Recursive linear estimation for discrete time systems in the presence of different multiplicative observation noises

C. Sánchez-González[1,*], T.M. García-Muñoz[1]

[1]Departamento de Métodos Cuantitativos para la Economía y la Empresa,

Facultad de Ciencias Económicas y Empresariales, Universidad de Granada,

Campus Universitario de Cartuja, s/n, 18011, Granada, Spain

Abstract

This paper describes a design for a least mean square error estimator in discrete time systems where the components of the state vector, in measurement equation, are corrupted by different multiplicative noises in addition to observation noise. We show how known results can be considered a particular case of the algorithm stated in this paper

Keywords : State estimation, multiplicative noise, uncertain observations

[*] Corresponding author. Tel.: +34 958 249909; Fax: +34 958 240620.
E-mails addresses: csanchez@ugr.es (C. Sanchez-Gonzalez); tgarciam@ugr.es (T.M. García-Muñoz).

1. Introduction

It was back in 1960 when R.E. Kalman [1] introduces his well known filter. Assuming the dynamic system is described through a state space model, Kalman considers the problem of optimum linear recursive estimation. From this event much other research work was developed including different hypothesis framework about system noises (Kalman [2], Meditch [3], Jazwinski [4], Kowalski and Szynal [5]).

In all studies above mentioned the estimated signal (state vector) in measurement equation is only corrupted by additive noise. Rajasekaran et al. [6] consider the problem of linear recursive estimation of stochastic signals in the presence of multiplicative noise in addition to measurement noise. When multiplicative noise is a Bernoulli random variable, the system is called system with uncertain observations. The estimation problem about these systems have been extensively treated (Nahi [7], Hermoso and Linares [8], Sanchez and García [9]).

This paper describes a design for a least mean square error (LMSE) estimator in discrete time systems where the components of the state vector, in measurement equation, are corrupted by different multiplicative noises in addition to observation noise. The estimation problems treated include one-stage prediction and filtering.

The presented algorithm can be considered as a general algorithm because, with particular specifications, this algorithm degenerates in known results as in Kalman [1], Rajasekaran and Szynal [6], Nahi [7], , Sanchez and García [10]. It can also be infered that if multiplicative noises are Bernoulli random variables, such situation is not, properly speaking, a system with uncertain observations because the components of the state can be present in the observation with different probabilities. Therefore, the presented algorithm solves the estimation problems in this new system specification with complete uncertainty about signal.

2. Statement and Notation

We now introduce symbols and definitions used across the paper. Let the following linear discrete-time dynamic system with $n \times 1$ elements be the state vector $x(k)$

$$\begin{aligned} x(k+1) &= \Phi(k+1,k)x(k) + \Gamma(k+1,k)\omega(k), \quad k \geq 0 \\ x(0) &= x_0 \end{aligned} \qquad \text{(State Equation)}$$

and $m \times 1$ observation vector $z(k)$ be given by

$$z(k) = H(k)\tilde{\gamma}(k)x(k) + v(k), \quad k \geq 0 \qquad \text{(Observation Equation)}$$

where $\Phi(k+1,k)$, $\Gamma(k+1,k)$ and $H(k)$ are known matrices with appropriate dimensions.

Usual and specific hypothesis regarding probability behavior for random variables are introduced to formalize the model as follows:

(H.1) x_0 is a centered random vector with variance-covariance matrix $P(0)$.

(H.2) $\{\omega(k), k \geq 0\}$ is centered white noise with $E\left[\omega(k)\omega^T(k)\right] = Q(k)$.

(H.3) $\tilde{\gamma}(k)$ is a diagonal matrix $\begin{pmatrix} \gamma_1(k) & & \\ & \ddots & \\ & & \gamma_n(k) \end{pmatrix}$ where $\{\gamma_i(k), k \geq 0\}$ is a scalar white sequence with nonzero mean $m_i(k)$ and variance $\sigma_{ii}(k)$, $\mathrm{i} = 1,...,n$. It is supposed that $\{\gamma_i(k), k \geq 0\}$ and $\{\gamma_j(k),\ k \geq 0\}$ are correlated in the same instant and $\sigma_{ij}(k) = Cov\left(\gamma_i(k), \gamma_j(k)\right)$, $\mathrm{i,j} = 1,...,n$. The next matrix will be used later on:

$$M(k) = \begin{pmatrix} m_1(k) & & \\ & \ddots & \\ & & m_n(k) \end{pmatrix}$$

(H.4) $\{v(k), k \geq 0\}$ is a centered white noise sequence with variance $E\left[v(k)v^T(k)\right] = R(k)$.

(H.5) x_0, $\{\omega(k),\ k \geq 0\}$, $\{v(k),\ k \geq 0\}$ are mutually independent.

(H.6) The sequences $\{\gamma_i(k), k \geq 0\}$, $i = 1,...,n$ are independent of initial state x_0, $\{\omega(k), k \geq 0\}$ and $\{v(k), k \geq 0\}$.

As we can observe, the components of the state vector, in the observation equation, are corrupted by multiplicative noise in addition to measurement noise.

Let $\hat{x}(k/l)$ be the LMSE estimate of $x(k)$ given observations $z(0),...,z(l)$. $e(k/l)=x(k)-\hat{x}(k/l)$ denote the estimation error, and the corresponding covariance matrix is $P(k/l)=\left[e(k/l)e^T(k/l)\right]$.

The LMSE linear filter and one-step ahead predictor of the state $x(k)$ are presented in the next section.

3. Prediction and filter algorithm

Theorem 1. *The one-step ahead predictor and filter are given by*

$$\hat{x}(k+1/k)=\Phi(k+1,k)\hat{x}(k/k),\ \ k\geq 0$$
$$\hat{x}(0/-1)=0$$
$$\hat{x}(k/k)=\hat{x}(k/k-1)+F(k)\left[z(k)-H(k)M(k)\hat{x}(k/k-1)\right],\ \ k\geq 0.$$

The filter gain matrix verifies

$$F(k)=P(k/k-1)M(k)H^T(k)\Pi^{-1}(k)$$

where

$$\Pi(k)=H(k)\tilde{S}(k)H^T(k)+H(k)M(k)P(k/k-1)M(k)H^T(k)+R(k)$$

with

$$\tilde{S}(k)=\begin{pmatrix}\sigma_{11}(k)S_{11}(k) & \sigma_{12}(k)S_{12}(k) & \cdots & \sigma_{1n}(k)S_{1n}(k)\\ \sigma_{12}(k)S_{21}(k) & \sigma_{22}(k)S_{22}(k) & \cdots & \sigma_{2n}(k)S_{2n}(k)\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{1n}(k)S_{1n}(k) & \sigma_{2n}(k)S_{2n}(k) & \cdots & \sigma_{nn}(k)S_{nn}(k)\end{pmatrix}$$

$$S_{ij}(k)=I_iS(k)I_j^T \text{ where } I_i=(0\quad 0\quad \cdots\quad 0\quad \underset{(i)}{1}\quad 0\quad \cdots\quad 0)_{1\times n}$$
$$S(k+1)=\Phi(k+1,k)S(k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q(k)\Gamma^T(k+1,k),\ \ k\geq 0$$
$$S(0)=P(0).$$

The prediction and filter error covariance matrices satisfy

$$P(k+1/k)=\Phi(k+1,k)P(k/k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q(k)\Gamma^T(k+1,k),\quad k\geq 0$$
$$P(0/-1)=P(0)$$
$$P(k/k)=P(k/k-1)-F(k)\Pi(k)F^T(k),\quad k\geq 0.$$

Proof.

By the state equation is easy to prove that the predictor $\Phi(k+1,k)\hat{x}(k/k)$ satisfies the Orthogonal Projection Lemma (OPL) [11]. In the initial instant, the estimate of $x(0)$ is its mean, so that $\hat{x}(0/-1)=0$.

As a consequence of the orthogonal projection theorem [11], the state filter can be written as a function of the one-step ahead predictor as

$$\hat{x}(k/k) = \hat{x}(k/k-1) + F(k)\delta(k), \quad k \geq 0$$

where $\delta(k) = z(k) - \hat{z}(k/k-1)$ is the innovation process. Its expression is obtained below.

Since $\hat{z}(k/k-1)$ is the orthogonal projection of $z(k)$ onto the subspace generated by observations $\{z(0),...,z(k-1)\}$, we know that this is the only element in that subspace verifying

$$E\left[z(k)z^T(\alpha)\right] = E\left[\hat{z}(k/k-1)z^T(\alpha)\right], \ \alpha = 0,...,k-1.$$

Then, by the observation equation and the hypotheses (H.3)-(H.6), it can be seen that $\hat{z}(k/k-1) = H(k)M(k)\hat{x}(k/k-1)$ and the innovation process for the problem we are solving is given by

$$\delta(k) = z(k) - H(k)M(k)\hat{x}(k/k-1).$$

To obtain the gain matrix $F(k)$, we observe that, given the OPL holds, $E\left[e(k/k)z^T(k)\right] = 0$, and we have

$$E\left[e(k/k-1)z^T(k)\right] = F(k)\Pi(k) \qquad (3.1)$$

where $\Pi(k)$ are the covariance matrices of the innovation. From the observation equation and the hypotheses (H.2)-(H.6), it can easily checked

$$E\left[e(k/k-1)z^T(k)\right] = P(k/k-1)M(k)H^T(k)$$

and therefore $F(k) = P(k/k-1)M(k)H^T(k)\Pi^{-1}(k).$

To obtain the covariance matrices of the innovation process, it can be seen that

$$\delta(k) = H(k)\tilde{\gamma}(k)x(k) + v(k) - H(k)M(k)\hat{x}(k/k-1)$$

and by adding and subtracting $H(k)M(k)x(k)$,

$$\delta(k) = H(k)\left(\tilde{\gamma}(k) - M(k)\right)x(k) + v(k) + H(k)M(k)e(k/k-1)$$

Then

$$\begin{aligned}\Pi(k) &= E\left[\delta(k)z^T(k)\right] \\ &= H(k)E\left[\left(\tilde{\gamma}(k) - M(k)\right)x(k)z^T(k)\right] + E\left[v(k)z^T(k)\right] + \\ &\quad H(k)M(k)E\left[e(k/k-1)z^T(k)\right].\end{aligned}$$

Let us work out each of the terms in previous expression. By the observation equation we have that

$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)z^T(k)\right]=H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)x^T(k)\tilde{\gamma}(k)\right]H^T(k)+$$
$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)v^T(k)\right]$$

and according to hypotheses in (H.4)-(H.6) the second term can be cancelled. Adding and subtracting $H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)x^T(k)M(k)\right]H^T(k)$,

$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)z^T(k)\right]=$$
$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)x^T(k)\left(\tilde{\gamma}(k)-M(k)\right)\right]H^T(k)+$$
$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)x^T(k)\right]M(k)H^T(k)$$

where the second term is zero by (H.3) and (H.6). According to (H.6), if we label $S_{ij}(k)=E\left[x_i(k)x_j(k)\right]$ for $i,j=1,\ldots,n$, we get

$$\tilde{S}(k)\equiv E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)x^T(k)\left(\tilde{\gamma}(k)-M(k)\right)\right]=$$
$$E\left[\begin{pmatrix}\left(\gamma_1(k)-m_1(k)\right)x_1(k)\\ \left(\gamma_2(k)-m_2(k)\right)x_2(k)\\ \vdots\\ \left(\gamma_n(k)-m_n(k)\right)x_n(k)\end{pmatrix}\begin{pmatrix}\left(\gamma_1(k)-m_1(k)\right)x_1(k)\\ \left(\gamma_2(k)-m_2(k)\right)x_2(k)\\ \vdots\\ \left(\gamma_n(k)-m_n(k)\right)x_n(k)\end{pmatrix}^T\right]=$$
$$\begin{pmatrix}\sigma_{11}(k)S_{11}(k) & \sigma_{12}(k)S_{12}(k) & \cdots & \sigma_{1n}(k)S_{1n}(k)\\ \sigma_{12}(k)S_{12}(k) & \sigma_{22}(k)S_{22}(k) & \cdots & \sigma_{2n}(k)S_{2n}(k)\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{1n}(k)S_{1n}(k) & \sigma_{1n}(k)S_{2n}(k) & \cdots & \sigma_{nn}(k)S_{nn}(k)\end{pmatrix}$$

Therefore

$$H(k)E\left[\left(\tilde{\gamma}(k)-M(k)\right)x(k)z^T(k)\right]=H(k)\tilde{S}(k)H^T(k).$$

On the other hand, by the observation equation and (H.4)-(H.6)

$$E\left[v(k)z^T(k)\right]=E\left[v(k)x^T(k)\tilde{\gamma}(k)\right]H^T(k)+E\left[v(k)v^T(k)\right]$$
$$=R(k).$$

By the same reasons

$$H(k)M(k)E\left[e(k/k-1)z^T(k)\right]=H(k)M(k)E\left[e(k/k-1)x^T(k)\right]M(k)H^T(k)$$
$$=H(k)M(k)P(k/k-1)M(k)H^T(k).$$

In short, the covariance matrices of the innovations process verify

$$\Pi(k)=H(k)\tilde{S}(k)H^T(k)+R(k)+H(k)M(k)P(k/k-1)M(k)H^T(k).$$

To obtain the components $S_{ij}(k)$ of the $\tilde{S}(k)$, we only need to observe that

$$S_{ij}(k)=I_iS(k)I_j^T$$

where $S(k)=E\left[x(k)x^T(k)\right]$ and $I_i=(0 \quad 0 \quad \cdots \quad 0 \quad \underset{(i)}{1} \quad 0 \quad \cdots \quad 0)_{1\times n}$. The next recursive expression of $S(k)$ is immediate given that $\{\omega(k),\, k\ge 0\}$ is a white noise sequence and independent of $x(0)$

$$\begin{aligned} S(k+1) &= \Phi(k+1,k)S(k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q(k)\Gamma^T(k+1,k),\ k\ge 0 \\ S(0) &= P(0). \end{aligned}$$

The expression of the prediction error covariance matrices

$$P(k+1/k)=\Phi(k+1,k)P(k/k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q(k)\Gamma^T(k+1,k).$$

is immediate since $e(k+1/k)=\Phi(k+1,k)e(k/k)+\Gamma(k+1,k)\omega(k)$.

In the other hand, given that $e(k/k)=e(k/k-1)-F(k)\delta(k)$ then

$$\begin{aligned} P(k/k) = P(k/k-1) &- E\left[e(k/k-1)\delta^T(k)\right]F^T(k) - \\ & F(k)E\left[\delta(k)e^T(k/k-1)\right]+F(k)\Pi(k)F^T(k). \end{aligned}$$

It can be observed that

$$E\left[\delta(k)e^T(k/k-1)\right]=E\left[z(k)e^T(k/k-1)\right]-H(k)M(k)E\left[\hat{x}(k/k-1)e^T(k/k-1)\right]$$

where the second term cancels according to OPL and by equation (3.1) it is obtained that

$$E\left[\delta(k)e^T(k/k-1)\right]=\Pi(k)F^T(k)$$

and then $P(k/k)=P(k/k-1)-F(k)\Pi(k)F^T(k)$.□

Next, we see how some known results can be considered as particular specifications of the general model proposed in this paper:

- If $\gamma_1(k)=\cdots=\gamma_n(k)=1$ the state vector are not corrupted by a multiplicative noise, then

$$\begin{aligned} \tilde{\gamma}(k) &= M(k)=I_{n\times n} \\ \sigma_{ij}(k) &= 0 \quad \forall i,j \end{aligned}$$

  and our algorithm degenerates in Kalman algorithm [1].
- If $\gamma_1(k)=\cdots=\gamma_n(k)=U(k)$ where $\{U(k),\ k\ge 0\}$ is a scalar white sequence with nonzero mean $m(k)$ and variance $n(k)$, we end up with Rajasekaran's [6] framework, $\tilde{\gamma}(k)=U(k)I_{n\times n}$, where the state vector (all components) is corrupted by multiplicative noise. In this case,

$$M(k) = m(k) I_{n\times n}$$
$$\sigma_{ij}(k) = n(k) \quad \forall i, j$$

and the presented algorithm collapses in Rajasekaran's.

- If $\gamma_1(k) = \cdots = \gamma_n(k) = \gamma(k)$ where $\{\gamma(k), k \geq 0\}$ is a sequence of Bernoulli independent random variable with $P[\gamma(k)=1] = p(k)$, then $\tilde{\gamma}(k) = \gamma(k) I_{n\times n}$ and we end up with Nahi's framework [7], where the state vector is present in the observation with probability $p(k)$. In this case,

$$M(k) = p(k) I_{n\times n}$$
$$\sigma_{ij}(k) = p(k)\left(1 - p(k)\right) \quad \forall i, j$$

and the new algorithm collapses in Nahi's.

- If $\gamma_1(k) = \cdots = \gamma_p(k) = 1$ and $\gamma_{p+1}(k) = \cdots = \gamma_n(k) = \gamma(k)$ where $\{\gamma(k), k \geq 0\}$ is a sequence of Bernoulli independent random variable with $P[\gamma(k)=1] = p(k)$, the observations can include some elements of the state vector not being ensure the presence of the resting others (Sanchez and García's framework [10]). In this case

$$M(k) = \begin{pmatrix} I_{p\times p} & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & I_{(n-p)\times(n-p)} \end{pmatrix}$$

$$\sigma_{ij}(k) = \begin{cases} 0, & i, j \leq p \\ 0, & i \leq p,\ j > p \\ 0, & i > p,\ j \leq p \\ p(k)\left(1 - p(k)\right), & i, j > p \end{cases}$$

and the new algorithm degenerates in Sanchez and García's.

Another interesting situation appears when some of the components in the state vector are present in the observation but appear with different probabilities. Such a situation is not a system with uncertain observations. The present algorithm solves estimation problems in this type of system, it is only necessary to suppose that the multiplicative noises are different Bernoulli random variables.

4. Some numerical simulation examples

We show now some numerical examples to illustrate the filtering and prediction algorithm presented in Theorem 1.

Example 1

We consider the following linear system described by the dynamic equation:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0.06 & 0.67 \\ 0.60 & 0.23 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0.02 \\ 0.24 \end{pmatrix} \omega(k), \ k \geq 0 \tag{4.1}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \tag{4.2}$$

$$z(k) = \begin{pmatrix} 0.85 & 0.42 \end{pmatrix} \begin{pmatrix} \gamma_1(k) & 0 \\ 0 & \gamma_2(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + v(k), \ k \geq 0 \tag{4.3}$$

where $\{\omega(k), \quad k \geq 0\}$ is centered Gaussian white noise with $Q(k) = 2.89$; x_{10} and x_{20} are centered Gaussian random variables with variances equal to 0.5; $\{\gamma_1(k), \quad k \geq 0\}$ and $\{\gamma_2(k), \quad k \geq 0\}$ are Gaussian white noise with means 2 and 3 and variances σ_{11} and σ_{22}, respectively; $\{\gamma_1(k), \quad k \geq 0\}$ and $\{\gamma_2(k), \quad k \geq 0\}$ are independent; $\{v(k), \quad k \geq 0\}$ is centered Gaussian white noise with variance $R = 0.1$.

Using the estimation algorithm of Theorem 1, we can calculate the filtering estimate $\hat{x}(k/k)$ of the state recursively. Fig. 1 and Fig. 2 illustrate the state $x_i(k)$ and the filter $\hat{x}_i(k/k)$, for $i = 1,2$, vs. k for the multiplicative Gaussian observation noises $\gamma_1 \to N\left(2, \sqrt{0.5}\right)$ and $\gamma_2 \to N\left(3, \sqrt{0.1}\right)$. The state is represented with black and the filter with red color.

Fig. 1. $x_1(k)$ **and** $\hat{x}_1(k/k)$ **vs.** k

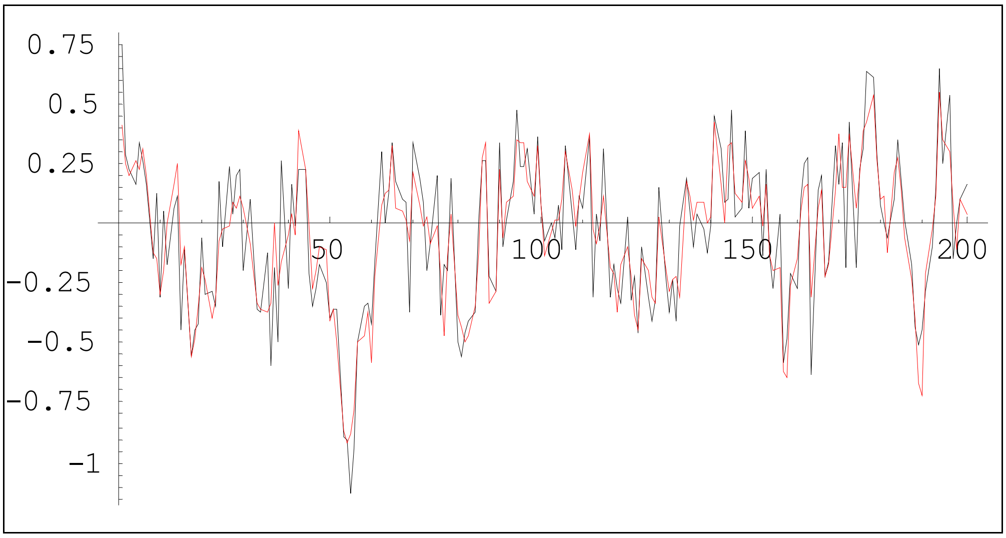
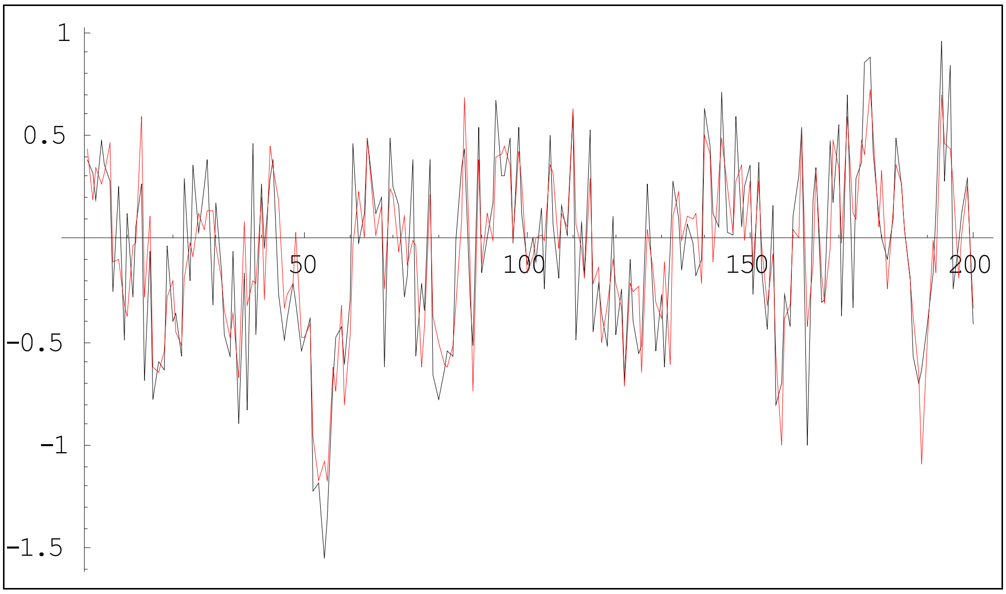


Fig. 2. $x_2(k)$ **and** $\hat{x}_2(k/k)$ **vs.** k



Tables 1 and 2 shows the mean-square values (MSVs) of the filtering errors $x_i(k)-\hat{x}_i(k/k)$ for $i=1,2$ and $k=1,2,\ldots,200$ corresponding to multiplicative white observation noises:

$$\gamma_1 : N(2,\sqrt{0.1}), N(2,\sqrt{0.5}), N(2,\sqrt{1})$$
$$\gamma_2 : N(3,\sqrt{0.1}), N(3,\sqrt{0.5}), N(3,\sqrt{1}).$$

Table 1. MSV of filtering errors $x_1(k)-\hat{x}_1(k/k),\ k=1,2,...,200$

	$\sigma_{22}=0.1$	$\sigma_{22}=0.5$	$\sigma_{22}=1$
$\sigma_{11}=0.1$	0.0171184	0.0194455	0.020916
$\sigma_{11}=0.5$	0.022236	0.0211678	0.022704
$\sigma_{11}=1$	0.0232388	0.023623	0.0237136

Table 2. MSV of filtering errors $x_2(k)-\hat{x}_2(k/k),\ k=1,2,...,200$

	$\sigma_{22}=0.1$	$\sigma_{22}=0.5$	$\sigma_{22}=1$
$\sigma_{11}=0.1$	0.0556318	0.0656372	0.0671069
$\sigma_{11}=0.5$	0.0698304	0.0703345	0.0690113
$\sigma_{11}=1$	0.0739651	0.075727	0.0730605

<u>Example 2</u>

We consider a linear system described by equations (4.1)-(4.3) where $\{\gamma_1(k),\quad k\geq 0\}$ and $\{\gamma_2(k),\quad k\geq 0\}$ are sequences of independent Bernoulli random variables being 1 with probabilities p_1 and p_2, respectively.

Fig. 3 and Fig. 4 illustrate the state $x_i(k)$ and the filter $\hat{x}_i(k/k)$, for $i=1,2$, vs. k for the multiplicative observation noises $\gamma_1 \to Bernoulli(0.5)$ and $\gamma_2 \to Bernoulli(1)$. The state is represented with blue and the filter with green color.

Fig. 3. $x_1(k)$ **and** $\hat{x}_1(k/k)$ **vs.** k

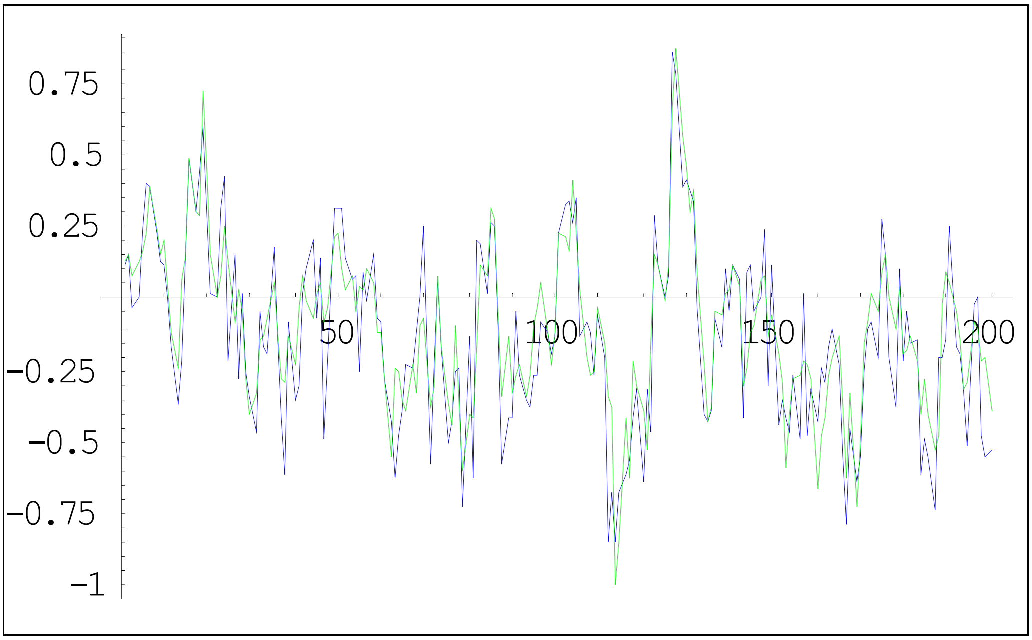
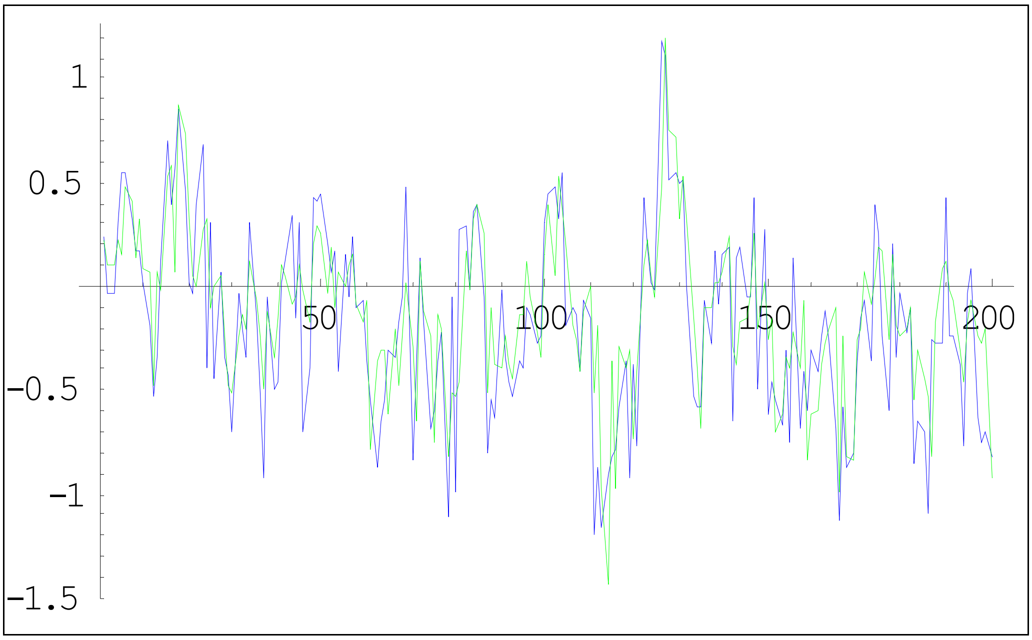


Fig. 4. $x_2(k)$ **and** $\hat{x}_2(k/k)$ **vs.** k



Tables 3 and 4 shows the mean-square values (MSVs) of the filtering errors $x_i(k)-\hat{x}_i(k/k)$ for $i=1,2$ and $k=1,2,...,200$ corresponding to multiplicative white observation noises:

$$\gamma_1 : Bernoulli(0.1), Bernoulli(0.5), Bernoulli(1)$$
$$\gamma_2 : Bernoulli(0.1), Bernoulli(0.5), Bernoulli(1).$$

Table 3. MSV of filtering errors $x_1(k) - \hat{x}_1(k/k),\ k = 1, 2, ..., 200$

	$p_2 = 0.1$	$p_2 = 0.5$	$p_2 = 1$
$p_1 = 0.1$	0.0948355	0.0696956	0.0273215
$p_1 = 0.5$	0.0616283	0.049987	0.0333839
$p_1 = 1$	0.013194	0.0197576	0.0154067

Table 4. MSV of filtering errors $x_2(k) - \hat{x}_2(k/k),\ k = 1, 2, ..., 200$

	$p_2 = 0.1$	$p_2 = 0.5$	$p_2 = 1$
$p_1 = 0.1$	0.211223	0.164514	0.0634171
$p_1 = 0.5$	0.184817	0.155435	0.09182
$p_1 = 1$	0.154539	0.130428	0.0708851

As we can observe, the simulation graphs and the MSV of the filtering in both examples show the effectiveness of the new algorithm.

References

[1] R.E. Kalman, *A new approach to linear filtering and prediction problems,* Trans. ASME Ser. D: J. Basic Eng., Vol. 82, 1960, pp.35-45.

[2] R.E. Kalman, *New methods in Wiener filtering theory, Proc. Symp. Eng. Appl. random function theory and probability (Wiley, New York),* 1963, pp.270-387.

[3] J.S. Meditch, *Stochastic optimal linear estimation and control,* McGraw-Hill, New York. 1969.

[4] A.H. Jazwinski, *Stochastic processes and filtering theory,* Academic Press, New York, 1970.

[5] A. Kowalski and D. Szynal, *Filtering in discrete-time systems with state and observation noises correlated on a finite time interval,* IEEE Trans. Automatic Control Vol. AC-31 (4), 1986, pp. 381-384.

[6] P.K. Rajasekaran, N. Satyanarayana and M.D. Srinath, *Optimum linear estimation of stochastic signals in the presence of multiplicative noise,*

IEEE Trans. on aerospace and electronic systems, Vol. AES-7, No. 3, 1971, pp. 462-468.

[7] N. Nahi, *Optimal recursive estimation with uncertain observations*, IEEE Transaction Information Theory, Vol. IT-15, 1969, pp. 457-462.

[8] A. Hermoso and J. Linares, *Linear estimation for discrete-time systems in the presence of time-correlated disturbances and uncertain observations*, IEEE Trans. Automatic Control, Vol. 39 (8), 1994, pp. 1636-1638.

[9] C. Sanchez-Gonzalez and T.M. García-Muñoz, *Linear estimation for discrete systems with uncertain observations: an application to the correction of declared incomes in inquiry*, Applied Mathematics and Computation, Vol. 156, 2004, pp. 211-233.

[10] C. Sanchez-Gonzalez and T.M. García-Muñoz, *Linear estimation of discrete dynamic systems with incomplete signals*, International Mathematical Journal, In press.

[11] A.P. Sage, and J.L. Melsa, *Estimation theory with applications to communications and control*, McGraw-Hill, New York, 1971.