TTU゙1918

# DUAL HESITANT FUZZY AGGREGATION OPERATORS 

Dejian $\mathrm{YU}^{\mathrm{a}}$, Wenyu $\mathrm{ZHANG}^{\text {a }}$, George HUANG ${ }^{\text {b }}$<br>${ }^{\text {a }}$ School of Information, Zhejiang University of Finance \& Economics, Hangzhou, China ${ }^{b}$ HKU-ZIRI Lab for Physical Internet, Department of Industrial and Manufacturing Systems Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong, China

Received 22 September 2012; accepted 25 August 2013


#### Abstract

Dual hesitant fuzzy sets (DHFSs) is a generalization of fuzzy sets (FSs) and it is typical of membership and non-membership degrees described by some discrete numerical. In this article we chiefly concerned with introducing the aggregation operators for aggregating dual hesitant fuzzy elements (DHFEs), including the dual hesitant fuzzy arithmetic mean and geometric mean. We laid emphasis on discussion of properties of newly introduced operators, and give a numerical example to describe the function of them. Finally, we used the proposed operators to select human resources outsourcing suppliers in a dual hesitant fuzzy environment.


Keywords: DHFSs, dual hesitant fuzzy arithmetic mean, dual hesitant fuzzy geometric mean, aggregation operator, human resources outsourcing suppliers.

JEL Classification: C02, C44, D81, O15.

## Introduction

Fuzzy set (FS) theory (Zadeh 1965) is a powerful technique for depicting indefiniteness. In order to give a more detailed description of an uncertain world, many extended forms of FS theory have been proposed. For example, Zadeh (1975) extended FSs and erected the theory interval-valued FSs (IVFSs). Years later, the type-2 fuzzy set was proposed by Dubois and Prade (1980). Yager (1986) introduced the fuzzy multiset as another generalization of FS. An intuitionistic FS (Atanassov 1986) has three main parts: a membership, non-membership and hesitancy (Xu 2007). Torra and Narukawa (2009) and Torra (2010) proposed another generalization of FS - the hesitant fuzzy set (HFS) - that allows the membership degree descried by a set of discrete numerical (Zhang, Xu 2015; Yu 2014a; Yu et al. 2013; Xia et al. 2013; Wei 2012; Xia, Xu 2011).

[^0]Dual hesitant fuzzy sets (DHFS) are a generalization of FS first proposed by Zhu et al. (2012a). These are characterized by membership and non-membership degrees that are represented by sets of possible values (Ye 2014; Yu 2014b). DHFS is an efficient mathematical approach for studying imprecise, uncertain, or incomplete information or knowledge. It is an invaluable aid in cases where there are troubles in establishing the membership and nonmembership of an element belongs to a set (Xia, Xu 2011). For example, three reviewers want to estimate the degrees to which a candidate satisfies the criterion of honesty. Because they have never seen each other before, the entire evaluation process is conducted in an uncertain environment. The first reviewer thinks that the degree of honesty for this candidate is 0.6 , and that $\mathrm{s} /$ he has a 0.3 possibility of being dishonest. Meanwhile, the second reviewer regards that the degree of honesty is 0.7 , and in his opinion, this candidate only has 0.2 possibility to be a dishonest man. Similarly, the third reviewer believes that the possibility of honesty is 0.5 while the contrary is 0.1 . We assume that the above three reviewers have the same degree of influence on the evaluation and that there is no mutual interference among them. In this circumstance, the integrated information of the candidate's honesty can be expressed as a dual hesitant fuzzy element (DHFE) $\{\{0.5,0.6,0.7\},\{0.1,0.2,0.3\}\}$. Another example, the review of a PhD thesis in China is always anonymously taken by three experts, this determines that those experts have no way to exchange ideas. Due to the complexity of reviewing a PhD thesis, it is very difficult for an expert to provide accurate evaluating values. The first expert thinks that the possibility of the PhD thesis meeting the requirements is 0.7 and that of it not being up to the standard is 0.3 . The second one believes that the chance that the PhD thesis meets the requirements is 0.6 while the contrary is 0.2 . The third expert regards the compliance to be 0.5 and the non-compliance to be 0.3 . In these situations, the degree to which the PhD thesis meets the requirements can be expressed as a DHFE $\{\{0.5,0.6,0.7\},\{0.2,0.3\}\}$. If we use a hesitant fuzzy element to represent this situation, the result is $\{0.5,0.6,0.7\}$. We found that the hesitant fuzzy element $\{0.5,0.6,0.7\}$ only expresses the membership degree but completely ignores the non-membership degree to which the PhD thesis meets the requirements. Therefore, it is far better to represent the situation by using a DHFE than a hesitant fuzzy element.

Information aggregation is one of the fields to which FS theory and extended FS theories have been applied extensively (Yager, Kacprzyk 1997; Calvo et al. 2002; Torra 2003; Xu, Da 2003; Bustince et al. 2007; Li 2010; Wei 2010; Fernando Umberto 2013; Kosareva, Krylovas 2013; Zhao, Wei 2013; Zhang 2013; Zhu et al. 2012b; Xu 2005, 2007, 2010, 2011; Yu 2015). However, there seem to have been no investigations on dual hesitant fuzzy information aggregation. This article aims at investigating aggregation methods for DHFEs. To achieve this target, we arranged the rest of this paper as follows. Section 1 reviews some fundamental theory about DHFS briefly. Section 2 develops the dual hesitant fuzzy weighted averaging (DHFWA) and dual hesitant fuzzy weighted geometric (DHFWG) operators, the desirable properties of which are also investigated in this section. Section 3 examines problems involving the selection of human resources outsourcing suppliers based on the proposed operators. The last Section carries on the summary to the whole paper.

## 1. Preliminaries

As a generalization of FS, the HFS was first put forwarded by Torra and Narukawa (2009).
Definition 1 (Torra, Narukawa 2009; Xia, Xu 2011). Suppose there is an objective set and marked by $X$, an HFS is defined as follows:

$$
\begin{equation*}
E=\left\{<x, h_{E}(x)>\mid x \in X\right\}, \tag{1}
\end{equation*}
$$

$h_{E}(x)$ in Eq. (1) is a real numbers set belongs to [0,1] and it shows the membership degree of the basic element $x \in X$.

Zhu et al. (2012a) proposed another generalization of an FS called DHFS.
Definition 2 (Zhu et al. 2012a). Suppose there is an objective set and marked byX. A DHFS $D$ is defined as:

$$
\begin{equation*}
D=\{\langle x, h(x), g(x)\rangle x \in X\} \tag{2}
\end{equation*}
$$

$h(x)$ and $g(x)$ in Eq. (1) are two real numbers set belongs to [0,1] and they convey the membership degree and non-membership degree of the basic element $x \in X$. Furthermore,

$$
\begin{equation*}
0 \leq \gamma, \eta \leq 1,0 \leq \gamma^{+}+\eta^{+} \leq 1, \tag{3}
\end{equation*}
$$

where $\gamma \in h(x), \eta \in g(x)$, and for any $x \in X, \quad \gamma^{+} \in h^{+}(x)=\bigcup_{\gamma \in h(x)} \max \{r\}$ and $\eta^{+} \in g^{+}(x)=\bigcup_{\eta \in g(x)} \max \{\eta\}$. We know from the concept of HFS (Torra, Narukawa 2009; Torra 2010) that $h(x)$ and $g(x)$ are two HFSs.

For convenience, Zhu et al. (2012a) defined the two dimensional arrays $d(x)=(h(x), g(x))$ as a DHFE, denoted by $d=(h, g)$, with the conditions $\gamma \in h, \eta \in g, \gamma^{+} \in h^{+}=\bigcup_{\gamma \in h} \max \{r\}$, $\eta^{+} \in g^{+}=\bigcup_{\eta \in g} \max \{\eta\}, 0 \leq \gamma, \eta \leq 1$, and $0 \leq \gamma^{+}+\eta^{+} \leq 1$.

To compare the DHFEs, Zhu et al. (2012a) introduced comparison laws as follows.
Definition 3 (Zhu et al. 2012a). Let $d_{1}=\left(h_{1}, g_{1}\right)$ and $d_{2}=\left(h_{2}, g_{2}\right)$ be any two DHFEs, $s\left(d_{i}\right)=\frac{1}{\# h} \sum_{\gamma \in h} \gamma-\frac{1}{\# g} \sum_{\eta \in g} \eta(i=1,2)$ the score function of $d_{i}(i=1,2)$, and $p\left(d_{i}\right)=\frac{1}{\# h} \sum_{\gamma \in h} \gamma+\frac{1}{\# g} \sum_{\eta \in g} \eta(i=1,2)$ the accuracy function of $d_{i}(i=1,2)$. The above mentioned $\# h$ and $\# g$ represented the quantity of components in $h$ and $g$, respectively. Furthermore, Zhu et al. (2012a) defined the following rules.

If the inequality $s\left(d_{1}\right)<s\left(d_{2}\right)$ holds, then $d_{1}$ is inferior to $d_{2}$, denoted as $d_{1} \prec d_{2}$.
If the equality $s\left(d_{1}\right)=s\left(d_{2}\right)$ holds, then:
i) $d_{1}$ is equivalent to $d_{2}$, denoted as $d_{1} \sim d_{2}$, if $h\left(d_{1}\right)=h\left(d_{2}\right)$, and
ii) $d_{1}$ is superior to $d_{2}$, denoted as $d_{1} \succ d_{2}$, if $h\left(d_{1}\right)>h\left(d_{2}\right)$.

Definition 4 (Zhu et al. 2012a). Suppose there is an objective set and marked by $X$, and let $d, d_{1}$ and $d_{2}$ be three any given DHFEs. Then:

$$
\begin{aligned}
& d_{1} \oplus d_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \eta_{1} \in g_{1}, \eta_{2} \in g_{2}}\left\{\left\{\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2}\right\},\left\{\eta_{1} \eta_{2}\right\}\right\} ; \\
& d_{1} \otimes d_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \eta_{1} \in g_{1}, \eta_{2} \in g_{2}}\left\{\left\{\gamma_{1} \gamma_{2}\right\},\left\{\eta_{1}+\eta_{2}-\eta_{1} \eta_{2}\right\}\right\} ; \\
& n d=\bigcup_{\gamma \in h, \eta \in g}\left\{\left\{1-(1-\gamma)^{n}\right\},\left\{\eta^{n}\right\}\right\}, \quad n>0 ; \\
& d^{n}=\bigcup_{\gamma \in h, \eta \in g}\left\{\left\{\gamma^{n}\right\},\left\{1-(1-\eta)^{n}\right\}\right\}, \quad n>0 .
\end{aligned}
$$

## 2. Aggregation operators for DHFEs

The weighted average (WA) and the weighted geometric (WG) operators are common aggregation operators used in information aggregation (Merigó 2012). They can be usefully employed in practical problems such as area of statistics, socioeconomic, and engineering world. Since their introduction, the WA and WG operators have been studied in a wide range of applications (Beliakov et al. 2007; Merigó, Casanovas 2011a, 2011b, 2011c, 2011d; Yager 1988, 2002, 2003, 2006, 2007, 2009a, 2009b; Zhao et al. 2010; Xu, Yager 2006; Wei 2009).

In this section, we have applied the WA and WG operators to dual hesitant fuzzy environment and introduced some aggregation operators to aggregate dual hesitant fuzzy information. To start with, we define the DHFWA operator and then propose the DHFWG operator. Based on the Definition 4, the DHFWA operator is defined as follows:
Definition 5. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. A DHFWA operator is a mapping $D^{n} \rightarrow D$ such that:

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\oplus_{j=1}^{n}\left(\omega_{j} d_{j}\right)=\omega_{1} d_{1} \oplus \omega_{2} d_{2} \oplus \cdots \oplus \omega_{n} d_{n}, \tag{4}
\end{equation*}
$$

where $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ is the measure of importance of $d_{j}$ and $\omega$ are standardized. In particular, if $\omega=\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{T}$, then the DHFWA operator degenerate into DHFA operator:

$$
\begin{equation*}
\operatorname{DHFA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\stackrel{n}{j=1}\left(\frac{1}{n} d_{j}\right)=\frac{1}{n} d_{1} \oplus \frac{1}{n} d_{2} \oplus \cdots \oplus \frac{1}{n} d_{n} . \tag{5}
\end{equation*}
$$

Theorem 1. Suppose there is family of DHFEs $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$, then:

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} \tag{6}
\end{equation*}
$$

where $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ is the measure of importance of $d_{j}$ and $\omega$ are standardized.
Proof: We first prove that Eq. (6) holds for $n=2$.

$$
\begin{align*}
& \omega_{1} d_{1}=\bigcup_{\gamma_{1} \in h_{1}, \eta_{1} \in g_{1}}\left\{\left\{1-\left(1-\gamma_{1}\right)^{n}\right\},\left\{\eta_{1}^{n}\right\}\right\} ;  \tag{7}\\
& \omega_{2} d_{2}=\bigcup_{\gamma_{2} \in h_{2}, \eta_{2} \in g_{2}}\left\{\left\{1-\left(1-\gamma_{2}\right)^{n}\right\},\left\{\eta_{2}^{n}\right\}\right\} . \tag{8}
\end{align*}
$$

Then,
$\operatorname{DHFWA}\left(d_{1}, d_{2}\right)=\omega_{1} d_{1} \oplus \omega_{2} d_{2}=$
$\bigcup_{\gamma_{1} \in h_{1}, \eta_{1} \in g_{1}, \gamma_{2} \in h_{2}, \eta_{2} \in g_{2}}\left\{\left\{2-\left(1-\gamma_{1}\right)^{\omega_{1}}-\left(1-\gamma_{2}\right)^{\omega_{2}}-\left(1-\left(1-\gamma_{1}\right)^{\omega_{1}}\right)\left(1-\left(1-\gamma_{2}\right)^{\omega_{2}}\right)\right\},\left\{\eta_{1}{ }^{\omega_{1}} \eta_{2}{ }^{\omega_{2}}\right\}\right\}=$
$\bigcup_{\gamma_{1} \in h_{1}, \eta_{1} \in g_{1}, \gamma_{2} \in h_{2}, \eta_{2} \in g_{2}}\left\{\left\{1-\prod_{j=1}^{2}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{2} \eta_{j}^{\omega_{j}}\right\}\right\}$.
If Eq. (6) is true when $n=k$, meaning:

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{k}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{k}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{k} \eta_{j}^{\omega_{j}}\right\}\right\} \tag{10}
\end{equation*}
$$

then, when $n$ increase single unit, we can get:

$$
\begin{align*}
& \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{k+1}\right)= \\
& \omega_{1} d_{1} \oplus \omega_{2} d_{2} \oplus \ldots \oplus \omega_{n} d_{n} \oplus \omega_{n+1} d_{n+1}=\left(\omega_{1} d_{1} \oplus \omega_{2} d_{2} \oplus \ldots \oplus \omega_{n} d_{n}\right) \oplus \omega_{n+1} d_{n+1}= \\
& U_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} \oplus \cup_{\gamma_{k+1} \in h_{k+1}, \eta_{k+1} \in g_{k+1}}\left\{\left\{1-\left(1-\gamma_{k+1}\right)^{\omega_{k+1}}\right\},\left\{\eta_{k+1} \omega_{k+1}\right\}\right\}= \\
& U_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{k}\left(1-\gamma_{j}\right)^{\omega_{j}}+\left(1-\left(1-\gamma_{k+1}\right)^{\left.\left.\omega_{k+1}\right)-\left(1-\prod_{j=1}^{k}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)\left(1-\left(1-\gamma_{k+1}\right)^{\left.\omega_{k+1}\right)}\right\},\left\{\prod_{j=1}^{k+1} \eta_{j}^{\omega_{j}}\right\}\right\}=}\right.\right.\right. \\
& \cup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{k+1}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{k+1} \eta_{j}^{\omega_{j}}\right\}\right\} . \tag{11}
\end{align*}
$$

In other words, Eq. (6) establishes when $n=k+1$. Therefore, Eq. (6) establishes for any given $n$, completing the proof of Theorem 1.

Now, let us look at all sorts of excellent properties of the DHFWA operator.
Theorem 2. Suppose $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ and $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. If for all $j, \gamma_{j}=\gamma, \eta_{j}=\eta$, where $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}$, $\gamma$ is the element of HFS $h$, and $\eta$ is the element of HFS $g$, then:

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=d \tag{12}
\end{equation*}
$$

Proof: By Theorem 1, we have:

$$
\begin{align*}
& \text { DHFWA }\left(d_{1}, d_{2}, \ldots, d_{n}\right)=U_{\gamma_{j} \in h_{j}, \mathbf{n}_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\}= \\
& \cup_{\gamma \in h, \eta \in g}\left\{\left\{1-\prod_{j=1}^{n}(1-\gamma)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta^{\omega_{j}}\right\}\right\}= \\
& \cup_{\gamma \in h, \eta \in g}\left\{\left\{1-(1-\gamma)_{j=1}^{n} \omega_{j}\right\},\left\{\sum_{j=1}^{n} \omega_{j}\right\}\right\}=U_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}=d, \tag{13}
\end{align*}
$$

completing the proof of Theorem 2.
Theorem 3. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. If $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ is a DHFE, $\gamma_{j}$ are elements of HFS $h_{j}$, and $\eta_{j}$ are elements of HFS $g_{j}$, then:

$$
\begin{equation*}
\text { DHFWA }\left(d_{1} \oplus d, d_{2} \oplus d, \ldots, d_{n} \oplus d\right)=\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus d \tag{14}
\end{equation*}
$$

Proof: Since for any $j$
$d_{j} \oplus d=\bigcup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{\gamma_{j}+\gamma-\gamma_{j} \gamma\right\},\left\{\eta_{j} \eta\right\}\right\}=\bigcup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-\left(1-\gamma_{j}\right)(1-\gamma)\right\},\left\{\eta_{j} \eta\right\}\right\}$,
according to Theorem 1, we have:
DHFWA $\left(d_{1} \oplus d, d_{2} \oplus d, \ldots, d_{n} \oplus d\right)=$

$$
\begin{align*}
& \cup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-\prod_{j=1}^{n}\left(\left(1-\gamma_{j}\right)(1-\gamma)\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j} \eta\right)^{\omega_{j}}\right\}\right\}= \\
& \cup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-(1-\gamma)^{\sum_{j=1}^{n}\left(\omega_{j}\right)} \prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\eta^{\sum_{j=1}^{n}\left(\omega_{j}\right)} \prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}}\right\}\right\}= \\
& U_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-(1-\gamma) \prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\eta \prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}}\right\}\right\} . \tag{16}
\end{align*}
$$

According to the operational laws of Definition 4, we can get:

$$
\begin{align*}
& \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus d= \\
& \cup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} \oplus \cup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}= \\
& \cup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-(1-\gamma)\left(1-\left(1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)\right)\right\},\left\{\eta \prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}}\right\}\right\}= \\
& \cup_{\gamma_{j} \in h_{j}, \gamma \in h, \eta_{j} \in g_{j}, \eta \in g}\left\{\left\{1-(1-\gamma) \prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\eta \prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}}\right\}\right\} . \tag{17}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1} \oplus d, d_{2} \oplus d, \ldots, d_{n} \oplus d\right)=\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus d \tag{18}
\end{equation*}
$$

completing the proof of Theorem 3.
Theorem 4. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a family of DHFEs. If $r>0$, then:

$$
\begin{equation*}
\operatorname{DHFWA}\left(r d_{1}, r d_{2}, \ldots, r d_{n}\right)=r \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \tag{19}
\end{equation*}
$$

Proof: According to Definition 4, we have:

$$
\begin{equation*}
r d_{j}=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\left(1-\gamma_{j}\right)^{r}\right\},\left\{\eta_{j}^{r}\right\}\right\} . \tag{20}
\end{equation*}
$$

According to Theorem 1, we have:
$\operatorname{DHFWA}\left(r d_{1}, r d_{2}, \ldots, r d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(\left(1-\gamma_{j}\right)^{r}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j}^{r}\right)^{\omega_{j}}\right\}\right\}=$
$\cup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{r \omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{r \omega_{j}}\right\}\right\} ;$
$r \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=r\left(\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\}\right)=$

$$
\begin{align*}
& \bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\left(1-\left(1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)^{r}\right\},\left\{\left(\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right)^{r}\right\}\right\}=\right. \\
& \bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\left(\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)^{r}\right\},\left\{\left(\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right)^{r}\right\}\right\}= \\
& \cup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{r \omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{r \omega_{j}}\right\}\right\} . \tag{22}
\end{align*}
$$

Thus:

$$
\begin{equation*}
\operatorname{DHFWA}\left(r d_{1}, r d_{2}, \ldots, r d_{n}\right)=r \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \tag{23}
\end{equation*}
$$

According to Theorems 3 and 4, we can get Theorem 5 easily.
Theorem 5. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a family of DHFEs. If $r>0$ and $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ is a DHFE, then:

$$
\begin{equation*}
\text { DHFWA }\left(r d_{1} \oplus d, r d_{2} \oplus d, \ldots, r d_{n} \oplus d\right)=r \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus d \tag{24}
\end{equation*}
$$

Theorem 6. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ and $l_{j}=\left(m_{j}, n_{j}\right)(j=1,2, \ldots, n)$ be two families of DHFEs, where $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}, \theta_{j}$ is the element of HFS $m_{j}$, and $\sigma_{j}$ is the element of HFS $n_{j}$, then:
DHFWA $\left(d_{1} \oplus l_{1}, d_{2} \oplus l_{2}, \ldots, d_{n} \oplus l_{n}\right)=\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus \operatorname{DHFWA}\left(l_{1}, l_{2}, \ldots, l_{n}\right)$.
Proof: According to the operational laws of Definition 4, we have:

$$
\begin{align*}
& d_{j} \oplus l_{j}=\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{\gamma_{j}+\theta_{j}-\gamma_{j} \theta_{j}\right\},\left\{\eta_{j} \sigma_{j}\right\}\right\}= \\
& \bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\left(1-\gamma_{j}\right)\left(1-\theta_{j}\right)\right\},\left\{\eta_{j} \sigma_{j}\right\}\right\} . \tag{26}
\end{align*}
$$

According to Theorem 1, we have:
DHFWA $\left(d_{1} \oplus l_{1}, d_{2} \oplus l_{2}, \ldots, d_{n} \oplus l_{n}\right)=$
$\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(\left(1-\gamma_{j}\right)\left(1-\theta_{j}\right)\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j} \sigma_{j}\right)^{\omega_{j}}\right\}\right\}=$
$\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}} \prod_{j=1}^{n}\left(1-\theta_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j} \sigma_{j}\right)^{\omega_{j}}\right\}\right\}=$
$\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\xi_{j}\right) \frac{T_{j}}{\sum_{j=1}^{n} T_{j}} \prod_{j=1}^{n}\left(1-\gamma_{j}\right) \frac{T_{j}}{\sum_{j=1}^{n} T_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}} \prod_{j=1}^{n}\left(\sigma_{j}\right)^{\omega_{j}}\right\}\right\} ;$

DHFWA $\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus \operatorname{DHFWA}\left(l_{1}, l_{2}, \ldots, l_{n}\right)=$
$\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} \oplus \cup_{\theta_{j} \in m_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\theta_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \sigma_{j}^{\omega_{j}}\right\}\right\}=$ $\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}$
$\left\{\left\{\left(1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)+\left(1-\prod_{j=1}^{n}\left(1-\theta_{j}\right)^{\omega_{j}}\right)-\left(1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right)\left(1-\prod_{j=1}^{n}\left(1-\theta_{j}\right)^{\omega_{j}}\right)\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}} \prod_{j=1}^{n}\left(\sigma_{j}\right)^{\omega_{j}}\right\}\right\}=$
$\bigcup_{\gamma_{j} \in h_{j}, \theta_{j} \in m_{2}, \eta_{j} \in g_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\xi_{j}\right) \frac{T_{j}}{\sum_{j=1}^{n} T_{j}} \prod_{j=1}^{n}\left(1-\gamma_{j}\right) \frac{T_{j}}{\sum_{j=1}^{n} T_{j}}\right\},\left\{\prod_{j=1}^{n}\left(\eta_{j}\right)^{\omega_{j}} \prod_{j=1}^{n}\left(\sigma_{j}\right)^{\omega_{j}}\right\}\right\}$.

Thus,
DHFWA $\left(d_{1} \oplus l_{1}, d_{2} \oplus l_{2}, \ldots, d_{n} \oplus l_{n}\right)=\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \oplus \operatorname{DHFWA}\left(l_{1}, l_{2}, \ldots, l_{n}\right)$,
completing the proof of Theorem 6 .
Theorem 7. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ and $l_{j}=\left(m_{j}, n_{j}\right)(j=1,2, \ldots, n)$ be two families of DHFEs, where $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}, \theta_{j}$ are elements of HFS $m_{j}$, and $\sigma_{j}$ are elements of HFS $n_{j}$. If for all $j, \gamma_{j} \geq \theta_{j}$ and $\eta_{j} \leq \sigma_{j}$, then,

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \geq \operatorname{DHFWA}\left(l_{1}, l_{2}, \ldots, l_{n}\right) \tag{30}
\end{equation*}
$$

Proof: Since,

$$
\begin{align*}
& \operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, n_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} ;  \tag{31}\\
& \operatorname{DHFWA}\left(l_{1}, l_{2}, \ldots, l_{n}\right)=\bigcup_{\theta_{j} \in m_{j}, \sigma_{j} \in n_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\theta_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \sigma_{j}^{\omega_{j}}\right\}\right\} . \tag{32}
\end{align*}
$$

Furthermore, since $\gamma_{j} \geq \theta_{j}$ and $\eta_{j} \leq \sigma_{j}$ for all $j$, then according to the definition of the comparison laws of DHFS, we know that Theorem 7 is true.

Aggregated geometric mean (Saaty 1980; Willet, Sharda 1991; Benjamin et al. 1992; Yu 2012; Yu et al. 2012) and DHFWA operator, we define here a DHFWG operator.
Definition 6. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a family of DHFEs. A DHFWG operator is:

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=d_{1}^{\omega_{1}} \oplus d_{2}{ }^{\omega_{2}} \oplus \cdots \oplus d_{n}^{\omega_{n}} \tag{33}
\end{equation*}
$$

where $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ is the measure of importance of $d_{j}$ and $\omega$ are standardized. In particular, if $\omega=\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{T}$, then the DHFWG operator degenerate into DHFG operator:

$$
\begin{equation*}
\stackrel{n}{\operatorname{DHFG}}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=d_{1}^{\frac{1}{n}} \oplus d_{2}^{\frac{1}{n}} \oplus \ldots \oplus d_{n}^{\frac{1}{n}} \tag{34}
\end{equation*}
$$

Theorem 8. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a family of DHFEs. Then,

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right\},\left\{1-\prod_{j=1}^{n}\left(1-\eta_{j}\right)^{\omega_{j}}\right\}\right\}, \tag{35}
\end{equation*}
$$

where $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ is the measure of importance of $d_{j}$ and $\omega$ are standardized.
Theorem 9. Let $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ and $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. If for all $j, \gamma_{j}=\gamma, \eta_{j}=\eta_{\text {,, where }} \gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}, \gamma$ is the element of HFS $h, \eta$ is the element of HFS $g$, then:

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=d \tag{36}
\end{equation*}
$$

Theorem 10. Suppose $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a family of any given DHFEs. If $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ is a DHFE, $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}$, then:

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1} \otimes d, d_{2} \otimes d, \ldots, d_{n} \otimes d\right)=\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \otimes d \tag{37}
\end{equation*}
$$

Proof: The proof of Theorem 10 is similar to that of Theorem 3.
Theorem 11. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. If $r>0$, then:

$$
\begin{equation*}
\operatorname{DHFWG}\left(r d_{1}, r d_{2}, \ldots, r d_{n}\right)=\left(\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)\right)^{r} . \tag{38}
\end{equation*}
$$

Proof: The proof of Theorem 11 is similar to that of Theorem 4.
Using Theorems 10 and 11 , we can get Theorem 12 easily.
Theorem 12. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. If $r>0$ and $d=(h, g)=\bigcup_{\gamma \in h, \eta \in g}\{\{\gamma\},\{\eta\}\}$ is a DHFE, then:

$$
\begin{equation*}
\operatorname{DHFWG}\left(\left(d_{1}\right)^{r} \otimes d,\left(d_{2}\right)^{r} \otimes d, \ldots,\left(d_{n}\right)^{r} \otimes d\right)=\left(\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)\right)^{r} \otimes d \tag{39}
\end{equation*}
$$

Theorem 13. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ and $l_{j}=\left(m_{j}, n_{j}\right)(j=1,2, \ldots, n)$ be two collections of DHFEs, where $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}, \theta_{j}$ is the element of HFS $m_{j}, \sigma_{j}$ is the element of HFS $n_{j}$, then:
$\operatorname{DHFWG}\left(d_{1} \otimes l_{1}, d_{2} \otimes l_{2}, \ldots, d_{n} \otimes l_{n}\right)=\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \otimes \operatorname{DHFWG}\left(l_{1}, l_{2}, \ldots, l_{n}\right)$.
Theorem 14. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ and $l_{j}=\left(m_{j}, n_{j}\right)(j=1,2, \ldots, n)$ be two collections of DHFEs, where $\gamma_{j}$ are elements of HFS $h_{j}, \eta_{j}$ are elements of HFS $g_{j}, \theta_{j}$ is the element of HFS $m_{j}$, and $\sigma_{j}$ is the element of HFS $n_{j}$. If for all $j, \gamma_{j} \geq \theta_{j}$ and $\eta_{j} \leq \sigma_{j}$, then,

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right) \geq \operatorname{DHFWG}\left(l_{1}, l_{2}, \ldots, l_{n}\right) . \tag{41}
\end{equation*}
$$

In order to understand the relationship between the DHFWA and DHFWG operators, we introduce the following Theorem.
Theorem 15. Let $d_{j}=\left(h_{j}, g_{j}\right)(j=1,2, \ldots, n)$ be a collection of DHFEs. Then,

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \cdots, d_{n}\right) \leq \operatorname{DHFWA}\left(d_{1}, d_{2}, \cdots, d_{n}\right) . \tag{42}
\end{equation*}
$$

## 3. Selection of human resources outsourcing suppliers

Consider a multi-criteria decision-making problem under uncertainty (Hu et al. 2013; Rolland 2013; Wang et al. 2013; Ertay et al. 2013). Let $Y=\left\{Y_{1}, Y_{2}, \ldots, Y_{m}\right\}$ be the bunch of alternative schemes and $C=\left\{C_{1}, C_{2}, \ldots, C_{n}\right\}$ be the family of criteria. Assuming that the experts provide the assessment information under the criterion $C_{j}$ for the alternative $Y_{i}$ using a DHFEs $\gamma_{i j}$, based on which, the matrix $D=\left(\gamma_{i j}\right)_{m \times n}$ can be constructed. Next, based on DHFWA and DHFWG operators, we give a decision-making procedure using DHFSs as follows:

Step 1. Aggregate the DHFEs $\gamma_{i j}$ for each alternative $Y_{i}$ using the DHFWA (or DHFWG) operator.

$$
\begin{equation*}
\operatorname{DHFWA}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\omega_{j}}\right\},\left\{\prod_{j=1}^{n} \eta_{j}^{\omega_{j}}\right\}\right\} \tag{43}
\end{equation*}
$$

or

$$
\begin{equation*}
\operatorname{DHFWG}\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\bigcup_{\gamma_{j} \in h_{j}, \eta_{j} \in g_{j}}\left\{\left\{\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right\},\left\{1-\prod_{j=1}^{n}\left(1-\eta_{j}\right)^{\omega_{j}}\right\}\right\} . \tag{44}
\end{equation*}
$$

Step 2. Sort the alternative schemes by Definition 3.

$$
\begin{equation*}
S\left(d_{i}\right)=\frac{1}{\# h} \sum_{\gamma \in h} \gamma-\frac{1}{\# g} \sum_{\eta \in g} \eta, i=1,2, \cdots, m . \tag{45}
\end{equation*}
$$

Then, the bigger the value of $S\left(\gamma_{i}\right)$, the larger the overall DHFE $\gamma_{i}$ will be, so choose the alternative $Y_{i}(i=1,2, \cdots, m)$.

It is quite common for enterprises to outsource human resource services from a thirdparty provider while concentrating on their core businesses. A company defines requirements for human resources, and the human resources outsourcing firm will attempt to provide associated services to meet such requirements. Some human resources outsourcing firms are generalists, providing all sorts of services, while others may be more specialized, focusing on specific areas such as payrolls and recruitments. Therefore, an enterprise can outsource all human resources tasks or only some of them depending on its business need and how much control it wish to retain over its human resources functions. Typical services provided by human resources outsourcing firms include organizational structure planning and personnel requirements, recruitment, training and development, and so on. Let us consider a foreign company ABC that recently started its core business in an industrial park. As a new comer in the region, ABC consider it is better to outsource HR services from an outsider provider. ABC views HR outsourcing as a strategic tool for getting the right people for its core business. ABC is now facing a decision which HR service provider should be selected to take its HR responsibilities.

After full consideration, they choose three evaluation criteria: enterprise size and background $\left(C_{1}\right)$, outsources service quantity $\left(C_{2}\right)$, and service quality $\left(C_{3}\right)$. The criterion weight vector is supposed as $w=(0.3,0.4,0.3)^{T}$. The evaluation information on the alternatives $x_{i}(i=1,2, \ldots, 4)$ under the criterion $C=\left\{C_{1}, C_{2}, C_{3}\right\}$ is represented by the DHFEs $d_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}, \eta_{i j} \in g_{i j}}\left\{\left\{\gamma_{i j}\right\},\left\{\eta_{i j}\right\}\right\}, 0 \leq \gamma_{i j}, \eta_{i j} \leq 1$ and $0 \leq \gamma_{i j}^{+}+\eta_{i j}^{+} \leq 1$. The dual hesitant fuzzy decision information given by experts is shown in Table 1.

Table 1. Evaluation information

|  | $C_{1}$ | $C_{2}$ | $C_{3}$ |
| :---: | :---: | :---: | :---: |
| $x_{1}$ | $\{\{0.5,0.6,0.7\},\{0.2,0.3\}\}$ | $\{\{0.6,0.7\},\{0.1,0.2\}\}$ | $\{\{0.7,0.8\},\{0.3\}\}$ |
| $x_{2}$ | $\{\{0.5,0.6\},\{0.1,0.2\}\}$ | $\{\{0.5,0.6\},\{0.1,0.2,0.3,0.4\}\}$ | $\{\{0.7,0.8,0.9\},\{0.1\}\}$ |
| $x_{3}$ | $\{\{0.5,0.6\},\{0.3,0.4\}\}$ | $\{\{0.4,0.5\},\{0.2,0.3,0.4,0.5\}\}$ | $\{\{0.7\},\{0.2,0.3\}\}$ |
| $x_{4}$ | $\{\{0.7,0.8\},\{0.1,0.2\}\}$ | $\{\{0.5,0.6,0.7\},\{0.2,0.3\}\}$ | $\{\{0.6,0.8\},\{0.1,0.2\}\}$ |

If we use the DHFWA operator, the main steps are as follows:
Step 1. Utilize the DHFWA operator (Eq. (6)) to fuse all the DHFEs $d_{i j}$ in the $i$ th line of $D$ and obtain the synthesized DHFEs $d_{i}$.
$d_{1}=\{\{0.6904,0.7485,0.7367,0.7862,0.7104,0.7648,0.7538,0.8000,0.7344,0.7842$, $0.7741,0.8165\},\{0.1866,0.2462,0.2297,0.3031\}\}$
$d_{2}=\{\{0.6134,0.7070,0.6860,0.7620\},\{0.2259,0.2980,0.3259,0.3505,0.2366,0.3121$, $0.3413,0.3671\}\}$
$d_{3}=\{\{0.7178,0.7862,0.7361,0.8000\},\{0.2980,0.3669,0.3259,0.4012,0.3728,0.4590$, $0.3933,0.4842,0.3249,0.4000,0.3552,0.4373,0.4064,0.5004,0.4287,0.5278\}\}$
$d_{4}=\{\{0.5988,0.6741,0.6331,0.7020,0.6730,0.7344,0.7115,0.7656,0.7361,0.7856$, $0.7648,0.8089\},\{0.1320,0.1625,0.1741,0.2144,0.1835,0.2259,0.2421,0.2980\}\}$
Step 2. Calculate the scores of $d_{i}(i=1,2,3,4)$, respectively, as

$$
s\left(d_{1}\right)=0.5169, s\left(d_{2}\right)=0.3849, s\left(d_{3}\right)=0.3549, s\left(d_{4}\right)=0.5117
$$

Since:

$$
s\left(d_{1}\right)>s\left(d_{4}\right)>s\left(d_{2}\right)>s\left(d_{3}\right)
$$

we have

$$
x_{1} \succ x_{4} \succ x_{2} \succ x_{3}
$$

The best option is candidate $x_{1}$.
If we use the DHFWG operator, the main steps are as follows:
Step 1'. Utilize the DHFWG operator (Eq. (37)) to fuse all the DHFEs $d_{i j}(i=1,2,3,4)$ in the $i$ th line of $D$ and obtain the synthesized DHFEs $d_{i}^{\prime}$.
$d_{1}{ }^{\prime}=\{\{0.6587,0.6823,0.6948,0.7198,0.6957,0.7207,0.7339,0.7602,0.7286,0.7548$, $0.7686,0.7962\},\{0.2307,0.2661,0.2944,0.3268\}\}$
$d_{2}{ }^{\prime}=\{\{0.3933,0.4122,0.4287,0.4494\},\{0.3268,0.4000,0.4422,0.4898,0.3825,0.4496$, $0.4883,0.5320\}\}$
$d_{3}{ }^{\prime}=\{\{0.6415,0.7198,0.6776,0.7602\},\{0.3150,0.3716,0.3631,0.4158,0.4808,0.5238$, $0.5586,0.5951,0.3459,0.4000,0.3919,0.4422,0.5043,0.5453,0.5785,0.6134\}\}$
$d_{4}{ }^{\prime}=\{\{0.5842,0.6369,0.6284,0.6850,0.6684,0.7286,0.6300,0.6867,0.6776,0.7387$, $0.7207,0.7857\},\{0.1414,0.1712,0.2347,0.2613,0.2038,0.2314,0.2903,0.3150\}\}$

Step 2'. Calculate the scores of $d_{i}^{\prime}(i=1,2,3,4)$, respectively, as:
$s\left(d_{1}^{\prime}\right)=0.4467, s\left(d_{2}^{\prime}\right)=-0.018, s\left(d_{3}^{\prime}\right)=0.2345, s\left(d_{4}{ }^{\prime}\right)=0.4498$.

Since

$$
s\left(d_{4}\right)>s\left(d_{1}\right)>s\left(d_{3}\right)>s\left(d_{2}\right)
$$

we have
$x_{4} \succ x_{1} \succ x_{3} \succ x_{2}$.
The best option is candidate $x_{4}$.
The optimal decision has changed, the sort result obtained using the DHFWG operator is different from that obtained using the DHFWA operator. The DHFWA operator focuses on the impact of the overall data while the DHFWG operator highlights the role of individual data.

## Concluding remarks

As a generalization of FSs, DHFSs give us an additional possibility for depicting imperfect knowledge. In this paper, we have developed a DHFWA operator and a DHFWG operator for information aggregation that extends two of the broadly applicable aggregation operators (the WA and WG operators) to accommodate situations, in which the input information is DHFEs. We also studied various properties of the proposed operators and have illustrated their application to the selection of human resources outsourcing suppliers in a dual hesitant fuzzy environment.

## Acknowledgements

The author would like to thank the anonymous reviewers. This paper is supported by the National Natural Science Foundation of China (No.71301142), Zhejiang Natural Science Foundation of China (No. LQ13G010004), Project funded by China Postdoctoral Science Foundation (No. 2014M550353) and the National Education Information Technology Research (No. 146242069).

## References

Atanasov, K. T. 1986. Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1): 87-96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3
Beliakov, G.; Pradera, A.; Calvo, T. 2007. Aggregation functions: a guide for practitioners. Berlin: Spring-er-Verlag.
Benjamin, C. O.; Ehie, L. C.; Omurtag, Y. 1992. Planning facilities at the University of Missourirolla, Interface 22(4): 95-105. http://dx.doi.org/10.1287/inte.22.4.95
Bustince, H.; Herrera, F.; Montero, J. 2007. Fuzzy sets and their extensions: representation, aggregation and models. Berlin, Germany: Springer Verlag.
Calvo, T.; Mayor, G.; Mesiar, R. 2002. Aggregation operators: new trendsand applications. Heidelberg, Germany: Physica-Verlag. http://dx.doi.org/10.1007/978-3-7908-1787-4

Dubois, D.; Prade, H. 1980. Fuzzy sets and systems: theory and applications. New York: Academic Press.
Ertay, T.; Kahraman, C.; Kaya, İ. 2013. Evaluation of renewable energy alternatives using MACBETH and fuzzy AHP multicriteria methods: the case of Turkey, Technological and Economic Development of Economy 19(1): 38-62. http://dx.doi.org/10.3846/20294913.2012.762950
Fernando, B.; Umberto, S. 2013. Aggregation operators for fuzzy ontologies, Applied Soft Computing 13(9): 3816-3830. http://dx.doi.org/10.1016/j.asoc.2013.05.008
Hu, J. H.; Zhang, Y.; Chen, X. H.; Liu, Y. M. 2013. Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number, Knowledge-Based Systems 43: 21-29. http://dx.doi.org/10.1016/j.knosys.2012.11.007
Kosareva, N.; Krylovas, A. 2013. Comparison of accuracy in ranking alternatives performing generalized fuzzy average functions, Technological and Economic Development of Economy 19(1): 162-187. http://dx.doi.org/10.3846/20294913.2012.763072
Li, D. F. 2010. Multi-attribute decision making method based on generalized OWA operators with intuitionistic fuzzy sets, Expert Systems with Applications 37(12): 8673-8678. http://dx.doi.org/10.1016/j.eswa.2010.06.062
Merigó, J. M. 2012. OWA operators in the weighted average and their application in decision making, Control and Cybernetics 41: 605-643.
Merigó, J. M.; Casanovas, M. 2011a. The uncertain induced quasi-arithmetic OWA operator, International Journal of Intelligent Systems 26(1): 1-24. http://dx.doi.org/10.1002/int. 20444
Merigó, J. M.; Casanovas, M. 2011b. Decision making with distance measures and induced aggregation operators, Computers and Industrial Engineering 60(1): 66-76.
http://dx.doi.org/10.1016/j.cie.2010.09.017
Merigó, J. M.; Casanovas, M. 2011c. Induced and uncertain heavy OWA operators, Computers and Industrial Engineering 60(1): 106-116. http://dx.doi.org/10.1016/j.cie.2010.10.005
Merigó, J. M.; Casanovas, M. 2011d. Induced aggregation operators in the Euclidean distance and its application in financial decision making, Expert Systems with Applications 38(6): 7603-7608. http://dx.doi.org/10.1016/j.eswa.2010.12.103
Rolland, A. 2013. Reference-based preferences aggregation procedures in multi-criteria decision making, European Journal of Operational Research 225(3): 479-486. http://dx.doi.org/10.1016/j.ejor.2012.10.013
Saaty, T. L. 1980. The analytic hierarchy process. New York: McGraw-Hill.
Torra, V. 2003. Information fusion in data mining. New York: Springer Verlag. http://dx.doi.org/10.1007/978-3-540-36519-8
Torra, V.; Narukawa, Y. 2009. On hesitant fuzzy sets and decision, in The $18^{\text {th }}$ IEEE International Conference on Fuzzy Systems, 20-24 August 2009, Jeju Island, Korea, 1378-1382.
Torra, V. 2010. Hesitant fuzzy sets, International Journal of Intelligent Systems 25(6): 529-539. http://dx.doi.org/10.1002/int. 20418
Wang, J. Q.; Nie, R. R.; Zhang, H. Y.; Chen, X. H. 2013. Intuitionistic fuzzy multi-criteria decisionmaking method based on evidential reasoning, Applied Soft Computing 13(9): 1823-1831. http://dx.doi.org/10.1016/j.asoc.2012.12.019
Wei, G.W. 2009. Some geometric aggregation functions and their application to dynamic multiple attribute decision making in intuitionistic fuzzy setting, International Journal of Uncertainty, Fuzziness and Knowledge- Based Systems 17(2): 179-196. http://dx.doi.org/10.1142/S0218488509005802
Wei, G. W. 2010. Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, Applied Soft Computing 10(2): 423-431.
http://dx.doi.org/10.1016/j.asoc.2009.08.009

Wei, G. W. 2012. Hesitant fuzzy prioritized operators and their application to multiple attribute decision making. Knowledge-Based Systems 31: 176-182. http://dx.doi.org/10.1016/j.knosys.2012.03.011
Willet, K.; Sharda, R. 1991. Using the analytic hierarchy process in water resources planning: selection of flood control projects, Socio-Economic Planning Sciences 25(2): 103-112. http://dx.doi.org/10.1016/0038-0121(91)90008-F
Xia, M. M.; Xu, Z. S. 2011. Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52(3): 395-407.
Xia, M. M.; Xu, Z. S.; Chen, N. 2013. Some hesitant fuzzy aggregation operators with their application in group decision making, Group Decision and Negotiation 22(2): 259-279. http://dx.doi.org/10.1007/s10726-011-9261-7
Xu, Z. S. 2000. On consistency of the weighted geometric mean complex judgment matrix in AHP, European Journal of Operational Research 126(3): 683-687. http://dx.doi.org/10.1016/S0377-2217(99)00082-X
Xu, Z. S.; Da, Q. L. 2003. An overview of operators for aggregating information, International Journal of Intelligent Systems 18(9): 953-969. http://dx.doi.org/10.1002/int. 10127
Xu, Z. S. 2005. An overview of methods for determining OWA weights, International Journal of Intelligent Systems 20(8): 843-865. http://dx.doi.org/10.1002/int.20097
Xu, Z. S.; Yager, R. R. 2006. Some geometric aggregation operators based on intuitionistic fuzzy sets, International Journal of General Systems 35(4): 417-433. http://dx.doi.org/10.1080/03081070600574353
Xu, Z. S. 2007. Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 15(6): 1179-1187. http://dx.doi.org/10.1109/TFUZZ.2006.890678
Xu, Z. S. 2010. Uncertain Bonferroni mean operators, International Journal of Computational Intelligence Systems 3(6): 761-769. http://dx.doi.org/10.1080/18756891.2010.9727739
Xu, Z. S. 2011. Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators, Knowledge Based Systems 24(6): 749-760. http://dx.doi.org/10.1016/j.knosys.2011.01.011
Yager, R. R. 1986. On the theory of bags, International Journal of General Systems 13(1): 23-37. http://dx.doi.org/10.1080/03081078608934952
Yager, R. R. 1988. On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Transactions on Systems, Man and Cybernetics, B 18(1): 183-190.
Yager, R. R.; Kacprzyk, J. 1997. The ordered weighted averaging operator: theory and applications. Boston, MA: Kluwer. http://dx.doi.org/10.1007/978-1-4615-6123-1
Yager, R. R. 2002. Heavy OWA operators, Fuzzy Optimization and Decision Making 1(4): 379-397. http://dx.doi.org/10.1023/A:1020959313432
Yager, R. R. 2003. Induced aggregation operators, Fuzzy Sets and Systems 137(1): 59-69. http://dx.doi.org/10.1016/S0165-0114(02)00432-3
Yager, R. R. 2006. Generalizing variance to allow the inclusion of decision attitude in decision making under uncertainty, International Journal of Approximate Reasoning 42(3): 137-158. http://dx.doi.org/10.1016/j.ijar.2005.09.001
Yager, R. R. 2007. Centered OWA operators, Soft Computing 11(7): 631-639. http://dx.doi.org/10.1007/s00500-006-0125-z
Yager, R. R. 2009a. Prioritized OWA aggregation, Fuzzy Optimization and Decision Making 8(3): 245262. http://dx.doi.org/10.1007/s10700-009-9063-4

Yager, R. R. 2009b. On the dispersion measure of OWA operators, Information Sciences 179(22): 39083919. http://dx.doi.org/10.1016/j.ins.2009.07.015

Ye, J. 2014. Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making, Applied Mathematical Modelling 38(2): 659-666.
http://dx.doi.org/10.1016/j.apm.2013.07.010
Yu, D. J. 2012. Group decision making based on generalized intuitionistic fuzzy prioritized geometric operator, International Journal of Intelligent Systems 27(7): 635-661.
http://dx.doi.org/10.1002/int. 21538
Yu, D. J.; Zhang, W. Y.; Xu, Y. J. 2013. Group decision making under hesitant fuzzy environment with application to personnel evaluation, Knowledge-Based Systems 52: 1-10. http://dx.doi.org/10.1016/j.knosys.2013.04.010
Yu, D. J. 2014a. Some hesitant fuzzy information aggregation operators based on Einstein operational laws, International Journal of Intelligent Systems 29(4): 320-340. http://dx.doi.org/10.1002/int.21636
Yu, D. J. 2014b. Some generalized dual hesitant fuzzy geometric aggregation operators and applications, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 22(3): 367-384. http://dx.doi.org/10.1142/S0218488514500184
Yu, D. J. 2015. Multi-attribute decision making based on intuitionistic fuzzy interaction average operators: a comparison, International Transactions in Operational Research 22(6): 1017-1032. http://dx.doi.org/10.1111/itor. 12115
Yu, D. J.; Wu, Y. Y.; Lu, T. 2012. Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making, Knowledge-Based Systems 30: 57-66.
http://dx.doi.org/10.1016/j.knosys.2011.11.004
Zadeh, L. A. 1965. Fuzzy sets, Information and Control 8: 338-353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X
Zadeh, L. A. 1975. The concept of a linguistic variable and its application to approximate reasoning-I, Information Sciences 8(3): 199-249.
Zhang, N. 2013. Method for aggregating correlated interval grey linguistic variables and its application to decision making, Technological and Economic Development of Economy 19(2): 189-202. http://dx.doi.org/10.3846/20294913.2012.763071
Zhao, X. F.; Wei, G. W. 2013. Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making, Knowledge-Based Systems 37: 472-479. http://dx.doi.org/10.1016/j.knosys.2012.09.006
Zhang, X. L.; Xu, Z. S. 2015. Hesitant fuzzy agglomerative hierarchical clustering algorithms, International Journal of Systems Science 46(3): 562-576. http://dx.doi.org/10.1080/00207721.2013.797037
Zhao, H.; Xu, Z. S.; Ni, M. F.; Liu. S. S. 2010. Generalized aggregation operators for intuitionistic fuzzy sets, International Journal of Intelligent Systems 25(1): 1-30. http://dx.doi.org/10.1002/int. 20386
Zhu, B.; Xu, Z. S.; Xia, M. M. 2012a. Dual Hesitant Fuzzy Sets, Journal of Applied Mathematics, vol. 2012, Article ID 879629.13 p. http://dx.doi.org/10.1155/2012/879629
Zhu, B.; Xu, Z. S.; Xia, M. M. 2012b. Hesitant fuzzy geometric Bonferroni means, Information Sciences 205: 72-85. http://dx.doi.org/10.1016/j.ins.2012.01.048

Dejian YU. He received the PhD degree in management science and engineering from Southeast University, Nanjing, China, in 2012. He is currently an associated professor with the School of Information, Zhejiang University of Finance and Economics, Hangzhou, China. He has authored more than 20 scientific articles. His current research interests include aggregation operators, information fusion, and multi-criteria decision making.

Wenyu ZHANG. He received the B.S. degree from the Zhejiang University, Hangzhou, China in 1989, and the PhD degree from the Nanyang Technological University, Singapore, in 2002. He is a full professor and dean at the School of Information, Zhejiang University of Finance \& Economics, Hangzhou, China. He has published more than 30 papers in International Journals and more than 20 papers in International Conference Proceedings in the recent ten years, covering a wide range of manufacturing automation, especially supply chain management, concurrent engineering, computer-aided manufacturing (CAM)/computer-aided process planning (CAPP)/computer integrated manufacturing (CIM), distributed manufacturing, multiagent technology, and Semantic Web.

George HUANG. Mr George Q. Huang is Professor and Head of Department in Department of Industrial and Manufacturing Systems Engineering, The University of Hong Kong. He gained his BEng and PhD in Mechanical Engineering from Southeast University (China) and Cardiff University (UK) respectively. He has conducted research projects in the field of Physical Internet (Internet of Things) for Manufacturing and Logistics with substantial government and industrial grants. He has published extensively including over two hundred refereed journal papers in addition to over 200 conference papers and ten monographs, edited reference books and conference proceedings. His research works have been widely cited in the relevant field. He serves as associate editors and editorial members for several international journals. He is a Chartered Engineer (CEng), a fellow of ASME, HKIE, IET and CILT, and member of IIE.


[^0]:    Corresponding author George Huang
    E-mail: gqhuang@hku.hk

