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CONFORMAL HOLOMORPHICALLY PROJECTIVE MAPPINGS SATISFYING A CERTAIN INITIAL CONDITION

HANA CHUDÁ AND MOHSEN SHIHA

Abstract. In this paper we study conformal holomorphically projective mappings between conformal e-Kähler manifolds $K_n=(M, g, F)$ and $\bar{K}_n=(\bar{M}, \bar{g}, \bar{F})$, i. e. diffeomorphisms $f\colon M \to \bar{M}$ satisfying $f = f_1 \circ f_2 \circ f_3$, where f_1, f_3 are conformal mappings and f_2 is a holomorphically projective mapping between e-Kähler manifolds (i. e. Kähler, pseudo-Kähler and hyperbolic Kähler manifolds).

Suppose that the initial condition $f^*\bar{g} = k \cdot g$ is satisfied at a point $x_0 \in M$ and that at this point the Weyl conformal tensor satisfies a certain inequality. We prove that the mapping f is then necessarily conformal.

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1. INTRODUCTION

One may say that the pioneering work in conformal and projective geometry was done by H. Weyl [19] and T. Thomas [17]. The topic of the holomorphically projective (*HP*) mappings was introduced (for classical, elliptic) Kähler manifolds $\mathbb{K}_n^-$ by T. Otsuki and Y. Tashiro [13], for hyperbolic Kähler manifolds $\mathbb{K}_n^+$ by M. Prvanović [14], and for parabolic Kähler manifolds $\mathbb{K}_n^o$ by V. V. Vishnevskij [18]. See, e. g., [1, 7, 10, 11, 16, 20].

Let us mention that geodesic, conformaly geodesic and holomorphically projective mappings were studied under a certain additional condition based on the proportionality of the metrics. It turns out that even under this condition, the mapping is a homothety. See, e. g., [2–6, 8, 12].

In this paper we consider the following question, whether properties of conformal geodesic mappings are applicable for the composition of the conformal and holomorphically projective mappings — *conformal holomorphically projective mappings* of conformal e-Kähler manifolds.

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An analysis of the HP mappings of e-Kähler manifolds in terms of differentiability is presented in paper by I. Hinterleitner [9]. If the contrary is not specified, consideration is given in the tensor form in the class of real sufficiently smooth functions, the dimension $n \geq 4$, and is not mentioned specially. All the spaces are assumed to be connected.

2. Main properties of Kähler and conformal Kähler manifolds

We introduce in the following definition generalizations of (pseudo-) Kähler, conformal Kähler and Hermitian manifolds.

Definition 1. An n-dimensional (pseudo-) Riemannian manifold (M, g) is called an *e-Hermite manifold* $H_n = (M, g, F)$ if besides the metric tensor g, a tensor field F $(\neq \mathrm{Id})$ of type (1,1) is given on manifold M_n, such that $F^2 = e\,\mathrm{Id}$, $e = \pm 1$, and $g(X, FX) = 0$ for all tangent vector X. Moreover, if $\nabla F = 0$ then H_n is *e-Kähler manifold* $K_n = (M, g, F)$.

We remark, that for $e = -1$ are manifolds H_n and K_n (pseudo-) Hermitian and (pseudo-) Kähler manifolds, respectively, F is (almost) complex structure. For $e = 1$ we get hyperbolic Hermitian and hyperbolic Kähler manifold, respectively, F is (almost) product structure. See [9, 12, 12, 16, 20].

We remind the fundamental knowledge of a conformal mapping, that to be found in many monographs, see [6, 7, 12, 16, 20].

Definition 2. A diffeomorphism f between pseudo-Riemannian manifolds $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ is called a *conformal mapping*, if f preserves angles between all (smooth) curves on V_n $(n \geq 2)$.

Equivalently, a mapping $f\colon V_n \to \bar{V}_n$ is conformal if and only if $\bar{g} = \rho \cdot g$, where ρ is a nowhere zero function on M and we will again suppose, $\bar{M} = M$.

From the equation $\bar{g} = \rho \cdot g$ it follows that

$$(\bar{\nabla} - \nabla)_X X = 2\sigma(X) \cdot X - g(X, X) \cdot \Sigma, \tag{2.1}$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections on V_n and $\bar{V}_n$, respectively, $\sigma(X) = \frac{1}{2}\nabla_X \ln|\rho|$, $\sigma(X) = g(X, \Sigma)$ and X is an arbitrary tangent vector.

Let us recall a definition of a *conformal Weyl tensor* C on V_n $(n \geq 3)$:

$$C^h_{ijk} = R^h_{ijk} + \delta^h_j L_{ik} - \delta^h_k L_{ij} + L^h_j g_{ik} + L^h_k g_{ij},$$

where $L_{ij} = -\frac{1}{n-2}\,(R_{ij} - \frac{R}{2\,(n-1)}\, g_{ij})$, $L^h_i = g^{h\alpha} L_{\alpha i}$, R^h_{ijk} are the components of the Riemannian tensor of (M, g), $R_{ij} = R^\alpha_{i\alpha j}$ are the components of the Ricci tensor, $R = R_{\alpha\beta}\, g^{\alpha\beta}$ is the scalar curvature and g^{ij} are components of the inverse matrix of g_{ij}.

If there is a conformal mapping $V_n \to \bar{V}_n$ $(n > 2)$, then the conformal Weyl tensor remains invariant (i. e. $\bar{C} = C$). The converse is not true.

For $n > 3$, a pseudo-Riemannian space is locally *conformally flat* if and only if, the conformal Weyl tensor vanishes ($C = 0$).

Definition 3. A *conformal e-Kähler manifold* is conformally equivalent to e-Kähler manifold.

Clearly, any conformal e-Kähler manifold K_n may be considered as an e-Hermite manifold and it may be characterised by an e-Hermite structure. This structure has the following properties (for $e = -1$, see [15]):

$$\nabla_Y F(X) = \varphi(X) \cdot Y - g(X, Y) \cdot \Phi + \varphi(FX) \cdot (FY) + g(FX, Y) \cdot F\Phi,$$

where $\varphi(X) = g(X, \Phi) = \nabla_X \mathcal{F}$, $\mathcal{F}$ is a function on M and X, Y are tangent vector fields.

3. Holomorphically projective mappings of e-Kähler manifolds

Assume the e-Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, the Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Likewise, as in [13]

Definition 4. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ ($\neq 0$), $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \mathrm{Span}\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies $\nabla_t \lambda = a(t)\lambda + b(t)F\lambda$, where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

Definition 5. A diffeomorphism $f \colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* if f maps any analytically planar curve in K_n onto an analytically planar curve in $\bar{K}_n$.

Let there exist a HP mapping $f \colon K_n = (M, g, F) \to \bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally $f \colon K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [12]

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X) \cdot Y + \psi(Y) \cdot X + e\psi(FX) \cdot FY + e\psi(FY) \cdot FX \quad (3.1)$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective*. Beside these facts it was proved [12], that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$.

The *holomorphically projective tensor*, which is defined by the following form,

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik} - eF^h_k R_{i\alpha} F^\alpha_j + eF^h_j R_{i\alpha} F^\alpha_k + 2eF^h_i R_{j\alpha} F^\alpha_k\right)$$

is invariant with respect to holomorphically projective mappings, i. e. $\bar{P} = P$.

It is known, that an e-Kähler K_n is a manifold of the constant holomorphically projective curvature if and only if the holomorphically projective tensor vanishes ($P = 0$).

4. Conformal holomorphically projective mappings

After we have sketched some basic properties of holomorphically projective and conformal mappings, let us focus our attention to the already mentioned conformal holomorphically projective ones. In papers [8, 9] by I. Hinterleitner so called conformal projective mappings were studied. These mappings are closely related to our subject. Inspired by her observations, we will derive some further results on them.

Definition 6. A diffeomorphism $f\colon K_n \to \bar{K}_n$ is caled a *conformal holomorphically projective mapping* if $f = f_1 \circ f_2 \circ f_3$, where

$$\begin{aligned}
f_1 :&\quad K_n = (M, g, F) \to {}^1K_n = (M, {}^1g, F) &&\text{is a conformal mapping,}\\
f_2 :&\quad {}^1K_n = (M, {}^1g, F) \to {}^2K_n = (M, {}^2g, F) &&\text{is a HP mapping and}\\
f_3 :&\quad {}^2K_n = (M, {}^2g, F) \to \bar{K}_n = (M, \bar{g}, F) &&\text{is a conformal mapping.}
\end{aligned}$$

Evidently, K_n and $\bar{K}_n$ are conformal e-Kähler manifolds, and ${}^1K_n$ and ${}^2K_n$ are e-Kähler manifolds. We will assume, that structures F are on the same manifold M. We have the following theorem.

Theorem 1. *A diffeomorphism $f\colon K_n = (M, g, F) \to \bar{K}_n = (M, \bar{g}, \bar{F})$ is a conformal holomorphically projective mapping if and only if for each vector field X the following condition holds*

$$(\bar{\nabla} - \nabla)_X X = 2\psi(X)\cdot X + e\,2\psi(FX)\cdot FX + g(X,X)\cdot \Sigma + \bar{g}(X,X)\cdot \Omega, \quad (4.1)$$

where ψ is a differential 1-form, Σ and Ω are vector fields and there exist the functions ϱ_1, ϱ_2 and ϱ_3 on the manifold M such that for each field X,

$$\nabla_X \varrho_1 = g(X, \Sigma), \quad \nabla_X \varrho_2 = \bar{g}(X, \Omega), \quad \nabla_X \varrho_3 = \psi(X).$$

Proof. The necessity of (4.1) and the existence of the functions ϱ_1, ϱ_2 and ϱ_3 follow from the relations (2.1) and (3.1). The conditions are sufficient due to the following observation.

Suppose the conditions (4.1) are satisfied. Then one may construct metrics ${}^1g = \exp(-2\,\varrho_1)\cdot g$ and ${}^2g = \exp(2\,\varrho_2)\cdot \bar{g}$. Computing the difference between the Levi-Civita connections associated to ${}^1g$ and ${}^2g$, we get formula, thus according to (3.1), the spaces ${}^1K_n$ and ${}^2K_n$ are in HP correspondence. □

It is evident that the relation of "being conformal holomorphically projective equivalent" is symmetric and reflexive. Unfortunately, the conformal holomorphically projective mappings do not form a group because of lack of transitivity — the relation is not an equivalence relation.

5. CONFORMAL HP MAPPINGS WITH INITIAL CONDITIONS

We generalized the following Theorem:

Theorem 2 (Chudá and Mikeš [4]). *Let f be a holomorphically projective mapping between e-Kähler manifolds (M, g, F) and $(M, \bar{g}, \bar{F})$, $x_0 \in M$ and $\bar{x}_0 = f(x_0)$. Suppose that the initial condition $\bar{g}(\bar{x}_0) = k \cdot g(x_0)$ is satisfied for a $k \in \mathbb{R}$. If the holomorphically projective tensor does not vanish at x_0, then the mapping f provides a homothety between (M, g, F) and $(M, \bar{g}, \bar{F})$, i. e. $\bar{g} = k \cdot g$, $k = const.$*

We introduce: $Q^h_{ijk} = \delta^h_j g_{ik} - \delta^h_k g_{ij} - \frac{n-1}{3e} \cdot (F^h_j F_{ik} - F^h_k F_{ij} + 2F^h_i F_{jk})$, where $F_{ik} = g_{i\alpha} F^\alpha_k$, and we prove the following lemma:

Lemma 1. *If x_0 be a fixed point on e-Kähler manifold K_n and $P(x_0) = 0$, then at the point x_0 the formula $C_{hijk} = \frac{R}{n(n+2)} \cdot Q_{hijk}$ holds.*

Proof. Let at the point x_0 the holomorphically projective tensor vanishes, i. e. $P^h_{ijk} = 0$. After contraction of g^{ij} we persuade, that $R_{ij} = \frac{R}{n} \cdot g_{ij}$, and finally we get $C_{hijk} = \frac{R}{n(n+2)} \cdot Q_{hijk}$. □

Theorem 3. *Let f be a conformal holomorphically projective mapping between two conformal e-Kähler manifolds (M, g, F) and $(M, \bar{g}, \bar{F})$. If the metrics are proportional at the point x_0, i. e. $\bar{g}_{ij}(x_0) = \mu \cdot g_{ij}(x_0)$, $\mu \in \mathbb{R}$ and there exists no $\alpha \in \mathbb{R}$ so that $C_{hijk}(x_0) = \alpha \cdot Q_{hijk}(x_0)$, then f is conformal.*

Proof. Let K_n admit a conformal holomorphically projective mapping f onto $\bar{K}_n$ and at the point $x_0 \in M$, the inequality $C_{hijk} \neq \alpha \cdot Q_{hijk}$ holds. Because the mapping f_1: $K_n \rightarrow {}^1K_n$ is a conformal, then for metrics ${}^1g_{ij} = f \cdot g_{ij}$ and for conformal Weyl tensor ${}^1C^h_{ijk} = C^h_{ijk}$ hold.

Therefore the inequality $C_{hijk} \neq \alpha \cdot Q_{hijk}$ from K_n has on manifold ${}^1K_n$ form: ${}^1C \neq {}^1\alpha \cdot {}^1Q$ Based on the contraposition of Lemma 1 we known, that the holomorphically projective tensor ${}^1P \neq 0$.

Consequently are satisfied the conditions from Theorem 2, the mapping f_2: ${}^1K_n \rightarrow {}^2K_n$ is homothetic. Moreover at the point x_0 the following assertion holds ${}^1g_{ij}(x_0) = \varrho \cdot {}^2g_{ij}(x_0)$, i. e. mapping $f = f_1 \circ f_2 \circ f_3$ is conformal. □

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Authors' addresses

Hana Chudá

Tomas Bata University in Zlín, Department of Mathematics, Faculty of Applied Informatics, nám. T. G. Masaryka 5555, 760 01 Zlín, Czech Republic

E-mail address: chuda@fai.utb.cz

Mohsen Shiha

University of Homs, Department of Mathematics, Homs, Syria

E-mail address: mohsen_sheha@yahoo.com