$ is called the *Ringel dual* of A. The algebra A will be called *weakly adapted* provided that every $T(\lambda)$, where $\lambda \in \Lambda$, has a finite standard filtration. The algebra A will be called *adapted* provided that the above $\mathbb{Z}$-grading on $R(A)$ is positive.

Proposition 22. *We have the following:*

(i) *Any adapted algebra is weakly adapted.*
(ii) *If A is weakly adapted, then $R(A)$ is locally finite.*

Proof. Because of Lemma 5 and the definition of tilting modules, every homomorphism from $T(\lambda)$ to $T(\mu)\langle j\rangle$ is induced from a homomorphism from some standard subquotient of $T(\lambda)$ to some proper standard subquotient of $T(\mu)\langle j\rangle$.

Since $\overline{\nabla}(\mu)\langle j\rangle$ is a (sub)quotient of $T(\mu)\langle j\rangle$, the condition that the above $\mathbb{Z}$-grading on $R(A)$ is positive implies that every standard subquotient of $T(\lambda)$, different from $\Delta(\lambda)$, must have the form $\Delta(\mu)\langle j\rangle$ for some $j > 0$. However, the vector space $\bigoplus_{j\leq 0} T(\lambda)_j$ is finite dimensional as $T(\lambda) \in A^{\downarrow}$-gmod, which yields that any standard filtration of $T(\lambda)$ must be finite. This proves statement (i).

Statement (ii) follows from the finiteness of a standard filtration of $T(\lambda)$ and the obvious fact that $\hom_A(\Delta(\lambda), M)$ is finite dimensional for any $M \in A$-gmod. □

Corollary 23. *Assume that A is adapted. Then every $M \in \mathcal{F}^b(\Delta)$, in particular, every indecomposable projective A-module, has a finite coresolution*

$$0 \to M \to T_0 \to T_1 \to \cdots \to T_k \to 0, \tag{22}$$

such that every T_i is a finite direct sum of indecomposable tilting A-modules.

Proof. It is enough to prove the claim for $M = \Delta(\lambda)$. The claim is obvious in the case λ is minimal as in this case we have $\Delta(\lambda) = T(\lambda)$. From Theorem 14(ii) we have the exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to \mathrm{Coker}$$

such that Coker has a standard filtration with possible subquotients $\Delta(\mu)\langle i\rangle$, where $\mu \prec \lambda$ and $i \in \mathbb{Z}$. By Proposition 22(i), the standard filtration of Coker is finite and hence the claim follows by induction (with respect to the partial preorder $\preceq$). □

A complex $\mathcal{X}^\bullet$ of A-modules is called *perfect* provided that it is bounded and every nonzero $\mathcal{X}^i$ is a direct sum of finitely many indecomposable modules. Let $\mathcal{P}(A)$ denote the homotopy category of perfect complexes of graded projective A-modules. As every indecomposable projective A-module has a finite standard filtration, it follows by induction that $\mathcal{F}^b(\Delta) \subseteq \mathcal{P}(A)$. Consider the contravariant functor

$$\mathrm{G} = \mathcal{R}\mathrm{hom}_A({}_-, \mathfrak{T})$$

(see [MOS] for details of hom-functors for $\Bbbk$-linear categories). As we will see in Theorem 24(iii), the functor G is a functor from $\mathcal{P}(A)$ to $\mathcal{P}(R(A))$. To distinguish A and $R(A)$-modules, if necessary, we will use A and $R(A)$ as superscripts for the corresponding modules.

Theorem 24 (Weak Ringel duality)**.** *Let A be an adapted standardly stratified algebra.*

(i) The algebra $R(A)$ is an adapted standardly stratified algebra with respect to $\preceq^{\mathrm{op}}$.
(ii) We have $R(R(A)) \cong A$.
(iii) The functor G is an antiequivalence from $\mathcal{P}(A)$ to $\mathcal{P}(R(A))$.
(iv) The functor G induces an antiequivalence between $\mathcal{F}^b(\Delta^{(A)})$ and $\mathcal{F}^b(\Delta^{(R(A))})$, which sends standard A-modules to standard $R(A)$-modules, tilting A-modules to projective $R(A)$-modules and projective A-modules modules to tilting $R(A)$-modules.

Proof. By construction, the functor G maps indecomposable tilting A-modules to indecomposable projective $R(A)$-modules. From Corollary 23 it follows that every indecomposable projective A-module M has a coresolution of the form (22), such that every T_i is a finite direct sum of indecomposable tilting A-modules. This implies that every object in $\mathcal{P}(A)$ can be represented by a perfect complex of tilting modules. This yields that G maps $\mathcal{P}(A)$ to $\mathcal{P}(R(A))$. As T is a tilting module, statement (iii) follows directly from the Rickard-Morita Theorem for $\Bbbk$-linear categories, see e.g. [Ke, Corollary 9.2] or [DM, Theorem 2.1].

The functor G is acyclic, in particular, exact on $\mathcal{F}^b(\Delta^{(A)})$ by Lemma 5. By construction, it maps tilting A-modules to projective $R(A)$-modules and thus projective $R(A)$-modules have filtrations by images (under G) of standard A-modules. By Proposition 22, these filtrations of projective $R(A)$-modules by images of standard A-modules are finite. As in the classical case (see [Ri]) it is easy to see that the images of standard A-modules are standard $R(A)$-modules (with respect to $\preceq^{\mathrm{op}}$). From Proposition 22(ii) and our assumptions it follows that the algebra $R(A)$ is positively graded. This implies that $R(A)$ is a graded standardly stratified algebra (with respect to $\preceq^{\mathrm{op}}$).

Because of our description of standard modules for $R(A)$, the functor G maps $\mathcal{F}^b(\Delta^{(A)})$ to $\mathcal{F}^b(\Delta^{(R(A))})$. In particular, projective A-modules are also mapped to some modules in $\mathcal{F}^b(\Delta^{(R(A))})$. Since G is a derived equivalence by (iii), for $i > 0$, $j \in \mathbb{Z}$ and $\lambda, \mu \in \Lambda$ we obtain

$$\mathrm{ext}^i_{R(A)}(\mathrm{G}\Delta(\lambda)\langle j\rangle, \mathrm{G}P(\mu)) = \mathrm{ext}^i_A(P(\mu), \Delta(\lambda)\langle j\rangle) = 0.$$

Hence $\mathrm{G}P(\mu)$ has a proper costandard filtration by Theorem 2(i), and thus is a tilting $R(A)$-module, which implies (ii). As projective A-modules have finite standard filtration, the algebra $R(A)$ is weakly adapted. It is even adapted as the grading on $R(R(A))$ coincides with the grading on A and is hence positive. This proves (i). Statement (iv) follows easily from the properties of G, established above. This completes the proof. □

Similarly to the above we consider the contravariant functors

$$\mathrm{F} = \mathcal{R}\mathrm{hom}_A(\mathfrak{T}, {}_-)^{\circledast} : \mathcal{D}^+(A^{\uparrow}\text{-gmod}) \to \mathcal{D}^-(R(A)^{\downarrow}\text{-gmod})$$
$$\tilde{\mathrm{F}} = \mathcal{R}\mathrm{hom}_A(\mathfrak{T}, {}_-)^{\circledast} : \mathcal{D}^-(A^{\downarrow}\text{-gmod}) \to \mathcal{D}^+(R(A)^{\uparrow}\text{-gmod}).$$

Although it is not obvious from the first impression, the following statement carries a strong resemblance with [MOS, Proposition 20]:

Theorem 25 (Strong Ringel duality). *Let A be an adapted standardly stratified algebra.*

(i) Both F *and* $\tilde{\mathrm{F}}$ *are antiequivalences.*

(ii) The functor F *induces an antiequivalence from the category* $\mathcal{F}^{\uparrow}(\overline{\nabla}^{(A)})$ *to the category* $\mathcal{F}^{\downarrow}(\overline{\nabla}^{(R(A))})$*, which sends proper costandard A-modules to proper costandard $R(A)$-modules, and injective A-modules to tilting $R(A)$-modules.*

(iii) The functor $\tilde{\mathrm{F}}$ *induces an antiequivalence from the category* $\mathcal{F}^{\downarrow}(\overline{\nabla}^{(A)})$ *to the category* $\mathcal{F}^{\uparrow}(\overline{\nabla}^{(R(A))})$*, which sends proper costandard A-modules to proper costandard $R(A)$-modules, and tilting A-modules to injective $R(A)$-modules.*

Proof. Consider the covariant versions of our functors:

$$\mathrm{H} = \mathcal{R}\mathrm{hom}_A(\mathfrak{T}, \_) : \mathcal{D}^+(A^{\uparrow}\text{-gmod}) \to \mathcal{D}^+(\text{gmod-}R(A)^{\uparrow})$$

$$\tilde{\mathrm{H}} = \mathcal{R}\mathrm{hom}_A(\mathfrak{T}, \_) : \mathcal{D}^-(A^{\downarrow}\text{-gmod}) \to \mathcal{D}^-(\text{gmod-}R(A)^{\downarrow}).$$

Every object in $\mathcal{D}^-(A^{\downarrow}\text{-gmod})$ has a projective resolution. Since T is a tilting module, every object in $\mathcal{D}^-(A^{\downarrow}\text{-gmod})$ is also given by a complex of tilting modules. As tilting modules are selforthogonal, for complexes of tilting modules the functor $\tilde{\mathrm{H}}$ reduces to the usual hom functor. Similarly every object in $\mathcal{D}^+(A^{\uparrow}\text{-gmod})$ has an injective resolution and for such complexes the functor H reduces to the usual hom functor.

The left adjoints H' and $\tilde{\mathrm{H}}'$ of H and $\tilde{\mathrm{H}}$, respectively, are thus given by the left derived of the tensoring with $\mathfrak{T}$. As T is a tilting module, these left adjoint functors can be given as a tensoring with a finite tilting complex of A-$R(A)$-bimodules, projective as right $R(A)$-modules, followed by taking the total complex.

Using the definition of proper costandard modules it is straightforward to verify that both H and $\tilde{\mathrm{H}}$ map proper costandard left A-modules to proper standard right $R(A)$-modules. Similarly, both H' and $\tilde{\mathrm{H}}'$ map proper standard right $R(A)$-modules to proper costandard left A-modules. Since proper (co)standard objects have trivial endomorphism rings, it follows by standard arguments that the adjunction morphism

$$\mathrm{Id}_{\mathcal{D}^+(\text{gmod-}R(A)^{\uparrow})} \to \mathrm{H}\mathrm{H}', \quad \mathrm{H}'\mathrm{H} \to \mathrm{Id}_{\mathcal{D}^+(A^{\uparrow}\text{-gmod})}$$

$$\mathrm{Id}_{\mathcal{D}^-(\text{gmod-}R(A)^{\downarrow})} \to \tilde{\mathrm{H}}\tilde{\mathrm{H}}', \quad \tilde{\mathrm{H}}'\tilde{\mathrm{H}} \to \mathrm{Id}_{\mathcal{D}^-(A^{\downarrow}\text{-gmod})}$$

induce isomorphisms, when evaluated on respective proper (co)standard objects. Therefore the adjunction morphism above are isomorphisms of functors on the categories, generated (as triangular categories) by proper (co)standard objects. Using the classical limit construction (see [Ric]) one shows that both H and $\tilde{\mathrm{H}}$ are equivalences of categories. This yields that both F and $\tilde{\mathrm{F}}$ are antiequivalences of categories. This proves statement (i) and statements (ii) and (iii) easily follow. □

5. Proof of the main result

If $M \in \{P(\lambda), I(\lambda), T(\lambda), \Delta(\lambda), \overline{\nabla}(\lambda)\}$, we will say that the centroid of the graded modules $M\langle j\rangle$, where $j \in \mathbb{Z}$, belongs to $-j$. Let $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ be two complexes of tilting modules, both bounded from the right. A complex $\mathcal{X}^\bullet$ of projective, injective, tilting, standard, or costandard modules is called *linear* provided that for every i centroids of all indecomposable summand of $\mathcal{X}^i$ belong to $-i$. A positively graded algebra B is called *Koszul* if all simple B-modules have linear projective resolutions. The Koszul dual $E(A)$ of a Koszul algebra A is just the Yoneda extension algebra of the direct sum of all simple A-modules. The algebra $E(A)$ is positively graded by the degree of extensions.

We will say that $\mathcal{X}^\bullet$ *dominates* $\mathcal{Y}^\bullet$ provided that for every $i \in \mathbb{Z}$ the following holds: if the centroid of an indecomposable summand of $\mathcal{X}^i$ belongs to j and the centroid of an indecomposable summand of $\mathcal{Y}^i$ belongs to j', then $j < j'$.

The aim of this section is to prove Theorem 1. For this we fix an algebra A satisfying the assumptions of Theorem 1 throughout (we will call such algebra *balanced*). For $\lambda \in \Lambda$ we denote by $\mathcal{S}^\bullet_\lambda$ and $\mathcal{C}^\bullet_\lambda$ the linear tilting coresolution of $\Delta(\lambda)$ and resolution of $\overline{\nabla}(\lambda)$, respectively. We will proceed along the lines of [Ma, Section 3] and do not repeat the arguments, which are similar to the ones from [Ma, Section 3].

Lemma 26. *The algebra A is adapted.*

Proof. Mutatis mutandis [Ma, Lemma 2]. □

Corollary 27. *We have* $\mathrm{hom}_A(T(\lambda)\langle i\rangle, T(\mu)) = 0$*, for all $\lambda, \mu \in \Lambda$ and $i \in \mathbb{N}$.*

Corollary 28. *Let $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ be two complexes of tilting modules, both bounded from the right. Assume that $\mathcal{X}^\bullet$ dominates $\mathcal{Y}^\bullet$. Then* $\mathrm{Hom}_{\mathcal{D}^-(A)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet) = 0$.

Proof. Mutatis mutandis [Ma, Corollary 4]. □

Proposition 29. *For every $\lambda \in \Lambda$ the module $L(\lambda)$ is isomorphic in $\mathcal{D}^-(A)$ to a linear complex $\mathcal{L}^\bullet_\lambda$ of tilting modules.*

Proof. Just as in [Ma, Proposition 5], one constructs a complex $\overline{\mathcal{P}}^\bullet$ of tilting modules in $\mathcal{D}^-(A)$, quasi-isomorphic to $L(\lambda)$ and such that for each i all centroids of indecomposable summands in $\overline{\mathcal{P}}^i$ belong to some j such that $j \geq -i$.

Let us now prove the claim by induction with respect to $\preceq$. If λ is minimal, then $L(\lambda) = \overline{\nabla}(\lambda)$ and we can take $\mathcal{L}_\lambda^\bullet = \mathcal{C}_\lambda^\bullet$. Otherwise, consider the short exact sequence

$$0 \to L(\lambda) \to \overline{\nabla}(\lambda) \to \mathrm{Coker} \to 0.$$

Since A is positively graded, we have $\mathrm{Coker}_j = 0$ for all $j \geq 0$. Moreover, Coker is finite dimensional (Lemma 4) and all simple subquotients of Coker correspond to some $\mu \in \Lambda$ such that $\mu \prec \lambda$. Using the inductive assumption, we can resolve every simple subquotient of Coker using the corresponding linear complexes of tilting modules and thus obtain that Coker is quasi-isomorphic to some complex $\mathcal{X}^\bullet$ of tilting modules such that for each i all centroids of indecomposable summands in $\mathcal{X}^i$ belong to some j such that $j \leq -i-1$. As $\overline{\nabla}(\lambda)$ has a linear tilting resolution, it follows that $L(\lambda)$ is quasi-isomorphic to some complex $\overline{\mathcal{Q}}^\bullet$ of tilting modules, such that for each i all centroids of indecomposable summands in $\overline{\mathcal{Q}}^i$ belong to some j such that $j \leq -i$.

Because of the uniqueness of the minimal tilting complex $\mathcal{L}_\lambda^\bullet$, representing $L(\lambda)$ in $\mathcal{D}^-(A^\downarrow\text{-mod})$, we thus conclude that for all $i \in \mathbb{Z}$ centroids of all indecomposable summands in $\mathcal{L}_\lambda^i$ belong to $-i$. This means that $\mathcal{L}_\lambda^\bullet$ is linear and completes the proof. □

Corollary 30. *The algebra A is Koszul.*

Proof. Mutatis mutandis [Ma, Corollary 6]. □

Corollary 31. *We have the following:*

(i) Standard A-modules have linear projective resolutions.
(ii) Proper costandard A-modules have linear injective coresolutions.

Proof. Assume that $\mathrm{ext}^i_A(\Delta(\lambda), L(\mu)\langle j\rangle) \neq 0$ for some $\lambda, \mu \in \Lambda$, $i \geq 0$ and $j \in \mathbb{Z}$. As A is positively graded we obviously have $j \leq -i$. On the other hand, this inequality yields an existence of a non-zero homomorphism (in $\mathcal{D}^-(A^\downarrow\text{-mod})$) from $\mathcal{S}_\lambda^\bullet$ to $\mathcal{L}_\lambda^\bullet[i]\langle j\rangle$. But both $\mathcal{S}_\lambda^\bullet$ and $\mathcal{L}_\lambda^\bullet$ are linear (Proposition 29) and hence from Corollary 28 it follows that $j \geq -i$. Therefore $j = -i$ and statement (i) follows. The statement (ii) is proved similarly. □

Corollary 32. *We have the following:*

(i) Standard $R(A)$-modules have finite linear projective resolutions.
(ii) Standard $R(A)$-modules have finite linear tilting coresolutions.
(iii) Proper costandard $R(A)$-modules have linear tilting resolutions.
(iv) Proper costandard $R(A)$-modules have linear injective coresolutions.

Proof. Using Theorem 24(iv) we see that the functor G maps a finite linear projective resolution of $\Delta^{(A)}$ (Corollary 31(i)) to a finite linear tilting coresolution of $\Delta^{(R(A))}$. It also maps a finite linear tilting coresolution of $\Delta^{(A)}$ to a finite linear projective resolution of $\Delta^{(R(A))}$.

Using Theorem 25(ii) we see that the functor F maps a linear injective coresolution of $\overline{\nabla}^{(A)}$ (Corollary 31(ii)) to a linear tilting resolution of $\overline{\nabla}^{(R(A))}$. Using Theorem 25(iii) we see that the functor $\tilde{\mathrm{F}}$ maps a linear tilting resolution of $\overline{\nabla}^{(A)}$ to a linear injective coresolution of $\overline{\nabla}^{(R(A))}$. The claim follows. □

Corollary 33. *The algebra $R(A)$ is Koszul.*

Proof. This follows from Corollaries 30 and Corollaries 32. □

Denote by $\mathfrak{LT}$ the full subcategory of $\mathcal{D}^{-}(A)$, which consists of all linear complexes of tilting A-modules. The category $\mathfrak{LT}$ is equivalent to gmod-$E(R(A))^{\uparrow}$ and simple objects of $\mathfrak{LT}$ have the form $T(\lambda)\langle -i\rangle[i]$, where $\lambda \in \Lambda$ and $i \in \mathbb{Z}$ (see [MOS]).

Proposition 34. *We have the following:*

- *(i) The objects $\mathcal{S}^{\bullet}_{\lambda}$, where $\lambda \in \Lambda$, are proper standard objects in $\mathfrak{LT}$ with respect to $\preceq$.*
- *(ii) The objects $\mathcal{C}^{\bullet}_{\lambda}$, where $\lambda \in \Lambda$, are costandard objects in $\mathfrak{LT}$ with respect to $\preceq$.*

Proof. Mutatis mutandis [Ma, Proposition 11]. □

Proposition 35. *For all $\lambda, \mu \in \Lambda$ and $i, j \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathcal{D}^b(\mathfrak{LT})}(\mathcal{S}^{\bullet}_{\lambda}, \mathcal{C}_{\mu}\langle j\rangle[-i]^{\bullet}) = \begin{cases} \Bbbk, & \lambda = \mu, i = j = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Proof. Mutatis mutandis [Ma, Proposition 12]. □

Corollary 36. *The algebra $E(R(A))$ is standardly stratified with respect to $\preceq$.*

Proof. Applying the duality to Propositions 34 and 35 we obtain that standard $E(R(A))$-modules are left orthogonal to proper costandard. Using this and the same arguments as in the proof of Theorem 2 one shows that projective $E(R(A))$-modules have a standard filtration.

Since standard $E(R(A))$-modules are left orthogonal to proper costandard modules, to prove that the standard filtration of an indecomposable projective $E(R(A))$-module is finite it is enough to show that the dimension of the full ungraded homomorphism space from any indecomposable projective $E(R(A))$-module to any proper costandard module is finite. In terms of the category $\mathfrak{LT}$ (which gives the dual picture), we thus have to show that the dimension N of the full ungraded homomorphism space from $\mathcal{S}^{\bullet}_{\lambda}$ to any injective object in $\mathfrak{LT}$ is finite. Realizing $\mathfrak{LT}$ as linear complexes of projective $R(A)$-modules, we know that injective objects of $\mathfrak{LT}$ are linear projective resolutions of simple $R(A)$-modules (see [MOS, Proposition 11]), while the proper standard objects are linear projective resolutions of standard $R(A)$-modules. We thus get that N is bounded by the sum of the dimensions of all extension from the corresponding standard module to the corresponding simple module. Now the claim follows from the fact that all standard $R(A)$-modules have finite linear resolutions (Corollary 32(i)). □

Corollary 37. *The complexes $\mathcal{L}^\bullet_\lambda$, where $\lambda \in \Lambda$, are tilting objects in $\mathfrak{LT}$.*

Proof. Mutatis mutandis [Ma, Corollary 14]. □

Corollary 38. *There is an isomorphism $E(A) \cong R(E(R(A)))$ of graded algebras, both considered with respect to the natural grading induced from $\mathcal{D}^-(A)$. In particular, we have $R(E(A)) \cong E(R(A))$.*

Proof. Mutatis mutandis [Ma, Corollary 15]. □

Corollary 39. *Both $E(A)$ and $R(E(A))$ are positively graded with respect to the natural grading induced from $\mathcal{D}^-(A)$.*

Proof. Mutatis mutandis [Ma, Corollary 16]. □

Lemma 40. *The algebra $E(R(A))$ is standard Koszul.*

Proof. Mutatis mutandis [Ma, Lemma 18]. □

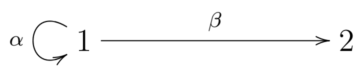
Proposition 41. *The positively graded algebras $E(A)$ and $R(E(A))$ are balanced.*

Proof. Mutatis mutandis [Ma, Proposition 17]. □

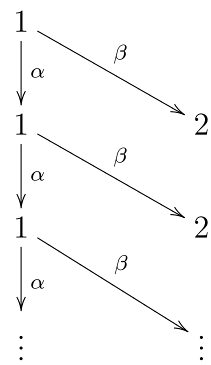
Proof of Theorem 1. Statement (i) follows from Corollaries 30 and 31. Statement (ii) follows from Corollary 32 and Proposition 41. Statement (iii) follows from Proposition 29. Finally, statement (iv) follows from Corollary 38. □

6. Examples

Example 42. Consider the path algebra A of the following quiver:

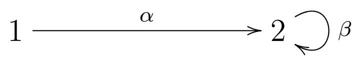


It is positively graded in the natural way (each arrow has degree one). We have $\Delta(2) = P(2) = L(2)$, while the projective module $P(1)$ looks as follows:

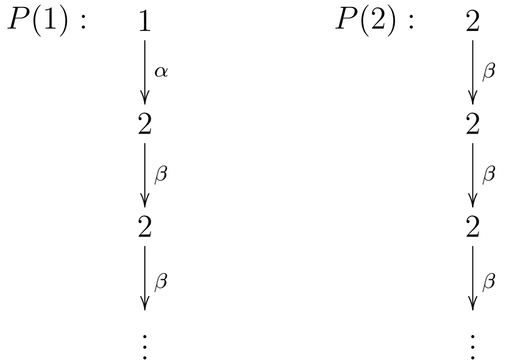


In particular, we have that the ungraded composition multiplicity of $L(2)$ in $P(1)$ is infinite and hence $P(1)$ has an infinite standard filtration. In particular, Lemma 15 fails in this case and hence the universal extension procedure does not have a starting point and can not give us a module from $A^{\downarrow}$-gmod.

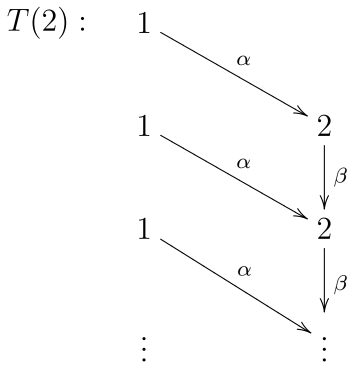
Example 43. Consider the path algebra B of the following quiver:



It is positively graded in the natural way (each arrow has degree one). We have $\Delta(1) = L(1) = T(1)$, $\Delta(2) = P(2)$ and the following projective B-modules:

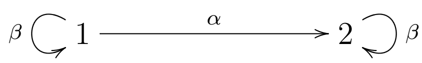


The module $T(2)$ looks as follows:

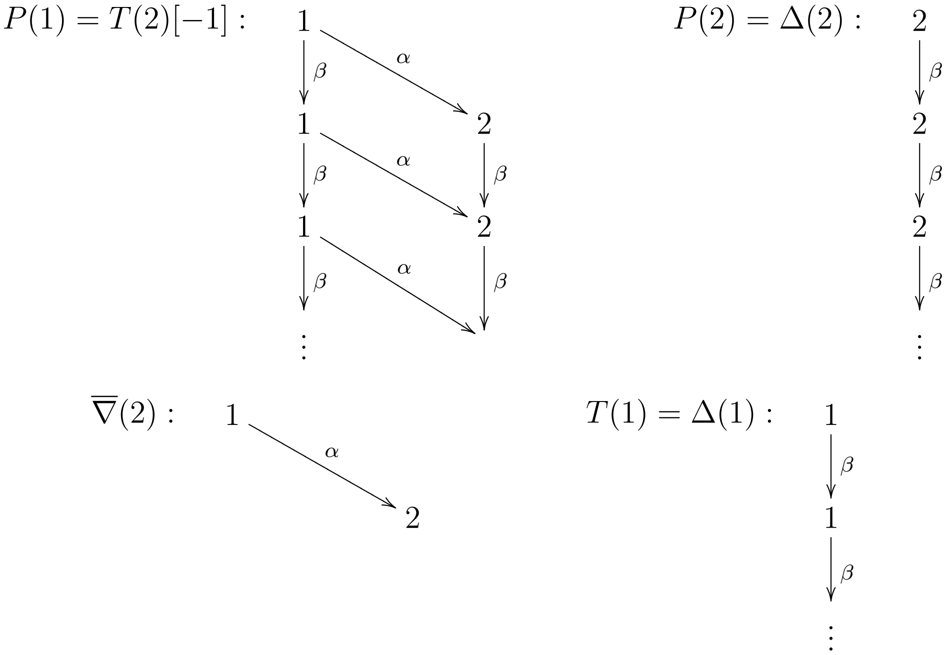


In particular, $T(2)$ has an infinite standard filtration and hence the algebra B is not weakly adapted.

Example 44. Consider the path algebra C of the following quiver:



modulo the ideal, generated by the relation $\alpha\beta = \beta\alpha$. It is positively graded in the natural way (each arrow has degree one). We have $\overline{\nabla}(1) = L(1)$ and also the following projective, standard, proper costandard and tilting C-modules:



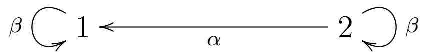
Standard and proper costandard C-modules have the following linear tilting (co)resolutions:

$$0 \to \Delta(1) \to T(1) \to 0$$
$$0 \to \Delta(2) \to T(2) \to T(1)[1] \to 0$$
$$0 \to T(1)[-1] \to T(1) \to \overline{\nabla}(1) \to 0$$
$$0 \to T(2)[-1] \to T(2) \to \overline{\nabla}(2) \to 0.$$

Hence C is balanced. The indecomposable tilting objects in $\mathfrak{LT}$ are:

$$0 \to T(1)[-1] \to T(1) \to 0$$
$$0 \to T(2)[-1] \to T(2) \oplus T(1) \to T(1)[1] \to 0.$$

We have $R(C) \cong C^{\mathrm{op}}$, $E(C)$ is the path algebra of the quiver:



modulo the ideal, generated by the relation $\alpha\beta = \beta\alpha$ and $\beta^2 = 0$, and $R(E(C)) \cong E(R(C)) \cong E(C)^{\mathrm{op}}$.

Example 45. Every Koszul positively graded local algebra algebra A with $\dim_{\Bbbk} A_0 = 1$ is balanced. Every Koszul positively graded algebra is balanced in the case when $\prec$ is the full relation.

Example 46. Directly from the definition it follows that if the algebra A is balanced, then the algebra $A/Ae_{\overline{\lambda}}A$ is balanced as well for any maximal λ. It is also easy to see that if A and B are balanced, then both $A \oplus B$ and $A \otimes_{\Bbbk} B$ are balanced.

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