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A SPECTRAL EXCESS THEOREM FOR DIGRAPHS WITH NORMAL LAPLACIAN MATRICES

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Abstract. The spectral excess theorem, due to Fiol and Garriga in 1997, is an important result, because it gives a good characterization of distance-regularity in graphs. Up to now, some authors have given some variations of this theorem. Motivated by this, we give the corresponding result by using the Laplacian spectrum for digraphs. We also illustrate this Laplacian spectral excess theorem for digraphs with few Laplacian eigenvalues and we show that any strongly connected and regular digraph that has normal Laplacian matrix with three distinct eigenvalues, is distance-regular. Hence such a digraph is strongly regular with girth $g = 2$ or $g = 3$.

1. Introduction

One of the most important concepts in combinatorics is *distance-regularity* of graphs. A *distance-regular* graph is a regular graph such that for any two vertices v and w at distance i, the number of vertices adjacent to w and at distance j from v only depends on i and j. Distance-regular graphs of diameter 2 are *strongly regular graphs* [2, 4]. For more information on distance-regular graphs, we refer the reader to [2, 3, 4, 6, 8] and [15]. Some authors extended the notion of distance-regularity of graphs to the directed graphs in different ways. The research on distance-regular digraphs was initiated by Damerell in 1981 [9]. A digraph is *strongly connected* if every pair of vertices can be

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joined by a path and is *connected* if its underlying graph is connected. The (directed) *distance* from a vertex u to a vertex v, $\partial(u,v)$, is the length of a shortest $u-v$ (directed) path in Γ. The maximum (directed) distance between any two distinct vertices of Γ is called the *diameter* of Γ and is denoted by D. A digraph is *geodetic* if the shortest path between any two vertices is unique. The *girth* $g(\Gamma)=g$ (resp. odd-girth $g_o(\Gamma)=g_o$) is the smallest length of a cycle (an odd cycle) in Γ. For any vertex $u\in V(\Gamma)$ we will denote $\Gamma_k^+(u)$ (respectively, $\Gamma_k^-(u)$) the set of vertices at distance k from u (respectively, the set of vertices from which to u is at distance k). If $k=1$, for any vertex u we have, $outdegree(u)=|\Gamma_1^+(u)|$ and $indegree(u)=|\Gamma_1^-(u)|$ and if the in-degree and the out-degree at each vertex of Γ are equal to k, we say Γ is regular with degree (valency) k.

The *adjacency matrix* $\mathbf{A}=(a_{ij})$ of Γ with vertex set $V(\Gamma)=\{v_1,v_2,\ldots,v_n\}$ is a $n\times n$ matrix indexed by the vertices of Γ, with entries $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. The spectrum of the *digraph* Γ is denoted by the multi-set

$$Spec(\Gamma)=\{\lambda_0^{m_0},\lambda_1^{m_1},\ldots,\lambda_d^{m_d}\},$$

where the superscripts m_i denote the multiplicities of the distinct eigenvalues $\lambda_i, i=0,1,\ldots,d$. Since $\mathbf{A}$ is not symmetric, the eigenvalues of $\mathbf{A}$ might not be real. Digraphs with symmetric adjacency matrix are precisely graphs. A digraph Γ is called *normal* if $\mathbf{A}$ is a normal matrix; that is, $\mathbf{A}\mathbf{A}^*=\mathbf{A}^*\mathbf{A}$, where $\mathbf{A}^*$ is the transpose of $\mathbf{A}$'s conjugate.

A strongly connected digraph Γ with diameter D is distance-regular if, for any two vertices $u,v\in V$ such that $\partial(u,v)=k$, for $0\le k\le D$, the numbers $a_{i1}^k(u,v)=|\Gamma_i^+(u)\bigcap\Gamma_1^+(v)|$, for each i such that $0\le i\le k+1$, only depend on their distance k. We write $a_{i1}^k(u,v)=a_{i1}^k$, for short that called intersection numbers. Every distance-regular digraph with girth g satisfies $\partial(u,v)+\partial(v,u)=g$ for any pair of vertices $u,v\in V(\Gamma)$ at distance $0<\partial(u,v)<g$ [9]. In this case we say the digraph Γ is stable. From the stability of distance-regular digraph Γ and the fact $g\in\{2,D,D+1\}$, we conclude that the digraph Γ is normal [5].

If we replace Γ_1^+ with Γ_1^- in the definition of a distance-regular digraph, the stability property dose not necessarily hold, and we obtain another digraph with weak structure but interesting properties that named as *weakly distance-regular* (see [5]). Another concept of weakly distance-regular digraph (that is equivalent to the previous concept) is: A digraph Γ with diameter D is weakly distance-regular if, for each nonnegative integer $\ell\le D$, the number a_{uv}^ℓ of walks of length ℓ from vertex u to vertex v only depends on their distance $\partial(u,v)=k$, for any $\ell=0,1,\ldots,D$. Weakly distance-regular digraphs of diameter 2 are *strongly regular digraphs* that were first investigated by Duval in [10] as an extension of strongly regular graphs to the directed case. A directed strongly regular graph with parameters (n,k,μ,λ,t) is a regular directed graph on n vertices with valency k, such that every vertex is incident with t undirected edges, and the number of walks of length 2 from a vertex u to another vertex v is λ, if there is an edge from u to v, and μ otherwise. In

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particular, for $t = k$ we have the undirected case and for $t = 0$ the digraph is a tournament. See Brouwer's website [2] and the references given there for more details. A weakly distance-regular digraph with adjacency matrix $\mathbf{A}$ is distance-regular if and only if Γ is stable or $\mathbf{A}$ is normal [5, 10].

So far, there have been some results on characterization of distance-regular graphs. Some of these results are based on the *spectral excess theorem*. This well-known theorem characterizes distance-regular graphs by their spectra and the average number of vertices at extremal distance. In fact this theorem states that a connected regular graph with $d + 1$ distinct eigenvalues is distance-regular if for every vertex, the mean of the numbers of vertices at distance d from every vertex (the excess) equals some given expression in terms of the spectrum of the graph. For more information about this theorem see [7, 8, 11, 13, 16, 17, 18, 19] and [20].

Distance-regularity of a digraph is in general not determined by the spectrum of the digraph. In [21] the author showed that distance-regularity of a strongly connected normal digraph Γ (with $d + 1$ distinct eigenvalues) is determined by its spectrum and the average of the numbers of vertices at distance d from every vertex. In this note we give the corresponding result by using the Laplacian spectrum for digraphs. We also state some applications of this result. In fact we illustrate this Laplacian spectral excess theorem for digraphs with few Laplacian eigenvalues. We show that any strongly connected and regular digraph that has normal Laplacian matrix with three distinct eigenvalues, is distance-regular. Hence such a digraph is strongly regular with girth $g = 2$ or $g = 3$. We next prove that if Γ is a strongly connected, regular and geodetic digraph that its Laplacian matrix is normal and has four distinct eigenvalues and with finite girth $g \geq 3$, then Γ is distance-regular. We also show that every strongly connected and geodetic digraph with normal Laplacian matrix $\mathbf{L}$ and finite girth $g \geq d + 1$ is distance-regular. In [21] also it is shown that every strongly connected normal digraph Γ with $d+1$ distinct eigenvalues and finite odd-girth at least $2d + 1$ is distance-regular. We think this result is obtained when the Laplacian matrix of any strongly connected digraph with $d+1$ distinct Laplacian eigenvalues and (finite) odd-girth at least $2d + 1$, is normal and we left it as a conjecture.

Conjecture. Let Γ be a strongly connected digraph with normal Laplacian matrix and with finite odd-girth $g_o \geq 2d + 1$. Then Γ is distance-regular and $g_o = 2d + 1$.

2. Preliminaries

In this section, we give the background and introduce some terminology and notations that occur in this note. A *directed* graph (or just *digraph*) $\Gamma = (V, E)$ consists of a non-empty finite set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ of elements called *vertices* or *nodes*, and a finite set $E(\Gamma)$ of ordered pairs of distict vertices called *arcs* or *directed edges*. For two distinct vertices u and v of Γ, we say that

u is adjacent to v if there is an edge (directed edge) from u to v. The digraph that loops, edges of the form (v, v), and multiple edges are not permitted in it is called *simple*. A closed directed walk in a digraph Γ is called *Eulerian* if it uses every edge exactly once. We say that Γ is Eulerian if it has such a walk. The distance-k matrix $\mathbf{A}_k$ of a digraph Γ with diameter D, where $0 \leq k \leq D$ is defined by $(\mathbf{A}_k)_{ij} = 1$ if $\partial(v_i, v_j) = k$ and $(\mathbf{A}_k)_{ij} = 0$, otherwise. In particular, $\mathbf{A}_0 = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix and $\mathbf{A}_1 = \mathbf{A}$. We observe that $\sum_{k=0}^{D} \mathbf{A}_k = \mathbf{J}$, where $\mathbf{J}$ is a matrix with all entries equal to 1. These matrices play an important role in the study of distance-regularity. We denote the *average excess* of digraph Γ with δ_d, where $\delta_d = \dfrac{\sum_{v \in V(\Gamma)} |\Gamma_d^+(v)|}{n}$.

The *Laplacian matrix* of a digraph Γ with the vertices $v_1, v_2, \ldots, v_n$ whose out-degrees are $d_1^+, d_2^+, d_3^+, \ldots, d_n^+$ respectively, is the matrix $\mathbf{L} = \mathbf{O} - \mathbf{A}$, where $\mathbf{O}$ is the (diagonal) matrix of vertex out-degree of digraph Γ. It generalizes the Laplacian matrix of an ordinary graph. Note that the Laplacian matrix is independent of the number of loops of Γ on each vertex. The Laplacian spectrum of the digraph Γ, denote by $spec_{\mathbf{L}}(\Gamma)$ consists of eigenvalues of $\mathbf{L}$, together with their (algebraic) multiplicities:

$$Spec_{\mathbf{L}}(\Gamma) = \{\mu_0^{m_0}, \mu_1^{m_1}, \ldots, \mu_d^{m_d}\}.$$

Since the graph is directed, $\mathbf{L}$ might be non-symmetric real. The polynomial $H(x) = n\dfrac{\mathbf{S}(x)}{\mathbf{S}(0)}$, where $m_\Gamma(x) = (x - \mu_0)\mathbf{S}(x)$ is the Laplacian minimal polynomial, is called *Hoffman-like polynomial*. The Laplacian algebra of Γ is defined by $\psi_L(\Gamma) = \{P(\mathbf{L}) : P \in C[x]\}$. The dimension of this space is equal to the degree of the Laplacian minimal polynomial of Γ. If Γ is strongly connected, then $\mathbf{L}$ has a simple zero eigenvalue and so from the equality $\mathbf{LH(L)} = \mathbf{H(L)L} = \mathbf{0}$, we see that each column of $\mathbf{H(L)}$ is a multiple of j and each row of $\mathbf{H(L)}$ is a multiple of j^t, where j denotes the all-1 matrix. So $\mathbf{H(L)} = cjj^t = c\mathbf{J}$. Now let the Laplacian matrix of Γ be normal with spectrum as above. For each μ_i, let $\mathbf{U}_i$ be the matrix whose columns form an orthonormal basis of the eigenspace $\varphi_i := Ker(\mathbf{L} - \mu_i I)$. Then the orthogonal projection onto φ_i is represented by the matrix

$$\mathbf{E}_i = \mathbf{U}_i \mathbf{U}_i^t = \frac{1}{\pi_i} \prod_{j \neq i} (\mathbf{L} - \mu_j I) \quad (0 \leq i \leq d),$$

where $\pi_i = \prod_{i=1, i \neq j}^{d} (\mu_i - \mu_j)$. In particular, $\mathbf{E}_0 = \dfrac{1}{n}\mathbf{J}$. For more information about digraphs, see [1, 14].

3. Laplacian pre-distance polynomials for digraphs

In this section, at first, we introduce Laplacian pre-distance polynomials for a given strongly connected digraph. Then we consider these polynomials for a strongly connected digraph that its Laplacian matrix is normal. Let $deg(m_\Gamma(x)) = \hat{D} + 1$. Consider the $(\hat{D} + 1)$-dimensional vector space $C_{\hat{D}}[x] \simeq C[x]/\tau$, where $\tau = \langle m_\Gamma(x) \rangle$ is the ideal generated by the Laplacian minimal polynomial of Γ. We define the following scalar product in this vector space, as

$$\langle f, g \rangle = \frac{1}{n} tr(f(\mathbf{L})g(\mathbf{L})^*) \tag{3.1}$$

with the norm defined by $\| f \| = \sqrt{\langle f, f \rangle}$ for $f, g \in C_{\hat{D}}[x]$. It is clear that $\| f \| = 0$ if and only if $f = 0$. By the defined scalar product (3.1), we have $\delta_k = \langle \mathbf{A}_k, \mathbf{A}_k \rangle = \frac{1}{n} tr(\mathbf{A}_k \mathbf{A}_k^*)$ for every $0 \leq k \leq D$. Let $\langle g, f \rangle$ be the scalar product in (3.1). Then the *projection* operator is defined by

$$\mathrm{Proj}_g(f) = \frac{\langle g, f \rangle}{\langle g, g \rangle} g.$$

This operator projects the polynomial $f(\mathbf{L})$ orthogonally onto the polynomial $g(\mathbf{L})$. The projection of f into $\{g_k\}_{k=0}^s$ is defined by $\sum_{k=0}^s \mathrm{Proj}_{g_k}(f)$. Notice that $\{1, x, x^2, \ldots, x^{\hat{D}}\}$ is a basis of $C_{\hat{D}}[x]$. By starting with this basis, we use the Gram-Schmidt method to produce an orthogonal set $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{\hat{D}}\}$ as follows:

Let $\mathcal{P}_0 = 1$, $\mathcal{P}_1 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{|E(\Gamma)|}{n}$ and for $0 \leq i \leq \hat{D} - 1$

$$\mathcal{P}_{i+1} = x^{i+1} - \sum_{k=0}^{i} \mathrm{Proj}_{\mathcal{P}_k}(x^{i+1}).$$

The polynomials $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{\hat{D}}\}$ is an orthogonal basis of $C_{\hat{D}}[x]$ that spans the same space $(\hat{D} + 1)$-dimensional as the basis $\{1, x, x^2, \ldots, x^{\hat{D}}\}$ and for each i, $\mathcal{P}_i$ is a polynomial that has degree i and leading coefficient 1. The polynomial $\mathcal{P}_i$ is the i-th *predistance polynomial* of Γ. By normalizing the set of predistance polynomials $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{\hat{D}}\}$, we obtain a new set of polynomials $\{p_0, p_1, \ldots, p_d\}$ such that $\|p_i\|^2 = \delta_i$ for each $0 \leq i \leq D$ and $p_i = \mathcal{P}_i$ for $D < i \leq \hat{D}$.

Throughout this note, we assume that $\Gamma = (V, E)$ is a strongly connected (finite) simple digraph on n vertices with $d+1$ distinct Laplacian eigenvalues, diameter D, minimal polynomial of degree $\hat{D} + 1$, distance matrices $\{\mathbf{A}_k\}_{k=0}^D$, predistance polynomials $\{\mathcal{P}_k\}_{k=0}^{\hat{D}}$ and normalized predistance polynomials $\{p_k\}_{k=0}^{\hat{D}}$.

Let two arbitrarily matrices A and B be normal. In this case, can we say AB and $A + B$ are normal? The following proposition gives an answer to this question.

Proposition 3.1. [14] *If A and B are normal with $AB = BA$, then both AB and $A + B$ are also normal.*

Notice that for a strongly connected digraph Γ if $\mathbf{AO} = \mathbf{OA}$, and Γ is normal, then Γ is regular. Thus by Proposition 3.1, if Γ is regular and normal, then $\mathbf{L}$ is normal.

Theorem 3.2. [1] *A digraph is Eulerian if and only if $|\Gamma_1^+(u)| = |\Gamma_1^-(u)|$ for every vertex u and the underlying graph has at most one nontrivial component.*

Suppose that an arbitrary digraph Γ is connected with Laplacian matrix $\mathbf{L}$ such that $\mathbf{H(L)} = \mathbf{J}$, thus we can write $\mathbf{LJ} = \mathbf{LH(L)} = \mathbf{H(L)L} = \mathbf{JL}$. So by Theorem 3.2 Γ is Eulerian and $j^t\mathbf{L} = \mathbf{0}$. Thus we have the following proposition.

Proposition 3.3. *Let Γ be a connected digraph and satisfies $\boldsymbol{H(L)} = \boldsymbol{J}$, then Γ is strongly connected.*

When Γ is a strongly connected digraph with normal Laplacian matrix and with Hoffman-like polynomial $\mathbf{H(x)} = \dfrac{n}{\pi_0} \prod_{i=1}^{d}(x-\mu_i)$, we have $\mathbf{H(L)} = n\mathbf{E}_0 = \mathbf{J}$. But the converse of this fact is not true (there is a strongly connected digraph such that $\mathbf{H(L)} = \mathbf{J}$, but its Laplacian matrix is not normal).

4. A Laplacian spectral excess theorem for digraphs

In this section, at first, we give some characterizations of weakly distance-regular digraphs. Then we give a Laplacian version of spectral excess theorem for strongly connected digraphs with normal Laplacian matrix $\mathbf{L}$ as a generalization of the Laplacian spectral excess theorem for graphs. Also we give some of the applications of this theorem. The following theorem may be proved in much the same way as [5, Theorem 2.2] and [13, Theorem 3].

Theorem 4.1. *For a strongly connected digraph Γ with diameter D, and distance matrices $\boldsymbol{A}_k$, where $0 \le k \le D$ the following statements are equivalent:*
(a) Γ is a weakly distance-regular digraph.
(b) The distance matrix $\boldsymbol{A}_i$ is a polynomial of degree i in the Laplacian matrix $\boldsymbol{L}$; that is, $\boldsymbol{A}_i = r_i(\boldsymbol{L})$ for each $i = 0, 1, \ldots, D$, where $r_i \in \boldsymbol{Q[x]}$.
(c) The set of distance matrices $\{\boldsymbol{A}_i : i = 0, 1, \ldots, D\}$ is a basis of the Laplacian algebra $\psi_L(\Gamma)$.

Now we state the following lemma that has an important role in the proof of Laplacian spectral excess theorem. Since the main idea of the proof of this lemma is similar to the idea of [21, the proof of Theorem 4.4], we state it without any proof.

Lemma 4.2. *Let Γ be a strongly connected digraph with normal Laplacian matrix $\boldsymbol{L}$, predistance polynomials $p_0, p_1, \ldots, p_d$, and distance matrices $\boldsymbol{A}_k, k = 0, 1, \ldots, d$. If $p_d(\boldsymbol{L}) = \boldsymbol{A}_d$ then $p_k(\boldsymbol{L}) = c_k\boldsymbol{A}_k$ for some constant c_k and for every $k = 0, 1, \ldots, d$.*

The following theorem gives a characterization of distance-regular digraphs. The proof is similar to that used in [21, Theorem 4.5].

Theorem 4.3. *For any strongly connected digraph Γ, we have $\varepsilon_\Gamma \le k_d$. Where, $\varepsilon_\Gamma = \frac{\langle \boldsymbol{A}_d, \boldsymbol{L}^d\rangle^2}{\delta_d}$ if $d \le D$ (ε_Γ is zero if $d > D$) and $k_d = \langle \mathcal{P}_d, \mathcal{P}_d\rangle$.*

For a strongly connected digraph Γ with normal Laplacian matrix the equality holds if and only if Γ is distance-regular.

Let Γ be digraph with normal Laplacian matrix. Then by considering the equation

$$\mathbf{H(L)} = \mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_d = \sum_{i=0}^{d} \frac{\mathcal{P}_i(0)}{\langle \mathcal{P}_i, \mathcal{P}_i \rangle} \mathcal{P}_i(\mathbf{L}), \tag{4.1}$$

since $\mathbf{H(L)}$ is a polynomial of degree d, thus $\langle H(\mathbf{L}), \mathcal{P}_d(\mathbf{L}) \rangle = \mathcal{P}_d(0) \neq 0$.

Now let $q_d = \frac{\mathcal{P}_d(0)}{\langle \mathcal{P}_d, \mathcal{P}_d \rangle} \mathcal{P}_d$ (notice that q_d is obtained by normalizing $\mathcal{P}_d$ so that $\langle q_d, q_d \rangle = q_d(0)$). It is clear that $q_d(0)$ is nonzero and are determined by the Laplacian spectrum of Γ. The important number $q_d(0)$ is called *the spectral excess* of Γ that gives a characterization of the distance-regular digraphs. Now we give a Laplacian spectral excess theorem for strongly connected digraph with normal Laplacian matrix.

Theorem 4.4. *For any strongly connected digraph Γ with normal Laplacian matrix $\boldsymbol{L}$ that has $d+1$ distinct eigenvalues, and Laplacian predistance polynomial $\mathcal{P}_d$, we have $\delta_d \leq q_d(0)$.*

And equality holds if and only if Γ is distance-regular.

Proof. By considering the equation 4.1, we have

$$\delta_d = \langle \mathbf{A}_d, H(\mathbf{L}) \rangle = \frac{\mathcal{P}_d(0)}{\langle \mathcal{P}_d, \mathcal{P}_d \rangle} \langle \mathbf{A}_d, \mathbf{L}^d \rangle. \tag{4.2}$$

If $d > D$, then $\mathbf{A}_d$ is the zero-matrix and so in this case we have $\delta_d = \varepsilon_\Gamma = 0$. Thus we are done instead. When $d \leq D$, from the equation 4.2 and the equation $q_d = \frac{\mathcal{P}_d(0)}{\langle \mathcal{P}_d, \mathcal{P}_d \rangle} \mathcal{P}_d$, it follows that $\varepsilon_\Gamma = \dfrac{\langle \mathbf{A}_d, \mathbf{L}^d \rangle^2}{\delta_d} = \frac{\langle \mathcal{P}_d, \mathcal{P}_d \rangle}{q_d(0)} \delta_d$ and so using Theorem 4.3, we have $\varepsilon_\Gamma = \frac{k_d}{q_d(0)} \delta_d \leq k_d$ and the proof is completed. □

Now we state and prove an application of Laplacian spectral excess theorem. In fact we show that any strongly connected and geodetic digraph Γ with normal Laplacian matrix $\mathbf{L}$ and finite girth $g \geq d+1$ is distance-regular.

Theorem 4.5. *Let Γ be a strongly connected and geodetic digraph with normal Laplacian matrix $\boldsymbol{L}$ and finite girth $g \geq d+1$. Then Γ is distance-regular.*

Proof. First, notice that the girth of any digraph is at most $D+1$, using this fact and the assumption we conclude that $D = d$. Also, since $\mathbf{L}$ is normal, by [5, Theorem 1.1] we have $\mathbf{L}^t = f(\mathbf{L}) \in \psi_L(\mathbf{L})$. So the equality $\mathbf{L}^t = \sum_{k=0}^{d} \dfrac{\langle \mathbf{L}^t, p_k(\mathbf{L}) \rangle}{\langle p_k(\mathbf{L}), p_k(\mathbf{L}) \rangle} p_k(\mathbf{L})$ implise that

$$\frac{\langle \mathbf{L}^t, \mathbf{A}_D \rangle}{\langle \mathbf{L}^t, p_D \rangle} = \frac{\langle p_D(\mathbf{L}), A_D \rangle}{\langle p_D(\mathbf{L}), p_D(\mathbf{L}) \rangle}.$$

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Now from $g = D + 1$ and the property of geodetic digraph, we deduce that $p_D(\mathbf{L}) = c\mathbf{A}_D$ for some constant c and so $\mathcal{P}_D(\mathbf{L}) = c'\mathbf{A}_D$ for some constant c'. Thus we have

$$q_d(0) = \frac{\mathcal{P}_d(0)}{\langle \mathcal{P}_d, \mathcal{P}_d \rangle}\mathcal{P}_d(0) = d\frac{\langle \mathcal{P}_d, \mathbf{J} \rangle \langle \mathcal{P}_d, \mathbf{J} \rangle}{\langle \mathcal{P}_d, \mathcal{P}_d \rangle} = \delta_d.$$

So Theorem 4.4 implies that *Gamma* is distance-regular. □

Now we consider Laplacian spectral excess theorem for digraphs with three Laplacian eigenvalues as an application of this theorem.

Theorem 4.6. *Let Γ be a strongly connected and regular digraph that its Laplacian matrix is normal and has three distinct eigenvalues. Then Γ is strongly regular.*

Proof. Since the Laplacian matrix of Γ has three distinct eigenvalues, it follows that $D \leq 2$. If the diameter of Γ is one, then Γ is a complete graph that its Laplacian matrix has two distinct eigenvalues. Hence we consider strongly connected and regular digraph Γ of diameter 2 that its Laplacian matrix is normal and has three distinct eigenvalues. So we can write

$$\mathbf{J} = H(\mathbf{L}) = \frac{n}{\pi_0}(\mathbf{L} - \mu_1\mathbf{I})(\mathbf{L} - \mu_2\mathbf{I}),$$

because L is normal and Γ is regular, we have $\mathcal{P}_1(\mathbf{L}) = -\mathbf{A}_1$. Thus $\langle \mathcal{P}_2, \mathbf{A}_1 \rangle = 0$. Therfore

$$\langle \mathcal{P}_2, \mathcal{P}_2 \rangle = \langle \mathcal{P}_2, \mathbf{L}^2 \rangle = \frac{\pi_0}{n}\langle \mathcal{P}_2, \mathbf{J} \rangle = \frac{\pi_0}{n}\langle \mathcal{P}_2, \mathbf{A}_2 \rangle = \frac{\pi_0^2}{n^2}\delta_2.$$

Hence we have

$$q_2(0) = \frac{\mathcal{P}_2(0)^2}{\langle \mathcal{P}_2, \mathcal{P}_2 \rangle} = \frac{n}{\pi_0}\mathcal{P}_2(0) = \delta_2.$$

So by Theorem 4.4, any strongly connected and regular digraph that has normal Laplacian matrix with three distinct eigenvalues, is distance-regular. Hence such a digraph is strongly-regular with girth $g = 2$ or $g = 3$ (see [10]). □

Motivated by the previous theorem, we give the following result.

Theorem 4.7. *Let Γ be a strongly connected, regular and geodetic digraph that its Laplacian matrix is normal and has four distinct eigenvalues and with finite girth $g \geq 3$. Then Γ is distance-regular.*

Proof. Since Γ is strongly connected with normal Laplacian matrix L that has four distinct eigenvalues, we conclude that

$$\mathbf{J} = H(\mathbf{L}) = \frac{n}{\pi_0}(\mathbf{L} - \mu_1\mathbf{I})(\mathbf{L} - \mu_2\mathbf{I})(\mathbf{L} - \mu_3\mathbf{I}). \tag{4.3}$$

Therefore

$$\langle \mathbf{A}_3, \mathcal{P}_3 \rangle = \langle \mathbf{A}_3, \mathbf{L}^3 \rangle = \frac{\pi_0}{n}\langle \mathbf{A}_3, \mathbf{J} \rangle = \frac{\pi_0}{n}\delta_3. \tag{4.4}$$

Now using 4.1 we obtain $\delta_3 = \langle \mathbf{J}, \mathbf{A}_3 \rangle = \dfrac{\mathcal{P}_3(0)}{\langle \mathcal{P}_3, \mathcal{P}_3 \rangle} \langle \mathcal{P}_3, \mathbf{A}_3 \rangle$. Hence it follows that

$$\langle \mathcal{P}_3, \mathcal{P}_3 \rangle = \frac{\pi_0}{n} \mathcal{P}_3(0). \tag{4.5}$$

On the other hand, since Γ is regular and geodetic digraph with $g \geqslant 3$, we can write

$$\langle \mathcal{P}_3, \mathbf{A}_2 \rangle + \langle \mathcal{P}_3, \mathbf{A}_1 \rangle + \langle \mathcal{P}_3, \mathbf{A}_0 \rangle = \langle \mathcal{P}_3, \mathcal{P}_2 + \frac{\langle \mathbf{L}^2, \mathcal{P}_1 \rangle}{\langle \mathcal{P}_1, \mathcal{P}_1 \rangle} \mathcal{P}_1 + \frac{\langle \mathbf{L}^2, \mathcal{P}_0 \rangle}{\langle \mathcal{P}_0, \mathcal{P}_0 \rangle} \mathcal{P}_0 \rangle + \langle \mathcal{P}_3, -P_1 \rangle + \langle \mathcal{P}_3, \mathcal{P}_0 \rangle = 0.$$

Hence

$$\mathcal{P}_3(0) = \langle \mathcal{P}_3, \mathbf{J} \rangle = \langle \mathcal{P}_3, \mathbf{A}_3 \rangle \tag{4.6}$$

Using the equations 4.4-4.6, we have $\dfrac{\langle \mathbf{A}_3, \mathbf{L}^3 \rangle^2}{\delta_3} = \langle \mathcal{P}_3, \mathcal{P}_3 \rangle = \dfrac{\pi_0^2}{n^2} \delta_3$. So Theorem 4.3 implies that Γ is distance-regular. $\square$

In [21], as an application of spectral excess theorem for normal digraphs, the author showed that distance-regularity of a strongly connected normal digraph is determined by the spectrum and the average excess of the digraph. Finally he showed that any connected normal digraph with finite odd-girth $g_o(\Gamma) \geq 2d+1$, is distance-regular and $g_o(\Gamma) = 2d+1$. This generalizes a result of van Dam and Fiol [12]. If we deal with any strongly connected digraph with normal Laplacian matrix, can we have the second result? It is clear that when Γ is regular, we get this result. But for general case, we think the answer of this question is positive and we left it as a conjecture.

Conjecture. Let Γ be a strongly connected with normal Laplacian matrix and finite odd-girth $g_o \geq 2d+1$. Then Γ is distance-regular and $g_o = 2d+1$.

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