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Locally adaptive density estimation on Riemannian manifolds

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Abstract

In this paper, we consider kernel type estimator with variable bandwidth when the random variables belong in a Riemannian manifolds. We study asymptotic properties such as the consistency and the asymptotic distribution. A simulation study is also considered to evaluate the performance of the proposal. Finally, to illustrate the potential applications of the proposed estimator, we analyse two real examples where two different manifolds are considered.

MSC: 62G07, 62G20

Keywords: Asymptotic results, density estimation, nonparametric, Riemannian manifolds.

1. Introduction

Let $X_1,\ldots,X_n$ be independent and identically distributed random variables taking values in $\mathbb{R}^d$ and having density function f. A class of estimators of f which has been widely studied since the work of Rosenblatt (1956) and Parzen (1962) has the form

$$f_n(x) = \frac{1}{nh^d}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h}\right),$$

where $K(u)$ is a bounded density on $\mathbb{R}^d$ and h is a sequence of positive number such that $h \to 0$ and $nh^d \to \infty$ as $n \to \infty$.

If we apply this estimator to data coming from long tailed distributions, with a small enough h to be appropriate for the central part of the distribution, a spurious noise ap-

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pears in the tails. With a large value of h for correctly handling the tails, we can not see the details occurring in the main part of the distribution. To overcome these defects, adaptive kernel estimators were introduced. For instance, a conceptually similar estimator of $f(x)$ was studied by Wagner (1975) who defined a general neighbour density estimator by

$$\widehat{f}_n(x) = \frac{1}{nH_n^d(x)} \sum_{j=1}^{n} K\left(\frac{x - X_j}{H_n(x)}\right),$$

where $H_n(x)$ is the distance between x and the k-nearest neighbour of x among $X_1, \ldots, X_n$, and $k = k_n$ is a sequence of non–random integers such that $\lim_{n\to\infty} k_n = \infty$. Through this adaptive bandwidth , the estimation in the point x has the guarantee that to be calculated using at least k points of the sample.

However, in many applications, the variables X take values on different spaces than $\mathbb{R}^d$. Usually these spaces have a more complicated geometry than the Euclidean space and this has to be taken into account in the analysis of the data. For example, if we study the distribution of the stars with luminosity in a given range it is natural to think that the variables belong to a spherical cylinder ($S^2 \times \mathbb{R}$) instead of $\mathbb{R}^4$. If we consider a region of the planet M, then the direction and the velocity of the wind in this region are points in the tangent bundle of M, that is a manifold of dimension 4. Other examples could be found in image analysis, mechanics, geology and other fields. They include distributions on spheres, Lie groups, among others, see for example Joshi et al. (2007), Goh and Vidal (2008). For this reason, it is interesting to study an estimation procedure of the density function that takes into account a more complex structure of the variables.

Nonparametric kernel methods for estimating densities of spherical data have been studied by Hall et al. (1987) and Bai et al. (1988). Pelletier (2005) proposed a family of nonparametric estimators for the density function based on kernel weighting when the variables are random objects valued in a closed Riemannian manifold. Pelletier's estimators are consistent with the kernel density estimators in the Euclidean case considered by Rosenblatt (1956) and Parzen (1962).

As we comment above, the importance of local adaptive bandwidth is well known in nonparametric statistics and this is even more true with data taking values in complex spaces. In this paper, we propose a kernel density estimator on a Riemannian manifold with a variable bandwidth defined by k-nearest neighbours.

This paper is organized as follows. Section 2 contains a brief summary of the necessary concepts of Riemannian geometry. In Section 2.1, we introduce the estimator. Uniform consistency of the estimator is derived in Section 3.1, while in Section 3.2 the asymptotic distribution is obtained under regular assumptions. Section 4 contains a Monte Carlo study designed to evaluate the proposed estimator. Finally, Section 5 presents two example using real data. Proofs are given in the Appendix.

2. Preliminaries and the estimator

Let (M,g) be a d-dimensional Riemannian manifold without boundary. We denote by d_g the distance induced by the metric g. With $B_s(p)$ we denote a normal ball with radius s centred at p. The injectivity radius of (M,g) is given by $inj_gM = \inf_{p\in M} \sup\{s \in \mathbb{R} > 0 : B_s(p)$ is a normal ball$\}$. It is easy to see that a compact Riemannian manifold has strictly positive injectivity radius. For example, it is not difficult to see that the d-dimensional sphere S^d endowed with the metric induced by the canonical metric g_0 of R^{d+1} has injectivity radius equal to π. If N is a proper submanifold of the same dimension than (M,g), then $inj_{g|_N}N = 0$. The Euclidean space or the hyperbolic space have infinite injectivity radius. Moreover, a complete and simply connected Riemannian manifold with non-positive sectional curvature has also this property.

Throughout this paper, we will assume that (M,g) is a complete Riemannian manifold, i.e. (M,d_g) is a complete metric space. Also we will consider that inj_gM is strictly positive. This restriction will be clear in the Section 2.1 when we define the estimator. For standard result on differential and Riemannian geometry we refer to the reader to Boothby (1975), Besse (1978), Do Carmo (1988) and Gallot, Hulin and Lafontaine (2004).

Let $p \in M$, we denote with 0_p and T_pM the null tangent vector and the tangent space of M at p. Let $B_s(p)$ be a normal ball centred at p. Then $B_s(0_p) = exp_p^{-1}(B_s(p))$ is an open neighbourhood of 0_p in T_pM and so it has a natural structure of differential manifold. We are going to consider the Riemannian metrics g' and g'' in $B_s(0_p)$, where $g' = exp_p^*(g)$ is the pullback of g by the exponential map and g'' is the canonical metric induced by g_p in $B_s(0_p)$. Let $w \in B_s(0_p)$, and $(\bar{U},\bar{\psi})$ be a chart of $B_s(0_p)$ such that $w \in \bar{U}$. We note by $\{\partial/\partial\bar{\psi}_1|_w,\ldots,\partial/\partial\bar{\psi}_d|_w\}$ the tangent vectors induced by $(\bar{U},\psi)$. Consider the matricial function with entries (i,j) are given by $g'\left(\left(\partial/\partial\bar{\psi}_i|_w\right),\left(\partial/\partial\bar{\psi}_j|_w\right)\right)$. The volumes of the parallelepiped spanned by $\{\left(\partial/\partial\bar{\psi}_1|_w\right),\ldots,\left(\partial/\partial\bar{\psi}_d|_w\right)\}$ with respect to the metrics g' and g'' are given by $|\det g'\left(\left(\partial/\partial\bar{\psi}_i|_w\right),\left(\partial/\partial\bar{\psi}_j|_w\right)\right)|^{1/2}$ and $|\det g''\left(\left(\partial/\partial\bar{\psi}_i|_w\right),\left(\partial/\partial\bar{\psi}_j|_w\right)\right)|^{1/2}$ respectively. The quotient between these two volumes is independent of the selected chart. So, given $q \in B_s(p)$, if $w = exp_p^{-1}(q) \in B_s(0_p)$ we can define the volume density function, $\theta_p(q)$, on (M,g) as

$$\theta_p(q) = \frac{|\det g'\left(\left(\partial/\partial\bar{\psi}_i|_w\right),\left(\partial/\partial\bar{\psi}_j|_w\right)\right)|^{1/2}}{|\det g''\left(\left(\partial/\partial\bar{\psi}_i|_w\right),\left(\partial/\partial\bar{\psi}_j|_w\right)\right)|^{1/2}}$$

for any chart $(\bar{U},\bar{\psi})$ of $B_s(0_p)$ that contains $w = exp_p^{-1}(q)$. For instance, if we consider a normal coordinate system (U,ψ) induced by an orthonormal basis $\{v_1,\ldots,v_d\}$ of T_pM then $\theta_p(q)$ is the function of the volume element $d\nu_g$ in the local expression with respect to chart (U,ψ) evaluated at q, i.e.

$$\theta_p(q) = \left|\det g_q\left(\frac{\partial}{\partial\psi_i}\Big|_q,\frac{\partial}{\partial\psi_j}\Big|_q\right)\right|^{\frac{1}{2}},$$

where $\frac{\partial}{\partial \psi_i}|_q = D_{\alpha_i(0)} exp_p(\dot{\alpha}_i(0))$ with $\alpha_i(t) = exp_p^{-1}(q) + tv_i$ for $q \in U$. Note that the volume density function $\theta_p(q)$ is not defined for all the pairs p and q in M, but it is if $d_g(p,q) < inj_g M$.

We finish the section showing some examples of the density function:

i) In the case of $(\mathbb{R}^d, g_0)$ the density function is $\theta_p(q) = 1$ for all $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$.

ii) In the 2-dimensional sphere of radius R, the volume density is

$$\theta_{p_1}(p_2) = R\,\frac{|\sin(d_g(p_1,p_2)/R)|}{d_g(p_1,p_2)} \quad \text{if } p_2 \neq p_1, -p_1 \text{ and } \theta_{p_1}(p_1) = 1.$$

where d_g induced is given by

$$d_g(p_1,p_2) = R \arccos\left(\frac{\langle p_1,\ p_2\rangle}{R^2}\right).$$

iii) In the case of the cylinder of radius 1 $\mathscr{C}_1$ endowed with the metric induced by the canonical metric of $\mathbb{R}^3$, $\theta_{p_1}(p_2) = 1$ for all $(p_1,p_2) \in \mathscr{C}_1 \times \mathscr{C}_1$, and the distance induced is given by $d_g(p_1,p_2) = d_2((r_1,s_1),(r_2,s_2))$ if $d_2((r_1,s_1),(r_2,s_2)) < \pi$, where d_2 is the Euclidean distance of $\mathbb{R}^2$ and $p_i = (\cos(r_i), \sin(r_i), s_i)$ for $i = 1,2$.

See also Besse (1978) and Pennec (2006) for a discussion on the volume density function.

2.1. The estimator

Consider a probability distribution with a density f on a d-dimensional Riemannian manifold (M,g). Let $X_1, \cdots, X_n$ be i.i.d random object taking values on M with density f. A natural extension of the estimator proposed by Wagner (1975) in the context of a Riemannian manifold is to consider the following estimator

$$\widehat{f}_n(p) = \frac{1}{nH_n^d(p)} \sum_{j=1}^{n} \frac{1}{\theta_{X_j}(p)} K\left(\frac{d_g(p,X_j)}{H_n(p)}\right),$$

where $K : \mathbb{R} \to \mathbb{R}$ is a non-negative function with compact support, $\theta_p(q)$ denotes the volume density function on (M,g) and $H_n(p)$ is the distance d_g between p and the k-nearest neighbour of p among $X_1, \ldots, X_n$, and $k = k_n$ is a sequence of non-random integers such that $\lim_{n\to\infty} k_n = \infty$.

As we mention above, the volume density function is not defined for all p and q. Therefore, in order to guarantee the well definition of the estimator we consider a modification of the proposed estimator. Using the fact that the kernel K has compact support, we consider as bandwidth $\zeta_n(p) = \min\{H_n(p), inj_gM\}$ instead of $H_n(p)$. Thus, the kernel only considers the points X_i such that $d_g(X_i, p) \leq \zeta_n(p)$ that are smaller than inj_gM and for these points, the volume density function is well defined. Hence, the k-nearest neighbour kernel type estimator is defined as follows,

$$\widehat{f}_n(p) = \frac{1}{n\zeta_n^d(p)} \sum_{j=1}^{n} \frac{1}{\theta_{X_j}(p)} K\left(\frac{d_g(p, X_j)}{\zeta_n(p)}\right), \tag{1}$$

where $\zeta_n(p) = \min\{H_n(p), inj_gM\}$.

Remark 2.1.1. If (M, g) is a compact Riemannian manifold and its sectional curvature is not bigger than $a > 0$, then we know by the Lemma of Klingerberg (see Gallot, Hulin, Lafontaine (2004)) that $inj_gM \geq \min\{\pi/\sqrt{a}, l/2\}$ where l is the length of the shortest closed geodesic in (M, g).

3. Asymptotic results

Denote by $C^\ell(U)$ the set of ℓ times continuously differentiable functions from U to $\mathbb{R}$ where U is an open set of M. We assume that the measure induced by the probability P and by X is absolutely continuous with respect to the Riemannian volume measure $d\nu_g$, and we denote by f its density on M with respect to $d\nu_g$. More precisely, let $\mathscr{B}(M)$ be the Borel $\sigma-$field of M (the $\sigma-$field generated by the class of open sets of M). The random variable X has a probability density function f, i.e. if $\chi \in \mathscr{B}(M)$, $P(X^{-1}(\chi)) = \int_\chi f d\nu_g$.

3.1. Uniform consistency

We will consider the following set of assumptions in order to derive the strong consistency results of the estimate $\widehat{f}_n(p)$ defined in (1).

H1. Let M_0 be a compact set on M such that:
   i) f is a bounded function such that $\inf_{p\in M_0} f(p) = A > 0$.
   ii) $\inf_{p,q\in M_0} \theta_p(q) = B > 0$.

H2. For any open set U_0 of M_0 such that $M_0 \subset U_0$, f is of class C^2 on U_0.

H3. The sequence k_n is such that $k_n \to \infty$, $\frac{k_n}{n} \to 0$ and $\frac{k_n}{\log n} \to \infty$ as $n \to \infty$.

H4. $K : \mathbb{R} \to \mathbb{R}$ is a bounded nonnegative Lipschitz function of order one, with compact support $[0, 1]$ satisfying: $\int_{\mathbb{R}^d} K(\|\mathbf{u}\|)d\mathbf{u} = 1$, $\int_{\mathbb{R}^d} \mathbf{u}K(\|\mathbf{u}\|)d\mathbf{u} = 0$ and $0 < \int_{\mathbb{R}^d} \|\mathbf{u}\|^2 K(\|\mathbf{u}\|)d\mathbf{u} < \infty$.

H5. The kernel $K(u)$ verifies $K(uz) \geq K(z)$ for all $u \in (0, 1)$.

Remark 3.1.1. The fact that $\theta_p(p) = 1$ for all $p \in M$ guarantees that H1 ii) holds. The assumption H3 is usual when dealing with nearest neighbor and the assumption H4 is standard when dealing with kernel estimators.

Theorem 3.1.2. Assume that H1 to H5 holds, then we have that

$$\sup_{p \in M_0} |\widehat{f}_n(p) - f(p)| \xrightarrow{a.s.} 0.$$

3.2. Asymptotic normality

To derive the asymptotic distribution of the regression parameter estimates we will need two additional assumptions. We will denote with $\mathscr{V}_r$ the Euclidean ball of radius r centered at the origin and with $\lambda(\mathscr{V}_r)$ its Lebesgue measure.

H5. $f(p) > 0$, $f \in C^2(V)$ with $V \subset M$ an open neighborhood of M and the second derivative of f is bounded.

H6. The sequence k_n is such that $k_n \to \infty$, $k_n/n \to 0$ as $n \to \infty$ and there exists $0 \leq \beta < \infty$ such that $\sqrt{k_n n^{-4/(d+4)}} \to \beta$ as $n \to \infty$.

H7. The kernel verifies

i) $\int K_1(\|\mathbf{u}\|)\|\mathbf{u}\|^2 d\mathbf{u} < \infty$ as $s \to \infty$ where $K_1(\mathbf{u}) = K'(\|\mathbf{u}\|)\|\mathbf{u}\|$.

ii) $\|\mathbf{u}\|^{d+1} K_2(\mathbf{u}) \to 0$ as $\|\mathbf{u}\| \to \infty$ where $K_2(\mathbf{u}) = K''(\|\mathbf{u}\|)\|\mathbf{u}\|^2 - K_1(\mathbf{u})$

Remark 3.2.1. Note that $div(K(\|\mathbf{u}\|)\mathbf{u}) = K'(\|\mathbf{u}\|)\|\mathbf{u}\| + d\,K(\|\mathbf{u}\|)$, then using the divergence Theorem, we get that $\int K'(\|\mathbf{u}\|)\|\mathbf{u}\| d\mathbf{u} = \int_{\|\mathbf{u}\|=1} K(\|\mathbf{u}\|)\mathbf{u}\frac{\mathbf{u}}{\|\mathbf{u}\|} d\mathbf{u} - d \int K(\|\mathbf{u}\|) d\mathbf{u}$. Thus, the fact that K has compact support in $[0,1]$ implies that $\int K_1(\mathbf{u}) d\mathbf{u} = -d$.

On the other hand, note that $\nabla(K(\|\mathbf{u}\|)\|\mathbf{u}\|^2) = K_1(\|\mathbf{u}\|)\mathbf{u} + 2K(\|\mathbf{u}\|)\mathbf{u}$ and by $H4$ we get that $\int K_1(\|\mathbf{u}\|)\mathbf{u}\, d\mathbf{u} = 0$.

Theorem 3.2.2. Assume H4 to H7. Then we have that

$$\sqrt{k_n}(\widehat{f}_n(p) - f(p)) \xrightarrow{\mathscr{D}} \mathscr{N}(b(p), \sigma^2(p))$$

with

$$b(p) = \frac{1}{2} \frac{\beta^{\frac{d+4}{d}}}{(f(p)\lambda(\mathscr{V}_1))^{\frac{2}{d}}} \int_{\mathscr{V}_1} K(\|\mathbf{u}\|) u_1^2 \, d\mathbf{u} \sum_{i=1}^{d} \frac{\partial f \circ \psi^{-1}}{\partial u_i \partial u_i}|_{u=0}$$

and

$$\sigma^2(p) = \lambda(\mathscr{V}_1) f^2(p) \int_{\mathscr{V}_1} K^2(\|\mathbf{u}\|) d\mathbf{u}$$

where $\mathbf{u} = (u_1, \ldots, u_d)$ and $(B_h(p), \psi)$ is any normal coordinate system.

In order to derive the asymptotic distribution of $\widehat{f}_n(p)$, we will study the asymptotic behaviour of $h_n^d/\zeta_n^d(p)$ where $h_n^d = k_n/(nf(p)\lambda(\mathscr{V}_1))$. Note that if we consider $\widetilde{f}_n(p) = k_n/(n\zeta_n^d(p)\lambda(\mathscr{V}_1))$, $\widetilde{f}_n(p)$ is a consistent estimator of $f(p)$ (see the proof of Theorem 3.1.2). The next Theorem states that this estimator is also asymptotically normally distributed as in the Euclidean case.

Theorem 3.2.3. Assume H4 to H6, and let $h_n^d = k_n/(nf(p)\lambda(\mathscr{V}_1))$. Then we have that

$$\sqrt{k_n}\left(\frac{h_n^d}{\zeta_n^d(p)} - 1\right) \xrightarrow{\mathscr{D}} N(b_1(p), 1)$$

with

$$b_1(p) = \left(\frac{\beta^{\frac{d+4}{2}}}{f(p)\mu(\mathscr{V}_1)}\right)^{\frac{2}{d}} \left\{\frac{\tau}{6d+12} + \frac{\int_{\mathscr{V}_1} u_1^2 d\mathbf{u}\, L_1(p)}{f(p)\mu(\mathscr{V}_1)}\right\}$$

where $\mathbf{u} = (u_1, \ldots, u_d)$, τ is the scalar curvature of (M, g), i.e. the trace of the Ricci tensor,

$$L_1(p) = \sum_{i=1}^{d} \left(\frac{\partial^2 f \circ \psi^{-1}}{\partial u_i u_i}\Big|_{u=0} + \frac{\partial f \circ \psi^{-1}}{\partial u_i}\Big|_{u=0} \frac{\partial \theta_p \circ \psi^{-1}}{\partial u_i}\Big|_{u=0} \right)$$

and $(B_h(p), \psi)$ is any normal coordinate system.

4. Simulations

This section contains the results of a simulation study designed to evaluate the performance of the estimator defined in the Section 2.1. We consider three models in two different Riemannian manifolds, the sphere and the cylinder endowed with the metric induced by the canonical metric of $\mathbb{R}^3$. We performed 1000 replications of independent samples of size $n = 200$ according to the following models:

Model 1 (in the sphere): The random variables X_i for $1 \leq i \leq n$ are i.i.d. Von Mises distribution $VM(\mu, \kappa)$ i.e.

$$f_{\mu,\kappa}(X) = \left(\frac{\kappa}{2}\right)^{1/2} I_{1/2}(\kappa)\exp\{\kappa X^{\mathsf{T}}\boldsymbol{\mu}\},$$

with $\boldsymbol{\mu}$ is the mean parameter, $\kappa > 0$ is the concentration parameter and $I_{1/2}(\kappa) = \left(\frac{\kappa\pi}{2}\right)\sinh(\kappa)$ stands for the modified Bessel function. This model has many important applications, as described in Jammalamadaka and Sengupta (2001) and Mardia and Jupp (2000). We generate a random sample following a Von Mises distribution with mean $(0,0,1)$ and concentration parameter 3.

Model 2 (in the sphere): We simulate i.i.d. random variables Z_i for $1 \leq i \leq n$ following a multivariate normal distribution of dimension 3, with mean $(0,0,0)$ and covariance matrix equal to the identity. We define $X_i = \frac{Z_i}{\|Z_i\|}$ for $1 \leq i \leq n$, therefore the variables X_i follow a uniform distribution in the two-dimensional sphere.

Model 3 (in the cylinder): We consider random variables $X_i = (\mathbf{y}_i, t_i)$ taking values in the cylinder $S^1 \times \mathbb{R}$. We generated the model proposed by Mardia and Sutton (1978) where,

$$\mathbf{y}_i = (\cos(\theta_i), \sin(\theta_i)) \sim VM((-1,0),5)$$

$$t_i|\mathbf{y}_i \sim N(1 + 2\sqrt{5}\cos(\theta_i), 1).$$

Some examples of variables with this distribution can be found in Mardia and Sutton (1978).

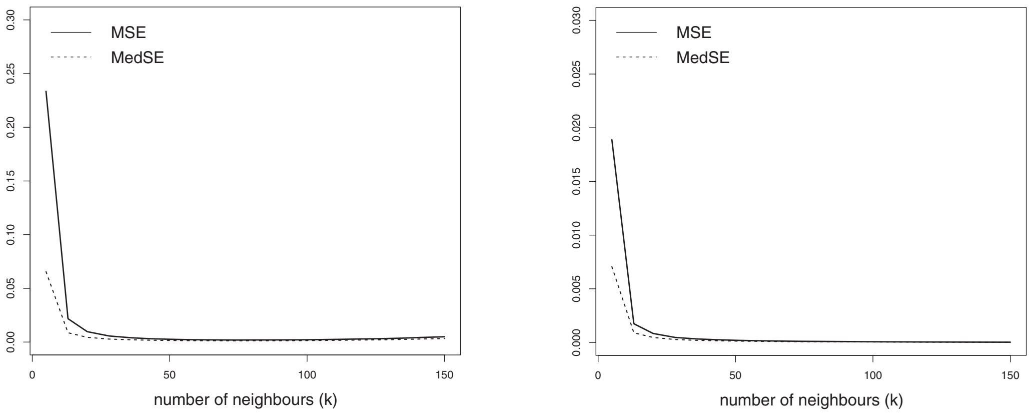
In all cases, for smoothing procedure, the kernel was taken as the quadratic kernel $K(t) = (15/16)(1-t^2)^2 I(|x| < 1)$. We have considered a grid of equidistant values of k between 5 and 150 of length 20.

To study the performance of the estimators of the density function f, denoted by $\widehat{f_n}$, we have considered the mean square error (MSE) and the median square error (MedSE), i.e,

$$\mathrm{MSE}(\widehat{f_n}) = \frac{1}{n}\sum_{i=1}^{n}[\widehat{f_n}(X_i) - f(X_i)]^2 .$$

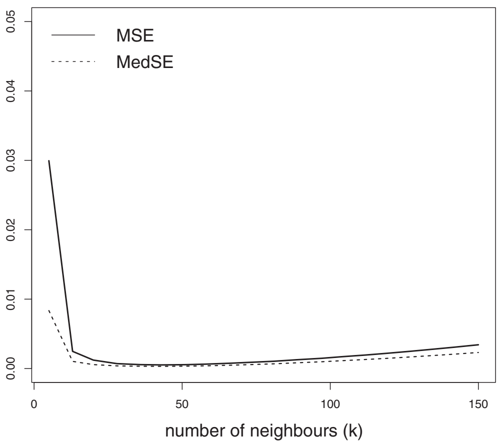
$$\mathrm{MedSE}(\widehat{f_n}) = \mathrm{median}\,|\widehat{f_n}(X_i) - f(X_i)|^2 .$$

Figure 1 gives the values of the MSE and MedSE of $\widehat{f_n}$ in the sphere model considering different numbers of neighbours, while Figure 2 shows the cylinder model. The simulation study confirms the good behaviour of k-nearest neighbour estimators, under the different models considered. In all cases, the estimators are stable under large numbers of neighbours. However, as expected, the estimators using a small number of neighbours have a poor behaviour, because in the neighborhood of each point there is a small number of samples.



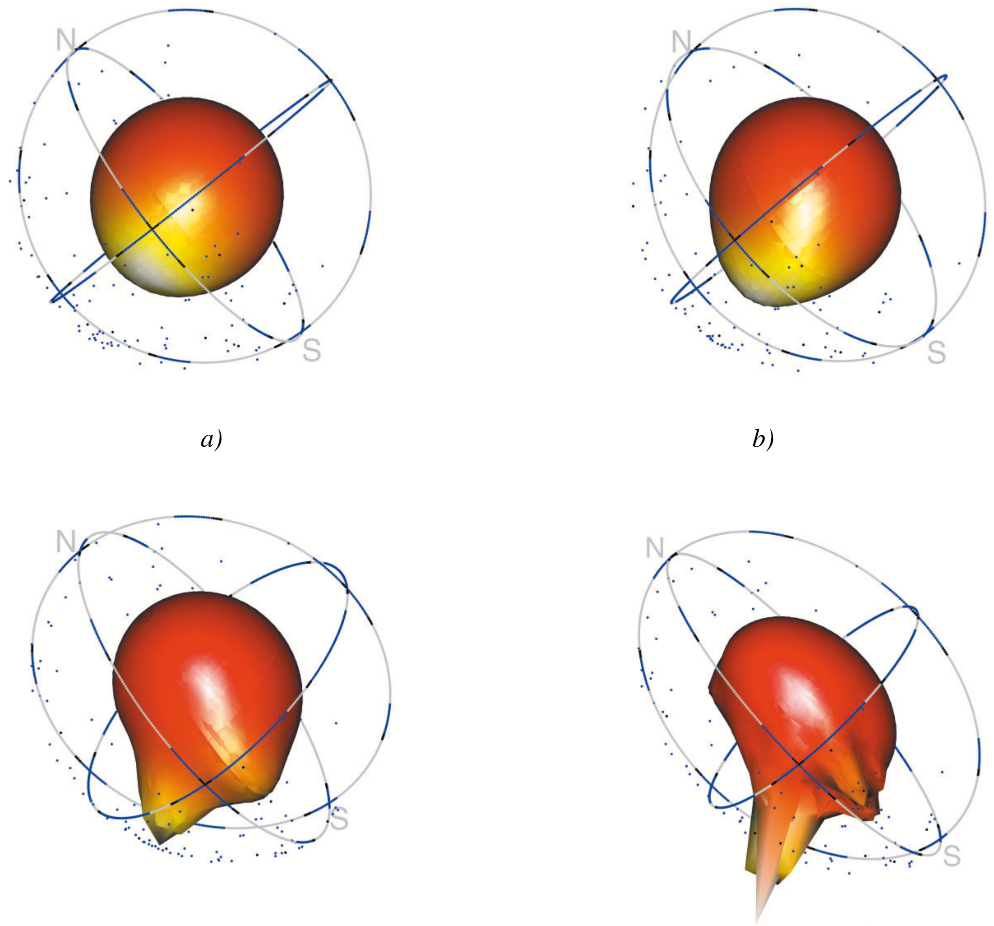
a) *b)*

Figure 1: *The nonparametric density estimator using different numbers of neighbours, a) the Von Mises model and b) the uniform model.*



c)

Figure 2: *The nonparametric density estimator using different numbers of neighbours in the cylinder.*



a) b)

c) d)

Figure 3: *The nonparametric density estimator using different numbers of neighbours, a)* $k = 75$*, b)* $k = 50$*, c)* $k = 25$ *and d)* $k = 10$*.*

5. Real Example

5.1. Paleomagnetic data

The need for statistical analysis of paleomagnetic data is well known. Since the work developed by Fisher (1953), the study of parametric families was considered a principal tools to analyse and quantify this type of data (see Cox and Doell (1960), Butler (1992) and Love and Constable (2003)). In particular, our proposal allows to explore the nature of directional dataset that include paleomagnetic data without making any parametric assumptions.

In order to illustrate the k-nearest neighbor kernel type estimator on the two-dimensional sphere, we illustrate the estimator using a paleomagnetic data set studied by Fisher, Lewis, and Embleton (1987). The data set consists of $n = 107$ sites from specimens of Precambrian volcanos with measurements of magnetic remanence. The data set contains two variables corresponding to the directional component on a longitude scale, and the directional component on a latitude scale. The original data set is available in the package sm of R.

To calculate the estimators the volume density function and the geodesic distance were taken as in Section 2 and we considered the quadratic kernel $K(t) = (15/16)(1-t^2)^2 I(|x| < 1)$. In order to analyse the sensitivity of the results with respect to the number of neighbours, we plot the estimator using different bandwidths. The results are shown in Figure 3.

The real data were plotted in blue and with a large radius in order to obtain a better visualization. The Equator line, the Greenwich meridian and a second meridian are in gray while the north and south poles are denoted with the capital letters N and S respectively. The levels of concentration of measurements of magnetic remanence are shown in yellow for high levels and in red for lowest density levels. Also, the levels of concentration of measurements of magnetic remanence were illustrated with relief on the sphere, which emphasizes high density levels and the form of the density function.

As in the Euclidean case a large number of neighbours produces estimators with small variance but high bias, while small values produce more wiggly estimators. This fact shows the need of the implementation of a method to select the adequate bandwidth for this estimator. However, this requires further careful investigation and is beyond the scope of this paper.

5.2. Meteorological data

In this section we consider a real data set collected in the meteorological station "Agüita de Perdiz", located in Viedma, province of Río Negro, Argentine. The data set consists of wind directions and temperatures during January 2011 and contains 1326 observations that were registered with a frequency of thirty minutes. We note that the considered variables belong to a cylinder with radius 1.

As in the previous section, we consider the quadratic kernel and we took the density function and the geodesic distance as in Section 2. Figure 4 shows the result of the estimation, the colour and form of the graphic was constructed as in the previous example.

It is important to remark that the measurement devices of wind direction do not present a sufficient precision to avoid repeated data. Therefore, we consider the proposal given in García-Portugués et al. (2011) to solve this problem. The proposal consists in perturbing the repeated data as follows $\widetilde{r}_i = r_i + \xi \varepsilon_i$, where r_i denotes the wind direction measurements and ε_i, for $i = 1, \ldots, n$ were independently generated from a von Mises distribution with $\mu = (1, 0)$ and $\kappa = 1$. The selection of the perturbation scale ξ was taken as $\xi = n^{-1/5}$, as in García-Portugués et al. (2011) where in this case $n = 1326$.

The work of García-Portugués et al. (2011) contains other nice real example where the proposed estimator can be applied. They considered a naive density estimator applied to wind directions and SO_2 concentrations, which allows one to explore high levels of contamination.

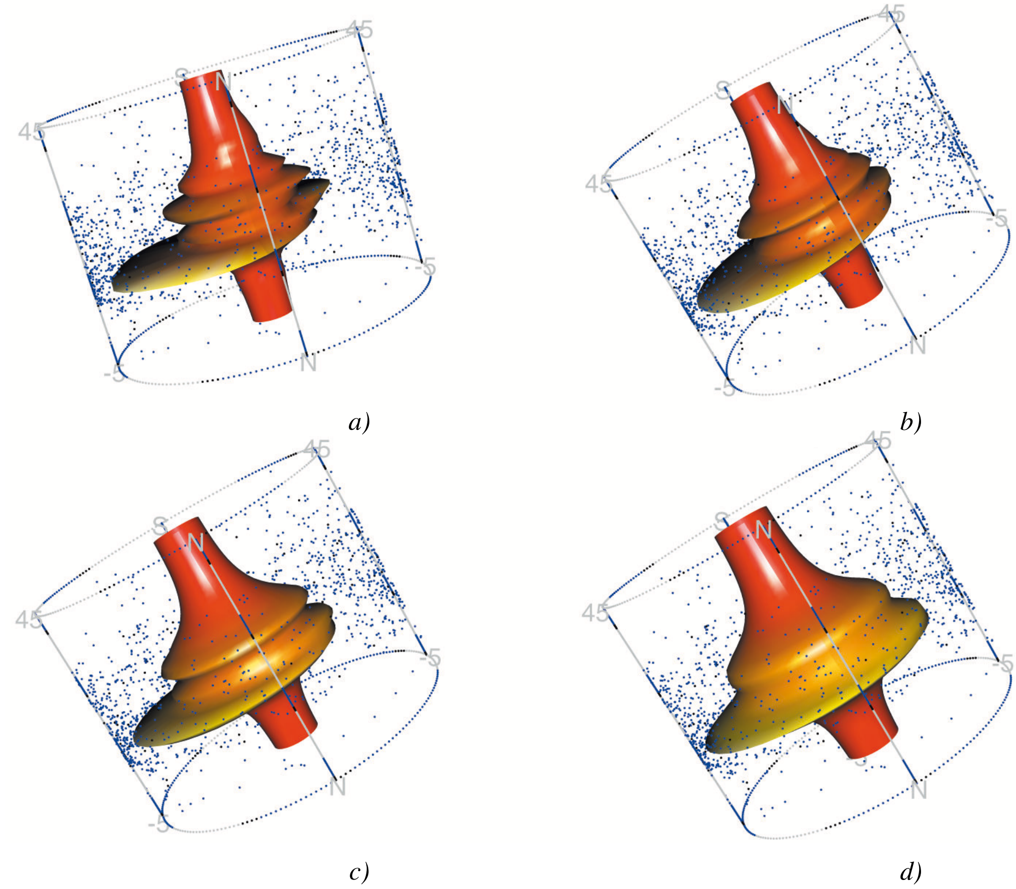


Figure 4: *The nonparametric density estimator using different numbers of neighbours, a)* $k = 75$*, b)* $k = 150$*, c)* $k = 300$ *and d)* $k = 400$*.*

In Figure 4 we can see that the lowest temperatures are more probable when the wind comes from an easterly direction. However, the highest temperature does not seem to have correlation with the wind direction. Also, note that in Figure 4 we can see two modes corresponding to the minimum and maximum daily temperatures.

These examples show the usefulness of the proposed estimator for the analysis and exploration of these type of data set.

Appendix

Proof of Theorem 3.1.2.

Let

$$f_n(p,\delta_n) = \frac{1}{n\delta_n^d}\sum_{i=1}^{n}\frac{1}{\theta_{X_i}(p)}K\left(\frac{d_g(p,X_i)}{\delta_n}\right).$$

Note that if $\delta_n = \delta_n(p)$ verifies $\delta_{1n} \le \delta_n(p) \le \delta_{2n}$ for all $p \in M_0$ where δ_{1n} and δ_{2n} satisfy $\delta_{in} \to 0$ and $\dfrac{n\delta_{in}^d}{\log n} \to \infty$ as $n \to \infty$ for $i = 1,2$ then by Theorem 3.2 in Henry and Rodriguez (2009) we have that

$$\sup_{p\in M_0} |f_n(p,\delta_n) - f(p)| \xrightarrow{a.s.} 0 \tag{2}$$

For each $0 < \beta < 1$ we define,

$$f_n^-(p,\beta) = \frac{1}{nD_n^+(\beta)^d}\sum_{i=1}^{n}\frac{1}{\theta_{X_i}(p)}K\left(\frac{d_g(p,X_i)}{D_n^-(\beta)}\right) = f_n^-(p,D_n^-(\beta)^d)\frac{D_n^-(\beta)^d}{D_n^+(\beta)^d}.$$

$$f_n^+(p,\beta) = \frac{1}{nD_n^-(\beta)^d}\sum_{i=1}^{n}\frac{1}{\theta_{X_i}(p)}K\left(\frac{d_g(p,X_i)}{D_n^+(\beta)}\right) = f_n^+(p,D_n^+(\beta)^d)\frac{D_n^+(\beta)^d}{D_n^-(\beta)^d}.$$

where $D_n^-(\beta) = \beta^{1/2d}h_n$, $D_n^+(\beta) = \beta^{-1/2d}h_n$ and $h_n^d = k_n/(n\lambda(\mathscr{V}_1)f(p))$ with $\lambda(\mathscr{V}_1)$ denote the Lebesgue measure of the ball in $\mathbb{R}^d$ with radius r centred at the origin. Note that

$$\sup_{p\in M_0} |f_n^-(p,\beta) - \beta f(p)| \xrightarrow{a.s.} 0 \text{ and } \sup_{p\in M_0} |f_n^+(p,\beta) - \beta^{-1} f(p)| \xrightarrow{a.s.} 0. \tag{3}$$

For all $0 < \beta < 1$ and $\varepsilon > 0$ we define

$$S_n^-(\beta,\varepsilon) = \{w : \sup_{p\in M_0} |f_n^-(p,\beta) - f(p)| < \varepsilon \},$$

$$S_n^+(\beta,\varepsilon) = \{w : \sup_{p\in M_0} |f_n^+(p,\beta) - f(p)| < \varepsilon \},$$

$$S_n(\varepsilon) = \{w : \sup_{p\in M_0} |\widehat{f}_n(p) - f(p)| < \varepsilon \},$$

$$A_n(\beta) = \{f_n^-(p,\beta) \le \widehat{f}_n(p) \le f_n^+(p,\beta)\}$$

Then, $A_n(\beta) \cap S_n^-(\beta,\varepsilon) \cap S_n^+(\beta,\varepsilon) \subset S_n(\varepsilon)$. Let $A = \sup_{p \in M_0} f(p)$. For $0 < \varepsilon < 3A/2$ and $\beta_\varepsilon = 1 - \frac{\varepsilon}{3A}$ consider the following sets

$$G_n(\varepsilon) = \left\{ w:\ D_n^-(\beta_\varepsilon) \leq \zeta_n(p) \leq D_n^+(\beta_\varepsilon) \ \text{ for all } p \in M_0 \right\}$$
$$G_n^-(\varepsilon) = \{ \sup_{p \in M_0} |f_n^-(p,\beta_\varepsilon) - \beta_\varepsilon f(p)| < \frac{\varepsilon}{3} \}$$
$$G_n^+(\varepsilon) = \left\{ \sup_{p \in M_0} |f_n^+(p,\beta_\varepsilon) - \beta_\varepsilon^{-1} f(p)| < \frac{\varepsilon}{3} \right\}.$$

Then we have that $G_n(\varepsilon) \subset A_n(\beta_\varepsilon)$, $G_n^-(\varepsilon) \subset S_n^-(\beta_\varepsilon,\varepsilon)$ and $G_n^+(\varepsilon) \subset S_n^+(\beta_\varepsilon,\varepsilon)$. Therefore, $G_n(\varepsilon) \cap G_n^-(\varepsilon) \cap G_n^+(\varepsilon) \subset S_n(\varepsilon)$.

On the other hand, using that $\lim_{r \to 0} V(B_r(p))/r^d \mu(\mathscr{V}_1) = 1$, where $V(B_r(p))$ denotes the volume of the geodesic ball centered at p with radius r (see Gray and Vanhecke (1979)) and similar arguments those considered in Devroye and Wagner (1977), we get that

$$\sup_{p \in M_0} \left| \frac{k_n}{n\lambda(\mathscr{V}_1) f(p) H_n^d(p)} - 1 \right| \xrightarrow{a.s.} 0.$$

Recall that $inj_g M > 0$ and $H_n^d(p) \xrightarrow{a.s.} 0$. Then for straightforward calculations we obtained that $\sup_{p \in M_0} \left| \frac{k_n}{n\lambda(\mathscr{V}_1) f(p) \zeta_n^d(p)} - 1 \right| \xrightarrow{a.s.} 0$. Thus, $I_{G_n^c(\varepsilon)} \xrightarrow{a.s.} 0$ and (3) imply that $I_{S_n^c(\varepsilon)} \xrightarrow{a.s.} 0$. □

Proof of Theorem 3.2.2.

A Taylor expansion of second order gives

$$\sqrt{k_n} \left\{ \frac{1}{n\zeta_n^d(p)} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} K\left(\frac{d_g(p,X_j)}{\zeta_n(p)} \right) - f(p) \right\} = A_n + B_n + C_n$$

where

$$A_n = (h_n^d/\zeta_n^d(p)) \sqrt{k_n} \left\{ \frac{1}{nh_n^d} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} K\left(\frac{d_g(p,X_j)}{h_n} \right) - f(p) \right\},$$

$$B_n = \sqrt{k_n}((h_n^d/\zeta_n^d(p)) - 1) \left\{ f(p) + \frac{[(h_n/\zeta_n(p)) - 1] h_n^d}{[(h_n^d/\zeta_n^d(p)) - 1] \zeta_n^d(p)} \ \frac{1}{nh_n^d} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} K_1\left(\frac{d_g(p,X_j)}{\zeta_n(p)} \right) \right\}$$

and

$$C_n = \sqrt{k_n}((h_n^d/\zeta_n^d(p)) - 1) \frac{[(h_n/\zeta_n(p)) - 1]^2}{2[(h_n^d/\zeta_n^d(p)) - 1]} \ \frac{1}{n\zeta_n^d(p)} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} K_2\left(\frac{d_g(p,X_j)}{\xi_n} \right) [\xi_n/h_n]^2$$

with $h_n^d = k_n/nf(p)\lambda(\mathscr{V}_1)$ and $\min(h_n,\zeta_n) \leq \xi_n \leq \max(h_n,\zeta_n)$. Note that H6 implies that h_n satisfies the necessary hypothesis given in Theorem 4.1 in Rodriguez and Henry (2009), in particular

$$\sqrt{nh_n^{d+4}} \to \beta^{\frac{d+4}{d}}(f(p)\lambda(\mathscr{V}_1))^{-\frac{d+4}{2d}}.$$

By the Theorem and the fact that $h_n/\zeta_n(p) \xrightarrow{p} 1$, we obtain that A_n converges to a normal distribution with mean $b(p)$ and variance $\sigma^2(p)$. Therefore it is enough to show that B_n and C_n converges to zero in probability.

Note that $\frac{(h_n/H_n(p))-1}{(h_n^d/\zeta_n^d(p))-1} \xrightarrow{p} d^{-1}$ and by similar arguments those considered in Theorem 3.1 in Pelletier (2005) and Remark 3.2.1 we get that

$$\frac{1}{nh_n^d}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}K_1\left(\frac{d_g(p,X_j)}{\zeta_n(p)}\right) \xrightarrow{p} \int K_1(\mathbf{u})d\mathbf{u}f(p) = -d\,f(p).$$

Therefore, by Theorem 3.2.3, we obtain that $B_n \xrightarrow{p} 0$. As ξ_n/h_n converges to one in probability, in order to concluded the proof, it remains to prove that

$$\frac{1}{n\zeta_n^d(p)}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}|K_2\left(d_g(p,X_j)/\xi_n\right)|$$

is bounded in probability.

By H7, there exits $r > 0$ such that $|t|^{d+1}|K_2(t)| \leq 1$ if $|t| \geq r$. Let $C_r = (-r,r)$, then we have that

$$\begin{aligned}\frac{1}{n\zeta_n^d(p)}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}\left|K_2\left(\frac{d_g(p,X_j)}{\xi_n}\right)\right| \;&\leq\; \frac{\sup_{|t|\leq r}|K_2(t)|}{n\zeta_n^d(p)}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}I_{C_r}\left(\frac{d_g(p,X_j)}{\xi_n}\right)\\ &+\; \frac{1}{n\zeta_n^d(p)}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}I_{C_r^c}\left(\frac{d_g(p,X_j)}{\xi_n}\right)\left|\frac{d_g(p,X_j)}{\xi_n}\right|^{-(d+1)}\end{aligned}$$

As $\min(h_n,\zeta_n(p)) \leq \xi_n \leq \max(h_n,\zeta_n(p)) = \widetilde{\xi}_n$ it follows that

$$\begin{aligned}\frac{1}{n\zeta_n^d(p)} \quad & \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}\left|K_2\left(\frac{d_g(p,X_j)}{\xi_n}\right)\right| \leq \\ \leq \quad & \left(\frac{h_n}{\zeta_n(p)}\right)^d \sup_{|t|\leq r}|K_2(t)|\,\frac{1}{nh_n^d}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}I_{C_r}\left(\frac{d_g(p,X_j)}{h_n}\right)\\ + \quad & \sup_{|t|\leq r}|K_2(t)|\,\frac{1}{n\zeta_n^d(p)}\sum_{j=1}^n \frac{1}{\theta_{X_j}(p)}I_{C_r}\left(\frac{d_g(p,X_j)}{\zeta_n(p)}\right)\end{aligned}$$

$$+ \left(\frac{h_n}{\zeta_n(p)}\right)^d \frac{1}{nh_n^d} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} I_{C_r^c}\left(\frac{d_g(p,X_j)}{h_n}\right) \left|\frac{d_g(p,X_j)}{h_n}\right|^{-(d+1)} \left|\frac{\widetilde{\xi}_n}{h_n}\right|^{(d+1)}$$

$$+ \frac{1}{n\zeta_n^d(p)} \sum_{j=1}^n \frac{1}{\theta_{X_j}(p)} I_{C_r^c}\left(\frac{d_g(p,X_j)}{\zeta_n(p)}\right) \left|\frac{d_g(p,X_j)}{\zeta_n(p)}\right|^{-(d+1)} \left|\frac{\widetilde{\xi}_n}{\zeta_n(p)}\right|^{(d+1)}$$

$$= C_{n1} + C_{n2} + C_{n3} + C_{n4}.$$

By similar arguments those considered in Theorem 3.1 in Pelletier (2005), we have that $C_{n1} \xrightarrow{p} f(p) \int I_{C_r}(s)ds$ and $C_{n3} \xrightarrow{p} f(p) \int I_{C_r^c}(s)|s|^{-(d+1)}ds$.

Finally, let $A_n^\varepsilon = \{(1-\varepsilon)h_n \le \zeta_n \le (1+\varepsilon)h_n\}$ for $0 \le \varepsilon \le 1$. Then for n large enough $P(A_n^\varepsilon) > 1-\varepsilon$ and in A_n^ε we have that

$$I_{C_r}\left(\frac{d_g(X_j,p)}{\zeta_n(p)}\right) \le I_{C_r}\left(\frac{d_g(X_j,p)}{(1+\varepsilon)h_n}\right),$$

$$I_{C_r^c}\left(\frac{d_g(X_j,p)}{\zeta_n(p)}\right) \left|\frac{d_g(X_j,p)}{\zeta_n(p)}\right|^{-(d+1)} \le I_{C_r^c}\left(\frac{d_g(X_j,p)}{(1-\varepsilon)h_n}\right) \left|\frac{d_g(X_j,p)}{(1-\varepsilon)h_n}\right|^{-(d+1)} \left|\frac{\zeta_n(p)}{(1-\varepsilon)h_n}\right|^{(d+1)}.$$

This fact and analogous arguments those considered in Theorem 3.1 in Pelletier (2005), allow to conclude the proof. □

Proof of Theorem 3.2.3.

Denote $b_n = h_n^d/(1+zk_n^{-1/2})$, then

$$P(\sqrt{k_n}(h_n^d/\zeta_n^d - 1) \le z) = P(\zeta_n^d \ge b_n) = P(H_n^d \ge b_n,\ inj_gM^d \ge b_n).$$

As $b_n \to 0$ and $inj_gM > 0$, there exists n_0 such that for all $n \ge n_0$ we have that

$$P(H_n^d \ge b_n,\ inj_gM^d \ge b_n) = P(H_n^d \ge b_n).$$

Let Z_i such that $Z_i = 1$ when $d_g(p,X_i) \le b_n^{1/d}$ and $Z_i = 0$ elsewhere. Thus, we have that $P(H_n^d \ge b_n) = P(\sum_{i=1}^n Z_i \le k_n)$. Let $q_n = P(d_g(p,X_i) \le b_n^{1/d})$. Note that $q_n \to 0$ and $nq_n \to \infty$ as $n \to \infty$, therefore

$$P\left(\sum_{i=1}^n Z_i \le k_n\right) = P\left(\frac{1}{\sqrt{nq_n}} \sum_{i=1}^n (Z_i - E(Z_i)) \le \frac{1}{\sqrt{nq_n}}(k_n - nq_n)\right).$$

Using the Lindeberg Central Limit Theorem we easily obtain that $(nq_n)^{-1/2}$ $\sum_{i=1}^{n}(Z_i - E(Z_i))$ is asymptotically normal with mean zero and variance one. Hence, it is enough to show that $(nq_n)^{-1/2}(k_n - nq_n) \xrightarrow{p} z + b_1(p)$.

Denote by $\mu_n = n\int_{B_{b_n^{1/d}}(p)} (f(q) - f(p))d\nu_g(q)$. Note that $\mu_n = n\, q_n - w_n$ with $w_n = n\, f(p)V(B_{b_n^{1/d}}(p))$. Thus,

$$\frac{1}{\sqrt{nq_n}}(k_n - nq_n) = w_n^{-1/2}(k_n - w_n)\left(\frac{w_n}{w_n+\mu_n}\right)^{1/2} + \frac{\mu_n}{w_n^{1/2}}\left(\frac{w_n}{w_n+\mu_n}\right)^{1/2}.$$

Let $(B_{b_n^{1/d}}(p), \psi)$ be a coordinate normal system. Then, we note that

$$\frac{1}{\lambda(\mathscr{V}_{b_n^{1/d}})}\int_{B_{b_n^{1/d}}(p)} f(q)d\nu_g(q) = \frac{1}{\lambda(\mathscr{V}_{b_n^{1/d}})}\int_{\mathscr{V}_{b_n^{1/d}}} f\circ\psi^{-1}(\mathbf{u})\theta_p\circ\psi^{-1}(\mathbf{u})d\mathbf{u}.$$

The Lebesgue's Differentiation Theorem and the fact that $\dfrac{V(B_{b_n^{1/d}}(p))}{\lambda(\mathscr{V}_{b_n^{1/d}})} \to 1$ imply that $\dfrac{\lambda_n}{w_n} \to 0$. On the other hand, from Gray and Vanhecke (1979), we have that

$$V(B_r(p)) = r^d\lambda(\mathscr{V}_1)(1 - \frac{\tau}{6d+12}r^2 + O(r^4)).$$

Hence, we obtain that

$$\begin{aligned} w_n^{-1/2}(k_n - w_n) &= \frac{w_n^{-1/2}\, k_n\, z\, k_n^{-1/2}}{1+zk_n^{-1/2}} + \frac{w_n^{-1/2}\,\tau b_n^{2/d}\, k_n}{(6d+12)(1+zk_n^{-1/2})} + w_n^{-1/2}\, k_n\, O(b_n^{4/d}) \\ &= A_n + B_n + C_n. \end{aligned}$$

It's easy to see that $A_n \to z$ and $w_n^{-1/2}\, b_n^{2/d}\, k_n = \frac{k_n n^{-1/2} b_n^{2/d-1/2}}{(f(p)\lambda(\mathscr{V}_1))^{-2/d}}\left(\frac{b_n\lambda(\mathscr{V}_1)}{V(B_{b_n^{1/d}}(p))}\right)^{1/2}$, since $H6$ we obtain that $B_n \to \tau\, \beta^{(d+4)/d}/(6d+12)\,(f(p)\mu(\mathscr{V}_1))^{-2/d}$. A similar argument shows that $C_n \to 0$ and therefore we get that $w_n^{-1/2}(k_n - w_n) \to z + \beta^{\frac{d+4}{d}}\frac{\tau}{6d+12}(f(p)\lambda(\mathscr{V}_1))^{-d/2}$.

In order to concluded the proof we will show that

$$\frac{\mu_n}{w_n^{1/2}} \to \frac{\beta^{\frac{d+4}{d}}}{(f(p)\lambda(\mathscr{V}_1))^{(d+2)/d}}\int_{\mathscr{V}_1} u_1^2\, d\mathbf{u}\, L_1(p).$$

We use a second-order Taylor expansion that leads to,

$$\int_{B_{b_n^{1/d}}(p)} (f(q)-f(p))d\nu_g(q) = \sum_{i=1}^{d} \frac{\partial f\circ\psi^{-1}}{\partial u_i}|_{u=0} b_n^{1+1/d} \int_{\mathscr{V}_1} u_i\, \theta_p\circ\psi^{-1}(b_n^{1/d}\mathbf{u})\, d\mathbf{u}$$

$$+ \sum_{i,j=1}^{d} \frac{\partial^2 f\circ\psi^{-1}}{\partial u_i \partial u_j}|_{u=0} b_n^{1+2/d} \int_{\mathscr{V}_1} u_i u_j\, \theta_p\circ\psi^{-1}(b_n^{1/d}\mathbf{u})\, d\mathbf{u}$$

$$+ O(b_n^{1+3/d}).$$

Using again a Taylor expansion on $\theta_p\circ\psi^{-1}(\cdot)$ at 0 we have that

$$\int_{B_{b_n^{1/d}}(p)} (f(q)-f(p))d\nu_g(q) = b_n^{1+2/d} \int_{\mathscr{V}_1} u_1^2\, d\mathbf{u}\, L_1(p) + O(b_n^{1+3/d})$$

and by H6 the theorem follows. □

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