Sborník vědeckých prací Vysoké školy báňské - Technické univerzity Ostrava

číslo 2, rok 2013, ročník XIII, řada stavební

článek č. 25

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MECHANICAL AND MATHEMATICAL MODELING OF VISCO-ELASTIC CONTINUA – CONSTITUTIVE EQUATIONS

MECHANO – MATEMATICKÉ MODELOVANIE VÄZKOPRUŽNÉHO KONTINUA – KONŠTITUTÍVNE ROVNICE

Abstract

Phenomenological models of continuum mechanics applied on the rigid body are more or less idealized. Experimental measuring showed there is a plastic flow, respectively relaxation in real rigid bodies, i.e. stress is the function of strain, strain velocity and the higher time derivatives. The paper deals with the rheological models based on the Hook elastic and Newton viscous masses. The corresponding constitutive equations are described.

Keywords

Linear visco-elasticity, relaxation, plastic flow, differential and integral representation of the constitutive equations for anisotropic media, rheological model, integral Laplace transformation.

Abstrakt

Fenomenologické modely mechaniky kontinua aplikované v tuhých telesách sú viac či menej idealizované. Experimentálne merania ukázali, že v tuhých telesách dochádza k dotvarovaniu, resp. relaxácii, t. j. že napätie je funkciou nielen deformácie, ale aj jej časových derivácií. Článok pojednáva o reologických modeloch, ktorých základom je Hookova pružná a Newtonova viskózna látka. Sú popísané aj príslušné konštitutívne rovnice.

Klíčová slova

Lineárna väzkopružnosť, relaxácia, dotvarovanie, diferenciálna a integrálna reprezentácia konštitutívnych rovníc v anizotropickom médiu, reologický model, Laplaceova transformácia.

1 INTRODUCTION

Classical elasticity theory studies the mechanical response of the perfectly elastic body to the ambient acting, where according to the Hook's law the stress is a linear function of the strain being independent on the deformation velocity. On the other hand in hydrodynamic problems, where Newtonian laws are valid, the stress is linearly proportional to the strain velocity, but independent from the strain itself [7, 8, 10].

Phenomenological models of continuum mechanics applied in mechanical and mathematical modeling by using boundary value problems are more or less idealized. Considering the experimental measuring [11, 12, 25, 29] it is easy to find out that in real rigid body the plastic flow, respectively relaxation of the stress is evident as the effect of the outer load. In another words, the stress can be the

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function of not only strain and strain velocity, but also the higher order time derivative of the strain, e.g. [3, 28].

The study of the materials, where both the rigid and the liquid properties are performed, is included in the visco-elasticity theory. [3, 13-15, 18, 20, 22]. The base of the visco-elasticity theory was introduced in the two last centuries in the papers of Maxwell [21], Boltzmann [2], Kelvin [32] and Voigt [35].

2 CONSTITUTIVE EQUATION AXIOMS

- Causal principle: The stage of the body Ω in the time t is determined by the only history and not by the future, e.g. [1,4]
- Determinism principle: The stress in the particle $\vec{X} \in \Omega$ in the time t is determined by the movement history $\vec{x}'$ of the movement of the body Ω until the time t

$$\boldsymbol{\sigma}(\vec{x},t) = \underset{t=-\infty}{\mathcal{F}}(\vec{x}'(\vec{X}',t';\vec{X},t)) \ , \tag{1}$$

  where $\mathcal{F}$ "constitutive operator" is a general operator expressing admissible functions of the body movement. The operator has to fulfill the conditions of the invariance, e.g. [2, 4].
- Local effect principle: According to the determinism principle the movement of the particle $\vec{Y} \in \Omega$ that is not situated "too near" the particle $\vec{X} \in \Omega$, $\vec{X} \neq \vec{Y}$ can influence to the stress in the particle $\vec{X}$. In the sense of contact stresses definition, the stresses are determined by the interactions of the particles in the infinitesimal neighborhood of the point $\vec{X}$. In terms of this definition we can further neglect the movement of the particles of the finite distance from $\vec{X}$ when calculating the stresses in the neighborhood of the point, i.e. [4]

$$\vec{x}' = (\vec{Y},s) = x', \quad s \geq 0, \quad \vec{Y} \in N(\vec{X}) \ , \tag{2}$$

  where $N(\vec{X})$ is a neighborhood of the point $\vec{X}$ so it is valid

$$\mathcal{F}(\vec{x}',\vec{Y},t) = \underset{t=-\infty}{\mathcal{F}}(\vec{x}',\vec{X},t)) \ . \tag{3}$$

- Objectivity principle: In the phenomenological theory of modeling we presume the independence of the various strain measures and stress velocities on the position and movement of observer. Also it is valid that the material properties expressed by the constitutive operator are independent on the observer, accordingly they are objective. That means when we would like to describe the real behavior of the materials, the constitutive equations have to be objective, i.e. if it holds [2, 4]

$$\boldsymbol{\sigma}(\vec{x},t) = \underset{t=-\infty}{\mathcal{F}}(\vec{x}(t);\vec{X},t) \ . \tag{4}$$

  Then the constitutive operator has to read

$$\boldsymbol{\sigma}^*(\dot{\vec{x}},\dot{t}) = \underset{t=-\infty}{\mathcal{F}}(\dot{\vec{x}}(\dot{t});\vec{X},\dot{t}) \ , \tag{5}$$

  where $\dot{\vec{x}}$, $\boldsymbol{\sigma}^*$ are the quantities dynamically equivalent to $\vec{x}$ and $\boldsymbol{\sigma}$.

3 THE FUNDAMENTAL SUBSTANCES AND THEIR RHEOLOGICAL MODELS

Obviously, we describe the rheological phenomenon by the working diagram which express the relationship between a two physical parameters; $\sigma \sim \varepsilon$, $\varepsilon \sim \sigma$, $\sigma \sim t$, $\varepsilon \sim t$. We often use also three dimensional working diagrams $\sigma \sim \varepsilon \sim t$, $\varepsilon \sim \sigma \sim t$, etc.

There are some essential rheological components [28]: solid material, flowable liquid, elastic material. By compounding of these components we can get more complex rheological models by using which we can concisely express the rheological properties of various real materials. In our case, we will start from the two basic rheological substances, (Fig. 1).

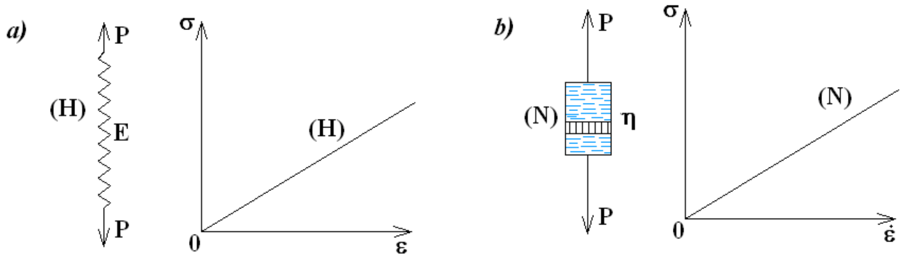

Fig. 1: Rheological models of elastic substance and viscous liquid, a) Hook elastic material, b) Newton viscous liquid

Working diagrams of both substances can be expressed by the known formulas

$$\text{(H)}: \ \boldsymbol{\sigma} = \hat{E}\boldsymbol{\varepsilon} \tag{6}$$

$$\text{(N)}: \ \boldsymbol{\sigma} = \hat{\eta}\dot{\boldsymbol{\varepsilon}}, \tag{7}$$

where $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are the stress and strain tensors, $\hat{E}$ elasticity module, $\hat{\eta}$ viscosity coefficient (in 3D represented by tensor operators). The dot above the tensor stands as the time derivative sign of it.

4 CONSTITUTIVE EQUATIONS FOR ANISOTROPIC MEDIUM

Let us consider a quasi static problem where we neglect the outer inertial loading influence on the infinitesimal strains of the body. In the case between the deformations and stresses there is a relation as follows:

$$\boldsymbol{\sigma} = \hat{\mathbf{H}}\boldsymbol{\varepsilon}, \tag{8}$$

where $\hat{\mathbf{H}} = H^{ijkl}$ is a tensor operator of the 4th order which is, according to the Onasger theory, symmetric for linear rheological models[24].

$$H^{ijkl} = H^{klij} = H^{klji} = H^{lkji} \tag{9}$$

and also positive definite – according to the 2nd thermodynamic law [5, 33]. The operator can be of three form of representation, e.g. [6, 7], integral, differential and integro – differential form.

(a) In the case of differential representation of equation (8) it holds [5, 6]

$$\mathbf{K}^{(\mathrm{r})}\boldsymbol{\sigma} = \hat{\mathbf{K}}_{(\mathrm{s})}\boldsymbol{\varepsilon} \tag{10}$$

respectively

$$\mathbf{Q}^{(\mathrm{r})}\boldsymbol{\varepsilon} = \hat{\mathbf{Q}}_{(\mathrm{s})}\boldsymbol{\sigma}, \tag{11}$$

where

$$\mathbf{K}^{(\mathrm{r})} = \prod_{n=1}^{r}\left(\frac{\partial}{\partial t} + \kappa_n\right), \quad \mathbf{K}^{(0)} = \mathbf{1} \tag{12}$$

$$\mathbf{Q}^{(\mathrm{r})} = \prod_{n=1}^{r}\left(\frac{\partial}{\partial t} + \lambda_n\right), \quad \mathbf{Q}^{(0)} = \mathbf{1} \tag{13}$$

are scalar operators and

$$\hat{\mathbf{K}}_{(s)} = \sum_{n=0}^{s} \hat{\mathbf{K}}_{(n)} \frac{\partial^n}{\partial t^n} \tag{14}$$

$$\hat{\mathbf{Q}}^{(s)} = \sum_{n=0}^{s} \hat{\mathbf{Q}}^{(n)} \frac{\partial^n}{\partial t^n} \tag{15}$$

are tensor operators, $\kappa_n \geq 0, \lambda_n \geq 0$ are inverse values of the relaxation time (plastic flow time). Equations (10) and (11) represent a generalization of the Lee equations for isotropic viscoelastic medium [5, 7, 20].

(b) In the case of integral representation of equation (8) we can write the equation (8) in the form, e.g. [6, 7]

$$\boldsymbol{\sigma} = \int_0^t \hat{\mathbf{G}}(t-\tau) \frac{\partial \boldsymbol{\varepsilon}}{\partial \tau} \mathrm{d}\tau \tag{16}$$

$$\boldsymbol{\varepsilon} = \int_0^t \hat{\mathbf{J}}(t-\tau) \frac{\partial \boldsymbol{\sigma}}{\partial \tau} \mathrm{d}\tau , \tag{17}$$

where $\hat{\mathbf{G}}(t)$ is a tensor operator of relaxation functions and $\hat{\mathbf{J}}(t)$ is a tensor operator of the plastic flow (both tensors are of the 4th order for anisotropic media). By using the Laplace transformation to the formulas (8), (10), (11) we get

$$\tilde{\boldsymbol{\sigma}} = \tilde{\hat{\mathbf{H}}}(p)\tilde{\boldsymbol{\varepsilon}} \tag{18}$$

$$\mathbf{K}^{(r)}(p)\tilde{\boldsymbol{\sigma}} = \tilde{\hat{\mathbf{K}}}_{(s)}(p)\tilde{\boldsymbol{\varepsilon}} \tag{19}$$

$$\mathbf{Q}^{(r)}(p)\tilde{\boldsymbol{\varepsilon}} = \tilde{\hat{\mathbf{Q}}}^{(s)}(p)\tilde{\boldsymbol{\sigma}} , \tag{20}$$

where for classical Laplace transform (with homogenous initial conditions) it holds

$$\tilde{F} = F(p) = \int_0^\infty f(t) e^{-pt} \mathrm{d}t , \tag{21}$$

where $\frac{\partial^n}{\partial t^n} \to p^n$, p is the parameter of Laplace transform. Similarly, taking the Laplace transform of (16), (17) we get

$$\tilde{\boldsymbol{\sigma}} = \tilde{\hat{\mathbf{G}}}(p) p \tilde{\boldsymbol{\varepsilon}} \tag{22}$$

$$\tilde{\boldsymbol{\varepsilon}} = \tilde{\hat{\mathbf{J}}}(p) p \tilde{\boldsymbol{\sigma}} . \tag{23}$$

5 CONSTITUTIVE EQUATIONS FOR SOME RHEOLOGICAL MODELS

Voigt rheological model

Structural formula is: $(V) = (H)\big|(N)$: (Hook's substance) $\big|$ (Newton's viscous liquid) in parallel connection; $\hat{\mathbf{E}} = E^{ijkl}$ is a tensor – operator of elastic modulus, $\hat{\boldsymbol{\eta}} = \eta^{ijkl}$ is a tensor – operator of viscous modulus.

In this case, the constitutive equation of the rheological model is of a form

$$\boldsymbol{\sigma} = (\hat{\mathbf{E}} + \hat{\boldsymbol{\eta}}\frac{\partial}{\partial t})\boldsymbol{\varepsilon}, \tag{24}$$

where stress and strain tensors read

$$\boldsymbol{\sigma} = {}_H\boldsymbol{\sigma} + {}_N\boldsymbol{\sigma}, \quad {}_H\boldsymbol{\varepsilon} = {}_H\boldsymbol{\varepsilon}. \tag{25}$$

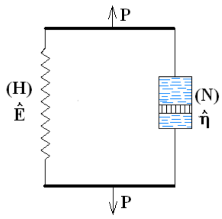

Fig. 2: Rheological model of Voigt material

After applying of the Laplace transform we get

$$\widetilde{\boldsymbol{\sigma}} = (\hat{\mathbf{E}} + p\hat{\boldsymbol{\eta}})\widetilde{\boldsymbol{\varepsilon}} \tag{26}$$

and, after inverting

$$\widetilde{\boldsymbol{\varepsilon}} = det^{-1}(\hat{\mathbf{E}} + p\hat{\boldsymbol{\eta}})\overline{\mathbf{A}}(p)\widetilde{\boldsymbol{\sigma}}, \tag{27}$$

where $\overline{\mathbf{A}}(p)$ is an adjoint p-matrix. Let us denote by the symbol $\Delta(p)$ the expression $\Delta(p) = det(\hat{\mathbf{E}} + p\hat{\boldsymbol{\eta}})$. By comparing (20) and (27) we get further

$$\mathbf{Q}^{(r)}(p) = \Delta(p), \quad \hat{\mathbf{Q}}^{(s)}(p) = \overline{\mathbf{F}}(p). \tag{28}$$

By the decomposition of the tensor operator in the equation (27) to the partial fractions we get

$$\hat{\boldsymbol{\varepsilon}} = \sum_{n=1}^{6} \overline{\mathbf{A}}(p)/(p+\lambda_n)\widetilde{\boldsymbol{\sigma}}, \tag{29}$$

where

$$\overline{\mathbf{A}}(p) = \overline{\mathbf{F}}(-\lambda_n)/\boldsymbol{\Delta}^{(1)}(-\lambda_n), \tag{30}$$

where $d\Delta(p)/dp\big|_{p=-\lambda_n} = \Delta^{(1)}(-\lambda_n)$ and λ_n are equal to the negative values of the roots of the determinant equation $\Delta(p) = 0$ and they represent inverse values of the viscous flow time. When we extend Voigt model by the Hook mass and Newtonian viscous liquid in the serial connection, we can rearrange the formula (29) into the form

$$\widetilde{\boldsymbol{\varepsilon}} = \left(\sum_{n=1}^{6} \overline{\mathbf{A}}(\lambda_n)/(p+\lambda_n) + \hat{\mathbf{C}} + p^{-1}\hat{\boldsymbol{\gamma}} \right)\widetilde{\boldsymbol{\sigma}}, \tag{31}$$

where $\hat{\mathbf{C}} = (\hat{\mathbf{E}})^{-1}, \quad \hat{\boldsymbol{\gamma}} = (\hat{\boldsymbol{\eta}})^{-1}$, and the corresponding structural formula will get the form

$$\{(H) - [(H) - (N)] - (N)\}. \tag{32}$$

By similar attempt we can derive the constitutive equations for a model with the structural formula

$$\{(H) - [(N_1) - (N_2) - \ldots - (N_m)]\}. \tag{33}$$

According to [3, 28] we can say that the process of the linear creep can by realized by a complex rheological model composed from the fundamental substances *(H)* and *(N)* with the following structure:

$$\{(H)-[(V_1)-(V_2)-...-(V_m)]-(N)\}\,, \tag{34}$$

where by the symbol (V_i) we have denoted the i-th Voigt rheological model. By using the inverse Laplace transform Λ^{-1} on the equation (29) we get a strain tensor in the form

$$\boldsymbol{\varepsilon}(t)=\sum_{n=1}^{6}\lambda_n^{-1}\overline{\mathbf{A}}(\lambda_n)\int_0^t[1-e^{\lambda_n(t-\tau)}]\frac{\partial\boldsymbol{\sigma}}{\partial\tau}\mathrm{d}\tau\,. \tag{35}$$

When we compare the equations (17) and (35), we can see that (35) is an equation of the Boltzmann type with the plastic flow tensor in the form

$$\hat{\mathbf{J}}(\mathrm{t})=\sum_{n=1}^{6}\lambda_n^{-1}\overline{\mathbf{A}}(\boldsymbol{\lambda}_\mathbf{n})[1-e^{-\lambda_n t}]\,. \tag{36}$$

Maxwell rheological model

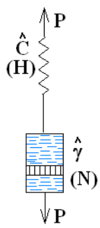

Fig. 3: Rheological model of the Maxwell type

The structural formula *(M) = (H) – (N)*: (Hook's mass) – (Newton viscous liquid) in the serial connection; $\hat{\mathbf{C}}=C^{ijkl}$ is a tensor-operator of the elastic modules, $\hat{\boldsymbol{\gamma}}=\gamma^{ijkl}$ tensor-operator of the viscous modules. With respect to equation (11) we have

$$Q^{(r)}=\frac{\partial}{\partial t},\quad \hat{\mathbf{Q}}=\hat{\mathbf{C}}\frac{\partial}{\partial t}+\hat{\boldsymbol{\gamma}}\,. \tag{37}$$

In this case the constitutive equation of the rheological model will be of the form

$$\frac{\partial}{\partial t}\boldsymbol{\varepsilon}=(\hat{\mathbf{C}}\frac{\partial}{\partial t}+\hat{\boldsymbol{\gamma}})\boldsymbol{\sigma}\,. \tag{38}$$

The mechanical response of this model is the fact that resulting strain equals the sum of strains of the fundamental masses *(H)* and *(N)*. The stresses in *(H)* and *(N)* are the same. Then

$${}_H\boldsymbol{\varepsilon}=\hat{\mathbf{C}}\boldsymbol{\sigma} \tag{39a}$$

$$\frac{\partial}{\partial t}{}_N\boldsymbol{\varepsilon}={}_N\dot{\boldsymbol{\varepsilon}}=\hat{\boldsymbol{\gamma}}\boldsymbol{\sigma} \tag{39b}$$

$$\boldsymbol{\varepsilon}={}_H\boldsymbol{\varepsilon}+{}_N\boldsymbol{\varepsilon} \tag{39c}$$

$$\frac{\partial}{\partial t}\boldsymbol{\varepsilon}=\frac{\partial}{\partial t}({}_H\boldsymbol{\varepsilon}+{}_N\boldsymbol{\varepsilon})=(\hat{\mathbf{C}}\frac{\partial}{\partial t}+\hat{\boldsymbol{\gamma}})\boldsymbol{\sigma}\,. \tag{40}$$

By using the Laplace transform we get

$$p\tilde{\boldsymbol{\varepsilon}}=(p\hat{\mathbf{C}}+\hat{\boldsymbol{\gamma}})\tilde{\boldsymbol{\sigma}}\,. \tag{41}$$

After inversing (41) we can express the stress tensor

$$\tilde{\boldsymbol{\sigma}} = det^{-1}(p\hat{\mathbf{C}} + \hat{\boldsymbol{\gamma}})p\overline{\mathbf{P}}(p)\tilde{\boldsymbol{\varepsilon}}, \tag{42}$$

where $det(p\hat{\mathbf{C}} + \hat{\boldsymbol{\gamma}}) = \Delta_M(p)$ is in general a polynom of the 6th degree of parameter p and $\overline{\mathbf{P}}(p)$ is an adjoint matrix.

In the following we attempt likewise in the case of Voigt rheological model. It means we decompose the expression $\overline{\mathbf{P}}/\Delta_M(p)$ to partial fractions and get

$$\tilde{\boldsymbol{\sigma}} = \sum_{n=1}^{6} \frac{p\overline{\mathbf{B}}(\kappa_n)}{p+\kappa_n}\tilde{\boldsymbol{\varepsilon}}, \tag{43}$$

where

$$\overline{\mathbf{B}}(\kappa_N) = \frac{\overline{\mathbf{P}}(-\kappa_n)}{\Delta_M^{(1)}(-\kappa_n)}, \tag{44}$$

where we denoted

$$\Delta_M^{(1)}(-\kappa_n) = \frac{d\Delta_M(p)}{dp}\bigg|_{p=\kappa_n} \tag{45}$$

and κ_n are negative values of the roots of the determinant equation $\Delta_M(p) = 0$ and they represent inverse values of the relaxation time. When we extend the Maxwell model by the Hook mass and Newton viscous liquid in the parallel connection, the relation (43) can be extended analogously as the relation (31), i.e.

$$\tilde{\boldsymbol{\sigma}} = \left(\sum_{n=1}^{6} \frac{p\overline{\mathbf{B}}(\kappa_n)}{p+\kappa_n} + \hat{\mathbf{E}} + p\hat{\boldsymbol{\eta}}\right)\tilde{\boldsymbol{\varepsilon}} \tag{46}$$

and the corresponding extended rheological model structural formula will be of the form

$$(H)|(N)|[(H)-(N)] = (H)|(M)|(N) \tag{47}$$

and after applying Λ^{-1} on the (43) we get

$$\boldsymbol{\sigma} = \sum \overline{\mathbf{B}}(\kappa_n)\int_0^t e^{-\kappa_n(t-\tau)}\frac{\partial\boldsymbol{\varepsilon}}{\partial\tau}\mathrm{d}\tau, \tag{48}$$

which is the representation of the Boltzmann type equation for the Maxwell rheological model with the tensor function of relaxation in the form

$$\hat{\mathbf{G}}(t) = \sum_{n=1}^{6} \overline{\mathbf{B}}(\kappa_n)e^{-\kappa_n t}. \tag{49}$$

Zener rheological model

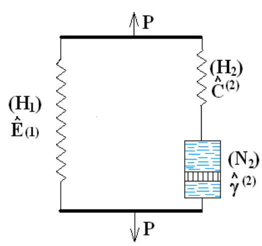

Fig. 4: Rheological model of the Zener mass

From the analyses it is evident that though the Voigt model represents the creep process well, it does not reflect immediate response of the instantaneous deformations. This drawback can be avoided by the parallel connection of the Hook mass with Maxwell rheological model. By doing this we obtain Zener rheological model, Fig. 4.

Structural formula $(Z)=(H_1)\|[(H_2)-(N_2)]$: (Hook's matter$_1$)$\|$ [(Hook's matter$_2$) – (Newton viscous liquid$_2$)]; $\hat{\mathbf{E}}_{(1)}=E_{(1)}^{ijkl}$ is a tensor – operator of elastic modules, $\hat{\mathbf{C}}^{(2)}=C_{ijkl}^{(2)}$ is a tensor – operator of elastic modules of the mass (M), $\hat{\boldsymbol{\gamma}}^{(2)}=\gamma_{ijkl}^{(2)}$ is a tensor – operator of viscous modules of the mass (M). For mechanical response of the Zener model we have

$$\boldsymbol{\sigma}={}_1\boldsymbol{\sigma}+{}_2\boldsymbol{\sigma}, \quad {}_1\boldsymbol{\varepsilon}={}_2\boldsymbol{\varepsilon}, \tag{50}$$

where ${}_2\boldsymbol{\sigma}$ is stress, ${}_2\boldsymbol{\varepsilon}$ is strain in Maxwell model. Considering (41) for Laplace transform of the stress we get

$$\tilde{\boldsymbol{\sigma}}=[p(p\hat{\mathbf{C}}^{(2)}+\boldsymbol{\gamma}^{(2)})^{-1}+\hat{\mathbf{E}}_{(1)}]\tilde{\boldsymbol{\varepsilon}}\,. \tag{51}$$

We can hereinafter rearrange the equation (51) in the sense of equation (43) and the first equation of (50) to the form

$$\tilde{\boldsymbol{\sigma}}=\left(\sum_{n=1}^{6}\frac{p\overline{\mathbf{B}}_{(2)}(\kappa_n)}{p+\kappa_n}+\hat{\mathbf{E}}_{(1)}\right)\tilde{\boldsymbol{\varepsilon}} \tag{52}$$

or after using Λ^{-1} where we get the original for the stress tensor

$$\boldsymbol{\sigma}=\int_0^t[\hat{\mathbf{E}}_{(1)}+\sum_{n=1}^{6}\overline{\mathbf{B}}_{(2)}(\kappa_n)e^{-\kappa_n(t-\tau)}]\frac{\partial\boldsymbol{\varepsilon}}{\partial\tau}\mathrm{d}\tau\,, \tag{53}$$

where

$$\hat{\mathbf{G}}(t)=\hat{\mathbf{E}}_{(1)}+\sum_{n=1}^{6}\overline{\mathbf{B}}_{(2)}(\kappa_n)e^{-\kappa_n t} \tag{54}$$

is a relaxation tensor. Of course, it is possible to use also the other types of rheological models, the various types can be found e.g. in [28] as Poynting – Thompson model, generalized Maxwell model, Voigt model with a finite number of fundamental matters, complex visco-elastic masses with several Voigt and Maxwell groups, etc.

While switch-over to the infinite number of fundamental masses the integro-differential models can be used.

6 CONCLUSION

Solid phase rheology, and especially its branch visco-elasticity and visco-plasticity, e.g. [3-6, 19, 23, 28] deals with deformation and stress analysis not only in steady state, but it observes also the time changes and time change velocities. It solves the relations between the stresses and strains, their time derivatives and time integrals. Various applications of the rheological process are synoptically presented in [19, 26, 34] where also the applications in industry, but also in medicine, in the diagnostic are introduced.

In this paper the differential operator form of constitutive equations are emphasized for linear visco-elastic anisotropic continuum with physical properties invariant in time. We focus to constitutive equations creation for material of so called 1st degree, where the stress tensor depends on the motion $\vec{x}(\vec{X},t)$ namely by means of strain gradients. Derived procedures are applicable for isothermal boundary value problems [16].

The weakness is the numerical realization of the inverse Laplace integral transform, which a lot of literature is devoted to [9, 25, 27]. We can mention [23], where Λ transform perform the improved Schapery – Erdélyi method for analysis of layered half-space.

A special case, so called "time invariant aging theory" with application of the Schwartz distribution theory for linear problems was elaborated by Kovařík [18]. Suitable methods for visco-elastic properties modeling of structures made from real materials is the application of "weakly singular kernels" elaborated e.g. by Koltunov [17], in Slovakia by Sumec and Lichardus [30], Sumec and Potúček [31], respectively for practical applications of space and planar building structural elements.

Dedication: This paper is devoted by the first author to the memory of not lived 85th birthday of his teacher, friend and colleague Prof. Dr. Ing. Jozef Brilla, DrSc. D.Sc, Dr.h.c mult.

ACKNOWLEDGEMENT

This work was supported by the grant APVV 0814-10.

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