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Refined Chern-Simons theory and (q,t)-deformed Yang-Mills theory: Semi-classical expansion and planar limit

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Abstract

We study the relationship between refined Chern-Simons theory on lens spaces $S^3/\mathbb{Z}_p$ and (q,t)-deformed Yang-Mills theory on the sphere S^2. We derive the instanton partition function of (q,t)-deformed $U(N)$ Yang-Mills theory and describe it explicitly as an analytical continuation of the semi-classical expansion of refined Chern-Simons theory. The derivations are based on a generalization of the Weyl character formula to Macdonald polynomials. The expansion is used to formulate q-generalizations of β-deformed matrix models for refined Chern-Simons theory, as well as conjectural formulas for the χ_y-genus of the moduli space of $U(N)$ instantons on the surface $\mathcal{O}(-p)\to\mathbb{P}^1$ for all $p\geq 1$ which enumerate black hole microstates in refined topological string theory. We study the large N phase structures of the refined gauge theories, and match them with refined topological string theory on the resolved conifold.

Contents

1 Introduction

Refined Chern-Simons theory has been of recent interest because of the rich structure of the new knot and three-manifold invariants that it computes, and also because of its connection to refined topological string theory and the refined topological vertex [1, 2, 3, 4, 5, 6, 7]. In this paper we explore its relation to refined topological string theory on the local Calabi-Yau threefold

$$X = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \longrightarrow \mathbb{P}^1 \ . \tag{1.1}$$

The usual topological string theory on this fibration reduces to a q-deformation of Yang-Mills theory on the sphere S^2 [8, 9]. The refined version of this correspondence was studied recently in [10], where it was shown that due to the refinement the reduction leads to a two-parameter (q,t)-deformation of $U(N)$ Yang-Mills theory on S^2.

The focus of the present work is on this (q,t)-deformed two-dimensional gauge theory. Starting with the refined version of the heat kernel expansion for the partition function of the theory, we show that it can be transformed in various insightful ways. From a three-dimensional perspective, we demonstrate explicitly that the (q,t)-deformed gauge theory defines an analytic continuation of refined Chern-Simons theory on the lens space $L(p,1) = S^3/\mathbb{Z}_p$. Via Poisson resummation we obtain the instanton expansion of the two-dimensional gauge theory, and also derive new β-deformed matrix model representations of the partition function.

The corresponding refined topological string partition function can also be studied by counting BPS bound states of D4-D2-D0 branes in X with angular momentum and R-charge, which is computed by the Hirzebruch χ_y-genus of the moduli space of $U(N)$ instantons in a topologically twisted $\mathcal{N}=4$ gauge theory on the worldvolume $D = \mathcal{O}(-p) \to \mathbb{P}^1$ of the D4-branes. Using our expansions of the two-dimensional gauge theory partition functions, we present new conjectural formulas for the contributions of curve classes to the χ_y-genus for arbitrary p.

We also describe the planar large N limit and the phase structure of the (q,t)-deformed two-dimensional Yang-Mills theory, both directly in terms of the heat kernel expansion and also from the point of view of the instanton expansion. We find an analogous phase behaviour to the unrefined q-deformed case [11, 12, 13]: The theory exhibits the usual phase transition which, as a function of the area $A = \tau_2\, p$ where τ_2 is identified with the 't Hooft coupling, occurs only for $p > 2$. We identify the critical curve and compute the free energy explicitly in the small area phase. Using this result we discuss the connection between the refined gauge theory and the emergent conifold geometry in refined topological string theory.

It would be interesting to find other large N limits, for instance involving a limit where one of the equivariant rotations vanishes as was studied in [14] in the context of refined topological string theory, but we have not found any other planar limit which leads to sensible results. To explore truly non-trivial effects of the β-deformation, one should proceed to construct the $\frac{1}{N}$-expansion of the theory based on a suitable double scaling limit; this has been studied in [15] for the β-deformation of the Chern-Simons matrix model, and it would be interesting to extend their results to include the full quantum group β-deformation appropriate to the refined Chern-Simons matrix model. It would also be interesting to understand the implications of this phase structure to the duality between (q,t)-deformed two-dimensional Yang-Mills theory and four-dimensional $\mathcal{N} = 2$ gauge theories with two superconformal fugacities [16, 17].

The structure of the paper is as follows. In §2 we discuss the formalism of the (q,t)-deformation of two-dimensional Yang-Mills theory, and the semiclassical expansion of its partition function. In §3 we describe the connection to refined Chern-Simons theory on the Lens space $S^3/\mathbb{Z}_p$, and its formulation in terms of β-deformed matrix models. In §4 we discuss the relationship between the two-dimensional gauge theory and the refined black hole partition function which enumerates spinning D4-D2-D0 brane bound states, and present the conjectural formulas for the χ_y-genus. In §5 we study the planar limit and phase structure of the (q,t)-deformed two-dimensional Yang-Mills theory, while §6 contains final remarks concerning the connection to the emergent geometry in the large N limit. Appendix A summarises the main features of the generalised Weyl denominator formula that is used in the main text to derive the pertinent semi-classical expansions.

2 (q,t)-deformed Yang-Mills theory on S^2

2.1 General aspects

The partition function for the (q,t)-deformation of $U(N)$ Yang-Mills theory on the sphere S^2 can be written as a generalization of the Migdal heat kernel expansion given by [10]

$$Z(q,t,Q;p) = \sum_R \frac{\dim_{q,t}(R)^2}{g_R}\, q^{\frac{p}{2}(R,R)}\, t^{p\,(\rho,R)}\, Q^{|R|} , \tag{2.1}$$

where the sum runs over all irreducible unitary representations of the $U(N)$ gauge group which are parametrized by partitions $R = (R_1, \dots, R_N)$, with $R_1 \geq R_2 \geq \cdots \geq R_N \geq 0$, such that R_i is the length of the i-th row of the corresponding Young diagram. Here the deformation parameters are

$$q = \mathrm{e}^{\,-\epsilon_1} \qquad \text{and} \qquad t = \mathrm{e}^{\,-\epsilon_2} , \tag{2.2}$$

where (ϵ_1, ϵ_2) may be regarded as parameterizing either the Ω-background in the corresponding five-dimensional gauge theory, the left/right angular momentum of BPS states of spinning M2-branes in M-theory, or the strength of the non-selfdual graviphoton background in topological string theory. For simplicity of presentation, below we shall write some formulas for the case when the refinement parameter

$$\beta = \frac{\epsilon_2}{\epsilon_1} \tag{2.3}$$

is a positive integer, and then analytically continue final results to arbitrary $\beta \in \mathbb{R}$. The parameter $Q = e^{-i\theta}$ is the contribution from the two-dimensional theta-angle. For a pair of weights $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_N)$, we define

$$(\lambda, \lambda') = \sum_{i=1}^{N} \lambda_i \, \lambda'_i \ , \tag{2.4}$$

and

$$\rho_i = \tfrac{1}{2} \, (N + 1 - 2i) \tag{2.5}$$

for $i = 1, \ldots, N$ are the components of the Weyl vector ρ for $U(N)$. We shall often assume that the rank N is odd, so that $\rho \in \mathbb{Z}^N$; this restriction is not necessary but it will simplify some of our analysis below. The quantity

$$|R| = \sum_{i=1}^{N} R_i \tag{2.6}$$

is the total number of boxes in the Young diagram associated to the representation R. The (q,t)-deformed dimension of the representation R is

$$\dim_{q,t}(R) = \prod_{m=0}^{\beta-1} \ \prod_{1 \le i < j \le N} \frac{\big[R_i - R_j + \beta \, (j-i) + m\big]_q}{\big[\beta \, (j-i) + m\big]_q} \ , \tag{2.7}$$

where

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q - q^{-1}} \tag{2.8}$$

for $x \in \mathbb{R}$ is a q-number. The Macdonald inner product normalization is given by

$$g_R = \prod_{m=0}^{\beta-1} \ \prod_{1 \le i < j \le N} \frac{\big[R_i - R_j + \beta \, (j-i) + m\big]_q}{\big[R_i - R_j + \beta \, (j-i) - m\big]_q} \ . \tag{2.9}$$

Our normalization differs from that of [10], wherein the refined quantum dimensions are multiplied by the factor

$$S_{00} = \prod_{m=0}^{\beta-1} \ \prod_{i=1}^{N-1} \big(q^{-m/2} \, t^{-i/2} - q^{m/2} \, t^{i/2}\big)^{N-i} \ . \tag{2.10}$$

This normalization factor will become important later on in our comparisons with refined Chern-Simons theory and its connection to refined topological string theory. We have also taken a different normalization for the q-number (2.8) such that $[x]_q = x + O(\log q)$ in the limit $q \to 1^-$; the present normalization is more useful for considering various limits below.

For our computations below we will require an explicit expression for the partition function (2.1) in terms of highest weight variables. For this, we define shifted weights n_i by

$$n_i = R_i + \beta \, \rho_i \tag{2.11}$$

for $i = 1, \ldots, N$. The range of these integers is $+\infty > n_1 > n_2 > \cdots > n_N > -\infty$. We can use the Weyl reflection symmetry of the summand of the partition function (2.1) to remove the restriction to the fundamental chamber of the summation over $n = (n_1, \ldots, n_N)$, and assume $n_i \neq n_j$ for all $i \neq j$. Then we can extend the summation range over all of $n \in \mathbb{Z}^N$ using the fact that $[n_i - n_j]_q = 0$

whenever $n_i = n_j$. Up to overall normalization, the partition function (2.1) can thus be written as a quantum β-deformation of the discrete Gaussian matrix model given by

$$Z(q,t,Q;p) = \sum_{n\in\mathbb{Z}^N} \Delta_{q,t}(\epsilon_1\, n)\, \Delta_{q,t}(-\epsilon_1\, n)\ \mathrm{e}^{-\frac{p\,\epsilon_1}{2}\,(n,n)-\mathrm{i}\,\theta\,|n|}\ , \tag{2.12}$$

where the Macdonald measure is given by

$$\Delta_{q,t}(x) = \prod_{m=0}^{\beta-1}\ \prod_{1\leq i<j\leq N} \big(q^{-m/2}\ \mathrm{e}^{\,(x_j-x_i)/2} - q^{m/2}\ \mathrm{e}^{\,(x_i-x_j)/2}\big) \tag{2.13}$$

for $x = (x_1,\dots,x_N) \in \mathbb{C}^N$. In the unrefined limit $\beta = 1$, the measure (2.13) reduces to the usual Weyl determinant

$$\Delta(x) = \Delta_{q,q}(x) = \prod_{1\leq i<j\leq N} 2\sinh\Big(\frac{x_i-x_j}{2}\Big) \tag{2.14}$$

which arises as a q-deformation of the Vandermonde determinant.

An interesting limit of this model is the one in which we send $\epsilon_1, \epsilon_2 \to 0$ and $p \to \infty$ with the parameters β and

$$a = \epsilon_1\, p \tag{2.15}$$

held fixed. In that case, all q-numbers reduce smoothly to ordinary numbers and the (q,t)-deformed Yang-Mills theory (2.1) reduces to a β-deformation of ordinary Yang-Mills theory given by

$$\mathcal{Z}_N^{\mathrm{YM},\beta}(a,\theta) = \sum_R \prod_{m=0}^{\beta-1}\ \prod_{1\leq i<j\leq N} \Big(\big(R_i - R_j + \beta\,(j-i)\big)^2 - m^2\Big)\ \mathrm{e}^{-\frac{a}{2}\,(R,R+2\,\beta\,\rho)}\ \mathrm{e}^{-\mathrm{i}\,\theta\,|R|}\ , \tag{2.16}$$

which for $\beta = 1$ coincides with ordinary (undeformed, unrefined) $U(N)$ Yang-Mills theory on the sphere S^2. As we discuss in §3, from a three-dimensional perspective the (q,t)-deformed gauge theory defines an analytical continuation of refined Chern-Simons theory [1] on the lens space $L(p,1) = S^3/\mathbb{Z}_p$ to arbitrary values of the Chern-Simons level k; the limit $p \to \infty$ of infinite degree of the Seifert fibration $S^3/\mathbb{Z}_p \to S^2$ thereby reduces the partition function of refined Chern-Simons theory to that of the β-deformation (2.16) of ordinary Yang-Mills theory. Alternatively, we can take this limit directly in (2.12) to obtain a β-deformation of the discrete Gaussian matrix model

$$\mathcal{Z}_N^{\mathrm{YM},\beta}(a,\theta) = \sum_{n\in\mathbb{Z}^N} \prod_{m=0}^{\beta-1}\ \prod_{1\leq i<j\leq N} \big((n_i-n_j)^2 - m^2\big)\ \mathrm{e}^{-\frac{a}{2}\,(n,n)-\mathrm{i}\,\theta\,|n|}\ . \tag{2.17}$$

From the point of view of topological string theory, the limit $p \to \infty$ should be understood as a singular limit of the underlying Calabi-Yau geometry (1.1). In this setting the partition function (2.17) defines a discrete version of the β-deformed Gaussian matrix ensemble considered in [18]; in §6 we show that the planar limit coincides with refined topological string theory on the conifold represented as the $c = 1$ string theory at a non-selfdual radius.

2.2 Semi-classical expansion

We will now derive the dual description of the refined q-deformed gauge theory in terms of instanton degrees of freedom, which is provided by performing a modular inversion of the series (2.12). For this, let us rewrite this series in the form

$$Z(q,t,Q;p) = \sum_{n\in\mathbb{Z}^N} \Delta(-\epsilon_1\, n)\, \widetilde{\Delta}_{q,t}(\epsilon_1\, n)\ \mathrm{e}^{-\frac{p\,\epsilon_1}{2}\,(n,n)-\mathrm{i}\,\theta\,|n|}\ , \tag{2.18}$$

where

$$\widetilde{\Delta}_{q,t}(x) := \frac{\Delta_{q,t}(x)\,\Delta_{q,t}(-x)}{\Delta(-x)} \ . \tag{2.19}$$

Substituting the (generalised) Weyl denominator formulas for $\Delta(x)$ and $\widetilde{\Delta}_{q,t}(x)$ from Appendix A, after some simple manipulations and dropping of overall normalisations throughout we can recast the partition function in the form

$$\begin{aligned} Z(q,t,Q;p) &= \sum_{w\in S_N} \varepsilon(w) \sum_{n\in\mathbb{Z}^N} \mathrm{e}^{-\frac{p\,\epsilon_1}{2}\,(n,n)-\mathrm{i}\,\theta\,|n|}\ \mathrm{e}^{\epsilon_1\,(w(\rho)-\beta\,\rho,n)} \\ &\quad \times \sum_{\mu\in\Lambda_\beta}\ \sum_{w'\in S_N} \mathrm{e}^{-\epsilon_1\,(\mu,w'^{-1}(n))}\ \Pi_\mu\big(\beta\,w\,w'(\rho);q,t\big) \ . \end{aligned} \tag{2.20}$$

The Poisson resummation of this series is now accomplished through an elementary Gaussian integration, and one finds

$$Z(q,t,Q;p) = \sum_{m\in\mathbb{Z}^N} \mathrm{e}^{-\frac{2\pi^2}{p\,\epsilon_1}\,(m,m)-\frac{2\pi\,\theta}{p\,\epsilon_1}\,|m|}\ \mathcal{W}_{q,t}(p;m) \tag{2.21}$$

where

$$\begin{aligned} \mathcal{W}_{q,t}(p;m) &= \sum_{w\in S_N} \varepsilon(w)\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{p}\,(m,w(\rho)-\beta\,\rho)}\ \mathrm{e}^{-\frac{\beta\,\epsilon_1}{p}\,(w(\rho),\rho)} \\ &\quad \times \sum_{\mu\in\Lambda_\beta}\ \sum_{w'\in S_N} \mathrm{e}^{-\frac{2\pi\,\mathrm{i}}{p}\,(m,w'(\mu))}\ \mathrm{e}^{\frac{\epsilon_1}{2p}\,(\mu,\mu-2w'^{-1}\,(w(\rho)-\beta\,\rho))}\ \Pi_\mu\big(\beta\,w\,w'(\rho);q,t\big) \ . \end{aligned} \tag{2.22}$$

To understand the meaning of this series, we note that at the classical level the refined two-dimensional gauge theory is just ordinary Yang-Mills theory on the sphere S^2 [10]. Using a gauge transformation, we can conjugate Yang-Mills connections of a $U(N)$ gauge bundle over S^2 so that they are valued in the Lie algebra of the maximal torus $U(1)^N\subset U(N)$; they correspond to sums of $U(1)$ Dirac monopole connections with topological charges $m_i\in\mathbb{Z}$ for $i=1,\dots,N$. The classical Yang-Mills action with theta-angle evaluated on such a configuration is given by

$$S_N^{\mathrm{YM}}(a,\theta;m) = \frac{2\pi^2}{a}\sum_{i=1}^N \Big(m_i^2+\frac{\theta\,m_i}{\pi}\Big) \ , \tag{2.23}$$

where a is the dimensionless Yang-Mills coupling constant on S^2. With the identification (2.15), we see that the exponential prefactors in the series (2.21) have a natural interpretation as the classical contributions $\mathrm{e}^{-S_N^{\mathrm{YM}}(a,\theta;m)}$ of instantons to the refined gauge theory path integral, while the sums (2.22) are the fluctuation determinants around each instanton.

Note that the residual Weyl symmetry S_N of the $U(N)$ gauge group after conjugation to the maximal torus permutes the different components of the classical monopole solutions. The classical field theory is invariant under this residual gauge symmetry. However, the path integral measure defining the quantum gauge theory differs from that of the unrefined case and is essentially determined by the Macdonald measure (2.13) [10]; this is reflected in the form of the quantum fluctuations $\mathcal{W}_{q,t}(p;m)$ which are not invariant under all gauge transformations in the Weyl group S_N. Whence the semi-classical expansion of the β-deformation of q-deformed Yang-Mills theory explicitly breaks a discrete part of the gauge symmetry; this is due to the way in which the quantum group nature of the gauge symmetry is manifested in the refined case which typically requires a notion of "twisted" invariance [19]. In the following we will find several interesting consequences of this symmetry breaking.

3 Refined Chern-Simons theory on $S^3/\mathbb{Z}_p$

3.1 Semi-classical expansion

We shall now describe the precise sense in which the (q,t)-deformed gauge theory on S^2 is an analytic continuation of the refinement of Chern-Simons theory on the lens space $S^3/\mathbb{Z}_p$ defined in [1, 10].

An expression for the path integral of $U(N)$ refined Chern-Simons gauge theory on $S^3/\mathbb{Z}_p$ at level $k\in\mathbb{Z}$ is derived in [1] using cutting and gluing rules of three-dimensional topological quantum field theory. The field theory depends on the parameters

$$q=\mathrm{e}^{\,g_s} \qquad \text{and} \qquad t=q^\beta \tag{3.1}$$

defined in terms of the genus expansion parameter

$$g_s:=\frac{2\pi\,\mathrm{i}}{k+\beta\,N}\ . \tag{3.2}$$

The partition function is [1, §5]

$$\mathcal{Z}_N^{\mathrm{CS},\beta}(g_s;p)=\sum_R\,(T_R)^p\ g_R^{-1}\,(S_{0R})^2\ , \tag{3.3}$$

where (up to overall normalization) $S_{0R}=S_{00}\,\dim_{q,t}(R)$ and

$$T_R=q^{\frac12\,(R,R)}\ t^{(R,\rho)}\ . \tag{3.4}$$

The series (3.3) is formally identical to (2.1), except that now the summation is finite and restricted to the integrable representations of $U(N)$ at level $k\in\mathbb{Z}$. With the same redefinition of weight vectors (2.11), we can write the sum over integrable representations as a sum over $n\in\mathbb{Z}^N_{k+\beta\,N}$. Using the generalised Weyl denominator formulas from Appendix A, we can then write the refined Chern-Simons partition function as a lattice Gauss sum

$$\begin{aligned}\mathcal{Z}_N^{\mathrm{CS},\beta}(g_s;p) &= \sum_{w\in S_N}\varepsilon(w)\sum_{n\in\mathbb{Z}^N_{k+\beta\,N}}\mathrm{e}^{\frac{\pi\,\mathrm{i}\,p}{k+\beta\,N}\,(n,n)}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{k+\beta\,N}\,(w(\rho)-\beta\,\rho,n)}\\ &\quad\times\sum_{\mu\in\Lambda_\beta}\ \sum_{w'\in S_N}\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{k+\beta\,N}\,(\mu,w'^{-1}(n))}\ \Pi_\mu\big(\beta\,w\,w'(\rho);q,t\big)\ .\end{aligned} \tag{3.5}$$

We now apply the quadratic reciprocity formula for Gauss sums to rewrite the sum over $n\in\mathbb{Z}^N_{k+\beta\,N}$ as a sum over $r\in\mathbb{Z}_p^N$, and again dropping irrelevant overall normalization factors we find

$$\mathcal{Z}_N^{\mathrm{CS},\beta}(g_s;p)=\sum_{r\in\mathbb{Z}_p^N}\mathcal{Z}_N^{\mathrm{CS},\beta}(g_s;p;r):=\sum_{r\in\mathbb{Z}_p^N}\mathrm{e}^{-\frac{\pi\,\mathrm{i}\,(k+\beta\,N)}{p}\,(r,r)}\ \mathcal{W}^\beta_{N,k}(p;r) \tag{3.6}$$

where

$$\begin{aligned}\mathcal{W}^\beta_{N,k}(p;r) &= \sum_{w\in S_N}\varepsilon(w)\ \mathrm{e}^{-\frac{2\pi\,\mathrm{i}}{p}\,(r,w(\rho)-\beta\,\rho)}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,\beta}{p\,(k+\beta\,N)}\,(w(\rho),\rho)}\\ &\quad\times\sum_{\mu\in\Lambda_\beta}\ \sum_{w'\in S_N}\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{p}\,(r,w'(\mu))}\ \mathrm{e}^{-\frac{\pi\,\mathrm{i}}{p\,(k+\beta\,N)}\,(\mu,\mu-2w'^{-1}(w(\rho)-\beta\,\rho))}\ \Pi_\mu\big(\beta\,w\,w'(\rho);q,t\big)\ .\end{aligned} \tag{3.7}$$

To understand the meaning of the sum (3.6), we note that at the classical level the refined gauge theory is identical to ordinary Chern-Simons theory on $S^3/\mathbb{Z}_p$ [1]. The critical points of the Chern-Simons action functional are flat connections. Gauge equivalence classes of flat $U(N)$ connections on the lens space $S^3/\mathbb{Z}_p$ are in one-to-one correspondence with isomorphism classes of N-dimensional unitary representations of the fundamental group $\pi_1(S^3/\mathbb{Z}_p)=\mathbb{Z}_p$. Using a gauge transformation, any such representation can be taken to have image in the maximal torus $U(1)^N\subset U(N)$. Thus the exponential prefactor in (3.6) is easily identified as $\mathrm{e}^{-S^{\mathrm{CS},\beta}_{N,k}(p;r)}$, where

$$S^{\mathrm{CS},\beta}_{N,k}(p;r)=\frac{\pi\,\mathrm{i}\,(k+\beta\,N)}{p}\,\sum_{i=1}^N r_i^2 \tag{3.8}$$

is the value of the classical Chern-Simons action on the lens space $S^3/\mathbb{Z}_p$ at the flat connection parameterized by the torsion vector $r=(r_1,\dots,r_N)\in\mathbb{Z}_p^N$ [20]. The expansion (3.6) is then evidently the semi-classical expansion of the refined Chern-Simons gauge theory, with the sums (3.7) representing the one-loop quantum fluctuation determinants about the classical solutions. In the unrefined limit $\beta=1$, the second line of (3.7) is equal to one (see Appendix A) and the expression (3.6) coincides with the semi-classical formula for the ordinary $U(N)$ Chern-Simons gauge theory partition function on $S^3/\mathbb{Z}_p$ [20].

By gauge invariance, the classical Chern-Simons action is invariant under the residual gauge symmetry generated by the action of the Weyl group of $U(N)$ which permutes the different components. On the other hand, as before the β-deformation breaks this gauge symmetry. Hence the refinement of Chern-Simons theory is generically sensitive to gauge equivalent flat connections which are related by a discrete gauge transformation in the subgroup S_N.

At the classical level, the equivalence between Chern-Simons theory on $S^3/\mathbb{Z}_p$ in the background of flat connections parametrized by $\mathbb{Z}_p^N$ and Yang-Mills theory on S^2 in the background of two-dimensional instantons parametrized by $\mathbb{Z}^N$ is well-known [13, 20]: Every flat connection on $S^3/\mathbb{Z}_p$ is the pullback by the bundle projection of the Seifert fibration $S^3/\mathbb{Z}_p\to S^2$ of a configuration of Dirac monopoles on the sphere S^2 with magnetic charges m_i, and the holonomy of this abelian gauge connection depends only on the values of the monopole numbers m_i modulo p. To state the equivalence at the quantum level, we note that the fluctuation factors (2.22) are quasi-periodic in m with period p in the sense that

$$\mathcal{W}_{q,t}(p;m+p\,n)=\mathrm{e}^{-2\pi\,\mathrm{i}\,(\beta-1)\,(n,\rho)}\,\mathcal{W}_{q,t}(p;m) \tag{3.9}$$

for all $n\in\mathbb{Z}^N$. It is thus natural to factorise the fluctuations by decomposing the sum over multi-monopole charges $m\in\mathbb{Z}^N$ as $m_i=p\,n_i+r_i$, where $n_i\in\mathbb{Z}$ and $r_i\in\mathbb{Z}_p$ for $i=1,\dots,N$. Then the instanton expansion (2.21) becomes

$$\begin{aligned} Z(q,t,Q;p) \;&=\; \sum_{r\in\mathbb{Z}_p^N}\,\mathcal{Z}_N^{\mathrm{CS},\beta}(-\epsilon_1;-p;r) \\ &\quad\times\,\sum_{n\in\mathbb{Z}^N}\,\mathrm{e}^{-\frac{2\pi^2\,p}{\epsilon_1}\,(n,n)-\frac{2\pi\,\theta}{p\,\epsilon_1}\,|r+p\,n|-\frac{4\pi^2}{\epsilon_1}\,(r,n)}\;\mathrm{e}^{-2\pi\,\mathrm{i}\,(\beta-1)\,(n,\rho)}\ . \end{aligned} \tag{3.10}$$

The meaning of the additional terms from the sum over $n\in\mathbb{Z}^N$ in (3.10) will be elucidated in §4.

3.2 Matrix models

In §5 we will treat the planar limit of the instanton partition function; for this, it is more convenient to have available a matrix integral representation of the refined Chern-Simons fluctuation terms

(3.7). Such a representation can be achieved by performing the Poisson resummation of the original series (2.12) in a different way to write the modular inversion as

$$Z(q,t,Q;p) = \sum_{m\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}x_i \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,m_i\,x_i-\mathrm{i}\,\theta\,x_i} \ \mathcal{F}_q(p;x)\,\mathcal{F}_{q,t}(p;x) \ , \tag{3.11}$$

where

$$\mathcal{F}_{q,t}(p;x) := \widetilde{\Delta}_{q,t}(\epsilon_1\,x) \ \mathrm{e}^{\,-\frac{p\,\epsilon_1}{4}\,|x|^2} \tag{3.12}$$

and $\mathcal{F}_q(p;x) := \mathcal{F}_{q,q}(p;-x)$. By using the ordinary Weyl denominator formula one shows [11, 13] that the Fourier transform of the function $\mathcal{F}_q(p;x)$ is given by

$$\widehat{\mathcal{F}}_q(p;m) := \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}x_i \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,m_i\,x_i} \ \mathcal{F}_q(p;x) = \Delta\big(-\tfrac{4\pi\,\mathrm{i}}{p}\,m\big) \ \mathrm{e}^{\,-\frac{4\pi^2}{p\,\epsilon_1}\,|m|^2} \ . \tag{3.13}$$

On the other hand, using the generalised Weyl denominator formula from Appendix A we compute the Fourier transformation

$$\begin{aligned}\widehat{\mathcal{F}}_{q,t}(p;m) \ &= \ \mathrm{e}^{\,-\frac{4\pi^2}{p\,\epsilon_1}\,|m|^2} \ \sum_{w\in S_N} \varepsilon(w) \ \mathrm{e}^{\,\frac{4\pi\,\mathrm{i}\,\beta}{p}\,(w(\rho),m)} \\ &\qquad \times \sum_{\mu\in\Lambda_\beta} \mathrm{e}^{\,\frac{4\pi\,\mathrm{i}}{p}\,(\mu,m)} \ \mathrm{e}^{\,\frac{\epsilon_1}{p}\,(\mu,\mu+2\beta\,w(\rho))} \ \Pi_\mu\big(\beta\,w(\rho);q,t\big) \ .\end{aligned} \tag{3.14}$$

Using the standard convolution formula to evaluate the product Fourier transformation in (3.11), we thus find that the fluctuation terms in (2.21) can be alternatively represented in the form of a matrix integral

$$\mathcal{W}_{q,t}(p;m) = \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}u_i \ \mathrm{e}^{\,-\frac{2\pi^2}{p\,\epsilon_1}\,u_i^2} \ \Delta\big(\tfrac{2\pi\,\mathrm{i}}{p}\,(u-m)\big) \, \widehat{\Delta}_{q,t,p}\big(\tfrac{2\pi\,\mathrm{i}}{p}\,(u+m)\big) \tag{3.15}$$

where

$$\widehat{\Delta}_{q,t,p}(x) := \sum_{w\in S_N} \varepsilon(w) \ \mathrm{e}^{\,\beta\,(w(\rho),x)} \ \sum_{\mu\in\Lambda_\beta} \mathrm{e}^{\,(\mu,x)} \ \mathrm{e}^{\,\frac{\epsilon_1}{p}\,(\mu,\mu+2\beta\,w(\rho))} \ \Pi_\mu\big(\beta\,w(\rho);q,t\big) \ . \tag{3.16}$$

Note that in the unrefined limit $\beta = 1$, only the $\mu = 0$ term contributes in the second sum of (3.16) with $\Pi_0(\lambda;q,t) = 1$ (see Appendix A), so that $\widehat{\Delta}_{q,q,p}(x) = \Delta(x)$ and (3.15) coincides with the standard fluctuation integral of the unrefined q-deformed gauge theory on S^2 [11, 13]. With the rescalings $u_i \to \mathrm{i}\,u_i/2\pi$ together with the analytic continuation $g_s = -\epsilon_1$ of the genus expansion parameter (3.2), we identify (3.15) as a β-deformed matrix model for refined Chern-Simons gauge theory on the lens space $S^3/\mathbb{Z}_p$. For $p = 1$ and $\beta \in \mathbb{Z}_{>0}$ a similar matrix integral is obtained in [1]; the equivalence between the discrete matrix model (2.12) and the β-deformed Stieltjes-Wigert matrix models for refined Chern-Simons theory is proven in [19].

From (3.15) we can equivalently cast the refined Chern-Simons partition function in the form of a β-deformed unitary matrix model. For this, we rescale $u \to p\,u$ and use quasi-periodicity (3.9) of the measure factor $\Delta\big(\frac{2\pi\,\mathrm{i}}{p}\,(p\,u-m)\big)\,\widehat{\Delta}_{q,t,p}\big(\frac{2\pi\,\mathrm{i}}{p}\,(p\,u+m)\big)$ under integer translations of the integration variables $u \in \mathbb{R}^N$. Hence we can truncate the integration domain to $u \in [0,1)^N$ by summing over all integer shifts of u_i, which we do for each $i = 1,\dots,N$ by using the modular transformation

$$\begin{aligned}\sum_{n_i\in\mathbb{Z}} \mathrm{e}^{\,-\frac{2\pi^2\,p}{\epsilon_1}\,(u_i-n_i)^2-2\pi\,\mathrm{i}\,(\beta-1)\,n_i\,\rho_i} \ &= \ \sqrt{\frac{\epsilon_1}{2\pi\,p}} \ \mathrm{e}^{\,\pi\,\mathrm{i}\,(\beta-1)\,\rho_i\,u_i+\frac{\epsilon_1}{2p}\,(\beta-1)^2\,\rho_i^2} \\ &\qquad \times \vartheta_3\big(\tfrac{\mathrm{i}\,\epsilon_1}{2\pi\,p}\,,\,2\pi\,u_i-\tfrac{\mathrm{i}\,\epsilon_1}{p}\,(\beta-1)\,\rho_i\big)\end{aligned} \tag{3.17}$$

of the Jacobi-Erderlyi elliptic function

$$\vartheta_3(\tau,z) := \sum_{n\in\mathbb{Z}} \, \mathrm{e}^{\,\pi\,\mathrm{i}\,n^2\,\tau+\mathrm{i}\,n\,z} \ . \tag{3.18}$$

Dropping irrelevant overall constants and rescaling $u_i = \frac{\phi_i}{2\pi}$ with $\phi_i \in [0,2\pi)$ for $i = 1,\dots,N$, we can rewrite the fluctuation factors (3.15) in the form of compact angular integrations

$$\begin{aligned} \mathcal{W}_{q,t}(p;m) \ &= \ \int_{[0,2\pi)^N} \, \prod_{i=1}^N \, \frac{\mathrm{d}\phi_i}{2\pi} \ \mathrm{e}^{\,\frac{\mathrm{i}}{2}\,(\beta-1)\,\rho_i\,\phi_i} \ \vartheta_3\big(\tfrac{\mathrm{i}\,\epsilon_1}{2\pi\,p}\,,\,\phi_i - \tfrac{\mathrm{i}\,\epsilon_1}{p}\,(\beta-1)\,\rho_i\big) \\ & \qquad\qquad \times\,\Delta\big(\mathrm{i}\,\phi - \tfrac{2\pi\,\mathrm{i}\,m}{p}\big)\,\widehat{\Delta}_{q,t,p}\big(\mathrm{i}\,\phi + \tfrac{2\pi\,\mathrm{i}\,m}{p}\big) \ . \end{aligned} \tag{3.19}$$

For $p = 1$ and $\beta \in \mathbb{Z}_{>0}$, a similar unitary matrix model is given in [1, Appendix B].

4 Refined black hole entropy and the χ_y-genus

Let us now discuss the precise relationship between the (q,t)-deformed Yang-Mills theory on S^2 and refined black hole partition functions in four dimensions. According to [10], refined black hole degeneracies are obtained by computing a protected spin character in four dimensions which enumerates spinning bound states of D4-D2-D0 brane systems with N D4-branes wrapped on the divisor $D = \mathcal{O}(-p) \to \mathbb{P}^1$ inside the ambient Calabi-Yau threefold (1.1), and D2-branes wrapping the base $\mathbb{P}^1$. The BPS degeneracies in this case are computed by the χ_y-genus of the moduli space of $U(N)$ instantons in a topologically twisted $\mathcal{N} = 4$ gauge theory on the D4-brane worldvolume D. Then the black hole partition function is given by

$$\mathcal{Z}_N^{\rm BH}(\phi_0,\phi_2,y;p) = \sum_{n,c\in\mathbb{Z}} \ \mathrm{e}^{\,-\phi_0\,n-\phi_2\,c} \ \chi_y\big(\mathfrak{M}_{n,c}(\mathcal{O}(-p))\big) \ , \tag{4.1}$$

where

$$\chi_y(\mathfrak{M}) = \sum_{i=0}^d (-y)^i \ \sum_{j=0}^d (-1)^j \ \dim H^j\big(\mathfrak{M}\,,\,\textstyle\bigwedge^i T^*\mathfrak{M}\big) \tag{4.2}$$

is the Hirzebruch χ_y-genus of the moduli space $\mathfrak{M} = \mathfrak{M}_{n,c}(\mathcal{O}(-p))$ of $U(N)$ instantons on the surface $\mathcal{O}(-p) \to \mathbb{P}^1$ of topological charge n and magnetic charge c; here $d = \dim_{\mathbb{C}} \mathfrak{M}$. The D0 and D2 brane chemical potentials ϕ_0 and ϕ_2 are related to the equivariant parameters of the Ω-deformation by

$$\phi_0 = \frac{4\pi^2}{\epsilon_1} \qquad \text{and} \qquad \phi_2 = \frac{2\,\pi\,\theta}{\epsilon_1} \ , \tag{4.3}$$

where here θ is interpreted as the four-dimensional theta-angle, while

$$y = \mathrm{e}^{\,-2\pi\,\mathrm{i}\,(\beta-1)} \ . \tag{4.4}$$

In the unrefined limit $\beta = 1$, $y = 1$, this is just the Vafa-Witten partition function which is the generating function for the Euler characteristics of instanton moduli spaces.

Following [20], we use our formalism to derive conjectural formulas for the χ_y-genus of the surfaces $\mathcal{O}(-p) \to \mathbb{P}^1$, which to the best of our knowledge are not known in closed form beyond the case $p = 1$ (where the geometry is simply that of $\mathbb{C}^2$ blown up at a point). For this, we keep only the classical contribution from the refined Chern-Simons partition function (3.6), which is associated to the boundary contribution to the four-dimensional instanton action, and drop the perturbative contribution represented by the sum over the Weyl group which should be absent from the partition function of the topologically twisted $\mathcal{N}=4$ gauge theory on D. This modifies (3.10) to the partition function

$$\widetilde{\mathcal{Z}}_N^{\rm BH}(\phi_0,\phi_2,y;p) = \sum_{r\in\mathbb{Z}_p^N}\ \sum_{n\in\mathbb{Z}^N} {\rm e}^{-\phi_0\,\frac{(r+p\,n,r+p\,n)}{2p} - \phi_2\,\frac{|r+p\,n|}{p}}\, y^{(n,\rho)} \ . \tag{4.5}$$

For $p=1$, this expression agrees with the contributions from fractional instantons to the anticipated generating function for the χ_y-genus of the instanton moduli space on $\mathcal{O}(-1)\to\mathbb{P}^1$, as discussed in [10, §5.5]. For $p>1$, we conjecture that it is the corresponding generating function for the surface $\mathcal{O}(-p)\to\mathbb{P}^1$. We see explicitly from (4.5) that the refinement keeps track of the contributions from each topological sector of fractional instantons with fixed holonomy $r\in\mathbb{Z}_p^N$ at infinity from the finite action requirement that the gauge fields be asymptotically flat; in the unrefined limit $y=1$, it can be resummed over $m=r+p\,n\in\mathbb{Z}^N$ to give the N-th power of the Jacobi theta-function (3.18) which computes the fractional instanton contributions to the usual $\mathcal{N}=4$ gauge theory partition function on $\mathcal{O}(-p)\to\mathbb{P}^1$ [20].

5 Planar limit

5.1 *Large N limit of (q,t)-deformed Yang-Mills theory*

In this section we take the $N\to\infty$ limit of the refined q-deformed Yang-Mills partition function (2.1); we set the theta-angle equal to zero from now on, i.e. $Q=1$. For this, we introduce the 't Hooft parameters τ_1 and τ_2 which are related to the deformation parameters ϵ_1 and ϵ_2 as

$$\tau_1 = \epsilon_1\,N \qquad \text{and} \qquad \tau_2=\epsilon_2\,N \ , \tag{5.1}$$

and we keep these couplings large but fixed when taking N large. In this limit the refinement parameter $\beta=\frac{\epsilon_2}{\epsilon_1}=\frac{\tau_2}{\tau_1}$ is also kept fixed. We introduce the continuous distribution $R(x)$, $x\in(0,1]$, of partitions as

$$R(x)=\frac{R_i}{N} \qquad \text{for} \quad x=\frac{i}{N} \ , \tag{5.2}$$

which obeys $R(x)\geq R(y)$ for $x\leq y$, and the shifted distribution

$$h(x) = -\frac{R(x)}{\beta}+x-\frac12 \tag{5.3}$$

which obeys $h(x)\leq h(y)$ for $x\leq y$ and

$$h(x)-h(y)\geq x-y \tag{5.4}$$

for $x\geq y$.

Writing the partition function (2.1) as

$$Z(q,t,Q;p)=\sum_R {\rm e}^{-N^2\,S_R(\tau_1,\tau_2;p)} \ , \tag{5.5}$$

we find for the planar free energy

$$\begin{aligned} S_R(\tau_1,\tau_2;p) \ &= \ -\beta\Big(\int_0^1{\rm d}x\,\int_0^1{\rm d}y\ \log\Big|2\sinh\frac{\tau_1\,\beta}{2}\,\big(h(x)-h(y)\big)\Big| + \frac{p\,\tau_1\,\beta}{2}\,\int_0^1{\rm d}x\ h(x)^2 \\ &\qquad -\frac{p\,\tau_1\,\beta}{24}+\frac{2}{(\tau_1\,\beta)^2}\,F_0^{\rm CS}(\tau_1\,\beta)\Big) \ , \end{aligned} \tag{5.6}$$

where the line $x=y$ is excluded from the domain of the double integral; we have used the fact that since β is finite, the sum over m coming from the dimension factors (2.7) is also finite and thus $\frac{m}{N}\to 0$ in the planar limit. Here

$$\frac{2}{t^2}\,F_0^{\rm CS}(t)=\int_0^1 {\rm d}x\,\int_0^1 {\rm d}y\ \log\Big|2\sinh\frac{t}{2}\,(x-y)\Big| \tag{5.7}$$

comes from the normalization factor (2.10), and it coincides with the planar free energy of Chern-Simons theory on S^3 with 't Hooft coupling t; it can be also expanded as

$$F_0^{\rm CS}(t)=\frac{t^3}{12}-\frac{\pi^2\,t}{6}-{\rm Li}_3\big({\rm e}^{\,-t}\big)+\zeta(3)\ , \tag{5.8}$$

where

$${\rm Li}_3(x)=\sum_{n=1}^\infty\frac{x^n}{n^3} \tag{5.9}$$

is the polylogarithm function of order 3. Apart from an overall factor of β, the (q,t)-deformed planar Yang-Mills free energy (5.6) is related to the q-deformed free energy $S_R(t;p)$ of [11] by the simple change of the 't Hooft coupling $t=\tau_1\,\beta=\tau_2$, where $t=g_s\,N$ is the 't Hooft coupling of the unrefined Yang-Mills theory, i.e. $S_R(\tau_1,\tau_2;p)=\beta\,S_R(t=\tau_2;p)$; the unrefined limit itself is of course obtained by setting $\beta=1$.

5.2 Phase transition

Using the simple relation between the unrefined and refined gauge theories in the planar limit, we can easily write down the one-cut solution and the corresponding density of eigenvalues. The saddle-point equation for the extrema $h(x)$ of the free energy (5.6) is

$$p\,h=⨍ {\rm d}h'\ \rho(h'\,)\ \coth\Big(\frac{\tau_1\,\beta}{2}\,(h-h'\,)\Big)\ , \tag{5.10}$$

where the spectral density

$$\rho(h):=\frac{{\rm d}x}{{\rm d}h} \tag{5.11}$$

is bounded as

$$0<\rho(h)\le 1 \tag{5.12}$$

and is normalized as $\int {\rm d}h\ \rho(h)=1$. This principal value integral equation coincides with the large N saddle-point equation of the Chern-Simons matrix model [21, 22] with 't Hooft coupling τ_2. More importantly, the planar limit of the β-deformed matrix model for refined Chern-Simons theory on the three-sphere [1]

$$Z_N^{{\rm CS},\beta}(g_s)=\int_{\mathbb{R}^N}\ \prod_{i=1}^N {\rm d}u_i\ {\rm e}^{\,-\frac{u_i^2}{2g_s}}\ \prod_{m=0}^{\beta-1}\ \prod_{i\neq j}\big({\rm e}^{\,(u_i-u_j)/2}-q^m\ {\rm e}^{\,(u_j-u_i)/2}\big) \tag{5.13}$$

gives the same saddle-point equation.

Following precisely the same steps as in [11] we obtain for the density functional

$$\rho(h)=\frac{p}{\pi}\ \arctan\left(\frac{\sqrt{{\rm e}^{\,A/p^2}-\cosh^2\big(\frac{A\,h}{2p}\big)}}{\cosh\big(\frac{A\,h}{2p}\big)}\right) \tag{5.14}$$

with the area parameter

$$A := \tau_1\, \beta\, p = \tau_2\, p \ . \tag{5.15}$$

The support of the spectral density is therefore $|h| < \frac{2p}{A}\,\mathrm{arccosh}\big(\,\mathrm{e}^{\,A/2p^2}\big)$ and its range is

$$\mathrm{im}(\rho) = \big[-\tfrac{p}{2}\,,\,\tfrac{p}{2}\big] \ . \tag{5.16}$$

Thus from (5.12) it follows that, similarly to the unrefined case, there is no phase transition for $p \leq 2$. For $p > 2$, a phase transition occurs when the density reaches its maximum value 1, which is on the critical line

$$A_*(p) = p^2\,\log\Big(\sec^2\big(\tfrac{\pi}{p}\big)\Big) \ . \tag{5.17}$$

Following [11] we can also write the refined Yang-Mills free energy (5.6) in the small area phase as

$$S_R(\tau_1,\tau_2;p) = \beta\,\Big(\frac{1}{\tau_2^2}\,\big(p^2\,F_0^{\mathrm{CS}}(\tfrac{\tau_2}{p}) - 2F_0^{\mathrm{CS}}(\tau_2)\big) + \frac{\tau_2}{12p} + \frac{p\,\tau_2}{24}\Big) \ , \tag{5.18}$$

where $F_0^{\mathrm{CS}}(t)$ is the planar Chern-Simons free energy (5.8). Note that in the limit $p \to \infty$ one has

$$A_*(p) \ \longrightarrow\ \pi^2 \qquad \text{and} \qquad \rho(h) \ \longrightarrow\ \rho_G\big(h,A^{-1}\big) := \tfrac{A}{2\pi}\,\sqrt{4A^{-1}-h^2} \ , \tag{5.19}$$

where $\rho_G(h,A)$ is the Wigner semicircle distribution of the Gaussian matrix model which governs the planar small area phase of ordinary Yang-Mills theory on S^2.

5.3 Instanton contributions

We now consider the large N phase transition from the point of view of the instanton expansion (2.21) of the (q,t)-deformed gauge theory. For this, we use the matrix integral representation (3.15) of the fluctuation factors to suitably perform the $N \to \infty$ limit. In the planar limit, the measure factor (3.16) simplifies drastically. Firstly, the factors $\mathrm{e}^{\frac{\tau_1}{p\,N}\,(\mu,\mu+2\beta\,w(\rho))} \to 1$ as $N \to \infty$, since the refinement parameter β is of order 1 in the limit and hence so are all root vectors $\mu \in \Lambda_\beta$; whence $\widehat{\Delta}_{q,t}(x) \to \widetilde{\Delta}_{q,t}(x)$. Secondly, in the Macdonald measure (2.13) one has $q^m = \mathrm{e}^{-\frac{m\,\tau_1}{N}} \to 1$ as $N \to \infty$; whence $\Delta_{q,t}(x) \to \Delta(x)^\beta$ as before. Altogether we get $\widehat{\Delta}_{q,t}(x) \to \Delta(x)^\beta\,\Delta(-x)^{\beta-1}$, and defining $y_i = 2\pi\,u_i$ for $i = 1,\dots,N$ we can write the fluctuation integral (3.15) up to overall normalization in the planar limit as

$$\mathcal{W}_{q,t}^\infty(p;m) = \int_{\mathbb{R}^N}\ \prod_{i=1}^N \mathrm{d}y_i\ \mathrm{e}^{-\frac{N\,\beta}{2A}\,y_i^2}\ \prod_{i<j}\sin\Big(\frac{\tau_1\,\beta}{2A}\,\big(y_{ij}-2\pi\,m_{ij}\big)\Big)\,\sin\Big(\frac{\tau_1\,\beta}{2A}\,\big(y_{ij}+2\pi\,m_{ij}\big)\Big)^{2\beta-1} \tag{5.20}$$

where we denote $x_{ij} := x_i - x_j$.

Following [11, 13], we look for a region in parameter space where the one-instanton contribution dominates the zero-instanton sector. Hence we define the function $\gamma(A,p)$ which measures the relative weight of these contributions in the $N \to \infty$ limit by

$$\exp\Big(-\frac{N\,\beta}{A}\,\gamma(A,p)\Big) = \mathrm{e}^{-\frac{2\pi^2\,N\,\beta}{A}}\ \frac{\mathcal{W}_{q,t}^\infty(p;e_i)}{\mathcal{W}_{q,t}^\infty(p;0)} \ , \tag{5.21}$$

where $e_i \in \mathbb{Z}^N$ is the vector with 1 in its i-th entry and 0 in all other components; at $N = \infty$ the precise choice of e_i is immaterial.

The partition function describing the zero-instanton sector of the gauge theory is defined by the β-deformed matrix integral

$$\mathcal{W}^{\infty}_{q,t}(p;0) = \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}y_i \ \mathrm{e}^{\,-\frac{N\,\beta}{2A}\,y_i^2} \ \prod_{i<j} \sin\Big(\,\frac{\tau_1\,\beta}{2A}\,(y_i-y_j)\Big)^{2\beta} \ . \tag{5.22}$$

The large N limit is dominated by solutions of the saddle-point equation

$$y = \tau_1\,\beta \,⨍\, \mathrm{d}y' \ \rho_{\rm inst}(y'\,) \ \cot\Big(\,\frac{\tau_1\,\beta}{2A}\,(y-y'\,)\Big) \tag{5.23}$$

for a suitable spectral distribution $\rho_{\rm inst}(y)$ for the matrix model (5.22). This equation is identical to the analogous saddle-point equation obtained for the unrefined case in [11, 13], and hence we can simply read off the solution by substituting $t=\tau_1\,\beta=\tau_2$ in their formulas. In particular, the spectral density is given by

$$\rho_{\rm inst}(y) = \frac{p}{\pi\,A}\,\mathrm{arccosh}\Big(\,\mathrm{e}^{\,A/2p^2}\ \cos\Big(\,\frac{y}{2p}\,\Big)\Big) \tag{5.24}$$

with support $|y|<2p\arccos\big(\,\mathrm{e}^{\,-A/2p^2}\big)$. In the limit $p\to\infty$, the density $\rho_{\rm inst}(y)\to\rho_G(y,A)$ is the Wigner semicircle distribution with area $A=\tau_1\,\beta\,p=\tau_2\,p$.

In the large N limit, the function $\gamma(A,p)$ is completely determined by the distribution (5.24) as

$$\begin{aligned}&\exp\Big(-\,\frac{N\,\beta}{A}\,\gamma(A,p)\Big) \qquad\qquad (5.25)\\ &= \int \mathrm{d}y\ \exp\bigg(-\,\frac{N\,\beta}{2A}\,y^2+N\int \mathrm{d}y'\ \rho_{\rm inst}(y'\,)\ \log\bigg(\frac{\sin\big(\frac{\tau_1\,\beta}{2A}\,(y-y'-2\pi)\big)\ \sin\big(\frac{\tau_1\,\beta}{2A}\,(y-y'+2\pi)\big)^{2\beta-1}}{\sin\big(\frac{\tau_1\,\beta}{2A}\,(y-y'\,)\big)^{2\beta}}\bigg)\bigg)\end{aligned}$$

and the integral over y can be evaluated in the saddle-point approximation by assuming that it is sharply peaked around $y=0$. Then we straightforwardly obtain

$$\gamma(A,p) = 2A\,\big(\mathcal{G}(0)-\mathcal{G}(2\pi)\big)\ , \tag{5.26}$$

where we have defined the function

$$\mathcal{G}(y) := \int \mathrm{d}y'\ \rho_{\rm inst}(y'\,)\ \log\Big|\sin\Big(\,\frac{\tau_1\,\beta}{2A}\,(y-y'\,)\Big)\Big| \tag{5.27}$$

and used reflection symmetry $\mathcal{G}(y)=\mathcal{G}(-y)$. This function is identical to that obtained in [11, 13] for the unrefined case, and hence we can simply copy their solution with the substitution $t=\tau_1\,\beta=\tau_2$ as before; see [11, eq. (4.22)] for the explicit form of the function (5.26). In particular, we obtain in this way the standard critical area curve (5.17) such that $\gamma(A_*(p),p)=0$; the instanton contributions to the gauge theory partition function are exponentially suppressed for $A<A_*(p)$ for all p, while at $A=A_*(p)$ the suppression ceases and they become the favourable configurations. Hence just as in the unrefined cases, the phase transition here is triggered by two-dimensional instantons.

6 Refined topological string theory

In this final section we discuss how the planar limit of the refined gauge theories is related to refined topological string theory. For this, we use the large N duality between $U(N)$ Chern-Simons theory on S^3 and topological string theory on the resolved conifold [23, 1]. The free energy computed in the weak coupling phase encodes information about the emergent geometry. In fact, the first piece of (5.18),

$$\beta \, \frac{p^2}{\tau_2^2} \, F_0^{\rm CS}\big(\tfrac{\tau_2}{p}\big) \ , \tag{6.1}$$

which is the relevant term when comparing with the instanton expansion, encodes the geometry of the resolved conifold with Kähler parameter $\frac{\tau_2}{p}$. As discussed in [10], for the refined topological string theory on the resolved conifold the partition function can be explicitly computed. Its degree zero parts are given by

$$\mathcal{Z}_0^{\rm top}(q,t;\kappa) = \big(M(q,t)\, M(t,q)\big)^{\chi/4} \ {\rm e}^{\, \frac{1}{\epsilon_1\, \epsilon_2} \, \frac{a\, \kappa^3}{6} + \beta \, \frac{b\, \pi^2 \kappa}{6}} \tag{6.2}$$

where the Kähler parameter is $\kappa = \frac{\tau_2}{p}$ in our case, and

$$M(q,t) = \prod_{n,m=1}^\infty \, \big(1-t^n\, q^{m-1}\big) \tag{6.3}$$

is the refined MacMahon function. Here $\chi = 2$ is the Euler characteristic of the conifold, while a and b are constants which are related to the triple intersection product of the Kähler class and to the second Chern class of the Calabi-Yau manifold, respectively. Since our manifold is non-compact, the choices for these constants are ambiguous; we choose them so that in the unrefined limit the expression (6.2) agrees with the usual free energy of the conifold [23]. Then the contribution to the genus zero free energy of the conifold is

$$\mathcal{F}_0^{\rm top}(q,t;\kappa) = \beta \, \Big(\zeta(3) + \frac{\kappa^3}{12} - \frac{\pi^2\, \kappa}{6}\Big) \ . \tag{6.4}$$

The non-trivial and unambiguous contribution to the partition function can be computed for example from the refined topological vertex [24], and is given by

$$\mathcal{Z}^{\rm top}(q,t;\kappa) = \exp\Big(-\sum_{n=1}^\infty \, \frac{\kappa^n}{n\, \big(q^{n/2}-q^{-n/2}\big)\, \big(t^{n/2}-t^{-n/2}\big)}\Big) \ . \tag{6.5}$$

Fixing $g_s = \epsilon_2$, this gives the additional genus zero contribution $-\beta \, {\rm Li}_3({\rm e}^{\,-\kappa})$, so that the total genus zero free energy of closed refined topological string theory on the conifold is given by

$$F_0^{\rm con}(q,t;\kappa) = \beta \, \Big(\zeta(3) + \frac{1}{12} \, \Big(\frac{\tau_2}{p}\Big)^3 - \frac{\pi^2}{6} \, \frac{\tau_2}{p} - {\rm Li}_3\big({\rm e}^{\,-\tau_2/p}\big)\Big) \ . \tag{6.6}$$

This expression agrees with the Chern-Simons contribution (6.1); the agreement is due to the geometric transition, and is related to the emergent conifold geometry seen in the strong coupling instantonic phase.

The reason why the conifold geometry emerges in the weak coupling phase can be understood by noting that in this phase the geometry is described by the β-deformed matrix model (5.22) of the zero-instanton sector; in the refined Chern-Simons description this corresponds to the contribution from the trivial flat connection. In fact, the full weak coupling geometry can be equivalently found from the planar limit of the corresponding Chern-Simons matrix model on S^3 as in [21, eq. (4.5)] with the Kähler parameter $t = \frac{\tau_2}{p}$, where it is also shown that its mirror geometry precisely describes the conifold. It is exactly this mirror conifold that one sees in the weak coupling phase of the gauge theory.

The free energy (5.18) also contains additional terms which do not appear to admit such an interpretation in terms of refined topological string theory. However, we must remember that, like in the unrefined case, the weak coupling phase is not expected to yield the correct description of the large N dual geometry. In the strong coupling phase multi-instantons contribute and the geometry is controlled by the more general matrix integrals (5.22) evaluated on torsion vectors $r \in \mathbb{Z}_p^N$ parametrizing non-trivial flat connections. We expect that the inclusion of all flat connections will restore the anticipated dual geometry to the cotangent bundle $T^*(S^3/\mathbb{Z}_p)$, which is an A_{p-1} fibration over $\mathbb{P}^1$ [21]. For this, one should construct the two-cut solution of the saddle-point equation (5.10) appropriate to the large area phase; we have refrained from attempting this, as even in the unrefined case only partial results are available from the two-cut analysis [11, 13]. Note that the fluctuation integrals $\mathcal{W}_{q,t}(p;r)$ from (3.15) for $r \in \mathbb{Z}_p^N$ have a natural interpretation in terms of a fixed configuration of N topological D3-branes wrapped on the cycles of $S^3/\mathbb{Z}_p$; however, the refined matrix model is sensitive to their insertion points because of the Weyl symmetry breaking, which disappears in the planar limit. It would be interesting to understand further the closed topological string theory emerging from the geometric transition through the matrix model geometry determined by (3.15).

As the refined conifold partition function emerges for any value of the degree p, it is natural to ask what becomes of the $p \to \infty$ limit. As we have seen, in this limit the small area phase is governed by the β-deformed Gaussian matrix model

$$\mathcal{Z}_\infty^{\mathrm{YM},\beta}(a) = \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}u_i \ \mathrm{e}^{-\frac{a}{2}\,u_i^2} \ \prod_{i<j} (u_i - u_j)^{2\beta} \ . \tag{6.7}$$

In the large N limit, it is shown in [18] that this matrix integral corresponds to the refined conifold geometry and coincides with the partition function of two-dimensional $c = 1$ string theory at radius $R = \beta$; β-deformed matrix ensembles are also used in [14] to discuss refined topological string theory. This result is in full agreement with the analysis of the planar limit of refined Chern-Simons theory on S^3 carried out in [25], where it is shown explicitly that the refinement replaces the virtual Euler characteristic of the moduli space of complex curves (which computes the perturbative free energy of the ordinary topological string theory on the resolved conifold [23]) with a parametrized Euler characteristic appropriate to the radius deformed $c = 1$ string theory.

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A Generalized Weyl denominator formula

A generalization of the Weyl character formula to Macdonald polynomials, seen as quantum group β-deformations of Schur polynomials, has been developed in [26, 27, 28, 29]. By considering the Macdonald polynomial corresponding to the trivial representation, which is equal to 1, we can extract a generalization of the Weyl denominator identity. In particular, from [29, Theorem 3.11] we infer the identity

$$\widetilde{\Delta}_{q,t}(x) := \frac{\Delta_{q,t}(x)\,\Delta_{q,t}(-x)}{\Delta(-x)} = \sum_{w\in S_N} \varepsilon(w)\,\Psi_{\beta\,w(\rho)}(x;q,t) \tag{A.1}$$

where $\varepsilon(w)$ is the sign of the Weyl group element $w \in S_N$ which acts on $\mathbb{C}^N$ by permutating components of N-vectors. The meromorphic function $\Psi_\lambda(x;q,t)$ of q and the $U(N)$ weights λ

is a complicated Laurent series. An explicit but involved combinatorial expansion can be found in [26, §8]; in [27, §5] it is described in terms of generalised characters, while in [28, 29] it is called a normalised Baker-Akhiezer function and constructed via applications of Macdonald difference operators to the function $\widetilde{\Delta}_{q,t}(x)$. Its main characteristics can be summarised as follows. Let us define the set

$$\Lambda_\beta^\circ := \Big\{ \mu = \sum_{\alpha>0} \mu_\alpha\, \alpha \ \Big|\ 0 \leq \mu_\alpha \leq \beta - 1 \Big\} \ , \tag{A.2}$$

where the sums run over positive roots of the Lie algebra of the unitary group $U(N)$. We may parametrize elements $\mu \in \Lambda_\beta^\circ$ by sequences of integers $\mu = \{\mu_{ij}\}_{1\leq i<j\leq N}$ with $0 \leq \mu_{ij} \leq \beta - 1$ and $(\mu, x) = \sum_{i<j} \mu_{ij}\,(x_i - x_j)$. Then for $t = q^{1-\beta}$, the function $\Psi_\lambda(x;q,t)$ can be expanded in the form

$$\Psi_\lambda(x;q,t) = \mathrm{e}^{\,(\lambda,x)} \sum_{\mu\in\Lambda_\beta^\circ} \mathrm{e}^{\,(\mu,x)}\ \Pi_\mu(\lambda;q,t) \ , \tag{A.3}$$

where the expansion coefficients $\Pi_\mu(\lambda;q,t)$ are normalised such that

$$\Pi_0(\lambda;q,t) = 1 \ . \tag{A.4}$$

It has the following properties:

- $\Psi_{w(\lambda)}\big(w(x);q,t\big) = \Psi_\lambda(x;q,t)$ for all $w \in S_N$.
- $\Psi_{-\lambda}(-x;q,t) = \Psi_\lambda(x;q,t)$.
- $\Psi_\lambda(x;q^{-1},t) = \Psi_\lambda(-x;q,t)$.

In the case of interest in this paper, wherein $t = q^\beta$, one must understand the expansion (A.3) as an analytical continuation by replacing Λ_β° with an infinite subset Λ_β of the root lattice of $U(N)$. In this case the series in (A.3) is infinite but still given by elementary functions, and it possesses the same properties as those listed above; see [29, §3.5], [27, §3] and [26, §8] for details.

In the unrefined limit $\beta = 1$, only the $\mu = 0$ contribution remains of the sum in (A.3). By (A.4), in this case

$$\Psi_\lambda(x;q,q) = \mathrm{e}^{\,(\lambda,x)} \tag{A.5}$$

is the usual character of the Verma module $\mathcal{M}_\lambda$ for $U(N)$, and the expansion (A.1) reduces to the usual Weyl denominator formula

$$\Delta(x) = \widetilde{\Delta}_{q,q}(x) = \sum_{w\in S_N} \varepsilon(w)\ \mathrm{e}^{\,(w(\rho),x)} \ . \tag{A.6}$$

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