ON THE LOGICAL STRUCTURE OF DE FINETTI'S NOTION OF EVENT

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Abstract. This paper sheds new light on the subtle relation between probability and logic by (i) providing a logical development of Bruno de Finetti's conception of events and (ii) suggesting that the subjective nature of de Finetti's interpretation of probability emerges in a clearer form against such a logical background. By making explicit the epistemic structure which underlies what we call *Choice-based probability* we show that whilst all rational degrees of belief must be probabilities, the converse doesn't hold: some probability values don't represent decision-relevant quantifications of uncertainty.

1. Introduction and motivation

This paper tackles the question as to whether the measure-theoretic concept of probability provides a satisfactory quantification of the uncertainty faced by an idealised "rational" agent who is presented with a well-defined choice problem. This is one of the most fundamental questions in the field of uncertain reasoning and as such it has been the focus of heated debates in various disciplines, from the foundations of probability and economic theory, to artificial intelligence. We do not aim at reproducing the many facets of this debate here. However, for the sake of putting our contribution into perspective, we begin by recalling briefly the relevant (to our purposes) interpretations of the concept of "probability" and the decision-theoretic argument supporting its use as a measure of "uncertainty".[1]

1.1. **Probability and uncertainty.** Frequenstist interpretations of probability take the notion of uncertainty as a primitive, and spell it out through the concept of *random-mass phenomena* (Pólya, 1954, Ch 14). The distinguishing feature of random-mass phenomena is that they are unpredictable in specific details, but predictable in the aggregate. A typical example is the sex ratio of newly born babies. The sex of the next baby to be born at a given hospital is unpredictable, but the country's ratio of males to females tends

[1]For a compact introduction to the interpretations of probability and the justification of its use in quantifying uncertainty, readers are referred to Williamson (2009).

to be very stable. The focus on random-mass phenomena leads naturally to defining probability as a theoretical limiting frequency. Under this interpretation, probability measures uncertainty as an objective, agent-independent, feature of the world. Subjective interpretations also take uncertainty to be a primitive notion, but refrain from assuming that uncertainty is an objective feature of the world. Hence uncertainty emerges as the psychological state of an agent who is facing a well-defined choice problem, say wether to buy or not an additional travel insurance prior to flying. Under this interpretation, probability is justified as a measure of a rational agent's degrees of belief by making reference to the agent's hypothetical choice behaviour (more on this in the next Section).

Measure-theoretic probability, on the other hand, introduces the concept of a probability measure from first principles –Kolmogorov's axioms– which do not refer directly to any underlying interpretation of uncertainty. Standard presentations of the subject (like, e.g. Billingsley, 1995; Ash, 1972) take the probability space Ω to contain all the possible outcomes of some unspecified experiment or observation, but insist that Ω is nothing but a set of points. It is therefore immaterial whether subsets of such points are interpreted as "repetitions" in a random-mass phenomenon (e.g. the sex ratio of newly born babies) or single cases (e.g. getting ill whilst abroad). This neutrality to interpretation may naturally suggest that measure-theoretic probability should be regarded as the unquestionable core of the mathematical representation of uncertainty, for it captures what two otherwise orthogonal interpretations of probability have in common. A consequence of this line of reasoning would then be that any remaining differences are not really of substantial consequence, but merely reflect personal philosophical taste. Whilst a mathematically unified perspective on uncertainty is no doubt appealing, we do believe that the interpretations of uncertainty are of substantial consequence for our formal models. Hence the main purpose of this paper is to show that a logical analysis of the foundations leads to discriminating among formal properties of probability functions *as* measures of uncertainty. Our conclusion will be that not all probability functions serve the purposes of quantifying decision-relevant uncertainty equally well. By articulating this in detail we will put ourselves in the footsteps of the subjective Bayesian tradition, especially Bruno de Finetti's.

From the point of view of de Finetti (1974), measure-theoretic probability offers no general justification for applying the *calculus* of probability to reasoning about the uncertainty of single, non-repeatable events. In addition, our ignorance of the boundary conditions of elementary "experiments" or "observations" make the very notion of a "repeatable event" dubious in the least.[2] In reaction to this, de Finetti points out that probability need not arise by making assumptions about the repetition of (independent) events, but is justified by imposing *coherence* to the degrees of belief of an agent who is in a state of uncertainty. Coherence captures all the logico-mathematical properties –essentially, additivity– that a probability function should satisfy to allow for an adequate representation of decision-relevant uncertainty, i.e. *rational degrees of belief*.[3] In short, subjective Bayesianism effectively reduces the meaning of probability to rational choice under uncertainty. This is a central point, which deserves further development.

1.2. **De Finetti's choice problem.** Throughout the paper we shall work with a finite propositional language L. $S(L)$ will denote the set of sentences recursively built up, as usual, from L.

Suppose $\Gamma \subseteq S(L)$. De Finetti's starting point is that (classical) logic can only help us distinguishing the conclusions which certainly (do not) follow from Γ from those which Γ licenses as *possible*. Uncertainty, in this picture, applies only to the domain of possibility. The *quantification* of uncertainty (i.e. the degree of belief that we attach to some relevant possibility being true) is motivated primarily by our need to make rational decisions in the face of such possibilities. Take again the travel insurance example. After careful reflection we all know that we might be in need of medical assistance abroad and our decision whether to buy the insurance that would cover for it clearly must depend on our estimate of "how serious" such a possibility is, *for us, in that specific trip, etc.*.

In an abstract (as we shall shortly see) framework in which rational degrees of belief are linked seamlessly to decision-making, a blatantly irrational decision must reveal irrational degrees of belief. This is the fulcrum de Finetti's justification for measuring uncertainty with probability, an argument anticipated (independently and with important differences) by Borel and Ramsey and brought to its full-fledged decision-theoretic representation by Savage (1972). Whilst de Finetti and Savage's theorems are clearly beyond dispute, the argument which rests on them –often referred to as the Dutch

[2]In the sex ratio example, for instance, it is very difficult to isolate the appropriate reference class for the ratio. The fact that we are observing one hospital gives us only partial information about the population.

[3]In addition, a simple condition like *exchangeability* would suffice to recover the mathematics of probability which naturally springs out of the frequentist definition. This is, in a nutshell, the import of de Finetti's celebrated Representation Theorem – see (Paris and Vencovská, 2014, Ch 9) for a probability logic version of the result and (Savage, 1972, Ch 4) for its decision-theoretic interpretation.

Book Argument– is not. To avoid unnecessary confusion, we will now recall the details of the set up which leads to identifying rational choice in de Finetti's choice problem, with rational degrees of belief. Our presentation follows closely Paris and Vencovská (2014), whilst making explicit all the assumptions originally made by de Finetti (1931).

The Argument consists of two parts. The first links the informal notion of degrees of belief to a numerical representation by identifying degrees of belief with willingness to bet on a suitably defined problem. The second step is the (formal) result to the effect that irrational degrees of belief (i.e. those leading to sure loss in the betting problem) are avoided exactly if degrees of belief are represented by probabilities.

1.2.1. *Belief as willingness to bet.* Let $\theta \in S(L)$, $p \in [0,1]$ and $S \in \mathbb{R}^+$. A gamble is a real-valued function on $S(L)$, that is to say a random variable which depends on how the elements of $S(L)$ are decided.[4] The choice between two gambles, $F(\cdot)$, and $A(\cdot)$ will play a central role in what follows.

The choice problem constructed by de Finetti for the purposes of his argument possesses the structure of a zero-sum game played by Bookmaker and Gambler. Bookmaker starts by choosing, for each sentence θ in a set $\Gamma \subseteq S(L)$, a number $p \in [0,1]$. Then, for a given sentence θ, Gambler must choose $S \in \mathbb{R}$ and one between $F_p(\theta)$ and $A_p(\theta)$, i.e. whether to "bet For or Against" θ. This leads to the payoff matrix for Gambler illustrated in Figure 1, where $v(\theta) = 1$ (resp. $v(\theta) = 0$) denotes the case where θ turns out to be true (resp. false).

	$v(\theta) = 1$	$v(\theta) = 0$
$F_p(\theta)$	$S(1-p)$	$-Sp$
$A_p(\theta)$	$-S(1-p)$	Sp

FIGURE 1. Gambler's payoff matrix.

Thus $F_p(\theta)$ is the payoff that Gambler secures by betting S on θ at "odds" p. Similarly, $A_p(\theta)$ is the payoff for betting S against θ at "odds" p.[5]

[4]This might give to some readers the impression of being unnecessarily general. This worry will be cleared as we keep laying down the details of de Finetti's choice problem. Our usage of "gamble" conforms to that of Walley (1991).

[5]Choosing $F_p(\theta)$ means, in the intended interpretation, that Gambler pays out Sp to Bookmaker. All monies, that is, are exchanged *before* (in an epistemic sense) the relevant events in the book are decided. Therefore the intended interpretation of betting here is that of horse races. We are grateful to Jon Williamson for drawing our attention to this point.

Remark 1.1. Note that if Gambler chooses to bet against θ, does so by paying to Bookmaker the (negative) amount $-S$, if θ is true. Thus, for given θ and p the choice between $F_p(\theta)$ and $A_p(\theta)$ amounts to choosing the sign of stake S, that is to say the "side" of the bet. Since the combined value of $F_p(\theta)$ and $A_p(\theta)$ is 0, Gambler's choice of a negative S unilaterally imposes a payoff-swap to Bookmaker.

De Finetti assumes that Bookmaker and Gambler are 'economically rational', i.e. prefer higher to lower payoffs. In addition they are idealised agents in the sense of not being subject to computational limitations. Under those assumptions, the above choice problem is sufficient to *reveal* an agent's willingness to bet on θ.

Let $p \in [0, 1]$. $F_p(\theta) \succsim A_p(\theta)$ abbreviates the expression "Gambler (weakly) prefers betting on θ to betting against it at odds p". It is immediate to note that if $p = 0$, betting on θ returns Gambler a gain of S if $v(\theta) = 1$ and a loss of 0 otherwise, so it must be that $F_p(\theta) \succsim A_p(\theta)$. Similarly, if $p = 1$, Gambler will always choose to bet against θ, i.e. $A_p(\theta) \succsim F_p(\theta)$, for doing otherwise would violate the assumption that she is 'economically rational'.

Suppose now $0 \leq p' < p \leq 1$ and that $F_p(\theta) \succsim A_p(\theta)$. Then it must still be the case that $F_{p'}(\theta) \succsim A_{p'}(\theta)$, for decreasing p can only increase Gambler's gain if θ is true, and decrease it otherwise. It follows that the values of p for which $F_p(\theta) \succsim A_p(\theta)$ form an initial segment of $[0, 1]$.

Let p_θ be the sup of the set $\{p \mid F_p(\theta) \succsim A_p(\theta)\}$. Note that p is fixed and p_θ can be read intuitively the highest price at which Gambler prefers betting "on" θ. In other words if Gambler's confidence in the occurrence of θ is greater than the price assigned by Bookmaker to θ in the book (i.e. p), then 'economically rational' Gambler will bet on θ because it is to her advantage. But if it is not, then Gambler will certainly find betting against θ to her advantage. Then we are justified in identifying Gambler's willingness to bet on θ with p_θ, for

- $F_p(\theta) \succsim A_p(\theta)$ if $p < p_\theta$
- $A_p(\theta) \succsim F_p(\theta)$ if $p_\theta < p$.

In the light of Remark 1.1 it is quite easy to anticipate that the only 'economically rational' solution to the interaction between Bookmaker and Gambler takes place when Bookmaker sets $p = p_\theta$.

1.2.2. *Coherence.* So far θ has been kept fixed. We are now interested in looking at how gambles on distinct events in $S(L)$ and with distinct stakes must (not) be combined. The leading intuition here is that a rational agent cannot accept a series of gambles which, taken jointly, may lead them to sure loss, i.e. loss under all valuations on $S(L)$. The goal is to show that conformity to the axioms of probability are necessary and sufficient for Bookmaker to guarantee that Gambler will not lead him into sure loss.

Let $\mathbb{V} = \{v \mid v \text{ is a } \{0,1\}\text{-valuation on } S(L)\}$ and let $v \in \mathbb{V}$. Recall that the game begins with Bookmaker choosing θ and p. Let p_θ be as above. If $p < p_\theta$ then Gambler will choose a positive S and pay p to secure a payoff $S(v(\theta)-p)$. If $p > p_\theta$, then Gambler will pick a negative S, thereby securing a payoff of $-S(v(\theta)-p)$.

De Finetti's choice problem is a device to define operationally *Bookmaker*'s degrees of belief in relevant uncertainties of interest. Let Γ be a finite subset of $S(L)$. A *book* is a map $B : \Gamma \to [0,1]$ which we interpret as the choice made by Bookmaker to assign a value in $[0,1]$ to every sentence in Γ.

Such values are often referred to as the "betting odds" for the events. The identification of degrees of belief with willingness to bet in the choice problem described in the previous section leads us to identifying the (ir)rationality of Bookmaker with the properties of the books he writes. In the light of this, the obvious property to be required is that the book B (on Γ) should not be *incoherent*, i.e. it should not expose Bookmaker to the logical possibility of sure loss. This is formalised by the following definition.

Definition 1.1. Let $\Gamma = \{\theta_1, \dots, \theta_k\}$ be a finite subset of $S(L)$. A book $B : \Gamma \to [0,1]$ is *coherent* iff for no $S_1, \dots, S_k \in \mathbb{R}$, Bookmaker's balance

$$\textstyle\sum_{i=1}^{k} S_i(B(\theta_i) - v(\theta_i))$$

is negative for all valuations $v \in \mathbb{V}$.

As shown by (de Finetti, 1931), this criterion is necessary and sufficient for the existence of a finitely additive measure P on the boolean algebra of formulas, such that $P(\theta_i) = B(\theta_i)$ for all $i = 1, \dots, k$.

Under the assumption that Bookmaker is 'economically rational', an incoherent book reveals irrational degrees of belief because it exposes Bookmaker (who chose it) to the possibility of sure loss. Under the modelling assumption that the choice problem (writing the book) is an exhaustive description of Bookmaker's behaviour, willingness to incur sure loss is certainly to be equated with irrationality.

Definition 1.2. Say that p is *a fair price for gamble F* if the Gambler is indifferent between betting "on" or "against" θ at price p, i.e., if $F_p(\theta) \succsim A_p(\theta)$ and $A_p(\theta) \succsim F_p(\theta)$. Equivalently, when $p = p_\theta$.

This leads to the Dutch Book theorem:

Theorem 1.1 (de Finetti (1931))**.** *Incoherent books are avoided if and only if prices are fair.*

The identification of the intuitive notion of irrationality with the mathematically precise one of writing an incoherent book captured by Theorem 1.1 requires a number of abstractions on the nature of the choice problem. Those abstractions mark a clear separation between de Finetti's choice problem and real-world gambling. More specifically, de Finetti (1931) assumes that the game takes place under the following contractual obligations

Completeness: The choice made by Bookmaker is forced for (boolean) combinations of gambles and, after the book has been written, Bookmaker is forced to accept all of a potentially infinite number of transactions with Gambler.

Swapping: After reading the published book, Gambler bets by paying to Bookmaker a real-valued stake of her choice. Since Gambler, as remarked above, can choose *negative* stakes (betting negative money), she can unilaterally impose a payoff-matrix swap to Bookmaker.

Rigidity: The transactions between Gambler and Bookmaker correspond to a *small* amount of money (in some currency).

Completeness is justified by de Finetti (1931) on the grounds that it provides the following modelling constraints. Were Bookmaker allowed to refuse selling certain bets, his choice behaviour (i.e. his choice of odds for a particular book) could not be claimed to *reveal* his sincere degrees of belief on the relevant events, and as a consequence, the betting problem would fail its fundamental purpose of connecting a rational agent's degrees of belief to their willingness to bet. As he would retrospectively notice, the betting problem is a "device to force the individual to make conscious choices, releasing him from inertia, preserving him from whim" (de Finetti, 1974, p.76).

In the presence of Completeness, Swapping entails that Bookmaker's degrees of belief should be *fair* betting odds. For suppose Bookmaker were to publish a book with non-zero expectation. Then he could be forced into sure loss by Gambler, who would put a negative stake on the book. Note that the abstraction leading to fair betting odds is justified only if the agents involved are *idealised*. This amounts to saying that Bookmaker confronts an individual who will exploit every logical possibility of leading him to sure loss, no matter how computationally demanding this might be.

Finally, Rigidity is a technical assumption which de Finetti endorses in order to avoid the potential complications arising from the diminishing marginal utility of money (de Finetti, 1974, p.77-78). By requiring payments to be small, he can effectively dispense with the notion of utility.

1.3. **Events and the epistemic structure of the betting problem.** Besides the above contractual obligations, the Dutch Book Argument rests on an *epistemic structure* which de Finetti mentions only in passing in his major contributions to this topic de Finetti (1931, 1937, 1974). A more direct, albeit very informal, reference to the point appears in de Finetti (2008). For reasons that will be apparent in a short while, the underlying epistemic structure of the betting problem is fundamental to understanding the notion of *event*.

An event is, for de Finetti, any random variable which takes values in the binary set and which, in addition, satisfies the properties of being a *single* and *well-defined* case . Single is opposed to "repeatable", and this marks the clear separation between the subjectivist and the frequentist interpretations of probability recalled above. The second requirement has attracted less attention despite the crucial role it plays in making probability subjective.

For de Finetti an event is "well-defined" when it stands for a question for which (a) neither Gambler nor Bookmaker have a definite answer –a question which pertains to the domain of possibility– and (b) Gambler and Bookmaker agree on the conditions under which this question will be answered:

> [T]he characteristic feature of what I refer to as an "event" is that the circumstances under which the event will turn out to be "verified" or "disproved" have been fixed in advance. (de Finetti, 2008, p.150)

This very informal characterisation echoes the characterisation de Finetti gives of random quantities –of which events are special cases. A random quantity is a "well-determined" unknown, namely one which is so formulated as "to rule out any possible disagreement on its actual value, for instance, as it might arise when a bet is placed on it." (de Finetti (1974), Section 2.10.4).

In addition to the above, de Finetti remarks that events must be the bearers of "genuine" uncertainty:

> I call "event" whatever is the object of an explicit question or curiosity. In other words, an event is something which has been previously figured out and subsequently checked in order to see whether it took place or not. (de Finetti, 2008, p. 151) (p. 151)

The formalisation of this idea will be the main object of our investigation for the rest of this paper.

In order to address the question as to whether a sentence $\theta \in S(L)$ is "well-defined" in the above sense –and hence it can be said to represent an event– we suppose that Gambler finds herself in a certain epistemic state $w \in W$ which is defined as a *partial* valuation on $S(L)$. We assume that epistemic states are dynamic in the sense that some propositional variables which are not decided at w may be decided at future states, until all propositional variables in L are eventually decided. Yet not all epistemic states in W may be accessible to Gambler at w. This allows for the possibility that Gambler may never be in a position to ascertain whether θ is decided positively or negatively. Similarly for Bookmaker.

This background allows us to put forward a definition of *events* which is relative to the *state of information* w of the individuals involved in the choice problem. Let $\theta \in S(L)$. We say that θ is

- a *w-fact* if the truth value of θ is decided at w;
- a *w-event* if it is not a *w-fact*, and every $w' \in W$ which determines the truth value of θ (up to redundancy) is w-accessible;
- *w-inaccessible* if no state w' which decides θ is accessible from w.

Section 4 will be devoted to a formalisation of the above which in turn will lead us to introduce *choice-based probabilities*, as partial functions

$$\mathrm{Cbp} : \mathcal{S}(L) \to [0, 1]$$

satisfying, among others, the following constraints

$$\mathrm{Cbp}(\theta) = \begin{cases} \epsilon \in \{0, 1\}, \text{ if } \theta \text{ is a fact} \\ 0, \text{ if } \theta \text{ is inaccessible} \\ x \in [0, 1], \text{ if } \theta \text{ is an event} \\ \text{undefined otherwise.} \end{cases}$$

The epistemic structure implicit in the betting framework clearly builds on the presupposition that at the time of betting, Bookmaker *and* Gambler ignore the truth value of the event on which they are betting, i.e. they agree that the truth value of θ is currently unknown. Hence the event belongs to the domain of possibility. Yet, for the bet to be meaningful, i.e. payable at all, the agents must also agree on the conditions which will *decide* the truth value of θ. This implies that a betting interpretation of probability is meaningful only for sentences which are undecided at the time of betting, but whose truth values will eventually be accessible to the agents. Now, w-inaccessible sentences are certainly well-formed formulas escaping this restriction, so probability functions defined on them cannot have a decision-relevant interpretation.

Our central result is a refinement of the classical representation theorem for probability functions (Theorem 2.1 below). Before doing that we show that whilst all coherent choice-based probabilities are indeed probability functions, *probabilities which are defined on sentences which are not events can coherently only be given trivial values*. Trivial, in this context, means one of two things. Either a sentence can (coherently) be given only its truth value (and this characterises betting on facts), or it should be given 0. This means that, given an epistemic state, the "uncertainty mass" must be concentrated only on events and on facts. Since events and facts are defined relative to the agents' own epistemic state, this determines to a crucial extent the subjective nature of their rational degrees of belief.

As a conclusion to our introductory remarks, let us pause for a second to appreciate that the epistemic structure of de Finetti's choice problem leads inevitably to a subjective interpretation of probability. For whether a sentence qualifies as an *event* depends on the *state of information* of the individuals involved in the betting problem. Compare this with the logical, measure-theory inspired, characterisation of probability functions which is derived under the tacit assumptions that the agent's state of information is empty and that all future "states of information" will be accessible, that is to say, that the set of events coincides with the algebra of sentences of a language L. By relaxing both assumptions our framework will lead to isolating the subset of all probability functions defined over a logical language that bear a meaningful interpretation with respect to de Finetti's choice problem.

2. Formal preliminaries

Recall that $L = \{p_1, \ldots, p_n\}$ is a finite set of propositional variables, and $S(L) = \{\theta, \phi, \ldots\}$ denotes the set of sentences built as usual from L in the language of classical propositional logic. Denote by AT^L be the set of maximally elementary conjunctions of L, that is the set of sentences of the form $\alpha = p_1^{\epsilon_1} \wedge p_2^{\epsilon_2} \wedge \ldots \wedge p_n^{\epsilon_n}$, with $\epsilon_i \in \{0, 1\}$ and where $p_i^1 = p_i$ and $p_i^0 = \neg p_i$, for $i = 1, \ldots, n$.

Since L is finite, the Lindenbaum algebra $\mathbf{Lind}(L)$[6] on L is a finite Boolean algebra and hence it is atomic, with atoms corresponding to AT^L. In addition, AT^L is in 1-1 correspondence with the set $\mathbb{V}$ of (classical) valuations on L. This implies that there is a unique valuation satisfying $v(\alpha) = 1$ namely $v_\alpha(p_i^{\epsilon_i}) = \epsilon_i$ for $1 \leq i \leq n$. Conversely, given a valuation $v \in \mathbb{V}$ there exists a unique atom $\alpha \in AT^L$ such that $v(\alpha) = 1$. Now let

$$M_\theta = \{\alpha \in AT^L \mid \alpha \models \theta\},$$

where $\models$ denotes the classical Tarskian consequence. Since there exists a unique valuation satisfying α, say v_α, by definition of $\models$ it must be the case that $v_\alpha(\theta) = 1$. Thus

$$M_\theta = \{\alpha \in AT^L \mid v_\alpha(\theta) = 1\}.$$

This framework is sufficient to provide a very general representation theorem for probability functions.

Theorem 2.1 (Paris 1994)**.**

(1) *Let P be a probability function on $S(L)$.*[7] *Then the values of P are completely determined by the values it takes on $AT^L = \{\alpha_1, \ldots, \alpha_J\}$, as fixed by the vector*

$$\langle P(\alpha_1), P(\alpha_2), \ldots, P(\alpha_J)\rangle \in \mathbb{D}^L = \{\vec{a} \in \mathbb{R}^J \mid \vec{a} \geq 0, \sum_{i=1}^{J} a_i = 1\}.$$

(2) *Conversely, fix $\vec{a} = \langle a_1, \ldots, a_J\rangle \in \mathbb{D}^L$ and let $P' : S(L) \to [0, 1]$ be defined by*

$$P'(\theta) = \sum_{i:\alpha_i \in M_\theta} a_i.$$

Then P' is a probability function.

In words, Theorem 2.1 shows that every probability function arises from distributing the unit mass of probability across the $J = 2^n$ atoms of the Lindenbaum algebra generated by $L = \{p_1, \ldots, p_n\}$.

Our goal is to refine this result by isolating a class of sentences on which, we argue, there should be no distribution of "epistemically significant" mass.

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[6] Recall that the Lindenbaum algebra $\mathbf{Lind}(L)$ over L is the quotient set $S(L)/\equiv$, where $\equiv$ is the logical equivalence relation (defined as $\theta_1 \equiv \theta_2$ iff $\models \theta_1 \leftrightarrow \theta_2$), with the operations induced by the classical conjunction, disjunction and negation connectives.

[7] $P : S(L) \to [0, 1]$ is a probability function on sentences if (i) $P(\top) = 1$ where $\top$ denotes any tautology, (ii) $P(\theta_1 \vee \theta_2) = P(\theta_1) + P(\theta_2)$ if $\models \neg(\theta_1 \wedge \theta_2)$, and (iii) $P(\theta_1) = P(\theta_2)$ if $\models \theta_1 \leftrightarrow \theta_2$.

More specifically, we aim at building a framework in which those probabilities which bear a meaning as *betting quotients* can be formally distinguished from those which do not. As illustrated informally in the previous section, we will achieve this by providing a rigorous definition of de Finetti's notion of *events*, which will be distinguished from the related notion of *facts* and *inaccessible* sentences.

3. Epistemic states and partial information

In what follows, we denote subsets of $S(L)$ by capital Greek letters $\Gamma, \Delta, \ldots$, and the classical Tarskian consequence is denoted by either $\models$ or Cn depending on whether its relational or operational definition is more suited to the specific context. Recall that a (total, classical) valuation is a function $v : L \to \{0, 1\}$ which extends uniquely to the sentences in $S(L)$ by truth-functionality. A total valuation represents a fully specified epistemic state since it allows agents to decide the truth-value (either 1 or 0) of any sentence in $S(L)$. However, an epistemic state determined by a set Γ of sentences (the ones known to be true), permits an assignment of truth-values only to some subset of sentences. More precisely, each Γ uniquely determines a three-valued map on $S(L)$, $e_\Gamma : S(L) \to \{0, 1, u\}$, defined as

$$e_\Gamma(\theta) = \begin{cases} 1 & \text{if } \theta \in Cn(\Gamma), \\ 0 & \text{if } \neg\theta \in Cn(\Gamma), \\ u & \text{otherwise.} \end{cases} \tag{1}$$

where the value u reads as *unknown*.

Notice that partial valuations are not truth-functional. Note also that, if $\Gamma \subseteq \Gamma'$ then $Cn(\Gamma) \subseteq Cn(\Gamma')$. From now on, we will say that a mapping $e : S(L) \to \{0, 1, u\}$ is a *partial valuation* whenever there exists $\Gamma \subseteq S(L)$ such that $e = e_\Gamma$.

Given two partial valuations e, e', we say that e' extends e, written $e \subseteq e'$, when the class of formulas which e sends into $\{0, 1\}$ is included into that one which e' sends into $\{0, 1\}$. Note that if $e = e_\Gamma$ and $e' = e_{\Gamma'}$ then

$$e \subseteq e' \Leftrightarrow \Gamma \subseteq \Gamma'. \tag{2}$$

By a *theory* we mean a deductively closed subset of $S(L)$. So, Γ is a theory if and only if $Cn(\Gamma) = \Gamma$. We denote the set of theories on L by $\mathbf{T}$. Let us finally recall that a theory $\Gamma \in \mathbf{T}$ is *maximally consistent* iff for every $\theta \in S(L)$, either $\Gamma \models \theta$, or $\Gamma \models \neg\theta$. Note also that for any maximally consistent $\Gamma \in \mathbf{T}$, there exists a (total) valuation $v \in \mathbb{V}$ such that for all $\theta \in S(L)$, $e_\Gamma(\theta) = v(\theta)$.

Definition 3.1 (Determined sentences)**.** We say that $\Gamma \subseteq S(L)$ *determines* $\theta \in S(L)$, written $\Gamma \gg \theta$, if and only if, for any propositional variable p_i appearing in θ, $e_\Gamma(p_i) \in \{0, 1\}$.

Definition 3.2 (Decided sentences)**.** We say that $\Gamma \subseteq S(L)$ *decides* $\theta \in S(L)$, written $\Gamma \rhd \theta$ if and only if $e_\Gamma(\theta) \in \{0, 1\}$.

The difference between the two notions can be rephrased by noting that θ may be decided even if not all of its propositional variables are determined (for instance, if it is a disjunction with at least a true disjunct). Conversely, it is immediate to see that for all $\Gamma \subseteq S(L)$ and $\theta \in S(L)$, if $\Gamma \gg \theta$ then $\Gamma \rhd \theta$. Furthermore, as remarked above, if $\Gamma \in \mathbf{T}$ is maximally consistent, then $\Gamma \gg \theta \Leftrightarrow \Gamma \rhd \theta$. The following are also immediate consequences of the definitions.

Proposition 3.1. *For all $\Gamma \subseteq S(L)$, and for all $\theta, \varphi \in S(L)$, the following hold:*

(1) $\Gamma \gg \theta$ *iff* $\Gamma \gg \neg\theta$; $\Gamma \rhd \theta$ *iff* $\Gamma \rhd \neg\theta$.
(2) *If* $\Gamma \rhd \theta$, *and* $\Gamma \rhd \varphi$, *then* $\Gamma \rhd \theta \circ \varphi$ *for all* $\circ \in \{\wedge, \vee, \rightarrow\}$.
(3) *If* $\Gamma \rhd \theta$, $\Gamma \not\rhd \varphi$, *and* $e_\Gamma(\theta) = 0$ *then* $\Gamma \not\gg \theta \circ \varphi$ *for every* $\circ \in \{\wedge, \vee, \rightarrow\}$, *but* $\Gamma \rhd \theta \wedge \varphi$ *and* $\Gamma \rhd \theta \rightarrow \varphi$, *and in particular* $e_\Gamma(\theta \wedge \varphi) = 0$, $e_\Gamma(\theta \rightarrow \varphi) = 1$.
(4) *If* $\Gamma \rhd \theta$, $\Gamma \not\rhd \varphi$, *and* $e_\Gamma(\theta) = 1$ *then* $\Gamma \not\gg \theta \circ \varphi$ *for every* $\circ \in \{\wedge, \vee, \rightarrow\}$, *but* $\Gamma \rhd \theta \vee \varphi$, $\Gamma \rhd \varphi \rightarrow \theta$ *and* $\Gamma \rhd \theta \rightarrow \varphi$, *and in particular* $e_\Gamma(\theta \vee \varphi) = e_\Gamma(\varphi \rightarrow \theta) = 1$.

We can associate naturally an agent's state of information with

(1) the sentences which are decided for the agent at that state
(2) those sentences which are undecided at the current state, but that can be decided in a future, reachable (for that agent) state.

The formalisation of this is an attempt to capture de Finetti's rather elusive remark to the effect that the "conditions of verification" of an event are known in advance to the agents. In doing this we will focus on decided rather than on determined sentences.

4. Information frames: facts, events and inaccessible sentences

4.1. **Information frames.**

Definition 4.1 (Information frame)**.** An *information frame* $\mathcal{F}$ is a pair $\langle W, R \rangle$ where W is a non-empty subset of partial valuations defined as in Equation (1) and R is a binary transitive relation on W.

Remark 4.1. Since each partial valuation is uniquely determined by a $\Gamma \subseteq S(L)$, we can freely use $w_1, w_2, \dots$ to denote either subsets of $S(L)$ or their associated partial valuations, depending on which interpretation suits best the specific context. As a consequence of Equation (2) the inclusion $w \subseteq w'$ is always defined.

We interpret $w_i \in W$ as the basic component of an agent's *state of information*, i.e. the sentences (equivalently, the partial valuation) which capture all and only the information available to an agent who finds itself in state w_i. Under this interpretation the relation R models the agent's *possible transitions* among information states. For reasons that will soon be apparent, we always require R to be transitive. As more structure is needed, further restrictions on R will be considered.

Definition 4.2. Let $\mathcal{F} = \langle W, R \rangle$ be an information frame. We say that $\mathcal{F}$ is

(1) *Monotone* if $(w, w') \in R$ implies $w \subseteq w'$.
(2) *Complete* if $w \subseteq w'$ implies $(w, w') \in R$.

Under our interpretation, monotonicity captures the idea that agents can only learn new information, but never "unlearn" the old one. In addition, monotonicity implies that the dynamics of information is stable in the sense that once a formula is either determined or decided at state w (i.e. it is given a binary truth-value), this remains fixed at any information state accessible from w. Hence if $w \rhd \phi$, there cannot exist $(w, w') \in R$ such that $w' \not\rhd \phi$. Moreover, by monotonicity, the truth-value of ϕ in w coincides with the truth-value of ϕ in w'. Completeness ensures that the agent will learn all the possible consistent refinements to its current information state. So, if $(w, w') \notin R$, there exists θ such that $w' \rhd \theta$ and $w \rhd \neg\theta$. Finally, note that if $\mathcal{F}$ is monotonic and complete then obviously R coincides with set-inclusion among states (equivalently, sets of sentences).

4.2. **Facts, events and inaccessible sentences.** We are now in a position to give formal definitions of facts, events and inaccessible sentences in monotone information frames.

Definition 4.3. Let $\langle W, R \rangle$ be a monotone information frame, let $w \in W$, and let $\theta \in S(L)$. We say that θ is a *w-fact* if $w \rhd \theta$.

On the other hand, if $w \not\rhd \theta$, we say that θ is:

- a *w-event* if for every (total) valuation v extending w there exists w' with $(w, w') \in R$ such that $w' \rhd \theta$ and $w'(\theta) = v(\theta)$.

- *w-inaccessible* if for every (total) valuation v and every world w' such that $w'(\theta) = v(\theta)$, $(w, w') \notin R$.

We shall respectively denote by $\mathcal{F}(w)$, $\mathcal{E}(w)$ and $\mathcal{I}(w)$ the class of w-facts, w-events, and w-inaccessible sentences, for some information frame $\langle W, R\rangle$ and some $w \in W$.

In addition, we shall denote by $\mathcal{C}(w)$ the class $\mathcal{F}(w) \cup \mathcal{E}(w) \cup \mathcal{I}(w)$. The class $\mathcal{C}(w)$ collects, for a given information frame $\mathcal{F}$ and a state w, all those elements of $S(L)$ which are well-defined in the choice-based sense illustrated in Section 1.2 above.[8] As we will shortly see $\mathcal{C}(w)$ is, in general, strictly contained in $S(L)$ and hence there exist elements of $S(L)$ which do not meet de Finetti's criterion of being well-defined. The following example clarifies the idea.

Example 4.1. Consider a Turing Machine TM and a finite input x. Let θ to be the *halting problem statement*:

$$\text{“}TM(x) \text{ will stop”}.$$

Let w be a state in which the agents know nothing about $TM(x)$. Suppose now that Bookmaker is writing a book involving θ. The undecidability of the halting problem forces any information frame $\mathcal{F}$ which claims to be adequate for this bet, to make inaccessible all those states in which $\neg\theta$ (i.e. "$TM(x)$ will run forever") is true. Let us assume that Bookmaker and Gambler consider those states in which θ is accessible. Let us call $W_\downarrow$ and $W_\uparrow$ the disjoint sets of partial states in which, respectively, $TM(x)$ stops and hence $w'(\theta) = 1$ for all $w' \in W_\downarrow$ and $TM(x)$ does not stop, i.e. $w''(\neg\theta) = 1$ for all $w'' \in W_\uparrow$. Then $\theta \notin \mathcal{C}(w)$. To see this note that

(1) Clearly θ is not a w-fact since $w \not\triangleright \theta$,
(2) In order for θ to be a w-event, by definition, every partial state deciding θ should be accessible from w. Yet every $w'' \in W_\uparrow$ is not accessible, hence θ is not a w-event.
(3) Finally, θ is not w-inaccessible either. In fact, in order for θ to be w-inaccessible, every partial state deciding θ should be inaccessible, whilst every $w' \in W_\downarrow$ is w-accessible in $\mathcal{F}$.

Notice that if every $w' \in W_\downarrow$ were w-inaccessible, θ *would be* in $\mathcal{I}(w)$. □

[8] It proves to be particularly hard to provide natural conditions under which $S(L)$ is partitioned by facts, events and inaccessible formulas. We hope to succeed in doing this in further work.

The following proposition sums up some key properties of the sets $\mathcal{F}(w)$, $\mathcal{E}(w)$ and $\mathcal{I}(w)$.

Proposition 4.1. *Let $\langle W, R\rangle$ be a monotone information frame, and let $w \in W$. Then the following hold:*

(1) *The structure $\langle \mathcal{F}(w), \wedge, \neg, \bot\rangle$ is a Boolean algebra.*
(2) *If w is a total valuation, then $S(L) = \mathcal{F}(w)$, while if $w = \emptyset$ is the empty valuation, then $\mathcal{F}(w) = \emptyset$.*
(3) *If $\langle W, R\rangle$ is complete, then $\langle \mathcal{E}(w), \wedge, \neg, \bot\rangle$ is a Boolean algebra.*
(4) *If $\langle W, R\rangle$ is complete, then for all $w \in W$, $S(L) = \mathcal{F}(w) \cup \mathcal{E}(w)$. Therefore, in particular, if $\langle W, R\rangle$ is complete, then $\mathcal{I}(w) = \emptyset$.*
(5) *If $\mathcal{I}(w) \neq \emptyset$, then for every w' such that its corresponding valuation is total, $(w, w') \notin R$.*

Proof. (1) follows by Proposition 3.1 (parts (1) and (2)). Also the claim (2) is also immediate.

In order to show (3), let $\langle W, R\rangle$ be a complete and monotone information frame, and let $\theta, \varphi \in \mathcal{E}(w)$. Let us assume by way of contradiction, that $\theta \wedge \varphi \notin \mathcal{E}(w)$. Then, since clearly $w \not\triangleright \theta \wedge \varphi$, it means that there exists a total valuation V which extends w, and a $w' \supseteq w$ such that $w' \triangleright \theta \wedge \varphi$, but $(w, w') \notin R$, against the completeness of $\langle W, R\rangle$.

(4) Assume that $\langle W, R\rangle$ is complete, and let w be any state in W. Since $\langle W, R\rangle$ is monotone by hypothesis, $R = \subseteq$, and hence, for every $w' \in W$, $(w, w') \in R$ iff $w \subseteq w'$. Then, for every $\theta \in S(L)$, either $\theta \in \mathcal{F}(w)$, or $w \not\triangleright \theta$. In the latter case, for every total valuation V and each w' such that $V(\theta) = w'(\theta)$, if $w' \supseteq w$, then $(w, w') \in R$ (by monotonicity and completeness). Hence $\theta \in \mathcal{E}(w)$.

(5) Assume that $\theta \in \mathcal{I}(w)$. Then, since every total valuation $V \triangleright \theta$, by definition of inaccessible formula, the world w' whose corresponding valuation is V cannot be accessible from w. And hence the claim holds. $\square$

Note that, for an arbitrary information frame $\langle W, R\rangle$ and for a $w \in W$, it is not always the case that the class $\mathcal{E}(w)$ is closed under logical connectives and hence, in particular, $\langle \mathcal{E}(w), \wedge, \neg, \bot\rangle$ may not be a Boolean algebra. For this reason, within our framework, and in contrast with the classical setting, we shall avoid, in general, to speak about the *algebra of events*, while we shall more frequently refer to the *class of events*. The following example illustrates the point.

Example 4.2. Let $L = \{p, q\}$ with the following intuitive interpretation:

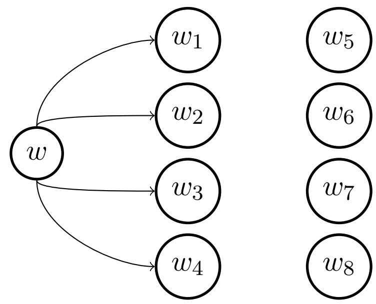

FIGURE 2. In accord with Heisenberg's principle the information frame is such that states w_1, w_2, w_3, w_4 are accessible from w, unlike those states in which both variables are decided, namely w_5, w_6, w_7 and w_8.

- p reads "the electron ε has position π";
- q reads "the electron ε has energy η".

Suppose further that our agent is in a state w such that the truth value of both p and q are unknown. In the usual quantum mechanics interpretation, an agent in w may either learn the position of ε, or its energy, but not both. This gives rise to the information frame depicted in Figure 2 where we may assume the following conditions hold:

$w_1 \rhd p,\ w_1 \not\rhd q$, and $w_1(p) = 0$; $\quad$ $w_2 \rhd p,\ w_2 \not\rhd q$, and $w_2(p) = 1$;
$w_3 \rhd q,\ w_3 \not\rhd p$, and $w_3(q) = 0$; $\quad$ $w_4 \rhd q,\ w_4 \not\rhd p$, and $w_4(q) = 1$;
$w_5 \rhd p, q$, and $w_5(p) = w_5(q) = 1$ $\quad$ $w_6 \rhd p, q$, and $w_5(p) = w_5(q) = 0$.
$w_7 \rhd p, q$, and $w_7(p) = 0, w_7(q) = 1$ $\quad$ $w_8 \rhd p, q$, and $w_8(p) = 1, w_8(q) = 0$.

It is immediate to see that p and q are w-events, but $p \wedge q$ is not. Indeed, due to the inaccessibility of, say w_5, the total valuation V mapping p and q to 1 has no correspondence in the worlds which are accessible from w. Analogously, $\neg p \wedge q$, $p \wedge \neg q$ and $\neg p \wedge \neg q$ are not w-events either. □

Examples 4.1 and 4.2 point out that in arbitrary monotone information frames one cannot ensure that the sets $\mathcal{F}(w)$, $\mathcal{E}(w)$, $\mathcal{I}(w)$ form a partition of $S(L)$. As we will discuss in further detail in the concluding section, it is surprisingly difficult to find natural properties on frames which ensure the rather desirable property that $S(L) = \mathcal{C}(w)$. When the information frame is also complete then we trivially get this condition since $\mathcal{I}(w) = \emptyset$.

5. De Finetti's choice problem revisited

We are now in a position to provide the epistemic refinement to the formalisation of de Finetti's betting problem, as anticipated in Section 1.2 above.

Definition 5.1. Let $\langle W, R\rangle$ be a monotone information frame, let $\Gamma = \{\theta_1, \ldots, \theta_k\} \subseteq S(L)$ and let $B : \Gamma \to [0, 1]$ be a book.

(1) For $w \in W$, the book B is said to be *w-coherent* iff
   (a) $\theta_1, \ldots, \theta_k \in \mathcal{C}(w)$,
   (b) there exist no $S_1, \ldots, S_k \in \mathbb{R}$ such that for every $w' \in W$ such that $(w, w') \in R$,
$$\sum_{i=1}^{k} Bal_i < 0$$
   where, for all $i = 1, \ldots, k$,
$$Bal_i = \begin{cases} S_i(B(\theta_i) - w'(\theta_i)), & \text{if } w' \rhd \theta_i \\ S_i B(\theta_i), & \text{otherwise.} \end{cases}$$
(2) The book B is said to be w-Dutch if B is not w-coherent.
(3) The book B is said to be a *w-book*, if every θ_i is a w-event.

Note that part (1.b) implies that inaccessible sentences do not enter the Bookmaker's balance. Hence even if Gambler placed bets on elements of $\mathcal{I}(w)$, Bookmaker would not be under a contractual obligation to pay anything to Gambler. Moreover, for w-books, being w-coherent is a notion that collapses to de Finetti's own definition of coherence (Definition 1.1). In fact if all the θ_i's are w-events, by definition, every state accessible from w decides θ and hence the book is w-coherent if and only if it is coherent. On the other hand, a w-coherent w-book can be extended to coherent w-books, as shown by the following result.

Theorem 5.1. *Let (W, R) be a monotone information frame, let $w \in W$ and let $B : \theta_i \in \Gamma \mapsto \beta_i \in [0, 1]$ be a w-coherent w-book. Let φ be a sentence in $\mathcal{C}(w)$ which is not a w-event and consider the book $B' = B \cup \{(\varphi, \alpha)\}$. Then:*

(1) *if φ is a w-fact, then B' is w-coherent iff $\alpha = w(\varphi)$,*
(2) *if φ is w-inaccessible, B' is w-coherent iff $\alpha = 0$,*

Proof: (1). ($\Rightarrow$). Suppose, to the contrary, that $\alpha \neq w(\varphi)$, and without loss of generality suppose that $w(\varphi) = 1$, so that $\alpha < 1$. Then, Gambler can get sure profit by betting a positive S on φ. Since the information frame is monotone, by the definition of w-book, $w(\varphi) = 1$ holds in every world w' accessible from w. Thus Gambler pays $S \cdot \alpha$ in order to surely receive S in any such w'. Conversely, suppose $w(\varphi) = 0$ and, for contradiction, that $\alpha > 0$. Then it is easy to see that Gambler can secure a win by swapping payoffs with Bookmaker, i.e. by betting a negative amount of money on φ.

($\Leftarrow$). Since B is w-coherent, there exists a w' accessible from w that decides every θ_i, and such that, for every $S_1, \ldots, S_n, S \in \mathbb{R}$,

$$\sum_{i=1}^{n} S_i(\beta_i - w'(\theta_i)) = 0.$$

Since φ is a w-fact and w' is accessible from w, it follows that $w'(\varphi) = w(\varphi) = \alpha$. Therefore for $S \in \mathbb{R}$, one also has

$$\left(\sum_{i=1}^{n} S_i(\beta_i - w'(\theta_i))\right) + S(\alpha - w'(\varphi)) = 0$$

and hence B' is also w-coherent.

(2). ($\Rightarrow$). Suppose $\alpha > 0$. By the "economic rationality" assumption described in Section 1.2 Gambler is willing to bet $S > 0$ on φ, i.e. to pay $\alpha \cdot S$ to Bookmaker. But since φ is w-inaccessible, this means sure loss for Gambler, contradicting her rationality.

($\Leftarrow$). Since B is w-coherent and since by hypothesis $\alpha = 0$, B' extends B in way which is trivial in the following sense: given any stakes $S_1, \ldots, S_n, S$ on B', the amount paid to Bookmaker is $\sum_i S_i\beta_i + S\alpha = \sum_i S_i\beta_i + 0$. Yet, since φ is w-inaccessible, in every world w' accessible from w, she will get $\sum_i S_i w'(\theta_i) + Sw'(\varphi)$. Now, since $\varphi \in \mathcal{I}(w)$, $w'(\varphi) = u$. The interpretation of the choice problem detailed in Section 1.2 forces us to put $Sw'(\varphi) = Su = 0$. For the Rigidity assumption (recalled at the end of Section 1.2.1 above) requires Su to be the *monetary* amount Gambler gets from Bookmaker after φ is decided. Hence the coherence of B' follows from the coherence of B. $\square$

The following example illustrates that w-coherent w-books cannot be characterised, in general, within the standard axiomatic framework for probability.

Example 5.1. Let $L = \{r, d\}$ which read as follows

> r : *"Tomorrow it will rain in Russell Square, London"*,
> d : *"Last winter α raindrops fell on Russell Square, London"*.

Let w_0 represent the agents' epistemic state in which neither r nor d are decided, i.e. $w_0 \not\triangleright r$, $w_0 \not\triangleright d$. Now, whilst they will certainly agree that tomorrow r will be easily decided, Bookmaker and Gambler will probably agree that d is impossible to decide. A suitable information frame to capture this is as follows. Let $W = \{w_0, \ldots, w_4\}$ be such that

$$w_1 \triangleright r,\ w_1 \not\triangleright d,\ w_1(r) = 1 \quad w_2 \triangleright r,\ w_2 \not\triangleright d,\ w_2(r) = 0$$
$$w_3 \triangleright d,\ w_3 \not\triangleright r,\ w_3(d) = 1 \quad w_4 \triangleright d,\ w_4 \not\triangleright r,\ w_4(d) = 0,$$

and let R be as in Figure 3.

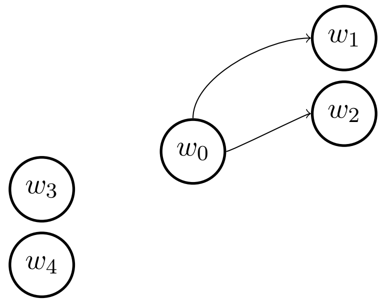

FIGURE 3. An information frame where, according to Example 5.1, the two states w_1 and w_2 in which r is decided are accessible from w_0 (the present state), while the two states w_3 and w_4 where d is decided are not.

Now, let $\mathcal{B}(X)$ be the Boolean algebra whose elements are Boolean combination of r and d and whose atoms are $r \wedge d$, $\neg r \wedge d$, $r \wedge \neg d$ and $\neg r \wedge \neg d$. Also, Let P be probability measure induced (for instance) by the uniform distribution on the above atoms.

Therefore, in particular, we have $P(r \wedge d) = 1/4$, $P(r) = P(d) = 1/2$ and hence, the book

$$B = \{(d, 1/2), (r, 1/2)\}$$

is coherent in the sense of de Finetti.

However, Theorem 5.1 shows that B is not w_0-coherent since the unique possible w_0-coherent value for w_0-inaccessible statement must be 0. $\Box$

The example illustrates that measure-theoretic probability is too wide a framework to represent the kind of uncertainty which motivates de Finetti's choice problem. The remainder of this paper is devoted to fleshing out a suitable refinement of probability functions $-$*choice-based probabilities*$-$ which are constrained by the notion of w-coherence.

6. Choice-based probability functions

Let $\langle W, R\rangle$ be a monotone information frame, and let $w \in W$ and let $X \subseteq S(L)$. Abusing the notation, let $S(X)$ denote the set of formulas $S(Var(X))$ built from the propositional variables appearing in the formulas of X. Finally, let $T_w = \{\varphi \in S(X) \mid w(\varphi) = 1\}$.

Definition 6.1. We say that a partial map

$$\mathrm{Cbp}_w : S(X) \to [0,1]$$

is a *w-choice based probability* with respect to X, if the following conditions are satisfied for any $\theta, \psi \in S(X)$:

C0. Cbp_w is defined on θ iff $\theta \in \mathcal{C}(w)$.
C1. if $\theta, \varphi, \theta \vee \varphi \in \mathcal{F}(w) \cup \mathcal{E}(w)$, and $T_w \models \neg(\theta \wedge \varphi)$, then $\mathrm{Cbp}_w(\theta \vee \varphi) = \mathrm{Cbp}_w(\theta) + \mathrm{Cbp}_w(\varphi)$.
C2. if $\theta \in \mathcal{F}(w)$, $\mathrm{Cbp}_w(\theta) = w(\theta)$.
C3. if $\theta \in \mathcal{I}(w)$, $\mathrm{Cbp}_w(\theta) = 0$.
C4. if $\theta, \psi \in \mathcal{C}(w)$ and $T_w \models \theta \leftrightarrow \psi$, then $\mathrm{Cbp}_w(\theta) = \mathrm{Cbp}_w(\psi)$.

Notice that although a w-choice based probability Cbp is defined over all formulas in the set of $\mathcal{C}(w)$, the additivity property C1 is only required on events and facts. Indeed, Cbp cannot be additive on $\mathcal{I}(w)$. To see this assume $\varphi \in \mathcal{I}(w)$. Then $\neg\varphi \in \mathcal{I}(w)$ as well. Now, by C3, we have $\mathrm{Cbp}(\varphi) = \mathrm{Cbp}(\neg\varphi) = 0$, but $\varphi \vee \neg\varphi \in \mathcal{F}(w)$ which moreover is a tautology. Hence by C2, it should be $\mathrm{Cbp}(\varphi \vee \neg\varphi) = 1$.

Definition 6.2. Let w, w' be partial valuations and let $X \subseteq S(L)$. We say that w and w' are *incompatible w.r.t.* X, written $w \perp_X w'$, if there exists $\theta \in X$ such that $w \rhd \theta, w' \rhd \theta$, and $w(\theta) \neq w'(\theta)$. We write $w \top_X w'$, to say that w, w' are not incompatible.

The intuition behind incompatible partial valuations is that, if $w \perp w'$, then w cannot be a refinement of w' since they do not assign the same truth value to the same decided sentence.

For fixed $w \in W$ and $X \subseteq S(L)$, let $d(X, w)$ be the set of all those accessible states $w' \in W$ deciding every (w-fact and) w-event in X. Formally,

$$d(X, w) = \{w' \in W : (w, w') \in R \text{ and } w' \rhd \theta \text{ for all } \theta \in X \cap (\mathcal{E}(w) \cup \mathcal{F}(w))\}.$$

Notice that we are excluding in the condition sentences of X which are neither w-events, nor w-facts, nor w-inaccessible.[9]

Finally let D be a set of states obtained from $d(X, w)$ in such a way that the states in $d(X, w)$ and those in D decide the same sentences, but for each $w', w'' \in D$, $w' \perp_X w''$. Sets D obtained in this way will be called *w-decisive for X*. More formally:

[9]Notice that $d(X, w)$ may be empty, see next Proposition 6.1.

Definition 6.3. Let $\langle W, R\rangle$ be a monotone information frame, let $w \in W$ and let $X \subseteq S(L)$ such that $d(X, w) \neq \emptyset$. A set $D \subseteq d(X, w)$ is called a *w-decisive set for X* whenever the following conditions hold:

- for all $\varphi \in X$, if $w' \rhd \varphi$ for all $w' \in D$, then $w'' \rhd \varphi$ for all $w'' \in d(X, w)$.
- for all $w', w'' \in D$, $w' \perp_X w''$.

We denote by $\mathcal{D}(X, w)$ the set of all the w-decisive sets for X.

Example 6.1. Let $L = \{p, q, r\}$, $X = \{p \vee q\}$ and let

$$W = \{w, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

be such that

(1) $w(p) = w(q) = u$, so $w \not\rhd p$ and $w \not\rhd q$;
(2) $w_1(p) = 1, w_1(q) = u$, $w_1(r) = u$ so $w_1 \rhd p$ and $w_1 \not\rhd q$;
(3) $w_2(p) = 0, w_2(q) = u$, $w_2(r) = u$, so $w_2 \rhd p$ and $w_2 \not\rhd q$;
(4) $w_3(p) = w_3(q) = 1$, $w_3(r) = u$, so $w_3 \rhd p$ and $w_3 \rhd q$;
(5) $w_4(p) = 1, w_4(q) = 0$, $w_4(r) = u$, so $w_4 \rhd p$ and $w_4 \rhd q$;
(6) $w_5(p) = 0, w_5(q) = 1$, $w_5(r) = u$, so $w_5 \rhd p$ and $w_5 \rhd q$;
(7) $w_6(p) = w_6(q) = 0$, $w_6(r) = u$, so $w_6 \rhd p$ and $w_6 \rhd q$;
(8) $w_7(p) = w_7(q) = w_7(r) = 0$, so $w_7 \rhd p$, $w_7 \rhd q$, $w_7 \rhd r$;
(9) $w_8(p) = w_8(q) = 0$, $w_8(r) = 1$, so $w_8 \rhd p$, $w_8 \rhd q$, $w_8 \rhd r$.

Let $\langle W, R\rangle$ be depicted as in Figure 4. Then $p \vee q$ is a w-event, because $w_3, w_4, w_5, w_6, w_7, w_8$ are the states whose associated partial valuations coincide with the restriction to $\{p, q\}$ of every total valuation V. Moreover $w_3, w_4, w_5, w_6, w_7, w_8$ are accessible from w. For a similar reason p is a w_1-event, and q is a w_2-event. Finally notice that whilst $p \vee q$ is a w_2-event, $p \vee q$ is a w_1-fact.

Moreover the following hold

(1) $D_1 = \{w_3, w_4, w_5, w_6\}$ is w-decisive for X, while
(2) $D_2 = \{w_1, w_5, w_6\}$ is not w-decisive for X because $(w, w_1) \notin R$;
(3) $D_3 = \{w_2, w_3, w_4\}$ is not w-decisive for X since $w_2 \not\rhd p \vee q$;
(4) $D_4 = \{w_1, w_5, w_7, w_8\}$ is again not w-decisive for X since $(w, w_1) \notin R$;
(5) $D_5 = \{w_3, w_4, w_5, w_6, w_7, w_8\}$ is not w-decisive for X because $w_6 \top_X w_7$, and $w_6 \top_X w_8$ as well;
(6) $D_6 = \{w_5, w_7, w_8\}$ is w_2-decisive for X.
(7) $D_7 = \{w_5, w_6\}$ is also w_2-decisive for X.

$\square$

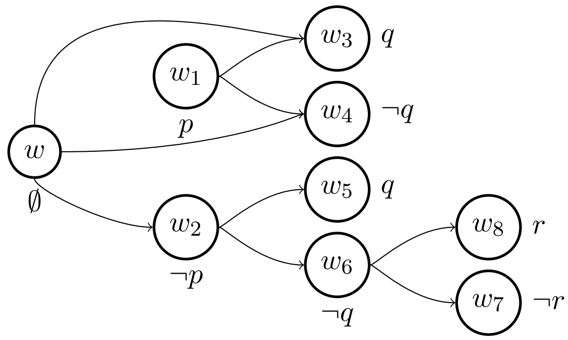

FIGURE 4. The information frame of Example 6.1. Note that each node represents a partial valuation in W and it is labelled (in various orientations) by the propositional variables it decides. Since the frame is transitive and monotone, we only label a state with the propositional variables which are decided at that state for the first time. A variable is undecided at w_i if it doesn't label any w_j with $j \leq i$.

The following properties will turn out to be useful for a description of Choice-based probability functions.

Proposition 6.1. *Let $\langle W, R\rangle$ be a monotone information frame, let $w \in W$ and let $X \subseteq S(L)$. Then the following properties hold true:*

(1) *For all $D \in \mathcal{D}(X, w)$ the following holds:*
 (a) *for all $w', w'' \in D$, $w' \perp_X w''$,*
 (b) *for every total valuation v, there exists $w' \in D$, such that $w'(\theta) = v(\theta)$ for all $\theta \in X \cap (\mathcal{F}(w) \cup \mathcal{E}(w))$.*
 (c) *If $\theta \in X$ is w-inaccessible then $\{w' \in D : w' \rhd \theta, (w, w') \in R\} = \emptyset$. In particular, if $X \cap \mathcal{I}(w) \neq \emptyset$ then $\mathcal{D}(X, w) = \{\emptyset\}$.*

(2) *Let $X' = X \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$. Then $\mathcal{D}(X, w) = \mathcal{D}(X', w)$.*

Proof. The properties in (1) follow by definition of w-decisive sets for X. Finally, (2) is a direct consequence of (c) of item (1) above. □

Remark 6.1. As we already stressed any decisive set D for a set X of formulas is made of incompatible partial valuations $w \in W$ deciding every formula in X. Decisive sets are then suitable domains for probability distributions. In particular, for $X \subseteq \mathcal{E}(w) \cup \mathcal{F}(w)$, although the information frame $\mathcal{F}$ might not allow us to reach some total valuations, it is easy to see that $\mathcal{D}(X, w) \neq \emptyset$, and therefore we are allowed to distribute a probability mass on at least a decisive set $D \in \mathcal{D}(X, w)$. Our next result shows that Choice-based probabilities arise in this way.

Theorem 6.1. *Let $\langle W, R\rangle$ be a monotone information frame, let $w \in W$ and let $X \subseteq S(L)$. Let $D \in \mathcal{D}(X, w)$ be w-decisive for X and let $\pi : D \to [0, 1]$ be a mapping satisfying*

$$\sum_{w \in D} \pi(w) = 1.$$

Then the map $\mathrm{Cbp} : S(X) \to [0, 1]$ defined for all $\theta \in \mathcal{C}(w) \cap S(X)$ by

$$\mathrm{Cbp}(\theta) = \sum_{w' \in D, w' \rhd \theta} \pi(w') \cdot w'(\theta),$$ [10]

is a w-choice based probability

Proof. Clearly Cbp ranges over $[0, 1]$, and it is likewise clear that Cbp is additive on $\mathcal{E}(w) \cup \mathcal{F}(w)$. Moreover, if $\models \theta \leftrightarrow \psi$, then for all $w' \in D$, $w' \rhd \theta$ iff $w' \rhd \psi$. Therefore, $\mathrm{Cbp}(\theta) = \mathrm{Cbp}(\psi)$. Hence C0, C1 and C4 of Definition 6.1 are immediately seen to hold.

To check C2, suppose $\theta \in \mathcal{F}(w)$. As we showed above, $w(\theta) = w'(\theta)$ for each $w' \in D$. So it suffices to check two cases. If $w(\theta) = 1$, then

$$\mathrm{Cbp}(\theta) = \sum_{w' \in D} \pi(w') \cdot w'(\theta) = \sum_{w' \in D} \pi(w') = 1 = w(\theta); \tag{3}$$

On the other hand, if $w(\theta) = 0$, then

$$\mathrm{Cbp}(\theta) = \sum_{w' \in D} \pi(w') \cdot 0 = 0 = w(\theta). \tag{4}$$

Therefore, $\mathrm{Cbp}(\theta) = w(\theta)$ for all $\theta \in \mathcal{F}(w)$, giving us $\mathrm{Cbp}(\top) = 1$, as required.

Finally C3. Let $\theta \in \mathcal{I}(w)$. In this case, from Proposition 6.1 (2), it follows that for no $w' \in D$, $w' \rhd \theta$, and hence $\mathrm{Cbp}(\theta) = 0$, completing the proof. □

The next result shows that Choice-based probability functions characterize w-coherence.

Theorem 6.2. *Let $\langle W, R\rangle$ be a monotone information frame. Let $X \subseteq S(L)$, and let $B : X \to [0, 1]$ be a book. Then the following are equivalent:*

(1) *B is w-coherent,*
(2) *There exists a w-choice based probability* Cbp *on $S(X)$ extending B.*

[10]We assume the sum is 0 in case the set of partial valuations w' satisfying the conditions $w' \in D$ and $w' \rhd \theta$ is empty.

Proof. (1) $\Rightarrow$ (2). If B is w-coherent, then so is the book B^- obtained by restricting B to the formulas in $X' = X \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$. Since X' does not contain w-inaccessible formulas, B^- is w-coherent and hence the obvious adaptation of de Finetti's Dutch Book theorem to our language shows that B' is w-coherent iff one can find a probability distribution π on D. Then the map Cbp_π defined as in Theorem (6.1) satisfies (2).

(2) $\Rightarrow$ (1). Let Cbp' the partial mapping on $S(L)$ defined by restricting Cbp to $\mathcal{E}(w)$. Then the claim easily follows from Theorem 5.1. $\square$

The above Theorem, together with Proposition 4.1 (4), shows that de Finetti's notion of coherence is a special case of w-coherence, which arises by imposing monotonicity and completeness to the information frame $\langle W, R\rangle$. More precisely, the following holds:

Corollary 6.1. *Let $\langle W, R\rangle$ be monotone and complete, with $w \in W$. Let $X \subseteq S(L)$, and let $B : X \to [0, 1]$. Then the following are equivalent*

(1) *B is w-coherent,*
(2) *B is coherent,*
(3) *There exists a w-choice based probability* Cbp *on $S(X)$ extending B,*
(4) *There exists a probability P on $S(X)$ extending B.*

Remark 6.2. In the light of the above Corollary 6.1, one can argue that Choice-based probability collapses to the usual notion of a probability measure whenever the agents who are engaging in de Finetti's betting problem have 'full access' to the sort of complete information provided by classical valuations on L. The fact that some valuations may not be accessible from a given state of information (i.e. that some sentences may not be w-events, for some $w \in W$ of interest) is ultimately what distinguishes Choice-based probability from its measure-theoretic counterpart. Consider again Example 4.1. Whilst it certainly makes "abstract" sense to ask the probability that a Turing Machine TM will halt on a given input x[11], there is no Choice-based probability which, coherently with the information frame described in the Example 4.1, will assign to the event "$TM(x)$ will stop" a positive value. Note that this does not lead to contradiction, but it clarifies how Choice-based probabilities are, in general, a strict refinement of probability measures. We will come back to a related point in the concluding section of the paper.

[11] G. Chaitin (cf. for instance Calude (2002)) introduces, for any Turing Machine TM the real number Ω_{TM} which is meant to represent the halting probability of TM. The number Ω_{TM} is clearly not-computable. We refer the interested reader to Calude (2002) for more details.

7. Choice-based probabilities as probability measures on quotient structures

We end this paper by comparing Choice-based probability functions, and their characterisation of w-coherence with the standard representation theorem for probability functions (Theorem 2.1 above.) In what follows we will always assume that the information frame we are working with is monotone and complete. Recall from Section 4.2 and Proposition 4.1, that among other things, this implies that (i) the accessibility relation R coincides with set-inclusion $\subseteq$ and that (ii) for all $w \in W$, $\mathcal{I}(w) = \emptyset$. Finally, recall that for monotone and complete frames, either $\theta \in \mathcal{E}(w)$ or $\theta \in \mathcal{F}(w)$, for all $\theta \in S(L)$. For this reason, instead of $S(L)$ as domain for a choice based probability, we work directly with the Lindenbaum algebra $\mathbf{Lind}(L)$ of formulas built up on the language L.

Let $w \in W$, and let $\mathcal{F}(w)$ be partitioned in $\mathcal{F}(w)^+$ and $\mathcal{F}(w)^-$ where for every $\theta \in \mathcal{F}(w)^+$, $w(\theta) = 1$ and for all $\psi \in \mathcal{F}(w)^-$, $w(\psi) = 0$. Then let

$$\mathcal{K}(w) = \mathcal{F}(w)^+ \cup \{\neg\psi : \psi \in \mathcal{F}(w)^-\}.$$

Notice that the set $\mathcal{K}(w)$ can be regarded as encoding the *knowledge* contained in the word w. By convention we identify each formula θ in $S(L)$ with its equivalence class (modulo equiprovability) $[\theta] \in \mathbf{Lind}(L)$, and hence each θ_i will be thought of as an element in the Lindenbaum algebra $\mathbf{Lind}(L)$. Let us denote by $\mathbb{I}(\mathcal{K}(w))$ the ideal of $\mathbf{Lind}(L)$ generated by $\mathcal{K}(w)$. In other words, let

$$\mathbb{I}(\mathcal{K}(w)) = \{\varphi \in \mathbf{Lind}(L) : \varphi \leq \bigvee\{\theta : \theta \in \mathcal{K}(w)\}\}.$$

Since $\mathcal{K}(w)$ will be always clear, we simply write $\mathbb{I}$ instead of $\mathbb{I}(\mathcal{K}(w))$. Since the quotient algebra $\mathbf{Lind}(L)/\mathbb{I}$ is finite, we denote by $At(\mathbf{Lind}(L)/\mathbb{I})$ the atoms of $\mathbf{Lind}(L)/\mathbb{I}$. More precisely, we denote the atoms of $\mathbf{Lind}(L)/\mathbb{I}$ by $a_1^{\mathbb{I}}, \ldots, a_k^{\mathbb{I}}$. Finally, by identifying each ideal of $\mathbf{Lind}(L)$ with its associated congruence (see (Burris et al., 1981, Theorem 3.5)), we denote by $[\theta]_{\mathbb{I}}$ the generic element of the quotient structure $\mathbf{Lind}(L)/\mathbb{I}$.

Lemma 7.1. *For every $w \in W$, $At(\mathbf{Lind}(X)/\mathbb{I}) \in \mathcal{D}(\mathbf{Lind}(L), w)$.*

Proof. (Sketch). Clearly, for each $a_i, a_j \in At(\mathbf{Lind}(X)/\mathbb{I})$, $a_i \perp_{\mathbf{Lind}(L)} a_j$. Moreover, to each atom $a_i \in At(\mathbf{Lind}(X)/\mathbb{I})$ we can associate a complete valuation as we explained in Section 2. Since the information frame we are considering is complete, all (total) valuations are accessible and hence the claim is settled. $\square$

The following lemma is straightforward.

Lemma 7.2. *Let A be an atomic Boolean algebra with atoms $\alpha_1, \ldots, \alpha_k$. If $\mathbb{I}$ is an ideal of A, and $\alpha_j \in \mathbb{I}$ for some j, then, in the quotient structure $A/\mathbb{I}$, it holds $[\alpha_j]_{\mathbb{I}} = [0]_{\mathbb{I}}$.*

In analogy with Theorem 2.1, let us define:

$$\mathbb{D}^{\mathbf{Lind}(L)/\mathbb{I}} = \{\vec{a} \in \mathbb{R}^k : k = |At(\mathbf{Lind}(L)/\mathbb{I})|, a_i \geq 0, \sum_{i=1}^{k} a_i = 1\}.$$

The following result offers a representation of Choice-based probabilities in terms of probability distributions on the atoms of suitably defined quotient structures.

Theorem 7.1. *For every $w \in W$, for every $\vec{a} \in \mathbb{D}^{\mathbf{Lind}(L)/\mathbb{I}}$, and for every $\theta \in \mathbf{Lind}(L)$, the map* Cbp *defined as*

$$\mathrm{Cbp}(\theta) = \sum_{i\colon \alpha_i \in M_\theta}^{k} a_i \tag{5}$$

is a Choice-based probability.

Conversely, for every Choice-based probability Cbp*, there exists a unique $\vec{a} \in \mathbb{D}^{\mathbf{Lind}(L)/\mathbb{I}}$ such that* Cbp *is defined by $\vec{a}$ through (5).*

Before proving the above theorem, let us notice that the function Cbp defined through (5) can be easily regarded as a probability measure P_{Bet} on the quotient structure $\mathbf{Lind}(L)/\mathbb{I}$, by setting $\mathrm{Cbp}(\theta) = P_{Bet}([\theta]_{\mathbb{I}})$.

Proof. Fix $w \in W$, let as usual $k = |At(\mathbf{Lind}(L)/\mathbb{I})|$, and, for the sake of a simpler notation, let us denote by $\alpha_1, \ldots, \alpha_k$ the atoms of $\mathbf{Lind}(L)/\mathbb{I}$. Let hence $\vec{a} \in \mathbb{D}^{\mathbf{Lind}(L)/\mathbb{I}}$, and let Cbp be defined as in (5). Then Cbp satisfies the following:

(1) If $\theta \in \mathcal{F}(w)$, then $\theta \in \mathbb{I}$, and hence from Lemma 7.2, $[\theta]_{\mathbb{I}} = [0]_{\mathbb{I}}$. Therefore, since Cbp is defined as a probability measure on the quotient structure $\mathbf{Lind}(L)/\mathbb{I}$, $\mathrm{Cbp}(\theta) = 0$.
(2) Clearly, if $\theta \in \mathcal{E}(w)$, by definition of Cbp, $\mathrm{Cbp}(\theta) \in [0, 1]$.

It follows from the monotonicity and completeness of $\langle W, R\rangle$ that $\mathcal{I}(w) = \emptyset$, and hence Cbp is a Choice-based probability.

Conversely, take any Cbp and let $\vec{a} = \langle a_1, \ldots, a_k\rangle \in \mathbb{R}^k$ be such that

$$a_i = \mathrm{Cbp}(\alpha_i), \quad i = 1, \ldots, k$$

where k denotes, as usual, the cardinality of $At(\mathbf{Lind}(L)/\mathbb{I})$. Then $a_i \in [0, 1]$, for $i = 1, \ldots, k$. Then the claim follows from Theorem 2.1 by observing that

$\mathrm{Cbp}(\alpha_i) = P_{Bet}([\alpha_i]_{\mathbb{I}})$, and that P_{Bet} is a probability measure on $\mathbf{Lind}(L)/\mathbb{I}$. □

Theorem 7.1 then generalises Theorem 2.1, which is recovered by adding the extra assumption that w corresponds to the empty valuation, i.e. ν_w is such that $\nu_w(p)$ undetermined for all $p \in V$.

Conversely it is easy to prove that, whenever w is such that its corresponding valuation ν_w is total (i.e. $\nu_w(p) \in \{0, 1\}$ for each $p \in V$), the unique possible w-choice based probability on $\mathbf{Lind}(L)$ is trivial and coincides with the canonical homomorphism $h_w : \mathbf{Lind}(L) \to \mathbf{Lind}(L)/\mathbb{I}$ since, in this particular case $\mathbf{Lind}(L)/\mathbb{I}$ is the two-element Boolean algebra $\mathbf{2}$.

8. Conclusions and future work

We fleshed out a logical framework which enabled us to show that some measure-theoretically sound probability values are trivial in the choice-based setting of subjective Bayesianism, as described in Section 1.2. We then argued that the restriction to the subclass of Choice-based probability functions developed in Section 6 arises naturally from the epistemic structure of events implicitly assumed by de Finetti (1931) in his operational definition of subjective probability. In addition, our formalisation of events (Definition 4.3) captures de Finetti's epistemological analysis to the effect that probability is the quantification of one's state of mind concerning genuine uncertainty, i.e. what pertains to the domain of what one coherently considers to be possible.

Whilst we focussed on the choice problem leading to the Dutch Book argument, it is interesting to ask if our framework can be applied also to the method of *scoring rules*.[12] Let us first recall a central observation by de Finetti (1981):

> it is clear that the condition of rejecting any Dutch Book cannot be violated in any fair betting situation, *but both competitors may be misled about the state of information of the other*.[13] For this reason betting, strictly speaking, does not pertain to probability but to the Theory of Games. Only under such a proviso can the argument be accepted.

In an attempt to keep the foundations of probability within the scope of Decision, rather than Game Theory, de Finetti developed the method of

[12] We are grateful to Gregory Wheeler for raising this point.

[13] Our emphasis.

"proper scoring rules". In a nutshell, the framework features a single agent who must assign probabilities to events of interest. In terms of the analysis put forward in this paper, it is natural to identify the agent with Bookmaker, whose task is to choose a point in $[0, 1]$ representing his degree of subjective belief in, say θ. Call this p_θ. The idea is to set up a device which gives Bookmaker an incentive to choose p_θ in such a way that this reflects his *sincere* degree of belief in θ, call this p^s_θ. In order to achieve this, de Finetti suggests that Bookmaker should be subject to a loss which amounts to the square of the (Euclidean) distance between his probabilistic forecast for θ and the truth-value that θ will eventually get in some $w \in W$. This loss function is also known as Brier's rule or score. A simple geometric argument shows that the expectation of loss under Brier's score is minimised when $p_\theta = p^s_\theta$. As shown in (de Finetti, 1974, Ch.3) the minimisation of loss under Brier's rule is equivalent to the criterion of avoiding sure loss in the Dutch Book setting recalled above. Now, it is intuitively clear that our restriction of events to the class of sentences that can be decided in all future developments of an agent's information states applies to Brier's scoring rule as well. To see this informally, note that if the forecast is not on an event (in the sense of Definition 4.3 above), Bookmaker is not effectively facing the prospect of an enforceable penalisation. For forecasting on an inaccessible sentence will make it impossible to compute the loss. Hence the restriction to elements of $\mathcal{E}(w)$ appears to be implicit in the method of scoring rules and appears not to depend on a particular choice of loss function. Further work is needed to turn this intuitive observation into a mathematical fact.

Besides putting de Finetti's analysis on a firm logical footing, we believe that the framework of Choice-based probabilities introduced in this paper will prove to be a fruitful tool for foundational clarification as well as formal advance in uncertain reasoning, broadly construed. We end the paper by sketching our vision for future research in this direction.

As our key motivation was to capture formally the informal characterisation of events put forward by de Finetti, we made a number of assumptions concerning the idealisation of the agents and the abstraction of the choice problem in accord with his version of the Dutch Book Argument. In addition, our refinement of the Argument investigated in detail in Section 5, depended on two assumptions on information frames, namely transitivity and monotonicity, reflecting two important idealisations on the nature of the agents. Further work will be devoted to exploring the significance, in terms of modelling uncertainty, of relaxing those assumptions. Relaxing transitivity amounts to making our agents capable of accessing only states which are immediately accessible from w_0. Modelling this sort of "short-sightedness" will shed interesting light on those problems in which accuracy is to be traded-off with speed. Reasoning based on heuristics and case-based reasoning are certainly cases in point. The relaxation of monotonicity would allow us to model cognitively limited agents who are subject to potentially imperfect recall. This strand of research could open fruitful interactions with the experimental literature on the effect of limited memory on logical inference.

In addition we will focus our future research on a slightly more general notion of information frame that we are now going to describe. Recall, that we defined the relation R featuring in information frames $\langle W, R\rangle$ in a binary way: either w^* is accessible from a state w, or it is not. In realistic scenarios however, it certainly makes sense to consider cases in which agents may attach a degree to w^* being accessible from w. This leads to defining $(w, w^*) \in R^\alpha$ (where $\alpha \in [0, 1]$) if they believe that the *probability* of reaching w^* from w is α. Notice that such a probability α would measure a higher order of uncertainty than the degree of belief the agent attaches to a w-event. This second-order uncertainty could be interpreted fruitfully as a measure of the reliability of the model, along the lines described in Hosni (2014).

Finally a glimpse at the applicability of our framework in the wider field of uncertain reasoning. We proved that whilst all Choice-based probabilities are normalised, monotonic and additive set-functions, the converse doesn't hold. Hence, the currently heated debate concerning the (in)adequacy of "probability" as a measure of rational belief under uncertainty, may greatly benefit from being framed in the context of Cbps. We claim that under the restrictions provided by w-coherence, the identification of "rational belief" with "probability" is beyond reasonable dispute. Normative shortcomings of Cbp's are thus to be found in the severe restrictions imposed by the abstraction of the underlying choice problem discussed in detail in Section 1.2 above. In future research we will investigate the relaxation of some of the abstraction of de Finetti's choice problem and its consequences for Cbp. The first such relaxation will follow the footsteps of Fedel et al. (2011) and drop the Swapping assumption recalled above. This will open to the investigation of *interval-valued* Choice-based probabilities.

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References

R.B. Ash. *Basic probability theory*, Dover Publications, 1972.

P. Billingsley. *Probability and Measure*. Wiley, 3rd edition, 1995.

S. Burris, and H.P. Sankappanavar *A course in Universal Algebra*. Springer-Velag, New York, 1981.

C. S. Calude. *Information and Randomness: An Algorithmic Perspective*. 2nd Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.

B. de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:289–329, 1931.

B. de Finetti. La prévision : ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré*, 7(1):1–68, 1937.

B. de Finetti. *Theory of Probability, Vol 1.* John Wiley and Sons, 1974.

B. de Finetti. *The Role of "Dutch Books" and of "Proper Scoring Rules."* British Journal for the Philosophy of Science, 32(1), 55–56.

B. de Finetti. Philosophical lectures on probability. Springer Verlag, 2008.

M. Fedel, H. Hosni, and F. Montagna. A logical characterization of coherence for imprecise probabilities. *International Journal of Approximate Reasoning*, 52(8):1147–1170, 2011.

T. Flaminio, L. Godo and H. Hosni. Zero-Probability and Coherent Betting: A Logical Point of View. In L.C. van der Gaag (Ed.), ECSQARU 2013, LNAI 7958, pp. 206-217, 2013.

H. Hosni Towards a Bayesian theory of second-order uncertainty: Lessons from non- standard logics. In S. O. Hansson (Ed.), *David Makinson on Classical Methods for Non-Classical Problems*, Outstanding Contributions to Logic Volume 3, Springer, 195–221, 2014

J.B. Paris. *The uncertain reasoner's companion: A mathematical perspective*. Cambridge University Press, 1994.

J.B. Paris and A. Vencovská. A model of belief. *Artificial Intelligence*, 64 (2):197–241, 1993.

J.B. Paris and A. Vencovská. *Pure Inductive Logic*. Cambridge University Press, 2014.

G. Parmigiani and L. Inoue. *Decision Theory: Principles and Approaches.* Wiley, 2009

G. Pólya. *Mathematics of Plausible Reasoning Volumes 1-2.* Princeton University Press, 1954.

L.J. Savage. *The Foundations of Statistics.* Dover, 2nd edition, 1972.

A. Turing. On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society*, Series 2, 42, pp. 230–265, 1936.

P. Walley. *Statistical Reasoning with Imprecise Probabilities*, Volume 42 of Monographs on Statistics and Applied Probability, Chapman and Hall, London 1991.

J. Williamson. Philosophies of probability. In A. Irvine, editor, *Handbook of the Philosophy of Mathematics.* Volume 4 of the Handbook of the Philosophy of Science, Elsevier, 2009.

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