Journal of Combinatorics and Number Theory
Volume 0, Issue 0, pp. 1-9
ISSN 1942-5600

A NEW CLASS OF THETA FUNCTION IDENTITIES IN TWO VARIABLES

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Received April 20, 2010; Accepted September 17, 2010.

Abstract

We describe a new class of identities, which hold for certain general theta series, in two completely independent variables. We provide explicit examples of these identities involving the Dedekind eta function, Jacobi theta functions, and various theta functions of Ramanujan.

2010 Mathematics Subject Classification: Primary 11F27; Secondary 11E39, 33E05.

1. Introduction

Let $z \in \mathcal{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$ and set $q = e(z)$ where $e(x) = \exp(2\pi i x)$. The Dedekind eta-function is defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

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Up to normalisation it is a 24-th root of the discriminant of an elliptic curve $\mathbb{C}/\Lambda$ where Λ is the lattice $[1, z]$. The Dedekind eta function also appears in the Chowla-Selberg formula, which gives a closed form expression for the value at 1 of the L-series associated to unramified Abelian extensions of imaginary quadratic number fields. As such, the Dedekind eta function is intimately tied up with the theory of complex multiplication and explicit class field theory.

Relations in one variable for the Dedekind eta function (known as modular equations) have played an important part in construction of class invariants (generators for class field extensions of imaginary quadratic number fields). See [Cox] and [We] for details.

In [BH] the following striking identity in two variables was proved using Jacobi's triple product identity.

$$\begin{aligned} 27\eta^3(3z)\eta^3(3w) &= \eta^3\left(\frac{z}{3}\right)\eta^3\left(\frac{w}{3}\right) + i\eta^3\left(\frac{z+1}{3}\right)\eta^3\left(\frac{w+1}{3}\right) \\ &\quad -\eta^3\left(\frac{z+2}{3}\right)\eta^3\left(\frac{w+2}{3}\right), \end{aligned} \tag{1.1}$$

for all $z, w \in \mathcal{H}$.

Köhler [Koh] subsequently also used the triple product identity to give another proof of the above.

Two variable relations such as the above can be viewed as giving rise to *permanent identities*, i.e. infinite families of one variable identities. For example, one may make the substitution $w = \tau$, $z = n\tau$. The identity then becomes a relation between $\eta(\tau)$ and $\eta(n\tau)$ which does not vary as n varies. In fact the above identity was discovered in the second author's thesis [H] by noting such a permanent family of identities for the eta function.

In this paper we generalize identity (1.1) for a very general class of theta functions. In particular we define a general theta function in the following way:

$$\Theta_{\nu,a,b,m,\psi}(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^{\nu} q^{\frac{(an+b)^2}{m}}, \tag{1.2}$$

where $z \in \mathcal{H}$, $\nu, a, b, m \in \mathbb{Z}$, $m > 0$, $(a, b) = 1$, and ψ is an additive character modulo a. Many well known functions in number theory, including $\eta^3(z)$, falls into this class of theta function. We will describe some examples in Section 3.

The general theta function bears some resemblance to the theta function defined by Shimura [S]. However we note that unlike Shimura's theta function, our definition includes functions which are not modular forms (see Example 5 below).

2. Generalized Identities

Fix two sets ν, a, b, m, ψ and ν', a', b', m', ψ', satisfying the above conditions and, for simplicity, denote

$$f(z) = \Theta_{\nu,a,b,m,\psi}(z), \quad g(z) = \Theta_{\nu',a',b',m',\psi'}(z).$$

The main result that we prove in this paper is the following identity.

Theorem 1. Let p be an odd prime such that $a \mid (p-1)$ and $a' \mid (p-1)$, then

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{z'-m'jk}{p}\right) = \psi(c)\psi'(c')\, p^{\nu+\nu'+1}\, f(pz)g(pz'),$$

for all $z, z' \in \mathcal{H}$, where k is any quadratic nonresidue modulo p, $c = b(p-1)/a$ and $c' = b'(p-1)/a'$.

In particular when $p \equiv 3$ *(mod* 4*)* then

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{z'+m'j}{p}\right) = \psi(c)\psi'(c')\, p^{\nu+\nu'+1}\, f(pz)g(pz'),$$

since -1 is a quadratic nonresidue.

When the theta function involved is a half integer weight modular form, the identity can be interpreted in terms of the half integer weight modular form being annihilated by the Hecke operator T_p. See [Kob] for details. However, as noted, our definition includes theta functions which are not modular forms, so no general proof along these lines is known to the authors.

Proof of Theorem 1. Let

$$f_j(z) = f\left(\frac{z+mj}{p}\right),$$

and note that

$$f_j(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^{\nu} e((an+b)^2 j/p) q^{\frac{(an+b)^2}{mp}}.$$

Thus f_j depends only on the congruence class of j modulo p. The vector

$$\alpha(z) = \big(f_0(z), f_1(z), \ldots, f_{p-1}(z)\big) \in \mathbb{C}^p,$$

lies in the linear subspace V_p of $\mathbb{C}^p$ spanned by the vectors

$$v_d = \big(1, e(d^2/p), e(2d^2/p), \ldots, e((p-1)d^2/p)\big),$$

where d runs through the integers.

There are $(p+1)/2$ distinct squares modulo p, so there are $(p+1)/2$ distinct v_d; these are linearly independent. Hence V_p has dimension $(p+1)/2$.

Now let

$$f_\infty(z) = \psi(c)p^{\nu+1/2} f(pz) = \psi(c)p^{\nu+1/2} \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^{\nu} q^{\frac{(an+b)^2 p}{m}},$$

where $c = b(p-1)/a$.

We express f_∞ in terms of f_j, i.e. we derive an identity involving only one variable z, for these functions. Consider

$$\begin{aligned}\sum_{j=0}^{p-1} f_j(z) &= \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^\nu q^{\frac{(an+b)^2}{mp}} \sum_{j=0}^{p-1} e((an+b)^2 j/p) \\ &= p \sum_{\substack{n=-\infty \\ p|(an+b)}}^{\infty} \psi(n)(an+b)^\nu q^{\frac{(an+b)^2}{mp}}.\end{aligned}$$

But $p \mid (an+b)$ if and only if $n = ps + c$ where $s \in \mathbb{Z}$. Thus

$$\begin{aligned}\sum_{j=0}^{p-1} f_j(z) &= p \sum_{s=-\infty}^{\infty} \psi(ps+c)(aps+bp)^\nu q^{\frac{(aps+bp)^2}{mp}} \\ &= p^{\nu+1}\psi(c) \sum_{s=-\infty}^{\infty} \psi(s)(as+b)^\nu q^{\frac{(as+b)^2 p}{m}} \\ &= \sqrt{p}\, f_\infty(z).\end{aligned}$$

Now we see that

$$F(z) = \big(f_0(z), f_1(z), \ldots, f_{p-1}(z), f_\infty(z)\big)$$

lies in the vector subspace W_p of $\mathbb{C}^{p+1}$ spanned by the vectors

$$w_0 = \big(1, 1, 1, \ldots, 1, \sqrt{p}\big)$$

and

$$w_d = \big(1, e(d^2/p), e(2d^2/p), \ldots, e((p-1)d^2/p), 0\big),$$

where d runs through the integers prime to p. Again W_p has dimension $(p+1)/2$.

Now, if we let

$$g_j(z') = g\left(\frac{z' - m'jk}{p}\right) \qquad \text{and} \qquad g_\infty(z') = \psi'(c')p^{\nu'+1/2} g(pz'),$$

a similar argument proves that

$$G(z') = \big(g_0(z'), g_1(z'), \ldots, g_{p-1}(z'), g_\infty(z')\big) \in W'_p$$

where W'_p is the subspace of $\mathbb{C}^{p+1}$ spanned by

$$w'_0 = w_0 = \big(1, 1, 1, \ldots, 1, \sqrt{p}\big)$$

and

$$w'_d = \big(1, e(-kd^2/p), e(-2kd^2/p), \ldots, e(-(p-1)kd^2/p), 0\big).$$

Next, for $v, u \in \mathbb{C}^{p+1}$, we define

$$B(v, u) = \sum_{j=0}^{p-1} v_j u_j - v_\infty u_\infty,$$

a bilinear form on $\mathbb{C}^{p+1}$.

Clearly for d_1, d_2 not both zero modulo p,

$$B(w_{d_1}, w'_{d_2}) = \sum_{j=0}^{p-1} e\left(\frac{j(d_1^2 - kd_2^2)}{p}\right) = 0,$$

since k is a quadratic nonresidue of p.

Also

$$B(w_{d_1}, w'_0) = \sum_{j=0}^{p-1} e(d_1^2 j/p) = 0$$

for d_1 nonzero modulo p. Similarly $B(w_0, w'_{d_2}) = 0$ for d_2 nonzero modulo p.

Finally,

$$B(w_0, w'_0) = \sum_{j=0}^{p-1} 1 - (\sqrt{p})^2 = 0.$$

Thus $B(v, u) = 0$ for all $v \in W_p, u \in W'_p$. In particular, $B(F(z), G(z')) = 0$, i.e.

$$\sum_{j=0}^{p-1} f_j(z) g_j(z') = \psi(c)\psi'(c')\, p^{\nu+\nu'+1} f(pz) g(pz'),$$

as was to be shown. □

Theorem 1 required that $a \mid (p-1)$ and $a' \mid (p-1)$. We can make a small modification to deal with the case where $a \mid (p+1)$. Indeed we simply note that for the original theta function we defined,

$$\Theta_{\nu,a,b,m,\psi}(z) = (-1)^\nu \sum_{n=-\infty}^{\infty} \psi(-n)(an-b)^\nu q^{\frac{(an-b)^2}{m}}.$$

Now our argument goes through much the same as before, except that we now require $\psi(n) = \psi(-n)$, etc., i.e. ψ and ψ' must now be real-valued characters taking only the values ± 1. We thus have the following.

Theorem 1.1. Let p be an odd prime such that $a \mid (p+1)$ and $a' \mid (p+1)$, then

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{z'-m'jk}{p}\right) = -\psi(c)\psi(c')\, (-p)^{\nu+\nu'+1} f(pz) g(pz'),$$

for all $z, z' \in \mathcal{H}$, where k is any quadratic nonresidue modulo p, $c = b(p+1)/a$ and $c' = b'(p+1)/a'$.

Theorem 1.2. Let p be an odd prime such that $a \mid (p-1)$ and $a' \mid (p+1)$, then

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{z'-m'jk}{p}\right) = (-1)^{\nu'} \psi(c)\psi(c')\, p^{\nu+\nu'+1} f(pz) g(pz'),$$

for all $z, z' \in \mathcal{H}$, where k is any quadratic nonresidue modulo p, $c = b(p-1)/a$ and $c' = b'(p+1)/a'$.

Remark. We note that the space V_p is essentially the same as $\mathcal{Q}$ in [Ch], which played a rôle as complex analogue of the quadratic residue code.

3. Examples

The theta function that we introduced in (1.2) is clearly a generalization of the null values of the classical Jacobi theta functions [WW, p. 463]:

$$\begin{aligned}
\theta_1'(q) &= \sum_{n=-\infty}^{\infty} i^{(2n-1)}(2n+1)q^{(2n+1)^2/4},\\
\theta_2(q) &= \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4},\\
\theta_3(q) &= \sum_{n=-\infty}^{\infty} q^{n^2},\\
\theta_4(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.
\end{aligned}$$

The Jacobi theta functions have several important applications to number theory. For example θ_3^3 can be used to compute class numbers of imaginary quadratic number fields, see [W] for details.

We shall describe below, several important functions satisfying the definition in (1.2).

Example 1: Jacobi's formula gives,

$$\eta^3(z) = \sum_{n=-\infty}^{\infty} (4n+1)q^{(4n+1)^2/8},$$

corresponding to $\Theta_{\nu,a,b,m,\psi}(z)$ with $\nu = 1$, $a = 4$, $b = 1$, $m = 8$ and ψ the trivial character.

Theorem 1 and 1.1, with $f(z) = \eta^3(z)$ and $g(z') = \eta^3(z')$, then yields:

$$p^3\eta^3(pz)\eta^3(pz') = \sum_{j=0}^{p-1} \eta^3\left(\frac{z+8j}{p}\right)\eta^3\left(\frac{z'-8jk}{p}\right), \tag{3.3}$$

for any odd prime p.

The case $p = 3$ and $k = -1$ reduce to identity (1.1) on using the fact that $\eta^3(z+1) = e(1/8)\eta^3(z)$.

Example 2: Let $f(z) = \eta(z)$ and $g(z') = \eta(z')$. Then by Euler's pentagonal number formula,

$$\eta(z) = q^{1/24}\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24},$$

corresponding to $\Theta_{\nu,a,b,m,\psi}(z)$ with $\nu = 0$, $a = 6$, $b = 1$, $m = 24$ and $\psi(n) = (-1)^n$.

It follows that for $p > 3$,

$$p\eta(pz)\eta(pz') = \sum_{j=0}^{p-1} \eta\left(\frac{z+24j}{p}\right)\eta\left(\frac{z'-24jk}{p}\right), \tag{3.4}$$

where k is a quadratic nonresidue modulo p.

Example 3: If we let $f(z) = \eta(z)$ and $g(z') = \eta^3(z')$. Since -3 is a quadratic nonresidue for primes $p \equiv 11 \pmod{12}$, we have

$$p^2\eta(pz)\eta^3(pz') = \sum_{j=0}^{p-1} \eta\left(\frac{z+24j}{p}\right)\eta^3\left(\frac{z'+24j}{p}\right). \tag{3.5}$$

Example 4: In [B, p.114] Entry 8 (ix) and (x), Ramanujan studied two theta functions which are consequences of the quintuple product identity [B, p.83], [Coo]. Using his notation for $\phi(q), \psi(q)$ and $f(-q)$, these are

$$\begin{aligned} q^{1/24}\phi^2(-q)f(-q) &= \frac{\eta^5(z)}{\eta^2(2z)} = \sum_{n=-\infty}^{\infty}(6n+1)q^{\frac{(6n+1)^2}{24}}, \\ q^{1/3}\psi(q^2)f^2(-q) &= \frac{\eta^2(4z)\eta^2(z)}{\eta(2z)} = \sum_{n=-\infty}^{\infty}(3n+1)q^{\frac{(3n+1)^2}{3}}. \end{aligned}$$

A simple transform of the latter function gives the following,

$$\frac{\eta^5(2z)}{\eta^2(z)} = \sum_{n=-\infty}^{\infty}(-1)^n(3n+1)q^{\frac{(3n+1)^2}{3}}.$$

All three clearly satisfy the definition given in (1.2).

Example 5: The preceding examples all involved the case where $\nu = 0$ or 1. Ramanujan [R, p. 369] , [AB, p.355–362] also studied theta functions involving higher values for ν. Let

$$\begin{aligned} U_{2\alpha+1}(q) &= \sum_{n=-\infty}^{\infty}(4n+1)^{2\alpha+1}q^{\frac{(4n+1)^2}{8}}, \\ V_{2\alpha}(q) &= \sum_{n=-\infty}^{\infty}(-1)^n(6n+1)^{2\alpha}q^{\frac{(6n+1)^2}{24}}. \end{aligned}$$

Ramanujan showed that

$$\begin{aligned} U_{2\alpha+1}(q) &= \eta^3(z)\sum_{i+2j+3k=\alpha} a_{ijk}E_2^iE_4^jE_6^k, \\ V_{2\alpha}(q) &= \eta(z)\sum_{i+2j+3k=\alpha} b_{ijk}E_2^iE_4^jE_6^k, \end{aligned}$$

where a_{ijk} and b_{ijk} are rational numbers and E_n are the normalized Eisenstein series of weight n. See [CCT] and references therein for more information about these two classes of theta functions.

Note that this example includes theta functions under our definition which are not modular forms, due to the presence of E_2.

Summary and Concluding Remarks

We have used elementary methods to prove our main identity for a general class of theta functions and given several well known examples of such theta functions.

Our general theorem can be used to construct infinite families of identities for such theta functions. It is unknown whether there is a link between such permanent identities and the identities of Nathan Fine [F].

Acknowledgements

Many thanks to Bruce Berndt for suggesting that we check the applicability of the method of this paper to the theta functions of Example 4, as defined by Ramanujan.

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