Representations of integers by the form $x^2 + xy + y^2 + z^2 + zt + t^2$

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We prove the formula for the number of representations of an integer by the positive definite quaternary form

$$\phi(x, y, z, t) = x^2 + xy + y^2 + z^2 + zt + t^2$$

by adapting the method of Spearman and Williams [2] in their elementary proof of the formula for the representations of an integer as a sum of four squares.

We introduce some notation. Let $\tau(n)$ denote the number of the positive integer divisors of n when n is a positive integer, and set
$tau(n) = 0$ whenever n is not a positive integer. Analogously let $\sigma(n)$ denote the sum of the positive integer divisors of n when n is a positive integer, and set $\sigma(n) = 0$ whenever n is not a positive integer. Also let $\chi(n)$ denote the nontrivial Dirichlet character modulo 3, that is

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Finally let $r(n)$ denote the number of quadruples $(x, y, z, t) \in \mathbf{Z}^4$ with $n = \phi(x, y, z, t)$.

Theorem 1 *Let n be a positive integer. Then*

$$r(n) = 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right).$$

Proof

Let $s(n)$ denote the number of representations of n as x^2+xy+y^2 with $(x,y)\in\mathbf{Z}^2$. Then

$$r(n)=\sum_{k=0}^{n} s(k)s(n-k). \qquad (1)$$

We use the classical formula of Lorenz [1] for the $s(n)$ namely

$$\sum_{n=0}^{\infty} s(n)x^n = 1+6\sum_{m=1}^{\infty}\chi(m)\frac{x^m}{1-x^m}. \qquad (2)$$

Expanding these gives

$$\sum_{n=0}^{\infty} s(n)x^n = 1+6\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\chi(m)x^{mr}$$

so that

$$s(n)=6\sum_{m|n}\chi(m) \qquad (3)$$

for $n>0$. We remark that (3) can be proved using elementary algebraic number theory. The number $s(n)$ is the number of elements of the ring $R=\mathbf{Z}[\exp(2\pi i/3)]$ having norm n. As R has unique factorization and exactly six units then $t_n=s(n)/6$ is the number of norm n ideals of R. It is easily proved that t_n is a multiplicative function of n, and so it suffices to prove (3) when n is a prime power. This follows from the splitting law for rational primes in R. From (3), (2) follows.

Define, for $n>0$ and $i\in\{0,1,2\}$

$$\tau_i(n)=\sum_{\substack{d|n\\ d\equiv i(3)}} 1.$$

Then

$$\tau(n)=\tau_0(n)+\tau_1(n)+\tau_2(n),$$
$$\tau_0(n)=\tau(n/3)$$

and

$$s(n)=6(\tau_1(n)-\tau_2(n)).$$

From (1) for $n>0$

$$r(n)-2s(n) = \sum_{k=1}^{n-1} s(k)s(n-k)$$

$$= 36\sum_{k=1}^{n-1}\sum_{a|k}\sum_{b|(n-k)}\chi(a)\chi(b)$$
$$= 36\sum_{k=1}^{n-1}\sum_{aA+bB=n}\chi(a)\chi(b).$$

Here and in the sequel the variables a, A, b, B, c, C run through the positive integers. Equivalently,

$$\frac{r(n)-2s(n)}{36} = \sum_{\substack{aA+bB=n\\ 3\nmid ab\\ a\equiv b(3)}} 1 - \sum_{\substack{aA+bB=n\\ 3\nmid ab\\ a\equiv -b(3)}} 1. \qquad (4)$$

For $\varepsilon = \pm 1$,

$$\sum_{\substack{aA+bB=n\\ 3\nmid ab\\ a\equiv \varepsilon b(3)}} 1 = \sum_{\substack{aA+bB=n\\ a\equiv \varepsilon b(3)}} 1 - \sum_{\substack{aA+bB=n\\ a\equiv b\equiv 0(3)}} 1.$$

Hence (4) becomes

$$\frac{r(n)-2s(n)}{36} = \sum_{\substack{aA+bB=n\\ a\equiv b(3)}} 1 - \sum_{\substack{aA+bB=n\\ a\equiv -b(3)}} 1. \qquad (5)$$

We wish to partition the summation ranges in a convenient way. To this end define

$$S_1 = \sum_{\substack{3aA+bB=n\\ A<B}} 1, \quad S_2 = \sum_{\substack{3aA+bB=n\\ A>B}} 1, \quad S_3 = \sum_{\substack{3aA+bB=n\\ A=B}} 1,$$

$$T_1 = \sum_{\substack{3aA+bB=n\\ 3a>b}} 1, \quad T_2 = \sum_{\substack{3aA+bB=n\\ 3a<b}} 1, \quad T_3 = \sum_{\substack{3aA+bB=n\\ 3a=b}} 1.$$

Then

$$\sum_{\substack{aA+bB=n\\ a\equiv b(3)}} 1 = \sum_{\substack{aA+bB=n\\ a>b\\ a\equiv b(3)}} 1 + \sum_{\substack{aA+bB=n\\ a<b\\ a\equiv b(3)}} 1 + \sum_{\substack{aA+bB=n\\ a=b\\ a\equiv b(3)}} 1$$
$$= 2\sum_{\substack{aA+bB=n\\ a>b\\ a\equiv b(3)}} 1 + \sum_{a(A+B)=n} 1$$
$$= 2\sum_{(3c+b)A+bB=n} 1 + \sum_{a|n}\sum_{(A+B)=n/a} 1$$

$$
\begin{aligned}
&= 2 \sum_{3cA+b(A+B)=n} 1 + \sum_{a|n} \left(\frac{n}{a} - 1\right) \\
&= 2 \sum_{\substack{3cA+bC=n \\ A<C}} 1 + \sigma(n) - \tau(n) \\
&= 2S_1 + \sigma(n) - \tau(n)
\end{aligned}
$$

making the substitutions $c = (a-b)/3$ as $a > b$ and $a \equiv b \pmod 3$, and $C = A + B > A$ Also

$$
\begin{aligned}
\sum_{\substack{aA+bB=n \\ a\equiv -b(3)}} 1 &= \sum_{\substack{aA+bB=n \\ A<B \\ a\equiv -b(3)}} 1 + \sum_{\substack{aA+bB=n \\ A>B \\ a\equiv -b(3)}} 1 + \sum_{\substack{aA+bB=n \\ A=B \\ a\equiv -b(3)}} 1 \\
&= 2 \sum_{\substack{aA+bB=n \\ A<B \\ a\equiv -b(3)}} 1 + \sum_{\substack{(a+b)A=n \\ a+b\equiv 0(3)}} 1 \\
&= 2 \sum_{\substack{aA+b(A+C)=n \\ a+b\equiv 0(3)}} 1 + \sum_{c|(n/3)} \sum_{a+b=3c} 1 \\
&= 2 \sum_{\substack{(a+b)A+bC=n \\ a+b\equiv 0(3)}} 1 + \sum_{c|(n/3)} (3c-1) \\
&= 2 \sum_{\substack{3cA+bC=n \\ 3c>b}} 1 + 3\sigma(n/3) - \tau(n/3) \\
&= 2T_1 + 3\sigma(n/3) - \tau(n/3)
\end{aligned}
$$

this time setting $B = A + C$ and $a + b = 3c$.

From (5) we now get

$$
\begin{aligned}
\frac{r(n) - 2s(n)}{36} &= 2(S_1 - T_1) + (\sigma(n) - 3\sigma(n/3)) - (\tau(n) - \tau(n/3)) \\
&= 2(S_1 - T_1) + (\sigma(n) - 3\sigma(n/3)) - (\tau_1(n) + \tau_2(n)).
\end{aligned}
$$

As $s(n) = 6(\tau_1(n) - \tau_2(n))$ then

$$
r(n) = 72(S_1 - T_1) + 36(\sigma(n) - 3\sigma(n/3)) - 24(\tau_1(n) + 2\tau_2(n)).
$$

To complete the proof it suffices to prove

$$
S_1 - T_1 = \frac{1}{3}(\tau_1(n) + 2\tau_2(n)) - \frac{1}{3}(\sigma(n) - 3\sigma(n/3)). \tag{6}
$$

Clearly

$$
S_1 + S_2 + S_3 = \sum_{3aA+bB} 1 = T_1 + T_2 + T_3
$$

and

$$S_2 = \sum_{3a(B+C)+bB=n} 1 = \sum_{3aC+(3a+b)B=n} 1 = T_2$$

and so

$$S_1 - T_1 = T_3 - S_3.$$

Now

$$\begin{aligned} S_3 &= \sum_{(3a+b)A=n} 1 = \sum_{A|n} \sum_{3a+b=n/A} 1 = \sum_{A|n} \left(\left\lceil \frac{n}{3A} \right\rceil - 1 \right) = \sum_{C|n} \left(\left\lceil \frac{C}{3} \right\rceil - 1 \right) \\ &= \sum_{\substack{C|n \\ C\equiv 0(3)}} \left(\frac{C}{3} - 1 \right) + \sum_{\substack{C|n \\ C\equiv 1(3)}} \frac{C-1}{3} + \sum_{\substack{C|n \\ C\equiv 2(3)}} \frac{C-2}{3} \\ &= \frac{\sigma(n)}{3} - \tau_0(n) - \frac{\tau_1(n)}{3} - \frac{2\tau_2(n)}{3} \end{aligned}$$

and

$$\begin{aligned} T_3 &= \sum_{3a(A+B)=n} 1 = \sum_{a|(n/3)} \sum_{A+B=n/(3a)} 1 \\ &= \sum_{a|(n/3)} \left(\frac{n}{3a} - 1 \right) = \sigma(n/3) - \tau(n/3). \end{aligned}$$

Hence, as $\tau(n/3) = \tau_0(n)$,

$$S_1 - T_1 = T_3 - S_3 = \frac{1}{3}(\tau_1(n) + 2\tau_2(n)) - \frac{1}{3}(\sigma(n) - 3\sigma(n/3))$$

which proves (6) and the theorem. □

We remark that the generating function of the sequence $r(n)$ is

$$\sum_{n=0}^{\infty} r(n)x^n = 1 + 12\sum_{m=1}^{\infty} \frac{mx^m}{1-x^m} - 36\sum_{m=1}^{\infty} \frac{mx^{3m}}{1-x^{3m}} = 1 + 12\sum_{3\nmid m} \frac{mx^m}{1-x^m}.$$

References

[1] L. Lorenz, 'Bidrag til tallenes theori', *Tidsskrift for Mathematik (3)* **1** (1871), 97–114.

[2] B. K. Spearman & K. S. Williams, 'The simplest arithmetic proof of Jacobi's four squares theorem' *Far East J. Math. Sci. (FJMS)* **2** (2000), 433–439.