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FINITE NILSEMIGROUPS WITH MODULAR CONGRUENCE LATTICES

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Abstract: This paper continues the joint work [2] of the author with P. Jones. We describe all finitely generated nilsemigroups with modular congruence lattices: there are 91 countable series of such semigroups. For finitely generated nilsemigroups a simple algorithmic test to the congruence modularity is obtained.

Key words: Semigroup, Nilsemigroup, Congruence lattice.

Introduction

In [2] the characterization of nilsemigroups with distributive and modular congruence lattices had been obtained. The basic notion in that result was *the width of a semigroup*, considered as a poset under division. Recall that *the width* of a poset is the maximal integer n such that the poset contains an antichain of n elements. It was proved in [2] that the congruence lattice of a nilsemigroup is distributive [modular and not distributive] if and only if it has the width 1 [the width 2].

A poset of the width 1 is a chain. Semigroups, whose congruence lattice form a chain, were investigated in the works [1, 3, 4]. There is no complete classification for such semigroups, in the same time some important cases (finite semigroups, commutative semigroups, permutative semigroups) were considered. It is known that finitely generated nilsemigroups whose congruence lattices form a chain are cyclic nilsemigroups. Thus we have a description of finitely generated nilsemigroups with distributive congruence lattices.

In this paper we describe all finitely generated nilsemigroups with the modular congruence lattice up to isomorphism or dual isomorphism. The set of all such semigroups has been splited into series (almost all of them are infinite), each of them has 4 or less natural parameters. The list of all series is given in the table below.

We prove the following theorem:

Theorem 1. *Let S be a finitely generated nilsemigroup. Then the following are equivalent:*

a) $\operatorname{Con} S$ *is modular and not distributive;*

b) S *is generated by two elements* a *and* b *and the poset* $\{a^2, ab, ba, b^2\}$ *under division has the width 2;*

c) S *is isomorphic or dually isomorphic to a suitable semigroup in the following table:*

N	Name	Presentation	Restrictions
1	$A(n)$	$a^2 = ab = ba = b^2$, $a^n = 0$	$n \geqslant 2$
2	$B_1(n)$	$a^2 = ab = b^2$, $a^n = 0$	$n \geqslant 3$
3	$B_{2.1}(m, n)$	$a^2 = b^2$, $ab = ba$, $a^m = ba^n = 0$	$m \geqslant 3$, $n \geqslant 2$, $\lvert m - n\rvert = 1$
4	$B_{2.2}(m, n)$	$a^2 = b^2$, $ab = ba$, $a^m = ba^{m-1}$, $a^n = 0$	$n \geqslant m \geqslant 3$
5	$B_{3.1}(m, n)$	$a^2 = ab = ba$, $a^m = b^n$	$n > m \geqslant 3$
6	$B_{3.2}(m, n)$	$a^2 = ab = ba$, $a^m = b^n = 0$	$n, m \geqslant 3$, $n \geqslant m - 1$, $n \neq m$
7	$B_{3.3}(m, k)$	$a^2 = ab = ba$, $a^m = b^m$, $a^k = 0$	$k \geqslant m \geqslant 3$
8	$B_{4.1}(m, n)$	$a^2 = ab$, $b^2 = ba$, $a^m = b^n = 0$	$\lvert m - n\rvert = 1$; $m, n \geqslant 3$
9	$B_{4.2}(m, n)$	$a^2 = ab$, $b^2 = ba$, $a^m = b^m$, $a^k = 0$	$k \geqslant m \geqslant 3$
10	C_1	$a^2 = ab$, $b^2 = ba = 0$	
11	C_2	$a^2 = b^2 = ab$, $ba = 0$	
12	C_3	$a^2 = b^2 = ab = 0$	
13	C_4	$a^2 = b^2$, $ab = ba = 0$	
14	C_5	$a^2 = ab = ba$, $b^2 = 0$	
15	C_6	$ab = ba$, $a^2 = b^2 = 0$	
16	$C_{7.1}(n)$	$a^2 = ab = ba = b^n$	$n \geqslant 3$
17	$C_{7.2}(n)$	$a^2 = ab = ba = b^n = 0$	$n \geqslant 3$
18	$D_{1.1}(m, n)$	$b^2 = ba = a^m = a^n b$	$m \geqslant 3$, $m - 1 \geqslant n \geqslant 2$
19	$D_{1.2}(m, n)$	$b^2 = ba = a^m$, $a^n b = 0$	$m \geqslant n \geqslant 2$, $m \geqslant 3$
20	$D_{2.1}(m, n, k)$	$ab = ba = a^m$, $a^n = b^k = 0$	$n > m \geqslant 3$, $k \geqslant 3$, $n \leqslant k(m-1)+1$
21	$D_{2.2}(m, n, k)$	$ab = ba = a^m$, $a^n = b^k = 0$, $a^{n-1} = b^{k-1}$	$m, k \geqslant 3$, $m \leqslant n-2 \leqslant (k-1)(m-1)$, $n \neq (m-1)(k-1)+1$
22	$D_{2.3}(m, n, q)$	$ab = ba = a^m$, $a^{(m-1)q} = b^q$, $a^n = 0$	$m \geqslant 3$,$q \geqslant 2$, $n \geqslant (m-1)q + 1$
23	$D_{3.1}(m, n, k)$	$ab = ba$, $b^2 = a^m$, $a^n = a^k b = 0$	$m \geqslant 3$, $k \geqslant 2$, $k + m \geqslant n \geqslant k$, $n \geqslant m + 1$
24	$D_{3.2}(n, k, q)$	$ab = ba$, $b^2 = a^{2(n-k)}$, $a^n = a^k b = 0$, $a^{q+n-k} = a^q b$	$n \geqslant 3$, $k \geqslant 2$, $n \geqslant k$, $n \geqslant 2n-2k+1$, $k > q \geqslant 2$
25	$D_{3.3}(m, n, k, q)$	$ab = ba$, $b^2 = a^m$, $a^{n-k+q} = a^q b$, $a^n = a^k b = 0$	$m \geqslant 3$, $k \geqslant 2$, $k + m \geqslant n \geqslant k$, $k \leqslant \min(n-k+q, q+m)$, $k > q \geqslant 2$
26	$D_4(n)$	$ba = b^n$, $a^2 = ab$	$n \geqslant 3$
27	$E_{1.1}(m, n, k)$	$b^2 = ba = a^m b$, $a^n = a^k b = 0$	$m \geqslant 2$, $2m \geqslant k \geqslant m$, $n \geqslant k$, $n \geqslant 3$
28	$E_{1.2}(m, n, k)$	$b^2 = ba = a^m b$, $a^n = a^k b$	$m \geqslant 2$, $2m \geqslant k \geqslant m$, $n > k$, $n \geqslant 3$
29	$E_{2.1}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = (ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$
30	$E_{2.2}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = 0$	$m \geqslant 3$
31	$E_{2.3}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$
32	$E_{2.4}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = (ba)^{\frac{m}{2}}$, $(ba)^{\frac{m+1}{2}} = 0$	$m \geqslant 3$
33	$E_{2.5}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$, m is odd
34	$E_{2.6}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ab)^{\frac{m+1}{2}} = (ba)^{\frac{m+1}{2}} = 0$	$m \geqslant 3$

N	Name	Presentation	Restrictions
35	$E_{3.1}(n,m)$	$ab = ba = a^n = b^m$	$n, m \geqslant 3$
36	$E_{3.2}(n,m)$	$ab = ba = a^n = b^m = 0$	$n, m \geqslant 3$
37	E_4	$a^2 = b^2$, $ba = 0$	
38	$E_{5.1}(m,n,k)$	$ab = ba$, $b^2 = a^m b$, $a^n = a^k b = 0$	$n \geqslant k \geqslant m \geqslant 2$
39	$E_{5.2}(m,n,k,q)$	$ab = ba$, $b^2 = a^m b$, $a^{n-k+q} = a^q b$, $a^k = a^n = 0$	$n \geqslant k \geqslant m \geqslant 2$, $n \geqslant 3$, $n - k \neq m$, $k \leqslant \min(n-k+q, q+m)$, $q \geqslant 2$
40	$E_{5.3}(m,n,q)$	$ab = ba$, $b^2 = a^m b$, $a^n = 0$, $a^{m+q} = a^q b$	$n \geqslant m \geqslant 2$, $n \geqslant 3$, $q \geqslant 2$
41	$E_{6.1}$	$a^2 = ab$, $b^2 = ba^2$	
42	$E_{6.2}$	$a^2 = ab$, $b^2 = 0$	
43	$E_7(n)$	$a^2 = ab$, $ba = 0$, $b^n = 0$	$n \geqslant 3$
44	$G(m,n)$	$ab = a^m$, $ba = b^n$	$m, n \geqslant 3$
45	$H_{1.1}(m,n,k)$	$ba = a^m b$, $b^2 = a^n$, $a^{n+1} = a^k b = 0$	$n \geqslant k > m \geqslant 2$
46	$H_{1.2}(m,n,k)$	$ba = a^m b$, $b^2 = a^n = a^{k-1} b$, $a^{n+1} = a^k b = 0$	$n \geqslant k > m \geqslant 2$
47	$H_{2.1}(m,n,k,l)$	$b^2 = ba^m$, $ab = a^n$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > n > m+1$, $m + n \geqslant k \geqslant l > m$
48	$H_{2.2}(m,n,k,l)$	$b^2 = ba^m$, $ab = a^n$, $a^{k-1} = ba^{l-1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > n > m+1$, $m + n \geqslant k \geqslant l > m$, $k \neq l + n - 1$
49	$H_{2.3}(m,n,k,q)$	$b^2 = ba^m$, $ab = a^n$, $a^{q+n-1} = ba^q$, $a^k = ba^l = 0$	$m \geqslant 2$, $n > m+1$, $m + n \geqslant k \geqslant q + n - 1$
50	$H_{2.4}(m,k,l)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k \geqslant l \geqslant m+1$
51	$H_{2.5}(m,k,l)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^{k-1} = ba^{l-1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > m+1$, $k \geqslant l > m$, $k \neq l + m$
52	$H_{2.6}(m,k,q)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^{q+m} = ba^q$, $a^k = 0$	$m \geqslant 2$, $n > m$, $k \geqslant q+n-1$, $q \geqslant 2$
53	$I_{1.1}(n)$	$a^2 = (ab)^{\frac{2n}{2}}$, $b^2 = (ba)^{\frac{2n}{2}}$	$n \geqslant 2$
54	$I_{1.2}(n)$	$a^2 = (ba)^{\frac{2n+1}{2}}$, $b^2 = (ab)^{\frac{2n+1}{2}}$	$n \geqslant 1$
55	$I_{1.3}(n)$	$a^2 = (ba)^{\frac{2n+1}{2}}$, $b^2 = (ab)^{\frac{2n+1}{2}}$, $(ab)^{\frac{2n+2}{2}} = (ba)^{\frac{2n+2}{2}} = 0$	$n \geqslant 1$
56	$I_{1.4}(n,m,k)$	$a^2 = (ab)^{\frac{2n+1}{2}}$, $b^2 = (ba)^{\frac{2n+1}{2}}$, $(ab)^{\frac{m}{2}} = (ba)^{\frac{k}{2}} = 0$	$n \geqslant 1$, $k, m > 2n+1$, $\lvert k - m \rvert \leqslant 1$
57	$J_{1.1}(n,m)$	$ab = ba = a^m$, $b^2 = a^n$	$m \geqslant 3$, $2m - 2 > n > m$
58	$J_{1.2}(n,m)$	$ab = ba = a^m$, $b^2 = a^n = 0$	$m \geqslant 3$, $2m - 2 > n > m$
59	$J_{1.3}(m,k)$	$ab = ba = a^m$, $b^2 = a^{2m-2}$, $a^k = 0$	$m \geqslant 3$, $k > 2m - 2$
60	$J_2(n)$	$b^2 = a^n$, $ab = ba = a^{n+1} = 0$	$n \geqslant 3$
61	$J_3(n)$	$b^2 = ab = a^n$, $ba = a^{n+1} = 0$	$n \geqslant 3$
62	$J_4(n)$	$ab = a^n$, $b^2 = ba = a^{n+1} = 0$	$n \geqslant 3$
63	$L_1(n,m)$	$b^2 = a^m$, $ba^n = 0$, $ab = 0$	$m \geqslant 3$, $m + 1 \geqslant n \geqslant 2$
64	$L_{2.1}(n,m)$	$ba = a^m$, $b^n = ab = 0$	$m \geqslant 3$, $n \geqslant 3$
65	$L_{2.2}(n,m)$	$ba = a^m = b^{n-1}$, $a^{m+1} = b^n = 0$	$m \geqslant 3$, $n \geqslant 4$
66	$L_{3.1}(m,k,n)$	$ab = a^m$, $b^2 = a^k$, $ba^{n+1} = 0$	$k > m \geqslant 3$, $k \neq 2m - 2$, $k \geqslant n \geqslant k - m$
67	$L_{3.2}(m,k,q)$	$ab = a^m$, $b^2 = a^k = ba^q$	$k > m \geqslant 3$, $k \neq 2m - 2$, $k > q \geqslant k - m + 1$

N	Name	Presentation	Restrictions
68	$L_{3.3}(m,k,q)$	$ab=a^m$, $b^2=a^k$, $a^{q+m-1}=ba^q$	$k>m\geqslant 3$, $k\neq 2m-2$, $k-m+1>q\geqslant k-2m+2$
69	$L_{3.4}(m,n,l)$	$ab=a^m$, $b^2=a^{2m-2}$, $ba^l=a^n=0$	$m\geqslant 3$, $l\geqslant m-1$, $l+m\geqslant n\geqslant 2m-1$
70	$L_{3.5}(m,n,l)$	$ab=a^m$, $b^2=a^{2m-2}$, $ba^l=a^n=0$, $a^{n-1}=ba^{l-1}$	$m\geqslant 3$, $l\geqslant m-1$, $l+m\geqslant n\geqslant 2m-1$
71	$L_{3.6}(m,n,q)$	$ab=a^m$, $b^2=ba^{2m-2}$, $a^{q+m-1}=ba^q$, $a^n=0$	$m\geqslant 2$, $q\geqslant 2$, $n\geqslant q+m$
72	$L_{3.7}(m,l,k,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=ba^k=0$	$2m-1\geqslant n\geqslant k>l\geqslant m\geqslant 3$
73	$L_{3.8}(m,l,k,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=ba^k=0$, $a^{n-1}=ba^{l-1}$	$2m-1\geqslant n\geqslant k>l\geqslant m\geqslant 3$
74	$L_{3.9}(m,l,q,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=0$, $ba^{q+m-1}=ba^q$	$2m-1\geqslant n>l\geqslant m\geqslant 3$, $q\geqslant 2$
75	$L_{3.10}(m,n,k)$	$ab=a^m$, $b^2=0$, $a^n=ba^k=0$	$n>m\geqslant 3$, $m+k\geqslant n\geqslant k\geqslant 2$
76	$L_{3.11}(m,n,k)$	$ab=a^m$, $b^2=0$, $a^n=ba^k=0$, $a^{n-1}=ba^{k-1}$	$n>m\geqslant 3$, $m+k\geqslant n\geqslant k\geqslant 2$
77	$L_{3.12}(m,n,q)$	$ab=a^m$, $b^2=0$, $a^n=0$, $a^{q+m-1}=ba^q$	$n>m\geqslant 3$, $q\geqslant 2$
78	$N_{1.1}(m,l,n,k)$	$b^2=a^mb$, $ba=a^lb$, $a^n=a^kb=0$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$, $2m>l$
79	$N_{1.2}(m,l,n,k)$	$b^2=a^mb$, $ba=a^lb$, $a^n=a^kb=0$, $a^{n-1}=a^{k-1}b$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$, $2m>l$
80	$N_{2.1}(m,l,n,k)$	$ba=a^mb$, $b^2=a^lb$, $a^n=a^kb=0$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$
81	$N_{2.2}(m,l,n,k)$	$ba=a^mb$, $b^2=a^kb$, $a^n=a^lb=0$, $a^{n-1}=a^{k-1}b$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$
82	$N_{3.1}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$	$k>2m+1$, $m>1$
83	$N_{3.2}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$, $b^2a=ab^2=b^3=0$	$k>2m$, $m>1$
84	$N_{3.3}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}=(ba)^{\frac{k}{2}}$	$k>2m$, $m>1$
85	$N_{3.4}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$, $(ba)^{\frac{k}{2}}=0$	$k>2m$, $m>1$
86	$N_{3.5}(m,k,n,l)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ba)^{\frac{2k+1}{2}}$ $(ab)^{\frac{n}{2}}=(ba)^{\frac{l}{2}}=0$	$k>m$, $m>1$, $n,l\geqslant k$, $\lvert n-l\rvert\leqslant 1$
87	$N_{3.6}(m,n,l)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{n}{2}}=(ba)^{\frac{l}{2}}=0$	$m>1$, $n,l\geqslant m$, $\lvert n-l\rvert\leqslant 1$
88	$N_{3.7}(m)$	$a^2=(ba)^{\frac{m}{2}}$, $b^2=(ba)^{\frac{m+1}{2}}$	$m\geqslant 3$
89	$N_{3.8}(m)$	$a^2=(ab)^{\frac{m}{2}}$, $b^2=0$	$m\geqslant 3$
90	$N_{3.9}(m)$	$a^2=(ab)^{\frac{m}{2}}=(ba)^{\frac{m}{2}}$, $b^2=0$	$n\geqslant 3$
91	$N_4(m,n)$	$ab=a^n=b^m$, $ba=0$	$m,n\geqslant 3$

We show later that every row in this table, with some constants fixed, gives us exactly one semigroup up to isomorphism or dually isomorphism. Some rows have no parameters, which means that such rows defines only one finite semigroup.

Let us note that any two semigroups in this table are not isomorphic and are not dually isomorphic. Indeed, every nilsemigroup has exactly one basis, i.e. a minimal set of generators. Every generator is a maximal element under division order $\leqslant$. Conversely, every maximal element of $(S,\leqslant)$ is an element of any basis. So, the set of maximal elements is the unique basis of S. Then every automorphism of S maps the basis onto itself, which means that it preserves the presentation of S. All semigroups in the table have distinct presentations, that can be revised by a careful check.

It is easy to check that all semigroups in this table have a width 2. It gives us the implication from c) to a). The implication from a) to b) is proved in [2]. The rest of the paper is directed to prove that b) leads c).

Theorem 1 provides a simple test to determine whether the congruence lattice of a finite nilsemigroup is modular by checking the condition (b) or by searching the corresponding semigroup in the Table.

Theorem 1 has an important corollary for the class of nilpotent semigroups. Every nilpotent semigroup S satisfy the ascending chain condition under $\leqslant$. Then S has a basis, which consists of maximal elements of S under $\leqslant$. This basis form an antichain, so by result of [2], it has 1 or 2 elements. From Theorem 1 we have the following corollary:

Corollary 1. *Every nilpotent semigroup with modular congruence lattice is finite. It is isomorphic or dually isomorphic to a suitable semigroup in the Table.*

1. Preliminaries

We consider the division relation $\leqslant$ on a semigroup S defined as $a \leqslant b$ iff there exist $s, t \in S^1$ such that $b = sat$. Since every nilsemigroup is $\mathscr{J}$-trivial, the relation $\leqslant$ is an order relation on a nilsemigroup.

Our starting point is the following statements that was proved in [1] as Corollary 2.

Proposition 1. *Let S be a nilsemigroup such that* $\operatorname{Con} S$ *is modular. If S is finitely generated, then it is finite. If S is not cyclic, then it is generated by two elements a, b and the poset $\{a^2, ab, ba, b^2\}$ has width at most two.*

We assume further in the paper that S is a finite nilsemigroup generated by two distinct elements a and b.

We say that an element $x \in S$ is *an atom*, if x covers 0, i.e. $x > 0$ and, for every $z \in S$, the condition $0 < z \leqslant x$ implies $z = x$. Put

$$x \gtrdot y \text{ iff there exist } s, t \in S^1 \text{ such that } y = sxt \text{ and } st \neq 1.$$

The relation $\gtrdot$ on S is antisymmetric and transitive. It is easy to see that, for $x, y \in S$, $x > y$ implies $x \gtrdot y$, and $x \gtrdot y$ implies $x \geqslant y$ (the converse is false, since $0 \gtrdot 0$, but $0 \not> 0$).

Lemma 1. *1) An element $x \in S$ is equal to zero if and only if $x \gtrdot x$.*
2) For every $s \in S$ either $s = s'a$ or $s = s'b$ for some $s' \in S^1$.
3) For every $t \in S$ either $t = at'$ or $t = bt'$ for some $t' \in S^1$.
4) If $x \in S$ satisfies $xa = ax = xb = bx = 0$, then x is an atom or a zero.

The proof is obvious.

Let u be a word of n letters. Define $u^{\frac{p}{n}}$ for $0 \leqslant p \leqslant n-1$ as a p-element prefix of u. For an arbitrary positive integer p, put $u^{\frac{p}{n}} = u^{[p/n]} u^{\frac{p \bmod n}{n}}$.

Lemma 2. *Let c, d be letters and let p be a positive integer. Then:*
1) $(cd)^{\frac{p}{2}} = c(dc)^{\frac{p-1}{2}}$.
2) $(cd)^{\frac{p}{2}} = (cd)^{\frac{p-1}{2}} c$, if p is odd.
3) $(cd)^{\frac{p}{2}} = (cd)^{\frac{p-1}{2}} d$, if p is even.
4) $(cd)^{\frac{p}{2}} (dc)^{\frac{q}{2}} = (cd)^{\frac{p+q}{2}}$, if p is odd.
5) $(cd)^{\frac{p}{2}} (cd)^{\frac{q}{2}} = (cd)^{\frac{p+q}{2}}$, if p is even.

The proof is obvious.

Lemma 3. *1) If $a^2 \lessdot ab$, then either $a^2 \lessdot b^2$ or $a^2 \lessdot ba$.*
2) If $ba \lessdot ab$, then either $ba \lessdot a^2$ or $ba \lessdot b^2$.
3) If $a^2 > b^2 > ab$, $ba \not> b^2$ and $ab \neq 0$, then $ab < ba$.
4) If $a^2 > ab > b^2$, $ba \not> ab$ and $b^2 \neq 0$, then $b^2 < ba$.
5) If $a^2 > ab > ba$ and $ba \neq 0$, then $b^2 > ba$.

P r o o f. 1) Let $a^2 \lessdot ab$. Then $a^2 = sabt$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $a^2 \lessdot a^2$, which implies $a^2 = 0 \lessdot b^2$. If $s = s'b$ for some $s' \in S^1$, then $a^2 = s'babt$ and $a^2 \lessdot ba$. Let $s = 1$ and $a^2 = abt$. If $t = at'$ for some $t' \in S^1$, then $a^2 \lessdot ba$. If $t = bt'$ for some $t' \in S^1$, then $a^2 \lessdot b^2$.

2) The proof is similar to 1).

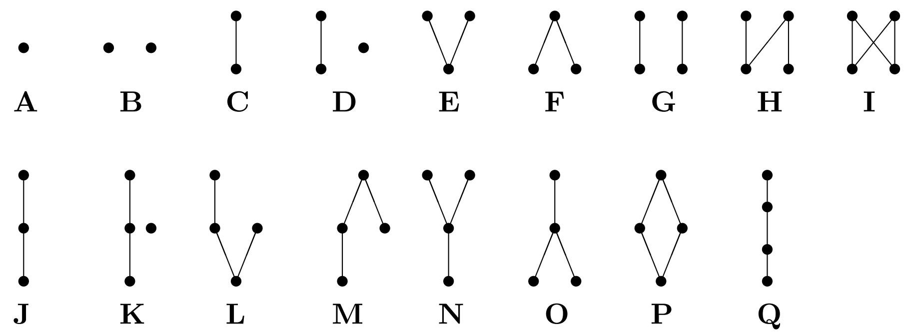
3) If $a^2 > b^2$, then $b^2 = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $b^2 < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $b^2 < ab$, a contradiction. So $b^2 = a^k$ for some $k \geqslant 3$. Then $ab < b^2 = a^k$, so $ab = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ab < a^{k+1} = ab^2$, i.e. $ab = 0$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $ab < ab$ and $ab = 0$. If $u = u'b$ for some $u' \in S^1$, then $ba > ab$.

4) If $a^2 > ab$, then $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $ab < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ab = 0$, contrary to $ab > b^2$. So $ab = a^k$ for some $k \geqslant 3$. Then $b^2 < ab = a^k$, so $b^2 = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ab < a^{k+1} = aba$, i.e. $ab < b^2$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $b^2 < a^kb = ab^2$ and $b^2 = 0$, a contradiction. If $u = u'b$ for some $u' \in S^1$, then $ba > b^2$.

5) If $a^2 > ab$, then $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $ab < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ab = 0$, contrary to $ab > ba$. So $ab = a^k$ for some $k \geqslant 3$. Then $ba < ab = a^k$, so $ba = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ba \leqslant a^{k+1} = aba \lessdot ba$, i.e. $ba = 0$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $ba \leqslant a^kb = ab^2 < b^2$. If $u = u'b$ for some $u' \in S^1$, then $ba \lessdot ba$, which means $ba = 0$, a contradiction. □

2. Finite nilsemigroups of width 2

The elements a^2, ab, ba, b^2 form a subposet of S. This subposet has no more than 4 elements and has no antichains with 3 elements. We enumerate all such posets in the following list.



For each poset **A**-**Q** we examine all possibilities of mapping the set $\{a^2, b^2, ab, ba\}$ onto the poset. We consider two cases be equal if one of them can be obtained from another either by replacing a to b and vice versa or by replacing ab to ba and vice versa. Indeed, these cases give us isomorphic or dually isomorphic semigroups. Some cases are forbidden by Lemma 3, we don't mention them.

Let us note that in cases **B**, **D**, **F**, **G**, **H**, **I**, **K**, **M**, **O** the elements a^2, b^2, ab, ba are not equal to zero, since every element of a nilsemigroup divides zero.

Series A. $a^2 = b^2 = ab = ba$. Then every element of S, except b, can be written as a^p for some positive p. Let n be the least positive integer such that $a^n = 0$. Then $S \cong A(n)$.

Series B. The following cases are possible:

$a^2 = b^2 = ab$ $\bullet$ $\quad$ ba $\bullet$

B1

$a^2 = b^2$ $\bullet$ $\quad$ $ab = ba$ $\bullet$

B2

$a^2 = ab = ba$ $\bullet$ $\quad$ b^2 $\bullet$

B3

$a^2 = ab$ $\bullet$ $\quad$ $b^2 = ba$ $\bullet$

B4

Case B1. Let x be an element of S. If a or b^2 is a left divisor for x, then $x = a^p$ for some p. If ba^2 is a left divisor for x, then $x = a^p$ for some p, since $ba^2 = b^3$. So, every element of S, except b and ba, can be written as a^p for some p. Let n be the least positive integer such that $a^n = 0$. We obtain the semigroup $B_1(n)$.

Case B2. It is easy to show that every element can be written as a^p or ba^p for some $p \geqslant 0$. Let n and l be the least positive integers such that $a^n = ba^l = 0$. Then $|n - l| \leqslant 1$ and $n \geqslant 3$, $l \geqslant 2$. If $|n - l| = 1$, then $S \cong B_{2.1}(n, l)$.

Let $a^m = ba^{m-1}$ for some $m \geqslant 3$. Then $a^p = ba^{p-1}$ for all $p \geqslant m$. Let n be the least positive integer such that $a^n = 0$. We obtain the semigroup $B_{2.2}(m, n)$.

Case B3. In this case every element can be written as ab^{p-1} or b^p for some $p \geqslant 1$. Let m be the least positive integer such that $ab^{m-1} = b^n$ for some $n \geqslant 3$, $m \geqslant 3$ and $n \geqslant m - 1$. The following cases are possible:

Case B3.1 $m \neq n$. Then $ab^m = a(ab^{m-1}) = ab^n$, so $ab^m = 0$. The element $ab^{m-1} = b^n$ is a single atom or a zero. Then $S \cong B_{3.1}(m, n)$ or $S \cong B_{3.2}(m, n)$ respectively.

Case B3.2. $m = n$. Then $ab^{p-1} = b^p$ for all $p \geqslant m$. Let k be the least positive integer such that $b^k = 0$. We have that $S \cong B_{3.3}(m, k)$.

Case B4. In this case every element can be written as $ab^{p-1} = a^p$ or b^p for some $p \geqslant 0$. Let m and n be the least positive integers such that $a^m = b^n$. If $m < n$ then $a^m \lessdot ba^m = b^{m+1} \leqslant b^n$, so $a^m = b^n = 0$ and $n = m + 1$. If $m > n$, then $b^n \lessdot ab^n = a^{n+1} \leqslant a^m$, so $a^m = b^n = 0$ and $m = n + 1$. We got $|m - n| = 1$ and $S \cong B_{4.1}$. If $m = n$, then $a^p = b^p$ for all $p \geqslant n$. Let k be the least positive integer such that $a^k = 0$. We deduce that $S \cong B_{4.2}(m, k)$.

Series C. The following cases are possible:

C1: $\bullet\, a^2 = ab$ above $\bullet\, b^2 = ba$

C2: $\bullet\, a^2 = b^2 = ab$ above $\bullet\, ba$

C3: $\bullet\, ba$ above $\bullet\, a^2 = b^2 = ab$

C4: $\bullet\, a^2 = b^2$ above $\bullet\, ab = ba$

C5: $\bullet\, a^2 = ab = ba$ above $\bullet\, b^2$

C6: $\bullet\, ab = ba$ above $\bullet\, a^2 = b^2$

C7: $\bullet\, b^2$ above $\bullet\, a^2 = ab = ba$

Case C1. Since $b^2 < a^2$, we have $b^2 = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $ba = b^2 = s'a^3t = s'abat$, which means that $ba \gtrdot ba$, so $b^2 = 0$. Cases $s = s'b$ and $t = at'$ for some $s', t' \in S^1$ are similar. If $t = bt'$ for some $t' \in S^1$, then $ba = sa^2bt' = sa^3t' = sabat'$, which implies $ba \gtrdot ba$. So, $b^2 = ba = 0$.

An element a^2 is a single atom. Indeed, $a^2b = a^3 = aba = 0$ and $ba^2 = 0$. Then $S \cong C_1$.

Case C2. Since $ba < a^2$, then $ba = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $ba = s'a^3t = s'abat$, which impies $ba = 0$. Cases $s = s'b$, $t = bt'$ and $t = at'$ for some $s', t' \in S^1$ are similar. So, $ba = 0$.

An element a^2 is a single atom. Indeed, $a^2b = a^3 = aba = 0$ and $ba^2 = 0$. Then $S \cong C_2$.

Case C3. By the same arguments as before, we have $a^2 = b^2 = ab = 0$. The element ba is a single atom. Then $S \cong C_3$.

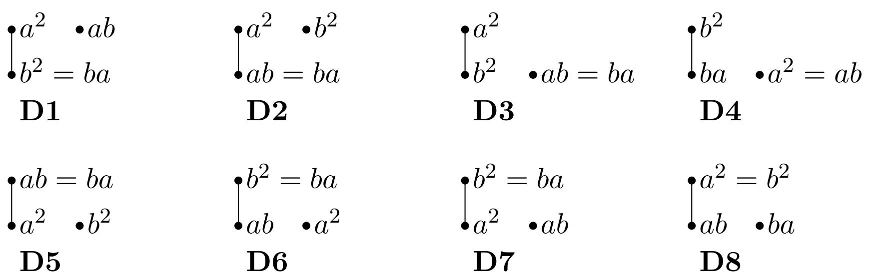
Case C4. Using arguments of case C1, we have $ab = ba = 0$. The element $a^2 = b^2$ is a single atom. Then $S \cong C_4$.

Case C5. Using arguments of case C2, we have $b^2 = 0$. The element $a^2 = ab = ba$ is a single atom. Then $S \cong C_5$.

Case C6. Using arguments of case C2, we have $a^2 = b^2 = 0$. The element $ab = ba$ is a single atom. Then $S \cong C_6$.

Case C7. We have $a^2 = ab = ba = b^n$ for some $n \geqslant 3$. The element a^2 is an atom or a zero, since $ab^2 = ba^2 = a^2b = a^3 = ab^k \lessdot ab^2$. If a^2 is an atom, then $S \cong C_{7.1}(n)$. If a^2 is a zero, then $S \cong C_{7.2}(n)$.

Series D. The following cases are possible:



Case D1. We have $b^2 < a^2$, so $b^2 = ba = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba < ab$ or $ba < ba$, a contradiction. So, $b^2 = ba = a^m$ for some $m \geqslant 3$. The element $a^{m+1} = ba^2 = b^2a = ba^m = a^{2m-1}$ with $m \neq 2$ divides itself, which implies that it is a zero. The elements $bab = b^3 = b^2a$ and $aba = a^{m+1}$ are also equal to zero, which means that ba is an atom. The element $a^{m-1}b$ is an atom or a zero.

Every element of S can be written as a^p or $a^{p-1}b$ for some $p \leqslant m$. Let $q \geqslant 1$ and $1 < n < m$ be the least positive integers such that $a^q = a^nb$. If $q < m$, then $a^m = a^qa^{m-q} = a^nba^{m-q} = a^na^ma^{m-q-1} < a^m$, so $a^m = ba = 0$, a contradiction. If $q = m$, then $a^{n+1}b = a^{m+1} = 0$, so a^m is a single atom and $S \cong D_{1.1}(m, n)$. If $q > m$, then $a^nb = 0$ and $S \cong D_{1.2}(m, n)$.

Case D2. We have $ab < a^2$, so $ab = ba = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba < ab$, a contradiction. So, $ab = ba = a^m$ for some $m \geqslant 3$. Then every element of S can be written as a^p or b^p for some p. Let n and k be the least positive integers such that $a^n = 0$ and $b^k = 0$. Then $n \leqslant k(m-1)+1$ and $k \geqslant 3$.

If $a^p = b^q$ implies $a^p = 0$, then $S \cong D_{2.1}(m, n, k)$. Let p, q be the least positive integers such that $a^p = b^q \neq 0$. Then $a^{p+1} = b^qa = a^{(m-1)q+1}$. If $p \neq (m-1)q$, then $a^{p+1} = 0$ and $p+1 = n$, $q+1 = k$. In this case $S \cong D_{2.2}(m, n, k)$. If $p = (m-1)q$, then $a^{r(m-1)} = b^r$ for all $r \geqslant p$, so $k = [n/(m-1)]$ and $S \cong D_{2.3}(m, n, q)$.

Case D3. We have $b^2 = a^m$ for some $m \geqslant 3$. Every element of S can be written in the form a^p or a^pb for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^kb = 0$. Since $a^kb^2 = a^{k+m}$, we have $k + m \geqslant n \geqslant k$.

If $a^p = a^q b$ for some p, q implies $a^p = 0$, then $S \cong D_{3.1}(m, n, k)$. Let p, q be the least positive integers such that $a^p = a^q b \neq 0$. Then $a^{p+r} = a^{q+r} b$ for all $r \geqslant 0$, which implies $n - p = k - q$, so $p = n - k + q$. We have $a^p b = a^q b^2 = a^{q+m}$, so either $q + m - p = p - q$ or $a^p b = 0$. In the former case $p = q + m/2$ and $m = 2(n - k)$, whence $S \cong D_{3.2}(n, k, q)$. In the latter case $k \leqslant \min(p, q + m)$ and $S \cong D_{3.3}(m, n, k, q)$.

Case D4. We have $ba = b^n$ for some $n \geqslant 3$. Then $bba = b^{n+1} = bab = baa = b^n a = b^{2n-1} \lessdot b^{n+1}$, so $bba = bab = baa = 0$. Also $a^3 = aba = ab^n = a^{n+1} \lessdot a^3$, so $aba = 0$. We got that ba is an atom. The element a^2 is also an atom, because $ba^2 = 0$, $a^3 = a^2 b = 0$. Elements of S are equal to a, or to a^2, or to b^i for $i = 1 \dots n$. We got a semigroup $D_4(n)$.

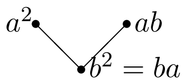
Case D5. We have $a^2 < ab = ba$, so $a^2 = sabt$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $a^2 \lessdot a^2$ and $a^2 = 0 < b^2$, a contradiction. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $a^2 < b^2$, a contradiction.

Case D6. We have $ab < a^2 = b^2$, so $ab = sa^2 t$ for some $s, t \in S^1$. If $s = s'a$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \lessdot ab$ and $ab = 0 < ba$, a contradiction. If $s = s'b$ or $t = at'$ for some $s', t' \in S^1$, then $ab < ba$, a contradiction.

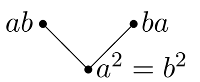
Case D7. We have $a^2 < b^2$, so $a^2 = sb^2 t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $a^2 < ab$, a contradiction. If $s = s'b$ for some $s' \in S^1$, then $a^2 = s'b^3 t = s'babt < ab$, a contradiction. Let $s = 1$. If $t = at'$ or $t = bt'$ for some $t' \in S^1$, then $a^2 = b^2 at' = b^3 t' = babt' < ab$, a contradiction.

Case D8. We have $ab < a^2$, so $ab = sa^2 t$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ab \leqslant a^3 = ab^2 < ab$, a contradiction. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \leqslant b^3 = a^2 b < ab$, a contradiction.

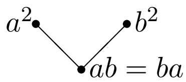
Series E. The following cases are possible:



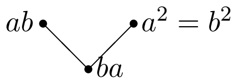
E1



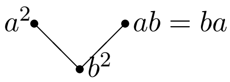
E2



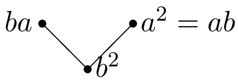
E3



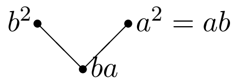
E4



E5



E6



E7

Case E1. We have $b^2 < ab$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ or $t = bt'$ for some $s', t' \in S^1$, then $b^2 \lessdot ba = b^2$, which implies $b^2 = 0$. Anyway, there exists $m \geqslant 2$ such that $b^2 = ba = a^m b$. Then every element of the semigroup S can be written as a^p or $a^p b$ for some p.

Let k and n be the minimal numbers such that $a^k b = a^n = 0$. Obviously, $n \geqslant k$ and $k \geqslant m$. We have $a^{2m} b = a^m b^2 = b^3 = b^2 a = ba^m b = b^2 a^{m-1} b$. Since $m \neq 1$, the element $b^2 a$ divides itself, which means $a^{2m} b = 0$ and $k \leqslant 2m$.

If $a^q = a^r b$ implies $a^q = 0$, then $S \cong E_{1.1}(m, n, k)$. Let $a^q = a^r b \neq 0$ for some $q > r > 0$. Then $a^{r+1} b = a^{q+1} = a^r ba = a^{r+m} b \lessdot a^{r+1} b$, so $a^{q+1} = a^{r+1} b = 0$, which means $q = n - 1$ and $r = k - 1$. Then $S \cong E_{1.2}(m, n, k)$.

Case E2. We have $a^2 < ab$ and $a^2 < ba$, so $a^2 = b^2 = (ab)^{m/2}$ or $a^2 = b^2 = (ba)^{m/2}$ for some $m \geqslant 3$. Without loss of generality we suppose that $a^2 = b^2 = (ab)^{m/2}$. Then every element of S can be written in the form $(ab)^{p/2}$ or in the form $(ba)^{p/2}$ for some p. The following cases are possible:

Case E2.1. $m = 2n + 1$ for some $n \geqslant 1$, so $a^2 = b^2 = (ab)^{\frac{2n+1}{2}}$. Then $a^3 = (ab)^{\frac{2n+1}{2}} a = (ab)^{\frac{2n}{2}} a^2 = (ab)^{\frac{2n}{2}} b^2 = (ab)^{\frac{2n-1}{2}} b^3 = (ab)^{\frac{2n-2}{2}} a^3 b$, so $a^3 = 0$. Therefore $a^2 b = b^3 = ba^2$, which means $(ab)^{\frac{2n+2}{2}} = (ba)^{\frac{2n+2}{2}}$. Then $(ab)^{\frac{2n+3}{2}} = (ab)^{\frac{2n+2}{2}} a = (ba)^{\frac{2n+2}{2}} a = b(ab)^{\frac{2n}{2}} a^2 = 0$ and, analogously,

$(ba)^{\frac{2n+3}{2}} = 0$. So, a^2b is an atom or a zero. If $a^2 = a^2b = (ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.1}(2n+1)$. If $a^2 = 0$ and $(ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.2}(2n+1)$. If $a^2 \neq 0$ and $(ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.3}(2n+1)$. If $a^2 = (ba)^{\frac{m}{2}} \neq 0$ and $a^2b = 0$, then $S \cong E_{2.4}(2n+1)$. If $a^2 \neq (ba)^{\frac{m}{2}}$ and $a^2b \neq 0$, then $S \cong E_{2.5}(2n+1)$. If $a^2 \neq 0$, $(ba)^{\frac{m}{2}} \neq 0$, $a^2 \neq (ba)^{\frac{m}{2}}$ and $a^2b = 0$, then $S \cong E_{2.6}(2n+1)$.

Case E2.2. $m = 2n$ for some $n \geqslant 2$, so $a^2 = b^2 = (ab)^{\frac{2n}{2}}$. Then $a^3 = a(ab)^{\frac{2n}{2}} = a^2(ba)^{\frac{2n-1}{2}} = b^3(ab)^{\frac{2n-2}{2}} = ba^3(ba)^{\frac{2n-3}{2}}$, so $a^3 = 0$. From here we obtain $ba^2 = b^3 = a^2b = (ab)^{\frac{2n}{2}}b = (ab)^{\frac{2n-2}{2}}ab^2 = (ab)^{\frac{2n-2}{2}}a^3 = 0$, which means that a^2 is an atom or a zero. If $a^2 = (ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.1}(2n)$. If $a^2 = 0$ and $(ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.2}(2n)$. If $a^2 \neq 0$ and $(ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.3}(2n)$. If $a^2 = (ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.4}(2n)$. If $a^2 \neq 0$, $(ba)^{\frac{m}{2}} \neq 0$ and $a^2 \neq (ba)^{\frac{m}{2}}$, then $S \cong E_{2.6}(2n)$.

Case E3. We have $ab < a^2$, whence $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \lessdot ab$ or $ba \lessdot ba$, which means $ab = ba = 0$. Anyway, $ab = a^m$ for some $m \geqslant 3$. Analogously, $ab = b^n$ for some $n \geqslant 3$. Every element of the semigroup S can be written in the form a^p or b^p for some p. Let n be the least positive integer such that $ab = ba = a^n$ and m be the least positive integer such that $ab = ba = b^m$. Then $aba = a^{n+1} = ab^m = a^nb^{m-1} = a^{2n-1}b^{m-2} \lessdot a^{n+1}$, so $aba = 0$. By the same arguments, $abb = 0$, which means that ab is either an atom or a zero. If ab is an atom, $S \cong E_{3.1}(n, m)$. If ab is a zero, $S \cong E_{3.2}(n, m)$.

Case E4. We have $ba < a^2 = b^2$, so $ba = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ba \leqslant a^3 = b^2a \lessdot ba$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba \leqslant b^3 = ba^2 \lessdot ba$, which implies $ba = 0$. Now we have $ab^2 = a^3 = b^2a = 0$, $a^2b = b^3 = ba^2 = 0$, $aba = bab = 0$, so the semigroup S consists only of five elements and $S \cong E_4$.

Case E5. We have $b^2 < ab = ba$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $b^2 \lessdot b^2$, which means $b^2 = 0$. Anyway, there exists m such that $b^2 = a^mb$. Let m be the minimal integer with such a property.

Every element of S can be written as a^p or a^pb for a suitable p. Let n and k be minimal positive integers such that $a^n = 0$ and $a^kb = 0$.

If $a^p = a^qb$ implies $a^p = 0$ for some p, q, then $S \cong E_{5.1}(m, n, k)$.

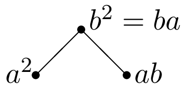
Let p, q be the least positive integers such that $a^p = a^qb \neq 0$. Then $p < n$, $q < k$ and $a^{p+r} = a^{q+r}b$ for all $r \geqslant 0$, so $n - p = k - q$, which means $p = n - k + q$. We have $a^{n-k+q} = a^qb$, so $a^{n-k+q}b = a^{q+m}b$. If $n - k \neq m$, then $a^{n-k+q}b = a^{q+m}b = 0$, so $k \leqslant \min(n-k+q, q+m)$. In this case $S \cong E_{5.2}(m, n, k, q)$. If $n - k = m$, then $S \cong E_{5.3}(m, n, q)$.

Case E6. Since $a^2 = ab$, every element of S can be written as a^p or b^qa^p for some p. Then $b^2 = 0$ or $b^2 = ba^n$ for some n. If $n \geqslant 3$, then $b^2 \lessdot a^3 = ab^2 \lessdot b^2$, which implies $b^2 = 0$. So, $b^2 = ba^2 \neq 0$ or $b^2 = 0$.

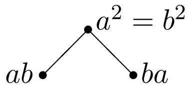
Let $b^2 = ba^2 \neq 0$. Then $a^3 = ab^2 = aba^2 = a^4$, so $a^3 = 0$. Thus, $b^2a = ba^3 = 0$, $b^3 = ba^2b = ba^3 = 0$ and $ab^2 = a^3 = 0$, so b^2 is an atom. Then $S \cong E_{6.1}$. If $b^2 = 0$, then $a^3 = 0$. Hence a^2 and ba are atoms and $S \cong E_{6.2}$.

Case E7. Using the same arguments as in case E6, for some $m \geqslant 2$, we have $ba = a^mb \lessdot a^3 = aba \lessdot ba$, so $ba = 0$. Let n be the index of b. Then S consist of elements $a, a^2, b, b^2, \dots, b^{n-1}, 0$ and is isomorphic to $E_7(n)$.

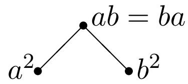
Series F. The following cases are possible:



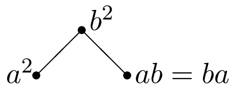
F1



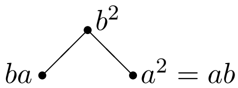
F2



F3



F4



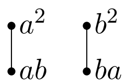
F5

Case F1. We have $a^2 < b^2 = ba$, but $a^2 \not< ab$. So, $a^2 \neq 0$ and $a^2 = b^k a$ for some $k \geqslant 2$. These arguments are true for ab, so $ab = b^l a$ for some $l \geqslant 2$. Then $a^2 \geqslant ab$ or $a^2 \leqslant ab$, a contradiction.

Case F2. $ab < a^2$, so $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for $s' \in S^1$, then $ab < ba$. If $t = bt'$ for $t' \in S^1$, then $ab < ab$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ab \leqslant a^3 = ab^2 < ab^2$. All the possibilities lead to a contradiction.

Cases **F3-F5** are analogous to **F1** or **F2**.

Series G. Only one case is possible:

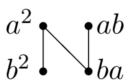


G1

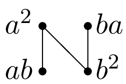
Case G1. We have $ab < a^2$, so $ab = a^m$ for some $m \geqslant 3$. Similarly, $ba = b^n$ for some $n \geqslant 3$. Then $a^{m+1} = a^2b = aba = ab^n = a^m b^{n-1} = a^{2m-1}b^{n-2} \lessdot a^{m+1}$, so $a^{m+1} = a^m b = ba^m = 0$. Similarly, $b^{n+1} = bab = b^2a = 0$. So, ab and ba are atoms.

Every element of the semigroup S can be written as a^p or b^p for a suitable p. Let $a^q = b^r$ for some $q < m$ and $r \leqslant n$. Then $a^{q+1} = b^r a = 0$ and $a^m = 0$, a contradiction. If $q = m$ and $r = n$, we have $ab = ba$, a contradiction. We obtain that $S \cong G_1(m, n)$.

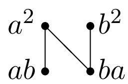
Series H. The following cases are possible:



H1



H2



H3

Case H1. We have $b^2 < a^2$, so $b^2 = a^n$ for $n \geqslant 3$. Since $ba < ab$ and $ba \not< b^2$, the equality $ba = a^m b$ holds for some $n > m \geqslant 2$. Every element of the semigroup S can be written as a^p or $a^p b$ for some p. Now $a^{n+1} = b^2 a = ba^m b = a^{m^2} b^2 = a^{m^2+n} \lessdot a^{n+1}$, so $b^2 = a^{n+1} = 0$, a contradiction. Therefore $b^2 = ba^n = a^{mn} = 0$ and b^2 is an atom. Let k be the least positive integer such that $a^k b = 0$. We have $a^n b = b^3 = 0$, so $m < k \leqslant n$.

If the equality $a^p = a^q b$ for some $p \leqslant n$ and $q \leqslant k$ implies $a^p = 0$, then $S \cong H_{1.1}(m, n, k)$. Let $a^p = a^q b$ for some $p \leqslant n$ and $q \leqslant k$. Then $a^{q+1} b = a^{p+1} = a^q ba = a^{q+m} b$, so $a^{p+1} = 0$. If $p < n$, then $b^2 = 0$, a contradiction. Let $p = n$ and $q > m$. We have $a^{q+1} b = 0$, so $q = k - 1$. We deduce $S \cong H_{1.2}(m, n, k)$.

Case H2. We have $ab = a^n$ for $n \geqslant 3$ and $b^2 = ba^m$ for $m \geqslant 2$. Since $b^2 \not> ab$, then $n \geqslant m + 1$. Every element of the semigroup S can be written as a^p or ba^p for some p.

Let $n > m+1$. Then $a^{n+m} = aba^m = ab^2 = a^n b = a^{2n-1} \lessdot a^{n+m}$, which implies $a^{n+m} = 0$. Let k and l be the minimal integers such that $a^k = 0$ and $ba^l = 0$. Therefore $n < k$ and $m < l \leqslant k \leqslant m+n$.

If $a^p = ba^q$ implies $a^p = 0$, then $S \cong H_{2.1}(m, n, k, l)$. Let $a^p = ba^q \neq 0$. Since $ab \not< ba$, then $p > n$. Therefore $ba^{q+1} = a^{p+1} = aba^q = a^{q+n}$. If $p + 1 \neq q + n$, then $a^{p+1} = 0$. This implies

$p+1=k$ and $q+1=l$, so $S\cong H_{2.2}(m,n,k,l)$. Let $p+1=q+n$. Then $a^{q+n-1+r}=ba^{q+r}$ for every $r\geqslant 0$, so $l=k-n+1$ and $k>q+n-1$. We obtain $S\cong H_{2.3}(m,n,k,q)$.

Let $n=m+1$. Let k and l be the minimal integers such that $a^k=0$ and $ba^l=0$. It is obvious that $m<k-1$, $m<l$ and $k\geqslant l$.

If $a^p=ba^q$ implies $a^p=0$, then $S\cong H_{2.4}(m,k,l)$. Let $a^p=ba^q\neq 0$. Since $ab\not< ba$, we have $p>m+1$. Therefore $ba^{q+1}=a^{p+1}=aba^q=a^{q+m+1}$. If $p+1\neq q+m+1$, then $a^{p+1}=0$. This means $p+1=k$ and $q+1=l$, so $S\cong H_{2.5}(m,k,l)$. Let $p+1=q+n$. Then $a^{q+m+r}=ba^{q+r}$ for every $r\geqslant 0$, so $l=k-m$ and $k>q+m$, whence we get $S\cong H_{2.6}(m,k,q)$.

Case H3. We have $ab=a^m$ for $m\geqslant 3$ and $ba=a^n$ for $n\geqslant 3$. Then $ab>ba$ or $ba>ab$, a contradiction.

Series I. The following cases are possible:

I1 **I2** **I3** **I4**

Case I1. We have $a^2<ab$, $a^2<ab$, but $a^2\not< b^2$, which means that $a^2=(ab)^{l/2}$ or $a^2=(ba)^{l/2}$ for some $l\geqslant 3$. We suppose without loss of generality that $a^2=(ab)^{l/2}$. Then $b^2=(ba)^{l/2}$. Every element can be written in the form $(ab)^p$ or $(ba)^p$ for some p. Two cases are possible:

Case I1.1: $a^2=(ab)^{\frac{2n}{2}}$ and $b^2=(ba)^{\frac{2n}{2}}$ for some $n\geqslant 2$. Then $(ab)^{\frac{2n+1}{2}}=a^2a=a^3=aa^2=a(ab)^{\frac{2n}{2}}=a^2(ba)^{\frac{2n-1}{2}}=(ab)^{\frac{2n}{2}}(ba)^{\frac{2n-1}{2}}=(ab)^{\frac{2n-1}{2}}b^2(ab)^{\frac{2n-2}{2}}=(ab)^{\frac{2n-1}{2}}(ba)^{\frac{2n}{2}}(ab)^{\frac{2n-2}{2}}=(ab)^{\frac{4n-1}{2}}(ab)^{\frac{2n-2}{2}}\lessdot(ab)^{\frac{2n+1}{2}}$, so $a^3=0$. Analogously, $b^3=0$. Then $a^2b=(ab)^{\frac{2n}{2}}b=(ab)^{\frac{2n-1}{2}}b^2=(ab)^{\frac{2n-1}{2}}(ba)^{\frac{2n}{2}}=(ab)^{\frac{4n-1}{2}}\lessdot a^3$, so $a^2b=0$. Similarly, $ba^2=ab^2=b^2a=0$, which means that a^2 and b^2 are atoms. Hence $S\cong I_{1.1}(n)$.

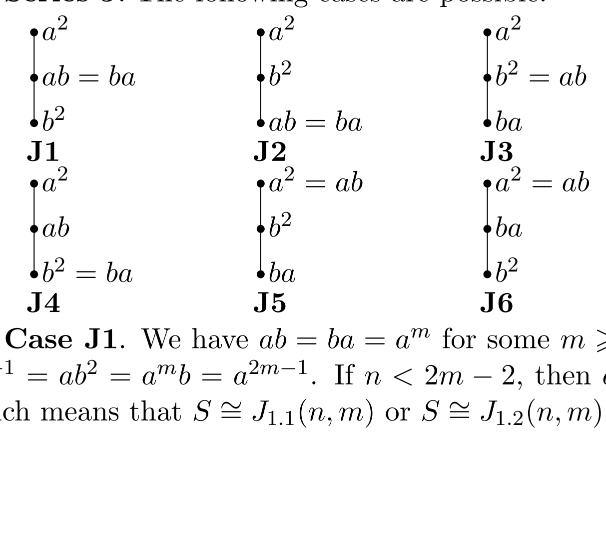
Case I1.2: $a^2=(ba)^{\frac{2n+1}{2}}$ and $b^2=(ab)^{\frac{2n+1}{2}}$ for some $n\geqslant 1$. Then $(ba)^{\frac{2n+2}{2}}=a^2a=aa^2=(ab)^{\frac{2n+2}{2}}$. It is easy to see that $(ba)^{\frac{2n+2}{2}}=(ab)^{\frac{2n+2}{2}}$ is either an atom or a zero. If it is an atom, then $S\cong I_{1.2}(n)$. If it is a zero, then $S\cong I_{1.3}$.

Case I1.3: $a^2=(ab)^{\frac{2n+1}{2}}$ and $b^2=(ba)^{\frac{2n+1}{2}}$ for some $n\geqslant 1$. Let m be the least positive integer such that $(ab)^{\frac{m}{2}}=0$ and k be the least positive integer such that $(ba)^{\frac{k}{2}}=0$. Then $|m-k|\leqslant 1$ and $m>2n+1$, $k>2n+1$. Hence $S\cong I_{1.4}(n,m,k)$.

Case I2. We have $b^2<ab$ and $b^2<a^2$, so $b^2=a^kb$ for some $k\geqslant 3$. By the same arguments $ba=a^lb$ for some $l\geqslant 3$. Then b^2 and ba are comparable, a contradiction.

Cases **I3** and **I4** lead to a contradiction in a similar way.

Series J. The following cases are possible:



Case J1. We have $ab=ba=a^m$ for some $m\geqslant 3$. Then $b^2=a^n$ for $m<n\leqslant 2m-2$. Hence $a^{n+1}=ab^2=a^mb=a^{2m-1}$. If $n<2m-2$, then $ab^2=0$ and $b^3=0$. So, b^2 is an atom or zero, which means that $S\cong J_{1.1}(n,m)$ or $S\cong J_{1.2}(n,m)$ respectively.

Let $n = 2m - 2$ and let k be the index of a. Then $S \cong J_3(m, k)$.

Case J2. We have $b^2 = a^n$ for some $n \geqslant 3$. Also, $ab < b^2$, so $ab = sb^2t$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for $s', t' \in S^1$, then $ab \leqslant a^{n+1} = ab^2 \lessdot ab$. Cases $s = s'b$ or $t = at'$ for $s', t' \in S^1$ are similar. So, $ab = ba = a^{n+1} = 0$ and $S \cong J_2(n)$.

Case J3. In this case $0 < b^2 = ab < a^2$ implies that $b^2 = ab = a^n$ for some $n \geqslant 3$. Then $a^{n+1} = b^2a < ba$ and $a^nb = b^3 = ba^n \lessdot ba$. So, $ba = a^{n+1} = 0$ and $S \cong J_3(n)$.

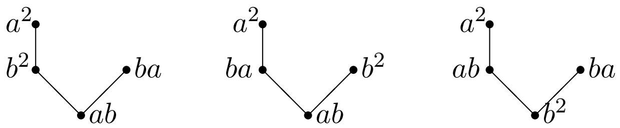
Case J4. We have $ab = a^n$ for some $n \geqslant 3$. Then $a^{n+1} = aba \lessdot ba$ and $a^nb = ab^2 \lessdot b^2 = ba$. But $ba < ab$, so $ba = 0$ and $S \cong J_4(n)$.

Case J5. The inequality $b^2 < a^2$ implies that $b^2 = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for $s' \in S^1$, then $b^2 < ba$, a contradiction. In other cases we have $b^2 < a^3 = ab^2$, which implies $b^2 = 0$ and $b^2 < ba$, a contradiction.

Case J6. As in the previous case, $ba \leqslant a^3 = aba$, whence $ba = 0$ and $ba < b^2$, a contradiction.

Series K. All cases from this series are impossible by Lemma 3.

Series L.



L1 **L2** **L3**

Case L1. We have $b^2 = a^m$ for some $m \geqslant 3$. If $ab = a^p$ for $p > m$, then $ab = a^p = a^{p-m}b^2 \lessdot ab$. If $ab = ba^q$ for $q \geqslant m$, then $ab = ba^q = b^3a^{q-m} = a^mba^{q-m} \lessdot ab$. Anyway, $ab = 0$. Every element of the semigroup S can be written in the form a^p or ba^p for some p.

Since $a^{m+1} = ab^2 = 0$ and $ba^m = b^3 = a^mb = 0$, the element b^2 is an atom. Let n be the minimal integer such that $ba^n = 0$. Then ba^{n-1} is an atom and $n \leqslant m + 1$. Every element of the semigroup S can be written in the form a^p or ba^p for some p.

Let $a^p = ba^q$ for $p, q \leqslant n$. Then $b^2 = a^n = a^{n-p}a^p = a^{n-p}ba^q = 0$, a contradiction. So, $S \cong L_1(m, n)$.

Case L2. We have $ba = a^m$ for some $m \geqslant 3$. Then $a^{m+1} = aba \lessdot ab$, $a^mb \lessdot ab$, $ba^m = a^{2m-1} \lessdot ab$, but $ab < ba$, so $ab = 0$ and $ba = a^m$ is an atom. Every element of the semigroup S can be written in the form a^p or b^p for some p. Let n be the least positive integer such that $b^n = 0$.

If $a^p = b^q$ for some p, q leads to $a^p = 0$, then $S \cong L_{2.1}(m, n)$. Let $a^p = b^q$ for $p \leqslant m$ and $q < k$. Then $a^{p+1} = ab^q = 0 = a^{m+1}$, so $p = m$ and $q = n - 1$. Then $S \cong L_{2.2}(m, n)$.

Case L3. $ab = a^m$ for some $m \geqslant 3$. Since $b^2 < ba$ and $b^2 < ab = a^m$, either $b^2 = a^n \neq 0$ for some $n \geqslant m + 1$ or $b^2 = ba^l$ for some $l \geqslant m$ or $b^2 = 0$.

Case L3.1. $b^2 = a^k \neq 0$ and $k \neq 2m - 2$. Then $b^2a = a^{k+1} = ab^2 = a^mb = a^{2m-1}$. Since $k + 1 \neq 2m - 1$, a^{k+1}=0. Then $b^2 \neq 0$, so b^2 is an atom. Every element of the semigroup can be written in the form a^p or ba^p for some p. Note that $ba^k = b^3 = a^kb = a^{m+k-1} = 0$. Let n be the least positive integer such that $ba^{n+1} = 0$. Then $k - m \leqslant n \leqslant k$, since $aba^l = a^{l+m}$ for all l.

If $a^p = ba^q$ implies $a^p = 0$ for all p, q, then $S \cong L_{3.1}(m, k, n)$. Suppose that $a^p = ba^q \neq 0$ for some p, q and let p, q be the least positive integers with such a property. Then $k \geqslant p > q$. We have $a^{p+1} = aba^q = a^{q+m}$, so either $p = k$ or $p = q + m - 1$. If $p = k$, then $a^k = ba^q$ and $a^{k+1} = a^{q+m}$, so $q \geqslant k - m + 1$ and $S \cong L_{3.2}(m, k, q)$. If $p = q + m - 1$, then $a^{r+m-1} = ba^r$ for all $r \geqslant q$. But $ba^{q+m-1} = b^2a^q = 0$, so $a^{q+2m-1} = 0$, which implies $q \geqslant k - 2m + 2$. We got $S \cong L_{3.3}(m, k, q)$.

Case L3.2. $b^2 = a^{2m-2} \neq 0$. Every element of the semigroup can be written in the form a^p or ba^p for some p. Then $ba^{2m-2} = b^3 = a^{2m-2}b = a^{3m-3}$ and $ba^{2m-2+r} = a^{3m-3+r}$ for all $r \geqslant 1$. Let

n be the least positive integer such that $a^n = 0$. Then $n \geqslant 2m-1$, since $b^2 \neq 0$. Let l be the least positive integer such that $ba^l = 0$. Then $l \geqslant m-1$ and $n \leqslant l+m$.

If $a^p = ba^q$ implies $a^p = 0$ for all p, q, then $S \cong L_{3.4}(m,n,l)$. Let $a^p = ba^q \neq 0$ for some p, q and let p, q be the least positive integers with such a property. Then $a^{p+1} = aba^q = a^{q+m}$, so either $a^{p+1} = a^{q+m} = 0$ or $p = q+m-1$. Let $a^{p+1} = a^{q+m} = 0$. Then $p+1 = n$, $q+1 = l$ and $q+m \geqslant n$. Therefore $S \cong L_{3.5}(m,n,l)$.

Let $p = q+m-1$. Then $q \geqslant 2$, $n \geqslant q+m$ and we have $S \cong L_{3.6}(m,n,q)$.

Case L3.3. $b^2 = ba^l \neq 0$ for some l. Since $ab > b^2$, $l \geqslant m$. Every element of the semigroup S can be written either in the form a^p or in the form ba^p for some p. Then $a^{2m-1} = a^m b = ab^2 = aba^l = a^{l+m}$. Since $l \geqslant m$, $a^{2m-1} = a^{m+l} = 0$. Let n be the least positive integer such that $a^n = 0$ and k be the least positive integer such that $ba^k = 0$. Then $l < k \leqslant n \leqslant 2m-1$.

If $a^p = ba^q$ for some p, q implies $a^p = ba^q = 0$, then $S \cong L_{3.7}(m,l,k,n)$. Let p, q be the least positive integers such that $a^p = ba^q \neq 0$. Obviously, $q \geqslant 2$. Then $a^{p+1} = aba^q = a^{q+m}$, so either $a^{p+1} = a^{q+m} = 0$ or $p = q+m-1$. In the former case we have $p = n-1$ and $q = l-1$, so $S \cong L_{3.8}(m,l,k,n)$. In the latter case we have $a^{r+m-1} = ba^r$ for all $r \geqslant q$, then $k = n-m+1$ and $S \cong L_{3.9}(m,l,q,n)$.

Case L3.4. $b^2 = 0$. Every element of the semigroup can be written in the form a^p or ba^p for some p. Let n be the least positive integer such that $a^n = 0$ and k be the least positive integer such that $ba^k = 0$. Then $k \leqslant n \leqslant k+m$.

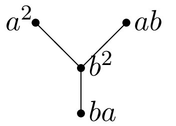
Let p, q be the least positive integers such that $a^p = ba^q$. Then one of the following possibilities holds:

1) $a^p = b^q = 0$;
2) $a^{p+1} = a^{q+m} = 0$;
3) $p = q+m-1$ and $n > p+1$.

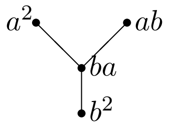
In the first case we have $p = n$ and $q = k$, so $S \cong L_{3.10}(m,n,k)$. In the second case we have $p+1 = n$ and $q+1 = k$, whence $S \cong L_{3.11}(m,n,k)$. In the third case we have $a^{r+m-1} = ba^r$ for all $r \geqslant q$, so $k = n-m+1$ and $S \cong L_{3.12}(m,n,q)$.

Series M. By Lemma 3, all cases lead to a contradiction.

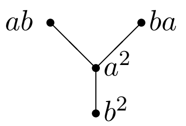
Series N.



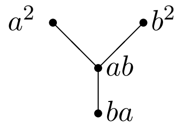
N1



N2



N3



N4

Case N1. We have $b^2 < ab$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ for $s', t' \in S^1$, then $b^2 < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $b^2 < b^2$, so $b^2 = 0 < ba$, a contradiction. Therefore $b^2 = a^m b$ for some $m \geqslant 2$. By the same arguments either $ba = 0$ or $ba = a^l b \neq 0$ for some $l > m$.

Case N1.1. $ba = a^l b \neq 0$ for some $l > m$. Every element of the semigroup can be written in the form a^p or $a^p b$ for some p. Note that $a^{2m} b = a^m b^2 = b^3 = ba^m b \lessdot ba$, so $l < 2m$. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $ba^k = 0$. Obviously, $l < k \leqslant n$. Since $a^{m+l} b = a^m ba = b^2 a = ba^l b = a^{l^2} b^2 = a^{l^2+m} b$, we obtain $k \leqslant m+l$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1} b = a^{p+1} = a^q ba = a^{q+l} b = 0$, so either $p = n$, $q = k$ and $S \cong N_{1.1}(m,l,n,k)$ or $p = n-1$ and $q = k-1$. This means that $S \cong N_{1.2}(m,l,n,k)$.

Case N1.2. $ba = 0$. Every element of the semigroup S can be written either in the form a^p or in the form $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $ba^k = 0$. Trivially, $k \leqslant n$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{p+1} = a^q ba = 0$, so either $p = n$, $q = k$ and $S \cong N_{1.1}(m, l, n, l)$ or $p = n - 1$ and $q = k - 1$, i.e. $S \cong N_{1.2}(m, l, n, l)$.

Case N2. We have $ba < ab$, so $ba = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ for some $s', t' \in S^1$, then $ba = 0 < b^2$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ba < b^2$, a contradiction. Therefore $ba = a^m b$ for some $m \geqslant 2$. By the same arguments either $b^2 = 0$ or $b^2 = a^l b \neq 0$ for some $l > m$.

Case N2.1. $b^2 = a^l b \neq 0$ for some $l > m$. Every element of the semigroup can written in the form a^p or $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^k b = 0$, then $l < k \leqslant n$. Since $a^{m+l}b = a^l ba = b^2 a = ba^m b = a^{m^2+l}b \lessdot a^{m+l}b$, we have $k \leqslant m + l$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1}b = a^{p+1} = a^q ba = a^{q+m}b$, so either $p = n$, $q = k$ and $S \cong N_{2.1}(m, l, n, k)$ or $p = n - 1$ and $q = k - 1$, i.e. $S \cong N_{2.2}(m, l, n, k)$.

Case N2.2. $b^2 = 0$. Every element of the semigroup can be written in the form a^p or $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^k b = 0$. Then $k \leqslant n$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1}b = a^{p+1} = a^q ba = a^{q+m}b = 0$, so either $p = n$, $q = k$ and $S \cong N_{2.1}(m, l, n, l)$ or $p = n-1$ and $q = k-1$, i.e. $S \cong N_{2.2}(m, l, n, l)$.

Case N3. We have $a^2 < ab$ and $a^2 < ba$, but $a^2 \not< b^2$. So $a^2 = (ab)^{\frac{p}{2}}$ or $a^2 = (ba)^{\frac{p}{2}}$ for some p. Obviously, every element of the semigroup can be written in the same form.

Case N3.1. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ab)^{\frac{2k+1}{2}} \neq 0$ for some $k > m > 1$. Then $(ab)^{\frac{2k+2}{2}} = (ab)^{\frac{2k+1}{2}} b = b^3 = b(ab)^{\frac{2k+1}{2}} = (ba)^{\frac{2k+2}{2}}$. So, $(ab)^{\frac{2k+3}{2}} = (ab)^{\frac{2k+2}{2}} a = (ba)^{\frac{2k+2}{2}} a = (ba)^{\frac{2k+1}{2}} a^2 = (ba)^{\frac{2k+1}{2}} (ab)^{\frac{2k+1}{2}} \lessdot (ab)^{\frac{2k+3}{2}}$, whence $(ab)^{\frac{2k+3}{2}} = 0$. Analogously, $(ba)^{\frac{2k+3}{2}} = 0$.

If $(ab)^{\frac{2k+1}{2}} \neq (ba)^{\frac{2k+1}{2}}$ and $(ab)^{\frac{2k+1}{2}} \neq 0$, then $S \cong N_{3.1}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} \neq (ba)^{\frac{2k+1}{2}} \neq 0$ and $(ab)^{\frac{2k+1}{2}} = 0$, then $S \cong N_{3.2}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} = (ba)^{\frac{2k+1}{2}}$, then $S \cong N_{3.3}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} \neq 0$ and $(ba)^{\frac{2k+1}{2}} = 0$, then $S \cong N_{3.4}(m, 2k+1)$.

Case N3.2. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ab)^{\frac{2k+2}{2}} \neq 0$ for some $k \geqslant m > 1$. Then $(ba)^{\frac{2k+3}{2}} = b(ab)^{\frac{2k+2}{2}} = b^3 = (ab)^{\frac{2k+2}{2}} b = (ab)^{\frac{2k+1}{2}} b^2 = (ab)^{\frac{2k+1}{2}} (ab)^{\frac{2k+2}{2}} = (ab)^{\frac{2k}{2}} a^2 (ba)^{\frac{2k+1}{2}} = (ab)^{\frac{2k}{2}} (ab)^{\frac{2m+1}{2}} (ba)^{\frac{2k+1}{2}} = (ab)^{\frac{4k+2m+2}{2}} \lessdot (ba)^{\frac{2k+2}{2}}$, so $(ba)^{\frac{2k+3}{2}} = b^3 = 0$. Therefore $b^2 ab \lessdot b^3$ and $b^2 ab = 0$.

If $b^2 a \neq 0$, then $S \cong N_{3.1}(m, 2k+2)$. If $b^2 a = 0$ and $(ba)^{\frac{2k+2}{2}} \neq 0$, then $S \cong N_{3.2}(m, 2k+2)$. If $(ab)^{\frac{2k+2}{2}} = (ba)^{\frac{2k+2}{2}} \neq 0$, then $S \cong N_{3.3}(m, 2k+2)$. If $(ab)^{\frac{2k+2}{2}} \neq 0$ and $(ba)^{\frac{2k+2}{2}} = 0$, then $S \cong N_{3.4}(m, 2k+2)$.

Case N3.3. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ba)^{\frac{2k+1}{2}}$ for some $m, k \geqslant 2$ and $k > m$. Let n and l be the least positive integers such that $(ab)^{\frac{n}{2}} = (ba)^{\frac{l}{2}} = 0$. Clearly, $n, l \geqslant 2k+1$ and $|n - l| \leqslant 1$. Then $S \cong N_{3.5}(m, k, n, l)$.

Case N3.4. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = 0$. Let n and l be the least positive integers such that $(ab)^{\frac{n}{2}} = 0$ and $(ab)^{\frac{l}{2}} = 0$. Obviously, $|n - l| \leqslant 1$. Then $S \cong N_{3.6}(m, n, l)$.

Case N3.5. $a^2 = (ab)^{\frac{2m}{2}}$ for some $m > 1$. Then $(ab)^{\frac{2m+1}{2}} = a^3 = a(ab)^{\frac{2m}{2}} = a^2 (ba)^{\frac{2m-1}{2}} = (ab)^{\frac{2m}{2}} (ba)^{\frac{2m-1}{2}} = (ab)^{\frac{2m-1}{2}} b^2 (ab)^{\frac{2m-2}{2}}$. It is easy to see that $(ab)^{\frac{2m-1}{2}} b^2 (ab)^{\frac{2m-2}{2}} \lessdot (ab)^{\frac{2m+1}{2}}$, so $(ab)^{\frac{2m+1}{2}} = 0$. Therefore $(ba)^{\frac{2m+2}{2}} = 0$. Then $b^2 = (ba)^{\frac{2m+1}{2}}$ or $b^2 = 0$.

If $b^2 = (ba)^{\frac{2m+1}{2}} \neq 0$, then $S \cong N_{3.7}(2m)$. If $(ba)^{\frac{2m+1}{2}} \neq 0$ and $b^2 = 0$, then $S \cong N_{3.8}(2m)$. If $b^2 = (ba)^{\frac{2m+1}{2}} = 0$, then $S \cong N_{3.9}(2m)$.

Case N3.6. $a^2 = (ba)^{\frac{2m+1}{2}}$. Then $(ab)^{\frac{2m+2}{2}} = a^3 = (ba)^{\frac{2m+2}{2}}$, so $(ab)^{\frac{2m+3}{2}} = (ba)^{\frac{2m+3}{2}} = 0$. Therefore $b^2 = (ab)^{\frac{2m+2}{2}}$ or $b^2 = 0$.

If $b^2 = (ab)^{\frac{2m+2}{2}} \neq 0$, then $S \cong N_{3.7}(2m+1)$. If $(ab)^{\frac{2m+2}{2}} \neq 0$ and $b^2 = 0$, then $S \cong N_{3.8}(2m+1)$. If $b^2 = (ab)^{\frac{2m+2}{2}} = 0$, then $S \cong N_{3.9}(2m+1)$.

Case N4. We have $ab = a^n = b^m$ for some $n, m \geqslant 3$. Then $bab \lessdot ba$, $a^2b = a^{n+1} = aba \lessdot ba$, $ab^2 = b^{m+1} = bab \lessdot ba$, but $ba < ab$. So, $ba = 0$ and ab is an atom. If $a^p = b^q$ for $1 < p < n$ and $1 < q < m$, then $a^{p+1} = 0$, which means $ab = a^n = 0 = ba$, a contradiction. Then $S \cong N_4(n, m)$.

Series O, P, Q leads to a contradiction by Lemma 3 or by arguments from series **F**.

Theorem 1 is now proved. □

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