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DIFFERENCE EQUATIONS WITH IMPULSES

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Abstract. Difference equations with impulses are studied focussing on the existence of periodic or bounded orbits, asymptotic behavior and chaos. So impulses are used to control the dynamics of the autonomous difference equations. A model of supply and demand is also considered when Li–Yorke chaos is shown among others.

Keywords: difference equations, impulses, stability, fixed points, Li–Yorke chaos.

Mathematics Subject Classification: 37B55, 39A23, 39A33, 39A60.

1. INTRODUCTION

Differential equations with impulses play an important role in many applications [9]. The purpose of this paper is to consider their discrete versions as follows. Let $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence of natural numbers. Let $f, g \in C(K, K)$ for a subset $K \subset \mathbb{R}^m$. Let us consider an impulsive difference equation (IDE) of the form

$$\begin{aligned} x_{n+1} &= f(x_n) \quad \text{for } n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}}, \\ x_{n_i+1} &= f(g(x_{n_i})) \quad \text{for } i \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. So we consider in (1.1) iterations $x_n \to x_{n+1} = f(x_n)$ for $n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}}$, then impulses $x_n \to \bar{x}_n = g(x_n)$ at $n \in \{n_i\}_{i\in\mathbb{N}}$ and continuing with iterations $\bar{x}_{n_i} \to x_{n_i+1} = f(\bar{x}_{n_i})$.

Another type of IDE, but a similar one has the form

$$x_{n+1} = \begin{cases} f(x_n), & n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}_0}, \\ x_n + \gamma, & n = n_i. \end{cases} \tag{1.2}$$

When g is identity, so there are no impulses in (1.1), we get a discrete dynamical system

$$x_{n+1} = f(x_n) \quad \text{for } n \in \mathbb{N}_0, \tag{1.3}$$

which is well studied [4]. The purpose of this paper is to control the dynamics of (1.3) by using the impulses $x_n \to \bar{x}_n = g(x_n)$ at $n \in \{n_i\}_{i\in\mathbb{N}}$.

The paper is organized as follows. In Section 2, we consider (1.1) with equidistant impulses to derive results on the existence, uniqueness and stability of periodic orbits. Section 3 is devoted to affine IDE focussing on asymptotically equidistant impulses and controlling their stability. In Section 4, conditions are derived for practical controllability of boundedness of solutions for affine IDE with shifted impulses. An impulsive supply and demand model of the form (1.2) with shifted impulses motivated by [11] is investigated in Section 5. When changing the impulse shift, it is shown a Li–Yorke chaos on one side, while globally asymptotically stable periodic orbits on the other side. The final Section 6 discusses periodic orbits and periodic points for general nonautonomous periodic difference equations. Several concrete numerical examples are presented to illustrate the theory. Related results are investigated in [1–3, 5, 8].

2. GENERAL EQUATIONS WITH EQUIDISTANT IMPULSES

The simplest case of (1.1) is for equidistant impulses, i.e., $n_i = i\Delta$ for a fixed $\Delta \in \mathbb{N}$. Then the dynamics of (1.1) is given by the mapping

$$F(x) = g(f^{\Delta}(x)), \tag{2.1}$$

since

$$\bar{x}_{n_i} = \bar{x}_{i\Delta} = F^i(x)$$

and then

$$x_n = f^{n-(\lceil \frac{n}{\Delta} \rceil - 1)\Delta}(F^{\lceil \frac{n}{\Delta} \rceil - 1}(x_0)), \quad n \in \mathbb{N}, \tag{2.2}$$

where $\lceil \cdot \rceil$ is the ceil function. In particular, (2.2) implies

$$x_{i\Delta+1} = f(F^i(x_0)) \tag{2.3}$$

for any $i \in \mathbb{N}$. Then (2.3) gives that any p-periodic orbit of F generates a $p\Delta$-periodic orbit of (1.1). Now we are ready to prove the next result.

Theorem 2.1. *Let L_f and L_g be global Lipschitz constants of f and g, i.e. $\|h(x_1) - h(x_2)\| \leq L_h \|x_1 - x_2\|$ for any $x_1, x_2 \in K$ and $h \in \{f, g\}$. If K is closed and*

$$L_g L_f^{\Delta} < 1, \tag{2.4}$$

then (1.1) *has a globally exponentially stable Δ-periodic orbit in K starting at the unique fixed point $\tilde{x}_0$ of F given by* (2.1) *in K.*

Proof. By (2.4), mapping $F : K \to K$ is a contraction with a constant $L_g L_f^{\Delta}$, so the Banach fixed point theorem gives a unique stable fixed point $\tilde{x}_0$ of F. For any $x_0 \in K$, (2.3) implies

$$\tilde{x}_{\Delta+1} = f(F(\tilde{x}_0)) = f(\tilde{x}_0) = \tilde{x}_1$$

as well as

$$\begin{aligned}\|x_n - \tilde{x}_n\| &= \left\| f^{n-(\lceil \frac{n}{\Delta}\rceil -1)\Delta}(F^{\lceil \frac{n}{\Delta}\rceil -1})(x_0) - f^{n-(\lceil \frac{n}{\Delta}\rceil -1)\Delta}(F^{\lceil \frac{n}{\Delta}\rceil -1})(\tilde{x}_0)\right\| \\ &\le L_f^{n-(\lceil \frac{n}{\Delta}\rceil -1)\Delta} \left\| F^{\lceil \frac{n}{\Delta}\rceil -1}(x_0) - F^{\lceil \frac{n}{\Delta}\rceil -1}(\tilde{x}_0)\right\| \\ &\le \max\{1, L_f^{\Delta}\}(L_g L_f^{\Delta})^{\lceil \frac{n}{\Delta}\rceil -1}\|x_0 - \tilde{x}_0\| \\ &\le \max\{1, L_f^{\Delta}\}(L_g L_f^{\Delta})^{\frac{n}{\Delta} -1}\|x_0 - \tilde{x}_0\| \\ &= \max\{1, L_f^{\Delta}\}(L_g L_f^{\Delta})^{\frac{n-\Delta}{\Delta}}\|x_0 - \tilde{x}_0\|,\end{aligned}$$

due to $\lceil r\rceil \ge r$ for any $r \in \mathbb{R}$. By (2.4) the proof is completed. □

Example 2.2. Let us consider the impulsed logistic map of the form (1.1) with $f(x) = \lambda x(1-x)$, $\lambda \in (0,4]$ and impulses $g(x) = \gamma x + \theta$, $\gamma \in (0,1)$, $\theta \in [0, 1-\gamma]$.

Now, $K = [0,1] \subset \mathbb{R}$, $L_f = \lambda$ and $L_g = \gamma$, condition (2.4) takes the form $\gamma\lambda^{\Delta} < 1$. Hence by Theorem 2.1 there is a globally exponentially stable Δ-periodic orbit of (1.1) in K. On the other hand, using [4, Corollary C.4], we can find another conditions for the existence of a globally asymptotically stable Δ-periodic orbit of (1.1) for the case $\theta \in (0, 1-\gamma]$. For instance, if $\lambda = 4$, then Theorem 2.1 gives that for $\gamma < \frac{1}{256}$, $\theta = 0$ and $\Delta = 4$ the zero orbit is a globally asymptotically stable 4-periodic orbit of (1.1) (see Figure 1a). While for $\gamma = \frac{1}{4^3}$, $\theta = 0.1$ and $\Delta = 4$, we apply [4, Corollary C.4] and it follows that, there is a unique globally asymptotically stable 4-periodic orbit of (1.1) starting from $x_0 \doteq 0.10942$ (see Figure 1b).

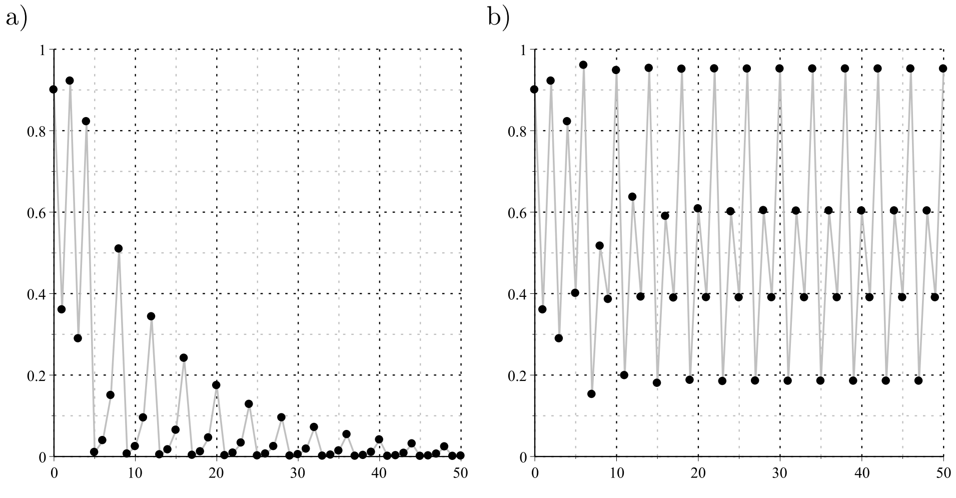

Fig. 1. Iterations of $x_0 = 0.9$ for the impulsed logistic map with $\Delta = 4$ and the parameters: a) $\lambda = 4$, $\gamma = 0.003$, $\theta = 0$; b) $\lambda = 4$, $\gamma = 1/4^3$, $\theta = 0.1$

Slightly modifying the above arguments we obtain a result for periodic set of impulse times, i.e., when $n_1 < n_2 < \ldots < n_s = \Delta$ for a fixed $\Delta \in \mathbb{N}$, and $n_k = n_{k-s} + \Delta$ for $k = s+1, s+2, \ldots$, or equivalently

$$n_k = \begin{cases} \left\lfloor \frac{k}{s} \right\rfloor \Delta + n_{k - \left\lfloor \frac{k}{s} \right\rfloor s} & \text{for } k \in \{s+1, s+2, \ldots\} \setminus s\mathbb{N}, \\ \frac{k}{s} \Delta & \text{for } k \in s\mathbb{N}, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function. Then the dynamics of (1.1) is ruled by

$$F(x) = \left(g \circ f^{\Delta - n_{s-1}} \circ g \circ f^{n_{s-1} - n_{s-2}} \circ \cdots \circ g \circ f^{n_2 - n_1} \circ g \circ f^{n_1} \right)(x) \tag{2.5}$$

and (2.2), (2.3) remain valid. We state a result analogous to Theorem 2.1.

Theorem 2.3. *Let L_f and L_g be global Lipschitz constants of f and g, i.e. $\|h(x_1) - h(x_2)\| \leq L_h \|x_1 - x_2\|$ for any $x_1, x_2 \in K$ and $h \in \{f, g\}$. If K is closed and*

$$L_g^s L_f^\Delta < 1, \tag{2.6}$$

then (1.1) *has a globally exponentially stable Δ-periodic orbit in K starting at the unique fixed point $\tilde{x}_0$ of F given by* (2.5) *in K.*

Proof. We omit the proof as it is the same as the proof of Theorem 2.1. □

More complicated dynamics can be derived from the next result.

Theorem 2.4. *Let $F : [0,1] \to [0,1]$ be a C^1-map with fixed points $\operatorname{Fix} F = \{0 = x_1 < x_2 < \ldots < x_k < 1\}$. Suppose $|F'(x_j)| \neq 1$ for any $j = 1, \ldots, k$, so fixed points are either attractors or repellers. Moreover $F(1) = 0$. If $F'(x_j) + 1 < 0$ for some $j > 1$ then there is a 2-periodic point of F in $(0, x_j)$ and also in $(x_j, 1)$. If $(F'(x_j) + 1)(F'(x_i) + 1) < 0$ for some $x_i < x_j$, then there is a 2-periodic point of F in (x_i, x_j).*

Proof. Consider the function $G(x) := \frac{F(F(x)) - x}{F(x) - x}$ for any $x \notin \operatorname{Fix} F$. On the other hand for any $x_0 \in \operatorname{Fix} F$, we have

$$\lim_{x \to x_0} G(x) = \lim_{x \to x_0} \frac{F'(F(x))F'(x) - 1}{F'(x) - 1} = F'(x_0) + 1,$$

by the l'Hospital rule. Hence G can be continuously extended on $[0,1]$. Next, $G(1) = 1$ and, since $F \in C^1([0,1],[0,1])$ and $F(0) = 0$, we get $F'(0) \geq 0$ and $G(0) = F'(0) + 1 \geq 1$. If $F'(x_j) + 1 < 0$ for some $j > 1$, then G changes the sign on $(0, x_j)$ and also on $(x_j, 1)$, so G has roots in these intervals and these roots are 2-periodic points of F, since $G(x_i) \neq 0$ for any $i = 1, \ldots, k$. So if $G(x) = 0$ for some $x \in [0,1]$, then $F(F(x)) = x$ but $F(x) \neq x$. Similarly, if $(F'(x_j) + 1)(F'(x_i) + 1) < 0$ for some $x_i < x_j$ then G changes the sign on (x_i, x_j), so G has a root in this interval and this root is a 2-periodic point of F. The proof is finished. □

Example 2.5. We can apply Theorem 2.4 to the above case $f(x) = \lambda x(1-x)$, $\lambda \in (0,4]$ and $g(x) = \gamma x$, $\gamma \in (0,1)$, such that for the given f, we can adjust $\gamma \in (0,1)$ so that (2.1) satisfies assumptions of Theorem 2.4. Consequently, (1.1) has many Δ-periodic as well as 2Δ-periodic solutions which are not Δ-periodic. On the other hand, a complementary result to Theorem 2.4 is [4, Theorem C.3]. For instance, from Figure 2 one can see that (1.1) with $f(x) = \lambda x(1-x)$, $\lambda = 4$, $g(x) = \gamma x + \theta$, $\gamma = 0.45$, $\theta = 0.42 \in [0, 1-\gamma]$, $\Delta = 2$, leads to the convergence of all orbits of (1.1), since there are no 2-periodic orbits of the corresponding F. Moreover, derivatives of F at its fixed points satisfy $F'(0.44305) = -0.79881$, $F'(0.74097) = 1.85808$, $F'(0.86721) = -0.41638$. Hence the fixed points 0.44305 and 0.86721 are stable while the fixed point 0.74097 is unstable.

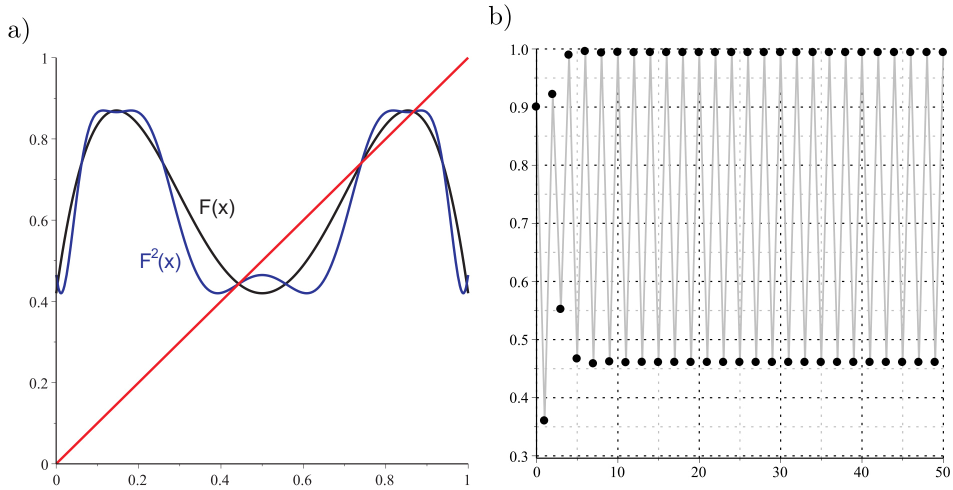

Fig. 2. Graphs of F and F^2 illustrating their fixed points (a); the orbit of $x_0 = 0.9$ for the impulsed logistic map with $\Delta = 2$ and the parameters $\lambda = 4$, $\gamma = 0.45$, $\theta = 0.42$ (b)

3. AFFINE IMPULSIVE DIFFERENCE EQUATIONS

In general, (1.1) is difficult to analyze. The simplest cases are affine ones with $f(x) = Ax + a$ and impulses $g(x) = x + Bx + b$ for matrices $A, B \in L(\mathbb{R}^m)$ and vectors $a, b \in \mathbb{R}^m$. Then by the superposition principle of solutions of (1.1), the stability of any solution of (1.1) is equivalent to the stability of the zero solution of the linear case

$$\begin{aligned} x_{n+1} &= Ax_n \quad \text{for } n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}}, \\ x_{n_i+1} &= A(\mathbb{I}+B)x_{n_i} \quad \text{for } i \in \mathbb{N}, \end{aligned} \tag{3.1}$$

so $f = A$ and $g = \mathbb{I} + B$ for the identity matrix $\mathbb{I} \in L(\mathbb{R}^m)$. By setting

$$i(n) = \#\{i \in \mathbb{N} \mid n_i < n\}$$

for any $n \in \mathbb{N}_0$, the general solution of (3.1) is given by

$$\begin{aligned} x_n &= X(n)x_0, \\ X(n) &= A^{n-n_{i(n)}}(\mathbb{I}+B)A^{n_{i(n)}-n_{i(n)-1}}\dots A^{n_2-n_1}(\mathbb{I}+B)A^{n_1}. \end{aligned} \tag{3.2}$$

This is rather similar to formula [9, (3.3)], so we can apply those results of [9] to (3.2). Moreover, assume that A and B are commutative, i.e., $AB = BA$, then formula (3.2) is simplified to

$$X(n) = A^n(\mathbb{I}+B)^{i(n)}, \tag{3.3}$$

which is rather similar to formula [9, (3.5)].

Remark 3.1. Before stating the next result, we need the following comment. Let $B \in L(\mathbb{R}^m)$ and suppose that its spectrum $\sigma(B)$ satisfies

$$\sigma(B) \cap \{z \in \mathbb{C} \mid \arg z = -\pi + \varepsilon\} = \emptyset$$

for some $\varepsilon \geq 0$ small. This is equivalent to the invertibility of B. Then following arguments of [6, Theorem 1.13, Problem 1.37] and [7, pp. 105–106], there is a unique logarithm matrix $\ln B$ of B such that

$$\sigma(\ln B) \subset \{z \in \mathbb{C} \mid -\pi + \varepsilon < \Im z < \pi + \varepsilon\}$$

which is independent of such ε. Moreover, if $A \in L(\mathbb{R}^m)$ with $AB = BA$, then $A \ln B = (\ln B)A$. Finally, we set $B^p = \mathrm{e}^{p \ln B}$ for any $p \in \mathbb{R}$ and note $AB^p = B^p A$.

Now we have the following result similar to [9, Theorem 34].

Theorem 3.2. *Consider* (3.1) *with* $AB = BA$, $\mathbb{I}+B$ *invertible, and assume*

$$\lim_{n\to\infty} \frac{i(n)}{n} = p \tag{3.4}$$

exists and is finite. Then it holds

(i) *if the spectrum radius* $r(A(\mathbb{I}+B)^p)$ *of* $A(\mathbb{I}+B)^p$ *satisfies* $r(A(\mathbb{I}+B)^p) < 1$ *then* (3.1) *is exponentially stable,*
(ii) *if* $r(A(\mathbb{I}+B)^p) > 1$ *then* (3.1) *is unstable.*

Proof. (i) There is a $K > 0$ such that

$$\|(A(\mathbb{I}+B)^p)^n\| \leq K\left(\frac{r(A(\mathbb{I}+B)^p)+1}{2}\right)^n, \quad \forall n \in \mathbb{N} \tag{3.5}$$

(see [7]). Set $\Lambda = \frac{r(A(\mathbb{I}+B)^p)+1}{2}$. By (3.3) and (3.5), we derive

$$\begin{aligned} \|X(n)\| &= \|A^n(\mathbb{I}+B)^{i(n)}\| = \|A^n \mathrm{e}^{i(n)\ln(\mathbb{I}+B)}\| \\ &= \|A^n \mathrm{e}^{np\ln(\mathbb{I}+B)}\, \mathrm{e}^{n\left(\frac{i(n)}{n}-p\right)\ln(\mathbb{I}+B)}\| \\ &\leq \|(A(\mathbb{I}+B)^p)^n\| \|\mathrm{e}^{n\left(\frac{i(n)}{n}-p\right)\ln(\mathbb{I}+B)}\| \\ &\leq K\Lambda^n \mathrm{e}^{n\left|\frac{i(n)}{n}-p\right|\|\ln(\mathbb{I}+B)\|} = K\,\mathrm{e}^{n\left(\ln\Lambda+\left|\frac{i(n)}{n}-p\right|\|\ln(\mathbb{I}+B)\|\right)}. \end{aligned} \tag{3.6}$$

Then using $\Lambda < 1$, there is $n_0 \in \mathbb{N}$ such that

$$\ln\Lambda + \left|\frac{i(n)}{n} - p\right| \|\ln(\mathbb{I}+B)\| \leq \frac{\ln\Lambda}{2} < 0 \tag{3.7}$$

for any $n \geq n_0$. Consequently, using (3.6) and (3.7), we arrive at

$$\|X(n)\| \leq K\,\mathrm{e}^{\frac{\ln\Lambda}{2}n} = K(\sqrt{\Lambda})^n$$

for any $n \geq n_0$, which proves (i).

(ii) There is a $K > 0$ and $x_0 \in \mathbb{R}^m$, $\|x_0\| = 1$, such that

$$\|(A(\mathbb{I}+B)^p)^n x_0\| \geq K\Lambda^n, \quad \forall n \in \mathbb{N} \tag{3.8}$$

(see [7]). By (3.3) and (3.8), we derive

$$\begin{aligned}
\|X(n)x_0\| &= \|A^n(\mathbb{I}+B)^{i(n)}x_0\| = \|A^n\,\mathrm{e}^{i(n)\ln(\mathbb{I}+B)}x_0\| \\
&= \|\,\mathrm{e}^{n\left(\frac{i(n)}{n}-p\right)\ln(\mathbb{I}+B)}A^n\,\mathrm{e}^{np\ln(\mathbb{I}+B)}x_0\| \\
&\geq \|\,\mathrm{e}^{-n\left(\frac{i(n)}{n}-p\right)\ln(\mathbb{I}+B)}\|^{-1}\|(A(\mathbb{I}+B)^p)^n x_0\| \\
&\geq \mathrm{e}^{-n\left|\frac{i(n)}{n}-p\right|\|\ln(\mathbb{I}+B)\|}K\Lambda^n = K\,\mathrm{e}^{n\left(\ln\Lambda-\left|\frac{i(n)}{n}-p\right|\|\ln(\mathbb{I}+B)\|\right)}.
\end{aligned} \tag{3.9}$$

Then using $\Lambda > 1$, there is $n_0 \in \mathbb{N}$ such that

$$\ln\Lambda - \left|\frac{i(n)}{n} - p\right| \|\ln(\mathbb{I}+B)\| \geq \frac{\ln\Lambda}{2} > 0 \tag{3.10}$$

for any $n \geq n_0$. Consequently, using (3.9) and (3.10), we arrive at

$$\|X(n)x_0\| \geq K\,\mathrm{e}^{\frac{\ln\Lambda}{2}n} = K(\sqrt{\Lambda})^n$$

for any $n \geq n_0$, which proves (ii), since $\|X(n)x_0\| \to \infty$ as $n \to \infty$. The proof is finished. □

The next step would be the study of semilinear (1.1), i.e., a nonlinear perturbation of (3.1) like in [9, Theorem 3.5, Section 3.2], but we do not go into details now.

4. AFFINE EQUATIONS WITH EQUIDISTANT AND SHIFTED IMPULSES

Even simpler cases than in Section 3 are affine ones

$$\begin{aligned}
x_{n+1} &= Ax_n + \alpha \quad \text{for } n \in \mathbb{N}_0 \setminus \{i\Delta\}_{i\in\mathbb{N}}, \\
x_{i\Delta+1} &= A(x_{i\Delta} + \gamma) + \alpha \quad \text{for } i \in \mathbb{N},
\end{aligned} \tag{4.1}$$

for a matrix $A \in L(\mathbb{R}^m)$ and vectors $\alpha, \gamma \in \mathbb{R}^m$, so $f(x) = Ax + \alpha$ and $g(x) = x + \gamma$. Then by the superposition principle of solutions of (4.1), the stability of any solution of (4.1) is equivalent to the stability of the zero solution of the linear case

$$x_{n+1} = Ax_n \quad \text{for } n \in \mathbb{N},$$

which is well-known [4]. Assume that A is hyperbolic and unstable, i.e., there is no eigenvalue of A in the unit circle and the spectral radius $r(A)$ of matrix A satisfies $r(A) > 1$. Then the affine system

$$x_{n+1} = Ax_n + \alpha \quad \text{for } n \in \mathbb{N} \tag{4.2}$$

has a unique hyperbolic unstable equilibrium $\bar{x} = (\mathbb{I} - A)^{-1}\alpha$ for the identity matrix $\mathbb{I} \in L(\mathbb{R}^m)$. If we are ensured to have iterations of (4.2) within a ball $B_r = \{x \in \mathbb{R}^m \mid \|x\| \leq r\}$ and $\|\bar{x}\| \gg r$, then generic iterations of (4.2) starting at B_r exponentially quickly leave B_r. On the other hand, consider (4.1) on the interval $n \leq n_0\Delta$ for a given $n_0 \in \mathbb{N}$. Then (2.1) has the form

$$F(x) = A^\Delta x + \sum_{j=0}^{\Delta-1} A^j\alpha + \gamma, \tag{4.3}$$

and (4.1) has a unique hyperbolic unstable Δ-periodic orbit starting at

$$\bar{x}_0 = (\mathbb{I} - A^\Delta)^{-1}\left(\sum_{j=0}^{\Delta-1} A^j\alpha + \gamma\right),$$

so

$$\gamma = \bar{x}_0 - A^\Delta\bar{x}_0 - \sum_{j=0}^{\Delta-1} A^j\alpha.$$

Taking

$$\gamma = -\sum_{j=0}^{\Delta-1} A^j\alpha, \tag{4.4}$$

we get $\bar{x}_0 = 0$, and the unique hyperbolic unstable Δ-periodic orbit is given by

$$\{\bar{x}_k\}_{k=1}^{\Delta} \quad \text{for } \bar{x}_k = \sum_{j=0}^{k-1} A^j\alpha, \tag{4.5}$$

which is inside the ball $B_{\bar{r}}$ for $\bar{r} = \sum_{j=0}^{\Delta-1}\|A\|^j\|\alpha\|$. Consequently, supposing

(C1) $\sum_{j=0}^{\Delta-1}\|A\|^j\|\alpha\| < r$,

the hyperbolic unstable Δ-periodic orbit (4.5) is in B_r. Summarizing, we get the following result.

Theorem 4.1. *Suppose* (C1) *and take* γ *by* (4.4). *Then for any* $x_0 \in \mathbb{R}^m$ *satisfying*

$$\|x_0\| \leq \frac{r - \bar{r}}{\|A\|^{n_0\Delta}}, \tag{4.6}$$

the iterations of (4.1) *starting at* x_0 *remain in the ball* B_r *for* $n \leq n_0\Delta$. *Moreover, the iterations of* (4.1) *starting at* $x_0 = 0$ *remain in the ball* B_r *for any* $n \in \mathbb{N}$.

Proof. Assuming (4.6), we derive

$$\|x_n\| \le \|\bar{x}_n\| + \|A\|^n \|x_0\| \le \bar{r} + \|A\|^{n_0\Delta} \|x_0\| \le r$$

for $n \le n_0\Delta$, where x_n is given by (4.1). The proof is finished. □

Roughly saying, Theorem 4.1 gives a practical controllability of boundedness of (4.2) using impulsive system (4.1).

Remark 4.2. 1. The above results can be extended to a semilinear case

$$\begin{aligned} x_{n+1} &= Ax_n + h(x_n) + \alpha \quad \text{for } n \in \mathbb{N}_0 \setminus \{i\Delta\}_{i\in\mathbb{N}}, \\ x_{i\Delta+1} &= A(x_{i\Delta} + \gamma) + h(x_{i\Delta} + \gamma) + \alpha \quad \text{for } i \in \mathbb{N}, \end{aligned}$$

for $h : \mathbb{R}^m \to \mathbb{R}^m$ with $\|h(y) - h(z)\| \le L_h \|y - z\|$ for any $y, z \in \mathbb{R}^m$ and for a suitable constant $L_h > 0$, but we do not go into details now.

2. By [7, Theorem 4.47], the lower bound of (C1) is $\sum_{j=0}^{\Delta-1} r(A)^j \|\alpha\|$, which also gives a lower bound for r. More precisely, for any $\varepsilon > 0$ there is a norm $\|\cdot\|$ on $\mathbb{R}^m$ such that $\sum_{j=0}^{\Delta-1} \|A\|^j \|\alpha\| \le \sum_{j=0}^{\Delta-1} r(A)^j \|\alpha\| + \varepsilon$.

Example 4.3. To illustrate our results, we consider a simple scalar case

$$\begin{aligned} x_{n+1} &= 1.01x_n + 1 \quad \text{for } n \in \mathbb{N}_0 \setminus \{10i\}_{i\in\mathbb{N}}, \\ x_{10i+1} &= 1.01(x_{10i} + \gamma) + 1 \quad \text{for } i \in \mathbb{N}, \end{aligned} \tag{4.7}$$

and $n_0 = 10$. The first 20 iterations of (see (4.2))

$$x_{n+1} = 1.01x_n + 1 \quad \text{for } n \in \mathbb{N} \tag{4.8}$$

with $x_0 = 0$ are given by

$$\begin{aligned} &\{0, 1, 2.01, 3.0301, 4.0604, 5.10101, 6.15202, 7.21354, \\ &\ 8.28567, 9.36853, 10.46221, 11.56683, 12.6825, 13.80933, \\ &\ 14.94742, 16.0969, 17.25786, 18.43044, 19.61475, 20.8109, 22.019\}. \end{aligned}$$

The hyperbolic unstable equilibrium of (4.8) is $\bar{x} = -100$. On the other hand, by (4.4) taking $\gamma = -10.46221$, in (4.7), it has a unique hyperbolic unstable 10-periodic orbit

$$\begin{aligned} &\{0, 1, 2.01, 3.0301, 4.0604, 5.10101, 6.15202, \\ &\ 7.21354, 8.28567, 9.36853, 10.46221\} \end{aligned}$$

lying in the ball B_{11}, so we take $r = 11$ and (C1) holds. Now $\bar{r} = 10.46221$ and (4.6) gives

$$\|x_0\| \le 0.198827. \tag{4.9}$$

Summarizing, we see that the iterations of (4.8) with $x_0 = 0$ leave earlier the ball B_{11}, but using

$$\begin{aligned} x_{n+1} &= 1.01x_n + 1 \quad \text{for } n \in \mathbb{N}_0 \setminus \{10i\}_{i\in\mathbb{N}}, \\ x_{10i+1} &= 1.01(x_{10i} - 10.46221) + 1 \quad \text{for } i \in \mathbb{N}, \end{aligned} \tag{4.10}$$

we get its 10-periodic orbit starting at x_1 and staying in B_{11} along with its first 100 iterations, if (4.9) holds. So we get a practical control of boundednes of iterations of (4.8) using (4.10) (see Figure 3). Note the 100th iteration of (4.8) with $x_0 = 0$ is 170.48138. Moreover, we worked with only 5 decimal digits. The closer the initial condition is to the right-hand side of (4.9), the higher precision is needed stay in B_{11}.

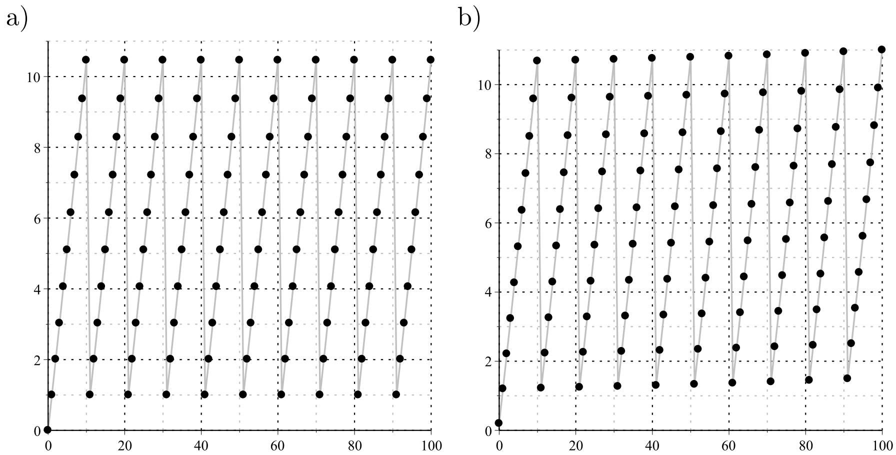

Fig. 3. Iterations of (4.10) with the initial condition: a) $x_0 = 0$; b) $x_0 = 0.1988$. In the latter case, $x_{100} = 10.99997$

5. IMPULSIVE SUPPLY AND DEMAND MODEL

Let $\{n_i\}_{i\in\mathbb{N}_0}$, $n_i = i\Delta$, $\Delta \in \mathbb{N}$ and $\Delta \geq 2$. Consider an IDE

$$x_{n+1} = \begin{cases} f(x_n), & n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}_0}, \\ x_n + \gamma, & n = n_i, \end{cases} \tag{5.1}$$

for a number $\gamma \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = (1-\lambda)x + \frac{a\lambda}{b} - \frac{\lambda}{b}\arctan(\mu x), \tag{5.2}$$

where $\lambda, a, b, \mu \in \mathbb{R}$ are parameters. IDE (5.1) is a supply and demand model presented in [11, pp. 178–183] with equidistant and constant impulses. Following [11], we assume that $b > 0$ and $\mu > 0$ and $\lambda \in (0, 1)$. Then the dynamics of (5.1) is given by the mapping

$$F(x) = f^{\Delta-1}(x+\gamma), \tag{5.3}$$

since

$$x_n = f^{n-1-(\lceil \frac{n}{\Delta} \rceil - 1)\Delta}(F^{\lceil \frac{n}{\Delta} \rceil - 1}(x_0) + \gamma), \quad n \in \mathbb{N}, \tag{5.4}$$

where $\lceil \cdot \rceil$ is the ceil function.

In particular, (5.4) implies

$$x_{i\Delta} = F^i(x_0) \tag{5.5}$$

for any $i \in \mathbb{N}$. Then (5.5) gives that any p-periodic orbit of F generates a $p\Delta$-periodic orbit of (5.1).

Let $k_- \le x_0 \le k_+$ for some $k_- < k_+$. Then

$$\begin{aligned} k_- + \gamma \le x_1 &\le k_+ + \gamma, \\ x_{i\Delta+1} &= x_{i\Delta} + \gamma, \\ (1-\lambda)x_{i\Delta+j} - A\lambda \le x_{i\Delta+j+1} &\le (1-\lambda)x_{i\Delta+j} + A\lambda \end{aligned}$$

for $j = 1, \dots, \Delta - 1$ and $i \in \mathbb{N}_0$, where

$$A = \frac{|a|}{b} + \frac{\pi}{2b}. \tag{5.6}$$

So we get

$$\begin{aligned} &(1-\lambda)^{j-1}(x_{i\Delta} + \gamma) - A(1 - (1-\lambda)^{j-1}) \\ &\le x_{i\Delta+j} \\ &\le (1-\lambda)^{j-1}(x_{i\Delta} + \gamma) + A(1 - (1-\lambda)^{j-1}) \quad \text{for } j = 1, \dots, \Delta, i \in \mathbb{N}_0. \end{aligned} \tag{5.7}$$

First, we derive from (5.7)

$$|x_{i\Delta+j}| \le (1-\lambda)^{j-1}|x_{i\Delta}| + |\gamma| + A \quad \text{for } j = 1, \dots, \Delta, i \in \mathbb{N}_0. \tag{5.8}$$

Then (5.8) gives

$$\begin{aligned} |x_{i\Delta}| &\le (1-\lambda)^{i(\Delta-1)}|x_0| + (|\gamma| + A)\,\frac{1 - (1-\lambda)^{i(\Delta-1)}}{1 - (1-\lambda)^{\Delta-1}} \\ &\le |x_0| + \frac{|\gamma| + A}{1 - (1-\lambda)^{\Delta-1}} \quad \text{for } i \in \mathbb{N}. \end{aligned}$$

Note that the last estimation holds also for $i = 0$. Then (5.8) implies

$$|x_n| \le |x_0| + \frac{|\gamma| + A}{1 - (1-\lambda)^{\Delta-1}} + |\gamma| + A \quad \text{for } n \in \mathbb{N}_0. \tag{5.9}$$

Consequently, iterations of (5.1) are bounded.

Next, by $x_0 \in [k_-, k_+]$, we obtain from (5.7),

$$\begin{aligned} &(1-\lambda)^{\Delta-1}(k_- + \gamma) - A(1 - (1-\lambda)^{\Delta-1}) \\ &\le F(x_0) \\ &\le (1-\lambda)^{\Delta-1}(k_+ + \gamma) + A(1 - (1-\lambda)^{\Delta-1}). \end{aligned} \tag{5.10}$$

Assuming

$$(k_- + A)\frac{1-(1-\lambda)^{\Delta-1}}{(1-\lambda)^{\Delta-1}} \leq \gamma \leq (k_+ - A)\frac{1-(1-\lambda)^{\Delta-1}}{(1-\lambda)^{\Delta-1}}, \tag{5.11}$$

(5.10) implies

$$k_- \leq F(x_0) \leq k_+ \quad \text{for } x_0 \in [k_-, k_+].$$

The Brouwer fixed point theorem gives the existence of a fixed point $x_0^* \in [k_-, k_+]$ of F, which implies the existence of Δ-periodic orbit of (5.1) starting from x_0^*. Since $F : [k_-, k_+] \to [k_-, k_+]$ then (5.1) may have much more sophisticated dynamics applying results of [4, 10], see Example 5.2 below. Note that (5.11) makes sense if and only if

$$2A \leq k_+ - k_-. \tag{5.12}$$

Summarizing, we arrive at the following result.

Theorem 5.1. *Consider* (5.1) *with* $b > 0$, $\lambda \in (0, 1)$ *and* f *given by* (5.2)*. Then all iterations of* (5.1) *are bounded on* $\mathbb{N}_0$*. If* (5.12) *holds with* (5.6)*, then taking* γ *satisfying* (5.11)*,* (5.1) *has a* Δ*-periodic orbit starting from* $[k_-, k_+]$*.*

Example 5.2. Let us consider (5.1) with $\lambda = 0.1$, $a = 0.2$, $b = 0.3$, $\mu = 23$, $\gamma = 0.1$ and $\Delta = 2$. The assumptions of Theorem 5.1 are fulfilled with $k_\pm = \pm 12$. So the existence of a Δ-periodic orbit (starting from $x_0 = -0.07664$) of (5.1) is obtained. Moreover, in this case F has 3-periodic orbits starting from the points -0.22767, -0.14817, -0.05909, -0.00656, 0.30216 and 0.36595 (see Figure 4). Consequently, (5.1) has $k\Delta$-periodic orbits for any $k \in \mathbb{N}$ by Sharkovskii's theorem [10, Theorem 11.2]. So it has Li–Yorke chaos [4, p. 37, p. 49, p. 243].

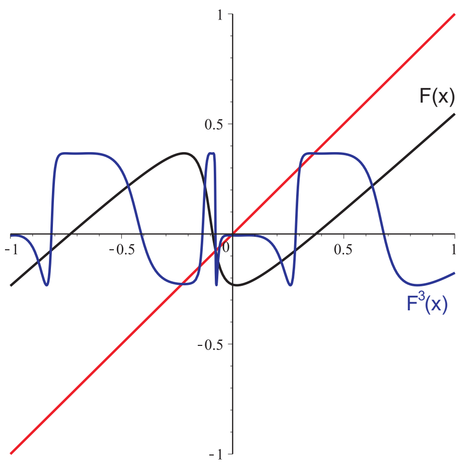

Fig. 4. Fixed point of F and its 3-periodic points for (5.1) with $\lambda = 0.1$, $a = 0.2$, $b = 0.3$, $\mu = 23$, $\gamma = 0.1$ and $\Delta = 2$

Next, similarly like above, assuming $x_0 \geq k_-$ and

$$(k_- + A)\frac{1-(1-\lambda)^{n-1}}{(1-\lambda)^{n-1}} \leq \gamma, \tag{5.13}$$

for all $n = 1, \dots, \Delta$, (5.10) implies

$$k_- \leq x_n \quad \text{for } n = 0, 1, \dots, \Delta.$$

When $k_- \geq 0$, then

$$(k_- + A)\frac{1-(1-\lambda)^{\Delta-1}}{(1-\lambda)^{\Delta-1}} \leq \gamma \tag{5.14}$$

implies (5.13) for all $n = 1, \dots, \Delta$. Furthermore, we derive

$$f'(x) = 1 - \lambda - \frac{\lambda\mu}{b(\mu^2 x^2 + 1)},$$

so

$$\max_{x\in\mathbb{R}} |f'(x)| = \max\left\{\left|1 - \lambda - \frac{\lambda\mu}{b}\right|, 1-\lambda\right\}.$$

Using

$$F'(x_0) = f'(x_{\Delta-1}) \dots f'(x_1), \tag{5.15}$$

we arrive at the following result.

Theorem 5.3. *Consider* (5.1) *with* $b > 0$, $\mu > 0$, $\lambda \in (0,1)$ *and* f *given by* (5.2). *If*

$$\left|1 - \lambda - \frac{\lambda\mu}{b}\right| < 1, \tag{5.16}$$

then there is a unique Δ*-periodic orbit of* (5.1) *which is in addition exponentially stable.*

Proof. By (5.15) and (5.16), $F : \mathbb{R} \to \mathbb{R}$ is contracting, so the Banach fixed point theorem gives the result. □

Note that (5.16) does not hold if and only if

$$1 - \lambda - \frac{\lambda\mu}{b} \leq -1. \tag{5.17}$$

Then there is a unique $x^* \geq 0$ solving $f'(x^*) = -1$ and we derive

$$x^* = \sqrt{\frac{b\lambda + \lambda\mu - 2b}{b(2-\lambda)\mu^2}}. \tag{5.18}$$

Moreover, $-1 < f'(\bar{x}) \leq f'(x) < 1 - \lambda$ for any $x \geq \bar{x} > x^*$. Hence taking $k_- = 2x^*$ in (5.14), so considering

$$\left(2\sqrt{\frac{b\lambda + \lambda\mu - 2b}{b(2-\lambda)\mu^2}} + A\right)\frac{1-(1-\lambda)^{\Delta-1}}{(1-\lambda)^{\Delta-1}} \leq \gamma, \tag{5.19}$$

for any $x_0 \in [2x^*, \infty)$ we get

$$x_n \in [2x^*, \infty) \quad \text{for } n = 0, 1, \dots, \Delta.$$

Consequently, $F : [2x^*, \infty) \to [2x^*, \infty)$ is a contraction. Summarizing, we obtain the following result.

Theorem 5.4. *Consider* (5.1) *with* $b > 0$, $\mu > 0$ *and* $\lambda \in (0, 1)$. *Suppose* (5.17) *and* (5.19). *Then* $[2x^*, \infty)$ *is invariant for* (5.1) *and there is a unique* Δ*-periodic orbit of* (5.1) *in* $[2x^*, \infty)$, *which is in addition exponentially stable. Here* A *and* x^* *are given by* (5.6) *and* (5.18), *respectively.*

Proof. The proof follows from the Banach fixed point theorem, like for Theorem 5.3. □

Example 5.5. Concerning (5.1) with $\lambda = 0.8$, $a = b = 1$, $\mu = 1.5$, $\gamma = 11$ and $\Delta = 2$ one can easily verify the assumptions of Theorem 5.4. Then $x^* = 0$ and we obtain the invariance of $[0, \infty)$ and the existence of exponentially stable 2-periodic orbit $x_{2k} = 2.22955$, $x_{2k+1} = 13.22955$ for each $k \in \mathbb{N}_0$ (see Figure 5).

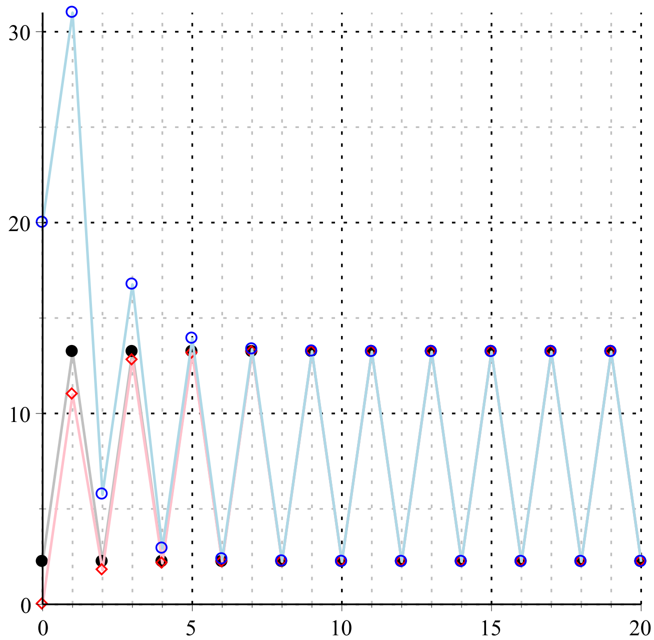

Fig. 5. Orbits of 0 (empty red diamond) and 20 (empty blue circle) for (5.1) with $\lambda = 0.8$, $a = b = 1$, $\mu = 1.5$, $\gamma = 11$ and $\Delta = 2$ tending to exponentially stable orbit of $x_0 = 2.22955$ (solid black circle)

Note when $\mu \to \infty$, then (5.1) should be considered as

$$x_{n+1} = \begin{cases} g(x_n), & n \in \mathbb{N}_0 \setminus \{n_i\}_{i\in\mathbb{N}_0}, \\ x_n + \gamma, & n = n_i, \end{cases} \tag{5.20}$$

with

$$g(x) = (1-\lambda)x + \frac{a\lambda}{b} - \frac{\lambda\pi}{2b}\operatorname{sgn} x,$$

which is discontinuous. Moreover $x^* \to 0$, (5.19) becomes

$$A\frac{1-(1-\lambda)^{\Delta-1}}{(1-\lambda)^{\Delta-1}} \leq \gamma, \tag{5.21}$$

and (5.3),

$$G(x) = g^{\Delta-1}(x+\gamma). \tag{5.22}$$

Then like above, for any $x_0 \in (0,\infty)$, we get

$$x_n \in (0,\infty) \quad \text{for } n = 0,1,\ldots,\Delta.$$

Consequently, $G : (0,\infty) \to (0,\infty)$ is a contraction possessing the form

$$G(x) = (1-\lambda)^{\Delta-1}(x+\gamma) + \bar{A}(1-(1-\lambda)^{\Delta-1}), \quad \bar{A} = \frac{a}{b} - \frac{\pi}{2b}.$$

Summarizing, we obtain the following result.

Theorem 5.6. *Consider* (5.20) *with* $b > 0$ *and* $\lambda \in (0,1)$. *Suppose* (5.21). *Then* $(0,\infty)$ *is invariant for* (5.20) *and there is a unique* Δ*-periodic orbit of* (5.20) *in* $(0,\infty)$ *with*

$$x_0 = \frac{(1-\lambda)^{\Delta-1}\gamma + \bar{A}(1-(1-\lambda)^{\Delta-1})}{1-(1-\lambda)^{\Delta-1}},$$

which is in addition exponentially stable.

6. NOTE ON THE PERIODICITY

When $n_i = i\Delta_0$ for a fixed $\Delta_0 \in \mathbb{N}$, then the both (1.1) and (5.1) are nonautonomous Δ-periodic difference equations of the form

$$x_{n+1} = h(x_n, n), \quad n \in \mathbb{N}_0 \tag{6.1}$$

with $h(x, n+\Delta) = h(x,n)$ for the corresponding Δ and any suitable x and n. Indeed, we take

$$h(x,n) = \begin{cases} f(x), & n \in \mathbb{N}_0 \setminus \{i(\Delta_0+1)-1\}_{i\in\mathbb{N}}, \\ g(x), & n = i(\Delta_0+1)-1, \end{cases}$$

for (1.1), so $\Delta = \Delta_0 + 1$, and

$$h(x,n) = \begin{cases} f(x), & n \in \mathbb{N}_0 \setminus \{i\Delta\}_{i\in\mathbb{N}_0}, \\ x+\gamma, & n = i\Delta, \end{cases}$$

for (5.1), so $\Delta = \Delta_0$. Thus we consider now a general $h : K \times \mathbb{N}_0 \to K$ with a subset $K \subset \mathbb{R}^m$. We introduce the following definition.

Definition 6.1. A point $x_0 \in \mathbb{R}^m$ of (6.1) is p-periodic for some $p \in \mathbb{N}\setminus\{1\}$ if $x_0 = x_{ip}$ for any $i \in \mathbb{N}$.

Of course any $k\Delta$-periodic orbit starts with a $k\Delta$-periodic point, and any p-periodic point determines a kp-periodic orbit for $k \in \mathbb{N}$ as the least one such that $kp \in \Delta\mathbb{N}$. Since (6.1) is Δ-periodic in n, it is natural to search for $k\Delta$-periodic orbits of (6.1) for some $k \in \mathbb{N}$. On the other hand, there can exist a p-periodic point x_0 of (6.1) of period different from $\Delta\mathbb{N}$. For instance, when $\lambda = 0.1$, $a = 0.1$, $b = 0.015$, $\gamma = 1$ and $\Delta = 2$ in (5.1), then taking $\mu = 4.49498$ one obtains a 3-periodic point $x_0 = -0.90631$ (see Figure 6). The corresponding 3Δ-periodic orbit can be seek as 3-periodic orbit of F.

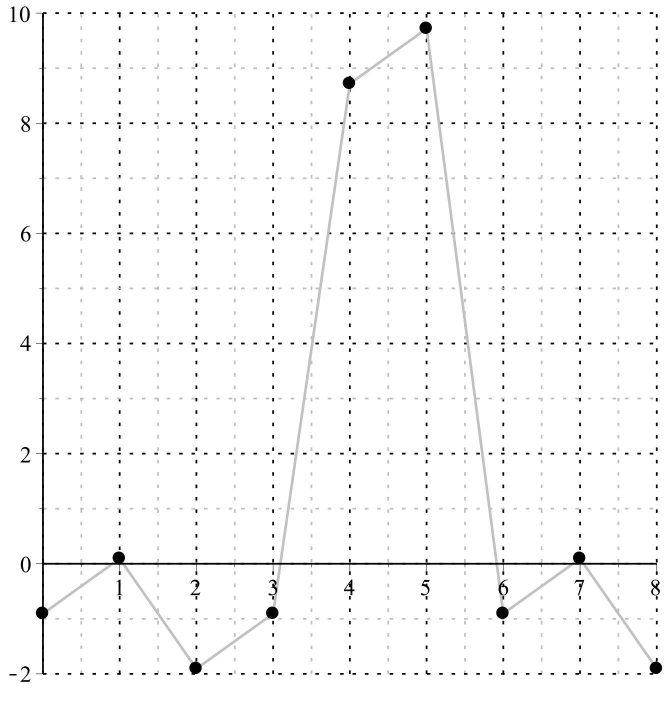

Fig. 6. Orbit of $x_0 = -0.90631$ for (5.1) with $\lambda = 0.1$, $a = 0.1$, $b = 0.015$, $\gamma = 1$, $\Delta = 2$ and $\mu = 4.49498$.

So, in general, a p-periodic point does not give a p-periodic orbit when $p \notin \Delta\mathbb{N}$. Furthermore, the existence of a p-periodic orbit for a general (6.1) of period different from $\Delta\mathbb{N}$ is rather non-generic. Indeed, let $k \in \mathbb{N}$ be the least one such that $kp \in \Delta\mathbb{N}$. Then we must solve the overdetermined system

$$\begin{aligned} x_{i+1} &= h(x_i, jp + i), & i &\in \{0, 1, \ldots, p-2\}, \\ x_0 &= h(x_{p-1}, jp + p - 1), & j &\in \{0, 1, \ldots, k-1\}. \end{aligned} \tag{6.2}$$

On the other hand, when $p = k\Delta$ then (6.2) is reduced to the existence of a k-periodic orbit of

$$F(x) = h(\cdot, \Delta - 1) \circ \cdots \circ h(x, 0).$$

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