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SOME HESITANT FUZZY GEOMETRIC OPERATORS AND THEIR APPLICATION TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING

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Abstract. Hesitant fuzzy set (HFS), a generalization of fuzzy set (FS), permits the membership degree of an element of a set to be represented as several possible values between 0 and 1. In this paper, motivated by the extension principle of HFs, we export Einstein operations on FSs to HFs, and develop some new aggregation operators, such as the hesitant fuzzy Einstein weighted geometric operator, hesitant fuzzy Einstein ordered weighted geometric operator, and hesitant fuzzy Einstein hybrid weighted geometric operator, for aggregating hesitant fuzzy elements. In addition, we discuss the correlations between the proposed aggregation operators and the existing ones respectively. Finally, we apply the hesitant fuzzy Einstein weighted geometric operator to multiple attribute group decision making with hesitant fuzzy information. Some numerical examples are given to illustrate the proposed aggregation operators.

Keywords: hesitant fuzzy set (HFS), Einstein t-norm, hesitant fuzzy Einstein weighted geometric (HFWG^e) operator, hesitant fuzzy Einstein ordered weighted geometric (HFOWG^e) operator, hesitant fuzzy Einstein hybrid weighted geometric (HFHWG^e) operator, multiple attribute group decision making (MAGDM).

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Introduction

Aggregation operators, usually taking the forms of mathematical functions, are common techniques to fuse all the input individual data into a single one. Due to their great importance in the information processing of extensive areas, such as decision making, pattern recognition, information retrieval, medical diagnosis, data mining, machine learning, etc., the investigation on aggregation operators has been receiving much attention from both researchers and practitioners over the last decades (Yager, Kacprzyk 1997; Calvo *et al.* 2002; Xu, Da 2003; Torra, Narukawa 2007; Beliakov *et al.* 2007; Grabisch *et al.* 2009). Three of the most common geometric operators for aggregating arguments are the weighted geometric (WG) operator (Saaty 1980; Aczl, Saaty 1983; Willett, Sharda 1991; Benjamin *et al.* 1992; Xu 2000; Xu, Da 2003), the ordered weighted geometric (OWG) operator (Herrera *et al.* 2001; Chiclana *et al.* 2001; Xu, Da 2003), which based on the ordered weighted geometric (OWG) operator (Yager 1988) and the geometric mean, and the hybrid weighted geometric (HWG) operator (Xu, Da 2003). The WG operator first weights all the given arguments and then aggregates all these weighted arguments into a collective one. The fundamental aspect of the OWG operator is the reordering step before aggregating all the ordered weighted arguments into a collective one. The HWG operator generalizes both the OWG and WG operators and reflects the importance degrees of both the given argument and the ordered position of the argument.

In the real-life world, due to the increasing complexity of the socioeconomic environment and the lack of knowledge or data about the problem domain, crisp data are sometimes unavailable. Thus, the input arguments may be vague or fuzzy in nature. Besides fuzzy sets (FSs) by Zadeh (1965), several extensions of this concept have been introduced in the literature, for example, intuitionistic fuzzy sets (Atanassov 1986), interval-valued fuzzy sets (Zadeh 1973), type 2 fuzzy sets (Mizumoto, Tanaka 1976; Dubois, Prade 1980), fuzzy multisets (Yager 1986; Chakrabarty, Despi 2007) and hesitant fuzzy sets (HFSs) (Torra, Narukawa 2009; Torra 2010). IFSs are equivalent to interval-valued fuzzy sets (Atanassov, Gargov 1989; Cornelis *et al.* 2004). The membership of an element to a type 2 fuzzy set is defined in terms of a FS on the domain of memberships. IFSs can be seen, from a mathematical point of view, as a particular case of type 2 fuzzy sets (Dubois *et al.* 2005). Fuzzy multisets, or fuzzy bags, permit us to have multiple occurrences of the elements. The basic elements of a HFS are hesitant fuzzy elements (HFEs) (Xia, Xu 2011), each of which is characterized by a membership degree consisting of a set of possible values. Although all HFSs can be represented as fuzzy multisets, the operations on fuzzy multisets do not apply properly on HFSs. Torra and Narukawa (2009) and Torra (2010) showed that the envelope of HFS is an IFS, and proved that the operations applied to the envelope of HFS are consistent with the ones of IFS. It can be proved that HFSs can also be represented as type 2 fuzzy sets and IFS is a particular case of HFS.

In many practical situations, particularly in the process of group decision making under uncertainty and anonymity, the experts may come from different research areas and thus have different backgrounds and levels of knowledge, skills, experience, and personality, the experts may not have enough expertise or possess a sufficient level of knowledge to precisely express their preferences over the objects, and then, they usually have some uncertainty in providing

their preferences. Moreover, the experts have only assigned a small and finite set in providing their preferences, where the difficulty may be caused by a doubt between a few different values. In such cases, the data or preferences given by the experts may be appropriately expressed in HFEs. For example, in multiple attribute group decision-making (MAGDM) problems, anonymity is required in order to protect the decision makers' privacy or avoid influencing each other, such as presidential election, blind peer review of thesis, etc., in which we do not know which attributes that the decision makers are respectively familiar with, and thus, leading us to consider all the situations in order to get more reasonable decision results. But the existing methods only consider the minor situations that each decision maker is good at evaluating all the attributes, which hardly happen. HFS is very useful in avoiding such issues in which each attribute can be described as a HFE defined in terms of the opinions of decision makers (Torra, Narukawa 2009). Up to now, some authors (Torra, Narukawa 2009; Torra 2010; Xu, Xia 2011a, b; Xia, Xu 2011; Xia *et al.* 2011; Rodriguez *et al.* 2012; Zhu *et al.* 2012a, b; Wei 2012; Yu *et al.* 2012) have paid attention to the HFS theory. Torra and Narukawa (2009) and Torra (2010) proposed the concept of HFS, which is deferent from other extensions exist for FSs, and also introduced some basic operations on HFSs. Torra and Narukawa (2009) presented an extension principle of HFSs, which permits to generalize existing operations on FSs to HFSs, and also discussed their use in decision making. Xu and Xia (2011a, b) proposed a variety of distance measures for HFSs, and particularly developed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures, which can alleviate the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. Xia and Xu (2011) developed a series of aggregation operators for hesitant fuzzy information, and applied them to solve decision making problems. Xia *et al.* (2011) developed several series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means, and gave a group decision making method under hesitant fuzzy environment based on the developed aggregation operators and the weight-determined technics. Rodriguez *et al.* (2012) introduced the concept of hesitant fuzzy linguistic term set (HFLTS) to increase the flexibility and richness of linguistic elicitation, and presented a multi-criteria linguistic decision-making model in which experts provide their assessments by using linguistic expressions based on comparative terms. Zhu *et al.* (2012a) introduced the dual hesitant fuzzy set (DHFS), and investigated some basic operations, properties and an extension principle for DHFSs. The results were illustrated by a practical example of group forecasting. Motivated by the ideal of prioritized aggregation operators (Yager 2008), Wei (2012) developed some prioritized aggregation operators for aggregating hesitant fuzzy information, and utilized these operators to develop some approaches to solve the hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level. Zhu *et al.* (2012b) extended the geometric Bonferroni mean (GBM) to hesitant fuzzy environment, and defined a hesitant fuzzy geometric Bonferroni mean, and applied it to multi-criteria decision making. Yu *et al.* (2012) extended the generalized Bonferroni mean (GBM) to hesitant fuzzy environment and proposed the generalized hesitant fuzzy Bonferroni mean (GHFBM). Then they proposed an approach based on proposed operator for multiple criteria group decision making under hesitant fuzzy environment.

The aforementioned hesitant aggregation operators (Xia, Xu 2011; Xu, Xia 2011b; Xia *et al.* 2011; Wei 2012; Zhu *et al.* 2012b; Yu *et al.* 2012) are mainly based on product triangular norm (t-norm) and its dual triangular conorm (t-conorm) (probabilistic sum), which are the most commonly used ones in decision making applications (Schweizer, Sklar 1983; Hájek 1998). The product and Einstein t-norms are two prototypical examples of the class of strict t-norms. For an intersection on FS, a good alternative to the product t-norm is the Einstein t-norm, which typically gives the same smooth approximations as the product t-norm. Equivalently, for an intersection on FS, a good alternative to probabilistic sum is the Einstein sum. Wang and Liu (2011, 2012) introduced Einstein operations on IFs, and studied some intuitionistic fuzzy aggregation operators with the help of Einstein operations. However, it seems that in the literature there is little investigation on aggregation techniques using the Einstein operations for aggregating a collection of HFEs. In this paper, we shall develop some geometric aggregation operators based on Einstein t-norm and its dual t-conorm, and give an application of these operators to MAGDM. To do so, this paper is structured as follows. In Section 1, we briefly review some basic concepts related to HFSs and the existing geometric operators for aggregating HFEs. In Section 2, we introduce some Einstein operations on HFSs, and develop some novel geometric aggregation operators, such as the hesitant fuzzy Einstein weighted geometric (HFWG^ε) operator, hesitant fuzzy ordered Einstein weighted geometric (HFOWG^ε) operator, and hesitant fuzzy Einstein hybrid weighted geometric (HFHWG^ε) operator, for aggregating a collection of HFEs. In addition, we make some comparisons between the proposed operator and ones proposed by Xia and Xu (2011). In Section 3, we apply the HFWG^ε operator to MAGDM with hesitant fuzzy information. In the last section, we have a conclusion.

1. Preliminary

The FS, an extension of the classical notion of set, was introduced by Zadeh (1965).

Definition 1. Let a set X be fixed, a FS F on X is defined as:

$$F = \{ \langle x, \mu_F(x) \rangle \mid x \in X \}, \quad (1)$$

where μ_F is a mapping from X to the closed interval $[0,1]$, and for each $x \in X$, $\mu_F(x)$ is called the degree of membership of x in X .

The set theoretical operations have had an important role since in the beginning of FS theory. Starting from Zadeh's operations \min and \max many other operators were introduced in the fuzzy set literature (Zadeh 1965). All types of the particular operators were included in the general concepts of t-norms and t-conorms (Schweizer, Sklar 1983; Hájek 1998), which satisfy the requirements of the conjunction and disjunction operators, respectively. They are the most general families of binary functions that map the unit square into the unit interval, i.e. $T : [0,1]^2 \rightarrow [0,1]$ and $S : [0,1]^2 \rightarrow [0,1]$, and they are related by the De Morgan duality: the t-conorm S can be defined as $S(x, y) = 1 - T(1 - x, 1 - y)$, where $(x, y) \in [0,1]^2$.

Here, we introduce some examples of the t-norms and t-conorms (Schweizer, Sklar 1983; Hájek 1998):

- Zadeh-intersection \min is a t-norm, Zadeh-union \max is a t-conorm;
- Algebraic product $\hat{\cdot}$ is a t-norm and Algebraic sum $\hat{+}$ is a t-conorm, where:

$$\hat{\cdot}(x, y) = xy, \hat{+}(x, y) = x + y - xy;$$

- Einstein product $\hat{\cdot}$ is a t-norm and Einstein sum $\hat{+}$ is a t-conorm, where:

$$\hat{\cdot}(x, y) = \frac{xy}{1 + (1-x)(1-y)}, \hat{+}(x, y) = \frac{x+y}{1+xy}.$$

The most accepted one extension of the FS is the notion of IFS (Atanassov 1986), which is characterized by a membership function and a non-member function.

Definition 2. Let a set X be fixed, an IFS A in X is defined as:

$$A = \left\{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \right\}, \tag{2}$$

where μ_A and ν_A are mappings from X to the closed interval $[0,1]$ such that $0 \leq \mu_A(x) \leq 1$, $0 \leq \nu_A(x) \leq 1$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, and they denote the degrees of membership and non-membership of element $x \in X$ to the set A , respectively. Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then it is usually called the intuitionistic fuzzy index of element $x \in X$ to A , representing the degree of indeterminacy or hesitation of x to A . It is obvious that $0 \leq \pi_A(x) \leq 1$ for $x \in X$.

However, when giving the membership degree of an element on FS, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra and Narukawa (2009) and Torra (2010) proposed another generation of FS as follows.

Definition 3. Let X be a reference set, then hesitant fuzzy set on X is defined in terms of a function h that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu (2011) express the HFS as follows:

Definition 4. Let X be a fixed set, a HFS E on X is defined as:

$$E = \left\{ \langle x, h_E(x) \rangle \mid x \in X \right\}, \tag{3}$$

where $h_E(x)$ is a set-valued function from X to the power set of the unit interval (i.e. $2^{[0,1]}$) and denotes the possible membership degrees of the element $x \in X$ to the set E . For convenience, let Ω be the set of all HFSs on X .

Given $x \in X$, $h_E(x)$ is called as a hesitant fuzzy element (HFE) (Xia, Xu 2011), which simply denoted as $h = h(x)$. For convenience, let H be the set of all HFEs on X .

For three HFEs $h, h_1, h_2 \in H$, Torra (2010) and Xia and Xu (2011) defined some operations on them, which can be described as:

- 1) $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$;
- 2) $h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;
- 3) $h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$;

- 4) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{ \gamma_1 + \gamma_2 - \gamma_1 \gamma_2 \};$
- 5) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{ \gamma_1 \gamma_2 \};$
- 6) $\lambda h = \cup_{\gamma \in h} \{ 1 - (1 - \gamma)^\lambda \};$
- 7) $h^\lambda = \cup_{\gamma \in h} \{ \gamma^\lambda \}.$

To compare the HFEs, Xia and Xu (2011) defined the following comparison laws.

Definition 5. For a HFE h , $s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ is called the score function of h , where $\#h$ is the number of the elements in h . For two HFEs h_1 and h_2 , if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Based on the above algebraic operational laws of HFEs and Definition 5, Xia and Xu (2011) proposed some geometric aggregation operators for aggregating HFEs as listed below:

For a collection of n HFEs $h_j (j=1, 2, \dots, n)$.

The hesitant fuzzy weighted geometric (HFWG) operator:

$$\begin{aligned} \text{HFWG}_\omega(h_1, h_2, \dots, h_n) &= h_1^{\omega_1} \otimes h_2^{\omega_2} \otimes \dots \otimes h_n^{\omega_n} = \\ & \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n} \left\{ \prod_{j=1}^n \gamma_j^{\omega_j} \right\}, \end{aligned} \tag{4}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j=1, 2, \dots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

The hesitant fuzzy ordered weighted geometric (HFLOWG) operator:

$$\begin{aligned} \text{HFLOWG}_w(h_1, h_2, \dots, h_n) &= h_{\sigma(1)}^{\omega_1} \otimes h_{\sigma(2)}^{\omega_2} \otimes \dots \otimes h_{\sigma(n)}^{\omega_n} = \\ & \cup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \dots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \prod_{j=1}^n \gamma_{\sigma(j)}^{\omega_j} \right\}, \end{aligned} \tag{5}$$

where $h_{\sigma(j)}$ is the j^{th} largest of $h_k (k=1, 2, \dots, n)$, and $w = (w_1, w_2, \dots, w_n)^T$ is the aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

The hesitant fuzzy hybrid weighted geometric (HFHWG^ε) operator:

$$\begin{aligned} \text{HFHWG}_{\omega, w}(h_1, h_2, \dots, h_n) &= \dot{h}_{\sigma(1)}^{\omega_1} \otimes \dot{h}_{\sigma(2)}^{\omega_2} \otimes \dots \otimes \dot{h}_{\sigma(n)}^{\omega_n} = \\ & \cup_{\dot{\gamma}_{\sigma(1)} \in \dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \dots, \dot{\gamma}_{\sigma(n)} \in \dot{h}_{\sigma(n)}} \left\{ \prod_{j=1}^n \dot{\gamma}_{\sigma(j)}^{\omega_j} \right\}, \end{aligned} \tag{6}$$

where $\dot{h}_{\sigma(j)}$ is the j^{th} largest of $\dot{h}_k (\dot{h}_k = h_k^{(n\omega_k)}, k=1, 2, \dots, n)$, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j=1, 2, \dots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. $w = (w_1, w_2, \dots, w_n)^T$ is the aggregation-associated vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Note that the HFWG, HFLOWG and HFHWG operators extend the WG, OWG, and HWG operators to aggregate HFEs, respectively.

2. Hesitant fuzzy Einstein geometric averaging aggregation operators

In this section, we first introduce the extension principle for extending functions to HFEs proposed by Torra and Narukawa (2009).

Definition 6. Let Θ be a function $\Theta: [0,1]^n \rightarrow [0,1]$ and $\tilde{h} = \{h_1, h_2, \dots, h_n\}$ be a set of n HFEs, then the extension of Θ on \tilde{h} is a function $\Theta_{\tilde{h}}: H^n \rightarrow H$,

$$\Theta_{\tilde{h}} = \bigcup_{\gamma \in h_1 \times h_2 \times \dots \times h_n} \{\Theta(\gamma)\}, \tag{7}$$

where $\Theta_{\tilde{h}}$ the extension of an operator Θ on a set of HFEs \tilde{h} , considers all the values in such sets and the application of Θ on them.

Naturally, this extension principle permits us to consider alternative operations for sum and product on HFEs, e.g. Einstein sum and Einstein product.

Let Θ be the Einstein operations on FSs, then the extensions of the Einstein sum and Einstein product on h_1 and h_2 (denoted by $h_1 \oplus_{\varepsilon} h_2$ and $h_1 \otimes_{\varepsilon} h_2$ respectively) are defined as follows respectively:

- 1) $h_1 \oplus_{\varepsilon} h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2} \right\};$
- 2) $h_1 \otimes_{\varepsilon} h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1 \gamma_2}{1 + (1 - \gamma_1)(1 - \gamma_2)} \right\}.$

Theorem 1. If n is any a positive integer and h is a HFE of H , then the power multiplication operation h^{ε^n} is a mapping from $Z^+ \times H$ to H :

$$h^{\varepsilon^n} = \bigcup_{\gamma \in h} \left\{ \frac{2\gamma^n}{(2-\gamma)^n + \gamma^n} \right\}, \tag{8}$$

where: $h^{\varepsilon^n} = \overbrace{h \otimes_{\varepsilon} h \otimes_{\varepsilon} \dots \otimes_{\varepsilon} h}^n$.

Proof. Mathematical induction can be used to prove that Eq. (8) holds for all positive integers n . Eq. (8) is called $P(n)$.

Basis: Show that the statement $P(n)$ holds for $n=1$. The statement $P(n)$ amounts to the statement $P(1): h^{\varepsilon^1} = \bigcup_{\gamma \in h} \left\{ \frac{2\gamma}{(2-\gamma) + \gamma} \right\}$.

In the left-hand side of the equation, $h^{\varepsilon^1} = h = \bigcup_{\gamma \in h} \{\gamma\}$. In the right-hand side of the equation, $\bigcup_{\gamma \in h} \left\{ \frac{2\gamma}{(2-\gamma) + \gamma} \right\} = \bigcup_{\gamma \in h} \{\gamma\}$. The two sides are equal, so the statement $P(n)$ is true for $n=1$. Thus it has been shown the statement $P(1)$ holds.

Inductive step: Show that if $P(n)$ holds, then also $P(n+1)$ holds. Assume $P(n)$ holds (for some unspecified value of n). It must then be shown that $P(n+1)$ holds, that is: $h^{\varepsilon^{(n+1)}} = \bigcup_{\gamma \in h} \left\{ \frac{2\gamma^{(n+1)}}{(2-\gamma)^{(n+1)} + \gamma^{(n+1)}} \right\}$. Using the induction hypothesis that $P(n)$ holds, the left-hand side can be rewritten to $h^{\varepsilon^n} \otimes_{\varepsilon} h$ and based on the Einstein product operation of two HFEs, we have:

$$h^{\varepsilon^n} \otimes_{\varepsilon} h = \bigcup_{\gamma \in h} \left\{ \frac{\frac{2\gamma^n}{(2-\gamma)^n + \gamma^n} \cdot \gamma}{1 + \left(1 - \frac{2\gamma^n}{(2-\gamma)^n + \gamma^n}\right) \cdot (1 - \gamma)} \right\} = \bigcup_{\gamma \in h} \left\{ \frac{2\gamma^{(n+1)}}{(2-\gamma)^{(n+1)} + \gamma^{(n+1)}} \right\}.$$

Thereby showing that indeed $P(n+1)$ holds. Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $P(n)$ holds for any positive integer n .

It can be easily verified that $0 \leq \frac{2\gamma^n}{(2-\gamma)^n + \gamma^n} \leq 1$ even if n is any positive real number. Thus, the operation h^{ε^n} defined above is a HFS for any positive real number n .

Theorem 2. Let h, h_1 and h_2 be three HFEs and let $h_3 = h_1 \otimes_{\varepsilon} h_2$ and $h_4 = h^{\varepsilon^\lambda}$, $\lambda > 0$, then both h_3 and h_4 are also HFEs.

In the following, let us look at h^{ε^λ} for some special cases of λ and h .

Proposition 1. Let h, h_1 and h_2 be three HFEs, $\lambda, \lambda_1, \lambda_2 > 0$, then:

- 1) $h_1 \otimes_{\varepsilon} h_2 = h_2 \otimes_{\varepsilon} h_1$;
- 2) $(h_1 \otimes_{\varepsilon} h_2)^{\varepsilon^\lambda} = h_1^{\varepsilon^\lambda} \otimes_{\varepsilon} h_2^{\varepsilon^\lambda}$;
- 3) $h^{\varepsilon^{\lambda_1}} \otimes_{\varepsilon} h^{\varepsilon^{\lambda_2}} = h^{\varepsilon^{(\lambda_1 + \lambda_2)}}$;
- 4) $h^{\varepsilon^{\lambda_2 \lambda_1}} = (h^{\varepsilon^{\lambda_2}})^{\lambda_1}$.

2.1. Hesitant fuzzy Einstein weighted geometric (HFWG^ε) operator

Similar to the HFWG operator (i.e. (4) described in Section 1), the HFWG^ε operator is defined as follows:

Definition 7. Let $h_j (j=1,2,\dots,n)$ be a collection of HFEs, then a hesitant fuzzy Einstein weighted geometric averaging (HFWG^ε) operator is a mapping $HFWG^{\varepsilon}: H^n \rightarrow H$,

$$HFWG_{\omega}^{\varepsilon}(h_1, h_2, \dots, h_n) = h_1^{\varepsilon^{\omega_1}} \otimes_{\varepsilon} h_2^{\varepsilon^{\omega_2}} \otimes_{\varepsilon} \dots \otimes_{\varepsilon} h_n^{\varepsilon^{\omega_n}}, \tag{9}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j=1,2,\dots,n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^n \omega_j = 1$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the HFWG^ε operator is reduced to a hesitant fuzzy Einstein geometric averaging HFG^ε operator of dimension n , which is defined as follows:

$$HFG_{\omega}^{\varepsilon}(h_1, h_2, \dots, h_n) = (h_1 \otimes_{\varepsilon} h_2 \otimes_{\varepsilon} \dots \otimes_{\varepsilon} h_n)^{\varepsilon^{\frac{1}{n}}}. \tag{10}$$

With the Einstein operational laws of HFEs, the HFWG^ε operator, i.e. Eq. (9), can be transformed into the following form by induction on n .

Theorem 3. Let $h_j (j=1,2,\dots,n)$ be a collection of HFEs, then their aggregated value by using the HFWG^ε operator is also a HFE and

$$HFWG_{\omega}^{\varepsilon}(h_1, h_2, \dots, h_n) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n} \left\{ \frac{2 \prod_{j=1}^n \gamma_j^{\omega_j}}{\prod_{j=1}^n (2 - \gamma_j)^{\omega_j} + \prod_{j=1}^n \gamma_j^{\omega_j}} \right\}, \tag{11}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j=1,2,\dots,n)$ such that $\omega_j \in [0,1]$, $j=1,2,\dots,n$ and $\sum_{j=1}^n \omega_j = 1$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the HFWG $^\varepsilon$ operator is reduced to the HFG $^\varepsilon$ operator of dimension n , which is expressed as follows:

$$\text{HFG}_\omega^\varepsilon(h_1, h_2, \dots, h_n) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n} \left\{ \frac{2 \prod_{j=1}^n \gamma_j^{\frac{1}{n}}}{\prod_{j=1}^n (2 - \gamma_j)^{\frac{1}{n}} + \prod_{j=1}^n \gamma_j^{\frac{1}{n}}} \right\}. \tag{12}$$

Proof. The first result follows quickly from Theorem 2. Below we prove Eq. (11) by using mathematical induction on n . We first prove that Eq. (11) holds for $n = 2$.

Since $h_1^{\varepsilon \omega_1} = \bigcup_{\gamma_1 \in h_1} \left\{ \frac{2\gamma_1^{\omega_1}}{(2-\gamma_1)^{\omega_1} + \gamma_1^{\omega_1}} \right\}$ and $h_2^{\varepsilon \omega_2} = \bigcup_{\gamma_2 \in h_2} \left\{ \frac{2\gamma_2^{\omega_2}}{(2-\gamma_2)^{\omega_2} + \gamma_2^{\omega_2}} \right\}$, and let $a_1 = \gamma_1^{\omega_1}$, $b_1 = (2 - \gamma_1)^{\omega_1}$, $a_2 = \gamma_2^{\omega_2}$ and $b_2 = (2 - \gamma_2)^{\omega_2}$, then:

$$\begin{aligned} \text{HFWG}_\omega^\varepsilon(h_1, h_2) &= h_1^{\varepsilon \omega_1} \otimes_\varepsilon h_2^{\varepsilon \omega_2} = \\ &= \bigcup_{\gamma_1 \in h_1} \left\{ \frac{2a_1}{b_1 + a_1} \right\} \otimes_\varepsilon \bigcup_{\gamma_2 \in h_2} \left\{ \frac{2a_2}{b_2 + a_2} \right\} = \\ &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2a_1 a_2}{b_1 b_2 + a_1 a_2} \right\} = \\ &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2\gamma_1^{\omega_1} \gamma_2^{\omega_2}}{(2 - \gamma_1)^{\omega_1} (2 - \gamma_2)^{\omega_2} + \gamma_1^{\omega_1} \gamma_2^{\omega_2}} \right\}, \end{aligned}$$

i.e. Eq. (11) holds for $n = 2$.

If Eq. (11) holds for $n = k$, i.e.:

$$\text{HFWG}_\omega^\varepsilon(h_1, h_2, \dots, h_k) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_k \in h_k} \left\{ \frac{2 \prod_{j=1}^k \gamma_j^{\omega_j}}{\prod_{j=1}^k (2 - \gamma_j)^{\omega_j} + \prod_{j=1}^k \gamma_j^{\omega_j}} \right\},$$

then, when $n = k + 1$, by the Einstein product, we have:

$$\begin{aligned} \text{HFWG}_\omega^\varepsilon(h_1, h_2, \dots, h_{k+1}) &= \\ &= h_1^{\varepsilon \omega_1} \otimes_\varepsilon h_2^{\varepsilon \omega_2} \otimes_\varepsilon \dots \otimes_\varepsilon h_k^{\varepsilon \omega_k} \otimes_\varepsilon h_{k+1}^{\varepsilon \omega_{k+1}} = \\ &= \text{HFWG}_\omega^\varepsilon(h_1, h_2, \dots, h_k) \otimes_\varepsilon h_{k+1}^{\varepsilon \omega_{k+1}} = \\ &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_k \in h_k} \left\{ \frac{2 \prod_{j=1}^k \gamma_j^{\omega_j}}{\prod_{j=1}^k (2 - \gamma_j)^{\omega_j} + \prod_{j=1}^k \gamma_j^{\omega_j}} \right\} \otimes_\varepsilon \bigcup_{\gamma_{k+1} \in h_{k+1}} \left\{ \frac{2\gamma_{k+1}^{\omega_{k+1}}}{(2 - \gamma_{k+1})^{\omega_{k+1}} + \gamma_{k+1}^{\omega_{k+1}}} \right\}, \end{aligned}$$

and let $a_1 = \prod_{j=1}^k \gamma_j^{\omega_j}$, $b_1 = \prod_{j=1}^k (2 - \gamma_j)^{\omega_j}$, $a_2 = \gamma_{k+1}^{\omega_{k+1}}$ and $b_2 = (2 - \gamma_{k+1})^{\omega_{k+1}}$, then:

$$\begin{aligned} \text{HFWG}_\omega^\varepsilon(h_1, h_2, \dots, h_{k+1}) &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_k \in h_k} \left\{ \frac{2a_1}{b_1 + a_1} \right\} \otimes_\varepsilon \bigcup_{\gamma_{k+1} \in h_{k+1}} \left\{ \frac{2a_2}{b_2 + a_2} \right\} = \\ &\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_k \in h_k, \gamma_{k+1} \in h_{k+1}} \left\{ \frac{2a_1 a_2}{b_1 b_2 + a_1 a_2} \right\} = \\ &\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_k \in h_k, \gamma_{k+1} \in h_{k+1}} \left\{ \frac{2 \prod_{j=1}^{k+1} \gamma_j^{\omega_j}}{\prod_{j=1}^{k+1} (2 - \gamma_j)^{\omega_j} + \prod_{j=1}^{k+1} \gamma_j^{\omega_j}} \right\}, \end{aligned}$$

i.e. Eq. (11) holds for $n = k + 1$.

Therefore, Eq. (11) holds for all n , which completes the proof of Theorem 3.

Lemma 4 (Torra, Narukawa 2007; Xu 2000). Let $a_j > 0$, $\omega_j > 0$, $j = 1, 2, \dots, n$ and $\sum_{j=1}^n \omega_j = 1$, then:

$$\prod_{j=1}^n a_j^{\omega_j} \leq \sum_{j=1}^n \omega_j a_j, \tag{13}$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Corollary 1. The HfWG operator and the HFWG $^\varepsilon$ operator have the following relation:

$$\text{HFWG}_\omega^\varepsilon(h_1, h_2, \dots, h_n) \geq \text{HFWG}_\omega(h_1, h_2, \dots, h_n), \tag{14}$$

where h_j ($j = 1, 2, \dots, n$) be a collection of HFEs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of h_j ($j = 1, 2, \dots, n$), with $\omega_j \in [0, 1]$, $j = 1, 2, \dots, n$ and $\sum_{j=1}^n \omega_j = 1$.

Proof. For $\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n$, based on Lemma 4, we have:

$$\prod_{j=1}^n (2 - \gamma_j)^{\omega_j} + \sum_{j=1}^n \gamma_j^{\omega_j} \leq \sum_{j=1}^n \omega_j (2 - \gamma_j) + \sum_{j=1}^n \omega_j \gamma_j = 2,$$

then:

$$\frac{2 \prod_{j=1}^n \gamma_j^{\omega_j}}{\prod_{j=1}^n (2 - \gamma_j)^{\omega_j} + \prod_{j=1}^n \gamma_j^{\omega_j}} \geq \prod_{j=1}^n \gamma_j^{\omega_j},$$

where that equality holds if and only if $\gamma_1 = \gamma_2 = \dots = \gamma_n$.

Corollary 1 shows that the values obtained by the HFWG $^\varepsilon$ operator are bigger than the ones obtained by the HFWG operator.

Example 1. Let $h_1 = \{0.2, 0.3, 0.5\}$, $h_2 = \{0.4, 0.6\}$ be two HFEs, and $\omega = (0.7, 0.3)^T$ be the weight vector of h_j ($j = 1, 2$), then by Theorem 3, we have:

$$\begin{aligned} \text{HFWG}_\omega^\varepsilon(h_1, h_2) &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2\gamma_1^{\omega_1} \gamma_2^{\omega_2}}{(2 - \gamma_1)^{\omega_1} (2 - \gamma_2)^{\omega_2} + \gamma_1^{\omega_1} \gamma_2^{\omega_2}} \right\} = \\ &\{0.2482, 0, 2856, 0.3276, 0.3744, 0.4683, 0.5288\}. \end{aligned}$$

If we use the HFWG operator developed by Xia and Xu (2011) (i.e. (4) described in Section 1), to aggregate the HFEs $h_j(j=1,2)$, then we have:

$$\begin{aligned} \text{HFWG}_\omega(h_1, h_2) &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \gamma_1^{\omega_1} \gamma_2^{\omega_2} \right\} = \\ & \{0.2462, 0, 2781, 0.3270, 0.3693, 0.4676, 0.5281\}. \end{aligned}$$

2.2. Hesitant fuzzy Einstein ordered weighted geometric (HFWG^ε) operator

Similar to the definition of the HFWG operator (i.e. Eq. (5) described in Section 1), in what follows, we develop a hesitant fuzzy Einstein ordered weighted geometric (HFWG^ε) operator.

Definition 8. Let $h_j(j=1,2,\dots,n)$ be a collection of HFEs, a HFWG^ε operator of dimension n is a mapping $\text{HFWG}_w^\varepsilon: H^n \rightarrow H$ that has an associated vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0,1], j=1,2,\dots,n$ and $\sum_{j=1}^n w_j = 1$, and

$$\text{HFWG}_w^\varepsilon(h_1, h_2, \dots, h_n) = h_{\sigma(1)}^{\varepsilon w_1} \otimes_\varepsilon h_{\sigma(2)}^{\varepsilon w_2} \otimes_\varepsilon \dots \otimes_\varepsilon h_{\sigma(n)}^{\varepsilon w_n}, \tag{15}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(j-1)} \geq h_{\sigma(j)}$ for all $j = 2, 3, \dots, n$. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the HFWG^ε operator is reduced to the HFG^ε operator of dimension n .

Based on Definition 5 and the Einstein operational laws of HFEs, the HFWG^ε operator (15) can be transformed into the following form.

Theorem 5. Let $h_j(j=1,2,\dots,n)$ be a collection of HFEs, then their aggregated value by using the HFWG^ε operator is also an HFE, and

$$\begin{aligned} \text{HFWG}_w^\varepsilon(h_1, h_2, \dots, h_n) &= \\ \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \dots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} & \left\{ \frac{2 \prod_{j=1}^n \gamma_{\sigma(j)}^{w_j}}{\prod_{j=1}^n (2 - \gamma_{\sigma(j)})^{w_j} + \prod_{j=1}^n \gamma_{\sigma(j)}^{w_j}} \right\}, \end{aligned} \tag{16}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(j-1)} \geq h_{\sigma(j)}$ for all $j = 2, 3, \dots, n$. $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of the HFWG^ε operator such that $w_j \in [0,1], j=1,2,\dots,n$ and $\sum_{j=1}^n w_j = 1$.

Note that the HFWG and HFWG^ε operators are developed based on the idea of the OWA operator (Yager 1988). The main characterization of the OWA operator is its reordering step. Several methods have been developed to obtain the OWA weights. Yager (1988) used linguistic quantifiers to compute the OWA weights. O’Hagan (1988) generated the OWA weights with a predefined degree of orness by maximizing the entropy of the OWA weights. Filev and Yager (1998) obtained the OWA weights based on the exponential smoothing. Yager and Filev (1999) got the OWA weights from a collection of samples with the relevant aggregated data. Xu and Da (2002) obtained the OWA weights under partial weight information by establishing a linear objective-programming model. Especially, based on the normal distribution (Gaussian distribution), Xu (2005) developed a method to obtain the

OWA weights, whose prominent characteristic is that it can relieve the influence of unfair arguments on the decision result by assigning low weights to those “false” or “biased” ones.

Corollary 2. The HFOWG and HFOWG^ε operators have the following relation:

$$\text{HFOWG}_w^\varepsilon(h_1, h_2, \dots, h_n) \geq \text{HFOWG}_w(h_1, h_2, \dots, h_n), \tag{17}$$

where $h_j (j=1, 2, \dots, n)$ are a collection of HFEs and $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $h_j (j=1, 2, \dots, n)$ with $w_j \in [0, 1], j=1, 2, \dots, n$ and $\sum_{j=1}^n w_j = 1$.

Proof. Similar to Corollary 1.

Example 2. Let $h_1 = \{0.1, 0.4\}$, $h_2 = \{0.3, 0.5\}$ and $h_3 = \{0.2, 0.5, 0.8\}$ be three HFEs. And the HFOWG^ε operator has an associated vector $w = (0.25, 0.4, 0.35)^T$.

By Definition 5, we calculate the scores $s(h_j)$ of $h_j (j=1, 2, 3)$:

$$s(h_1) = 0.25, s(h_2) = 0.4, s(h_3) = 0.5.$$

Since $s(h_3) > s(h_2) > s(h_1)$, then:

$$h_{\sigma(1)} = h_3, h_{\sigma(2)} = h_2, h_{\sigma(3)} = h_1.$$

Thus by Eq. (16), it follows that:

$$\begin{aligned} &\text{HFOWG}_w^\varepsilon(h_1, h_2, h_3) = \\ &\cup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}} \left\{ \frac{2 \prod_{j=1}^3 \gamma_{\sigma(j)}^{w_j}}{\prod_{j=1}^3 (2 - \gamma_{\sigma(j)})^{w_j} + \prod_{j=1}^n \gamma_{\sigma(j)}^{w_j}} \right\} = \\ &\{0.1866, 0.2344, 0.2386, 0.2775, 0.2974, 0.3016, 0.3440, 0.3727, 0.3789, 0.4350, 0.4632, 0.5277\}. \end{aligned}$$

If we use the HFOWG operator, developed by Xia and Xu (2011) (i.e. (5) described in Section 1), to aggregate the HFEs $h_j (j=1, 2, 3)$, then we have:

$$\begin{aligned} &\text{HFOWG}_w(h_1, h_2, h_3) = \\ &\cup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}} \left\{ \prod_{j=1}^3 \gamma_{\sigma(j)}^{w_j} \right\} = \\ &\{0.1845, 0.2264, 0.2321, 0.2610, 0.2847, 0.2998, 0.3202, 0.3678, 0.3770, 0.4240, 0.4624, 0.5201\}. \end{aligned}$$

2.3. Hesitant fuzzy Einstein hybrid weighted geometric (HFHWG^ε) operator

The HFHWG^ε operator weights the hesitant fuzzy argument itself. The HFOWG^ε operator weights the values instead of weighting the arguments. This is so because each ω_j is attached to the j^{th} value in decreasing order without considering from which argument the value comes. Therefore, weights represent different aspects in both aggregation operators. However, although both points of view are meaningful in a single problem, both aggregation operators present the drawback of considering only one of them. To solve this drawback, it

is necessary to introduce a hybrid aggregation operator for hesitant fuzzy arguments, which considers the importance of the arguments and the importance of the values of the arguments.

Definition 9. Let $h_j (j = 1, 2, \dots, n)$ be a collection of HFEs, the HFHWG $^\epsilon$ operator of dimension n is a mapping HFHWG $^\epsilon: H^n \rightarrow H$ that has an associated vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1], j = 1, 2, \dots, n$ and $\sum_{j=1}^n w_j = 1$, then:

$$\text{HFHWG}_{\omega, w}^\epsilon(h_1, h_2, \dots, h_n) = \dot{h}_{\sigma(1)}^{\epsilon w_1} \otimes_\epsilon \dot{h}_{\sigma(2)}^{\epsilon w_2} \otimes_\epsilon \dots \otimes_\epsilon \dot{h}_{\sigma(n)}^{\epsilon w_n}, \tag{18}$$

where: $\dot{h}_{\sigma(j)}$ is the j^{th} largest of the weighted HFEs $\dot{h}_i (\dot{h}_i = h_i^{\epsilon(n\omega_i)}, i = 1, 2, \dots, n)$; $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j = 1, 2, \dots, n)$ with $\omega_j \in [0, 1], j = 1, 2, \dots, n$ and $\sum_{j=1}^n \omega_j = 1$, and n is the balancing coefficient.

Similar to Theorem 5, the HFHWG $^\epsilon$ operator (18) can be transformed into the following form.

Theorem 6. Let $h_j (j = 1, 2, \dots, n)$ be a collection of HFEs. A hesitant fuzzy Einstein hybrid averaging HFHWG $^\epsilon$ operator of dimension n is a mapping HFHWG $^\epsilon: H^n \rightarrow H$ that has an associated vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1], j = 1, 2, \dots, n$ and $\sum_{j=1}^n w_j = 1$, then:

$$\text{HFHWG}_{\omega, w}^\epsilon(h_1, h_2, \dots, h_n) = \bigcup_{\dot{\gamma}_{\sigma(1)} \in \dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \dots, \dot{\gamma}_{\sigma(n)} \in \dot{h}_{\sigma(n)}} \left\{ \frac{2 \prod_{j=1}^n \dot{\gamma}_{\sigma(j)}^{w_j}}{\prod_{j=1}^n (2 - \dot{\gamma}_{\sigma(j)})^{w_j} + \prod_{j=1}^n \dot{\gamma}_{\sigma(j)}^{w_j}} \right\}, \tag{19}$$

where: $\dot{h}_{\sigma(j)}$ is the j^{th} largest of the weighted HFEs $\dot{h}_i (\dot{h}_i = h_i^{\epsilon(n\omega_i)}, i = 1, 2, \dots, n)$; $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $h_j (j = 1, 2, \dots, n)$ with $\omega_j \in [0, 1], j = 1, 2, \dots, n$ and $\sum_{j=1}^n \omega_j = 1$; and n is the balancing coefficient.

Corollary 3. The HFWG $^\epsilon$ operator is a special case of the HFHWG $^\epsilon$ operator.

Proof. If $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then:

$$\begin{aligned} \text{HFHWG}_{\omega, w}^\epsilon(h_1, h_2, \dots, h_n) &= \dot{h}_{\sigma(1)}^{\epsilon w_1} \otimes_\epsilon \dot{h}_{\sigma(2)}^{\epsilon w_2} \otimes_\epsilon \dots \otimes_\epsilon \dot{h}_{\sigma(n)}^{\epsilon w_n} = \\ &\dot{h}_{\sigma(1)}^{\epsilon \frac{1}{n}} \otimes_\epsilon \dot{h}_{\sigma(2)}^{\epsilon \frac{1}{n}} \otimes_\epsilon \dots \otimes_\epsilon \dot{h}_{\sigma(n)}^{\epsilon \frac{1}{n}} = \dot{h}_{\sigma(1)}^{\epsilon \omega_1} \otimes_\epsilon \dot{h}_{\sigma(2)}^{\epsilon \omega_2} \otimes_\epsilon \dots \otimes_\epsilon \dot{h}_{\sigma(n)}^{\epsilon \omega_n} = \\ &\text{HFWG}_\omega^\epsilon(h_1, h_2, \dots, h_n), \end{aligned}$$

which completes the proof of Corollary 3.

Corollary 4. The HFOWG $^\epsilon$ operator is a special case of the HFHWG $^\epsilon$ operator.

Proof. If $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then:

$$\begin{aligned} \text{HFHWG}_{\omega, w}^\epsilon(h_1, h_2, \dots, h_n) &= \dot{h}_{\sigma(1)}^{\epsilon w_1} \otimes_\epsilon \dot{h}_{\sigma(2)}^{\epsilon w_2} \otimes_\epsilon \dots \otimes_\epsilon \dot{h}_{\sigma(n)}^{\epsilon w_n} = \\ &h_{\sigma(1)}^{\epsilon w_1} \otimes_\epsilon h_{\sigma(2)}^{\epsilon w_2} \otimes_\epsilon \dots \otimes_\epsilon h_{\sigma(n)}^{\epsilon w_n} = \\ &\text{HFOWG}_w^\epsilon(h_1, h_2, \dots, h_n), \end{aligned}$$

which completes the proof of Corollary 4.

Example 3. Let $h_1 = \{0.2, 0.4, 0.5\}$, $h_2 = \{0.2, 0.6\}$ and $h_3 = \{0.1, 0.3, 0.4\}$ are three HFEs, and $\omega = (0.15, 0.30, 0.55)^T$ be the weight vector of $h_j (j = 1, 2, 3)$, the HFHWG^ε operator has an associated vector $w = (0.3, 0.4, 0.3)^T$.

We first get the weighted hesitant fuzzy values $\dot{h}_j = h_j^{\cdot \varepsilon(3\omega_j)} (j = 1, 2, 3)$:

$$\begin{aligned} \dot{h}_1 &= h_1^{\cdot \varepsilon(3\omega_1)} = \{0.5423, 0.6978, 0.7577\}, \\ \dot{h}_2 &= h_2^{\cdot \varepsilon(3\omega_2)} = \{0.2432, 0.6362\}, \\ \dot{h}_3 &= h_3^{\cdot \varepsilon(3\omega_3)} = \{0.0154, 0.1081, 0.1843\}, \end{aligned}$$

and the scores $s(\dot{h}_j)$ of $\dot{h}_j (j = 1, 2, 3)$:

$$s(\dot{h}_1) = 0.6659, s(\dot{h}_2) = 0.4397, s(\dot{h}_3) = 0.1026.$$

Since $s(\dot{h}_1) > s(\dot{h}_2) > s(\dot{h}_3)$, then:

$$\dot{h}_{\sigma(1)} = \dot{h}_1, \dot{h}_{\sigma(2)} = \dot{h}_2, \dot{h}_{\sigma(3)} = \dot{h}_3.$$

By Eq. (19), it follows that:

$$\begin{aligned} \text{HFHWG}_{\omega, w}^{\varepsilon}(h_1, h_2, h_3) &= \bigcup_{\dot{\gamma}_{\sigma(1)} \in \dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \dot{\gamma}_{\sigma(3)} \in \dot{h}_{\sigma(3)}} \left\{ \frac{2 \prod_{j=1}^3 \dot{\gamma}_{\sigma(j)}^{w_j}}{\prod_{j=1}^3 (2 - \dot{\gamma}_{\sigma(j)})^{w_j} + \prod_{j=1}^3 \dot{\gamma}_{\sigma(j)}^{w_j}} \right\} = \\ &\{0.1455, 0.1610, 0.1668, 0.2262, 0.2491, 0.2499, 0.2577, 0.2749, 0.2842, \\ &0.2901, 0.3183, 0.3289, 0.3768, 0.4115, 0.4243, 0.4324, 0.4707, 0.4848\}. \end{aligned}$$

3. Decision making based on hesitant fuzzy information

The MAGDM problems are widespread in real life decision situations. In some practical problems, such as presidential election or the blind peer review of thesis, the experts propose the preferences or opinions for the alternatives with anonymous in order to protect their privacy or avoid influencing each other. In such situations, HFE permits us to represent the rating of the alternative on the attribute given by several experts, so we use a hesitant fuzzy decision matrix to describe the group decision making problems. Suppose there are n alternatives $X = \{x_1, x_2, \dots, x_n\}$ and n attributes $G = \{g_1, g_2, \dots, g_m\}$ with the attribute weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ such that $\omega_j \in [0, 1], j = 1, 2, \dots, m$ and $\sum_{j=1}^m \omega_j = 1$. The ratings of alternatives $x_i (i = 1, 2, \dots, n)$ on attributes $g_j (j = 1, 2, \dots, m)$ given by the experts are HFEs $h_{ij} (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$, where h_{ij} indicates the set of the degrees that the alternative x_i satisfies the attribute g_j given by the experts. In the case where more than one experts provide the same value, then the value emerges only once in h_{ij} . Hence, a fuzzy MAGDM problem can be concisely expressed in matrix format: $D = (h_{ij})_{n \times m}$ (Table 1).

Table 1. The hesitant fuzzy decision matrix D

	g_1	g_2	\dots	g_m
x_1	h_{11}	h_{12}	\dots	h_{1m}
x_2	h_{21}	h_{22}	\dots	h_{2m}
\vdots	\vdots	\vdots	\ddots	\vdots
x_n	h_{n1}	h_{n2}	\dots	h_{nm}

In what follows, we apply the HFWG^ε and HFWG operators proposed by Xia and Xu (2011) to hesitant fuzzy MAGDM, which involves the following steps.

Step 1. Obtain the normalized hesitant fuzzy decision matrix. In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set G can be divided into two subsets: G_1 and G_2 , which are the subset of benefit attributes and cost attributes, respectively. Furthermore, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \Phi$, where Φ is empty set. The decision matrix D needs to be normalized besides all the attributes $g_j (j = 1, 2, \dots, m)$ are of the same type. In this paper we choose the following normalization formula to update the hesitant fuzzy decision matrix D :

$$h_{ij} = \begin{cases} h_{ij} & j \in G_1 \\ h_{ij}^c & j \in G_2 \end{cases}, \tag{20}$$

where h_{ij}^c is the complement of h_{ij} . Hence, we obtain the normalized hesitant fuzzy decision matrix $D = (h_{ij})_{n \times m}$.

Step 2. Compute the overall ratings of alternatives. Utilize the HFWG^ε operator (or the HFWG operator) to aggregate all the rating values $h_{ij} (j = 1, 2, \dots, m)$ of the i^{th} line and get the overall rating value h_i corresponding to the alternative $x_i (i = 1, 2, \dots, n)$, i.e.:

$$h_i = \text{HFWG}_w^\epsilon(h_{i1}, h_{i2}, \dots, h_{im}), \quad i = 1, 2, \dots, n. \tag{21}$$

Step 3. Rank the order of all alternatives. Utilize the method in Definition 5 to compute the scores of the overall rating values $h_i (i = 1, 2, \dots, n)$, and rank all the alternatives $x_i (i = 1, 2, \dots, n)$ in accordance with $h_i (i = 1, 2, \dots, n)$ in descending order, finally select the most desirable alternative(s) with the largest overall rating value.

We consider a MAGDM problem involves the prioritization of a set of information technology improvement projects (adapted from (Ngwenyama, Bryson 1999)) is used to illustrate the developed HFWG^ε operator.

Example 4. The information management steering committee of Midwest American Manufacturing Company must prioritize for development and implementation a set of four information technology improvement projects $X = \{x_1, x_2, x_3, x_4\}$: x_1 – Quality Management Information, x_2 – Customer Order Tracking, x_3 – Inventory Control and x_4 – Budget Analysis.

The committee is concerned that the projects are prioritized from highest to lowest potential contribution to the firm’s strategic goal of gaining competitive advantage in the industry. In assessing the potential contribution of each project, three factors are considered as follows (it should be noted that three factors are benefit attributes):

- Productivity (g_1): The productivity factor assesses the potential of a proposed project to increase effectiveness and efficiency of the firm’s manufacturing and service operations;
- Differentiation (g_2): The differentiation factor assesses the potential of a proposed project to fundamentally differentiate the firm’s products and services from its competitors, and to make them more desirable to its customers;
- Management (g_3): The management factor assesses the potential of a proposed project to assist management in improving their planning, controlling and decision-making activities.

Additionally suppose that the weight vector of the attributes is:

$$\omega = (\omega_1, \omega_2, \omega_3)^T = (0.45, 0.25, 0.3)^T.$$

In order to avoid influencing each other, the members of the committee are required to provide their preferences in anonymity, so the committee, which includes five members, represents the characteristics of the projects $x_i (i = 1, 2, 3, 4)$ by the HFEs $h_{ij} (i = 1, 2, 3, 4; j = 1, 2, 3)$ with respect to the factors $g_j (j = 1, 2, 3)$, listed in Table 2.

Table 2. The hesitant fuzzy decision matrix D

	g_1	g_2	g_3
x_1	{0.2, 0.4, 0.7}	{0.2, 0.6, 0.8}	{0.2, 0.3, 0.6, 0.7}
x_2	{0.2, 0.4, 0.7, 0.9}	{0.1, 0.2, 0.4, 0.5}	{0.3, 0.4, 0.6, 0.9}
x_3	{0.3, 0.5, 0.6, 0.7}	{0.2, 0.4, 0.5, 0.6}	{0.3, 0.5, 0.7, 0.8}
x_4	{0.3, 0.5, 0.6}	{0.2, 0.4}	{0.5, 0.6, 0.7}

In the following, we use the developed HFWG^ε operator to get the optimal project.

Step 1. Because all the factors are benefit attributes, the hesitant fuzzy decision matrix D need not be updated.

Step 2. Utilize the HFWG^ε operator to aggregate all the rating values $h_{ij} (j = 1, 2, 3)$ of the i^{th} line and get the overall rating value h_i corresponding to the alternative $x_i (i = 1, 2, 3, 4)$. Due to vast amounts of data, we only give h_4 as an instance, the results of h_1, h_2 and h_3 are not listed.

$$h_4 = \text{HFWG}_{\omega}^{\varepsilon}(h_{41}, h_{42}, h_{43}) = \bigcup_{\gamma_{41} \in h_{41}, \gamma_{42} \in h_{42}, \gamma_{43} \in h_{43}} \left\{ \frac{2 \prod_{j=1}^3 \gamma_{4j}^{\omega_j}}{\prod_{j=1}^3 (2 - \gamma_{4j})^{\omega_j} + \prod_{j=1}^3 \gamma_{4j}^{\omega_j}} \right\} = \{0.3197, 0.3404, 0.3602, 0.3779, 0.4016, 0.4042, 0.4240, 0.4290, 0.4419, 0.4526, 0.4684, 0.4735, 0.4934, 0.5013, 0.5156, 0.5275, 0.5450, 0.5725\}.$$

Step 3. Compute the scores of the overall rating values $h_i(i=1,2,3,4)$ by Definition 5, and rank all the alternatives $x_i(i=1,2,3,4)$ in accordance with $h_i(i=1,2,3,4)$ in descending order. The scores of the overall rating values and the rankings of alternatives are listed in Table 3. Finally select the most desirable alternative x_2 with the largest overall rating value.

Table 3. The overall scores and rankings of alternatives

	x_1	x_2	x_3	x_4	Ranking
HFWG ^ε	0.4354	0.4503	0.4995	0.4472	$x_3 \succ x_2 \succ x_4 \succ x_1$
HFWG	0.4252	0.4343	0.4924	0.4405	$x_3 \succ x_4 \succ x_2 \succ x_1$

In Step 2, if we use the HFWG operator proposed by Xia and Xu (2011) (i.e. (4) described in Section 1) to aggregation the values of the alternatives, then the scores of the overall rating values $h_i(i=1,2,3,4)$ and the rankings of the alternatives $x_i(i=1,2,3,4)$ are also listed in Table 3.

From Table 3, we can see that the values obtained by the HFWG^ε operator are always greater than the ones obtained by the HFWG operator for the same aggregation rating values corresponding to the alternatives, but the rankings of alternatives by both different aggregation operators are slightly different, and the best projects are the same, i.e. the project x_3 .

Remark 1. Using the HFWG^ε operator rather than the HFWG operator, the decision-maker has more optimistic attitude in aggregation process. Using different techniques reflects the decision maker’s optimistic (or pessimistic) attitude in aggregation process. Therefore, in general, the different aggregation operators do not always return the same ranking orders and the same alternatives.

Conclusion

Motivated by the extension principle of HFSs, we have extended the Einstein operations on FSs to HFSs, and have developed some new hesitant fuzzy aggregation operators, including the HFWG^ε operator, HFOWG^ε operator, and HFHWG^ε operator. Then, we have applied the HFWG^ε to the DMGDM problem with anonymity. It is worth point out that these aggregation operators are the same effective tools as the aggregation operators proposed by Xia and Xu (2011), for aggregating hesitant fuzzy information. It is worth pointing that the proposed aggregation operators not only have the same good nature as those proposed by Xia and Xu (2011), but also can reflect the decision-makers more optimistic attitude in aggregation process.

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