www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 6 No. 1 (2017), pp. 1-11.

www.ui.ac.ir

ON ANNIHILATOR GRAPHS OF A FINITE COMMUTATIVE RING

SANGHITA DUTTA* AND CHANLEMKI LANONG

Communicated by Dariush Kiani

Abstract. The annihilator graph $AG(R)$ of a commutative ring R is a simple undirected graph with the vertex set $Z(R)^*$ and two distinct vertices are adjacent if and only if $ann(x) \cup ann(y) \neq ann(xy)$. In this paper we give the sufficient condition for a graph $AG(R)$ to be complete. We characterize rings for which $AG(R)$ is a regular graph, we show that $\gamma(AG(R)) \in \{1, 2\}$ and we also characterize the rings for which $AG(R)$ has a cut vertex. Finally we find the clique number of a finite reduced ring and characterize the rings for which $AG(R)$ is a planar graph.

1. Introduction

The study of rings using the properties of graphs lead to many interesting results. The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph with the vertex set $Z(R)^* = Z(R)\backslash\{0\}$ and two distinct vertices x, y are adjacent if and only if $xy = 0$. The concept of a zero divisor graph goes back to I. Beck [8], who considered all elements of R as the set of vertices and was mainly interested in coloring of a graph. The zero-divisor graph $\Gamma(R)$ was introduced by David F. Anderson and Philip S. Livingston [2], where it was shown among other results that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $girth(\Gamma(R)) \in \{3, 4\}$. Many mathematicians have studied the zero divisor graph of a ring and obtained many interesting results regarding ring theoretic properties as well as graph theoretic properties of this graph. Badawi [7] defined a graph associated with a commutative ring called the annihilator graph of a ring R, denoted by $AG(R)$. The vertex set of this graph is $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $ann(x) \cup ann(y) \neq ann(xy)$. Badawi [7] proved that $AG(R)$ is a connected graph, diameter of $AG(R)$ is atmost two, girth of $AG(R)$ is atmost four if it has a cycle and if R is a

MSC(2010): Primary: 05C69; Secondary: 13H05.

Keywords: Annihilator, Clique number, Domination Number.

Received: 02 July 2015, Accepted: 16 February 2016.

*Corresponding author.

reduced ring then $AG(R)$ is identical to $\Gamma(R)$ if and only if the ring R has exactly two minimal prime ideals. D.A Mojdeh et al. [10] found the domination number of a zero divisor graph, zero divisor graph with respect to an ideal of a ring R and T. Tamish Chelvam et al. [9] found the domination number of total graph of a ring. M. Axtell et al. [6] have found the condition for a vertex x to be a cut vertex of $\Gamma(R)$.

In section 2, we discuss about the existence of a vertex which is adjacent to all vertices of $AG(R)$, sufficient condition for $AG(R)$ to be a complete graph and a regular graph and we show that the domination number of $AG(R)$ is less than or equal to 2 for any finite ring. We find that if R is a finite ring and $AG(R)$ has a cut vertex then $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field with $\mathbb{F} \ncong \mathbb{Z}_2$. We also compute $\alpha(AG(R))$ and $\omega(AG(R))$ for some classes of rings. We show that $AG(R)$ is Hamiltonian if $R \cong A \times A$ where A is a finite local ring with identity. In section 3, we characterize rings for which $AG(R)$ is planar.

Throughout the paper, all rings are assumed to be commutative ring with unity $1 \neq 0$. A ring R is said to be *reduced* if R has no non-zero nilpotent element. Let $Z(R)$ denote the set of zero-divisors of a ring R. If X is either an element or a subset of R, then $ann(X)$ denotes the annihilator of X in R, i.e., $ann(X) = \{r \in R |\ rX = 0\}$. For any subset X of R let $X^* = X \backslash \{0\}$. A ring R is said to be *decomposable* if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise R is said to be *indecomposable*.

All graphs considered in this paper are simple graphs. For a graph G, the degree of a vertex v in G, denoted by $deg(v)$ is the number of edges incident to v. A graph G is said to be *regular* if the degrees of all vertices of G are same. A graph G is said to be *complete* if every pair of distinct vertices are connected by an edge. A *bipartite graph* is a graph whose set of vertices can be partitioned into two sets U and V such that every edge is between a vertex of U and a vertex of V. We denote the complete graph with n vertices and complete bipartite graph with two sets of sizes m and n by K_n and $K_{m,n}$ respectively. The complete bipartite graph $K_{1,n}$ is called a *star graph*. The *diameter* of a graph G is $diam(G) = sup\{d(x, y) : x$ and y are distinct vertices of $G\}$. A vertex a in a connected graph G is a *cut-vertex* if G can be expressed as a union of two sub graphs X and Y such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $X \backslash \{a\} \neq \emptyset$, and $Y \backslash \{a\} \neq \emptyset$. A subset D of the set of vertices $V(G)$ of a graph G is called a *dominating set*, if every vertex of $V(G) \backslash D$ is adjacent to some vertex of D. The minimum size of such a subset is called the *domination number* of G and is denoted by $\gamma(G)$. A set $S \subseteq V(G)$ is *independent set* of G, if no two vertices of S are adjacent. The *independence number* of a graph G denoted by $\alpha(G)$ is the size of the maximum independent set in G. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G.

A *Hamiltonian* cycle (resp. path) in a graph is a cycle (resp. path) including all the vertices of the graph. Similarly, an *Eulerian* tour or circuit(resp. trail) in a graph is a closed walk (resp. walk) including all the edges of the graph. A graph is *Hamiltonian* if it has a Hamiltonian cycle and it is *Eulerian* if it has an Eulerian tour or circuit. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

2. **Properties of** $AG(R)$

In this section, we find for which ring R there exist a vertex which is adjacent to all vertices of $AG(R)$ and then find some more properties of $AG(R)$. We note here the following proposition from Axtell et.al [6] which will be used frequently in this paper.

Proposition 2.1. [6] *Let R be a finite commutative ring with identity. Then the following are equivalent:*
(1) *$Z(R)$ is an ideal;*
(2) *$Z(R)$ is a maximal ideal;*
(3) *R is local;*
(4) *Every $x \in Z(R)$ is nilpotent.*

The following two propositions give criterion for existence of a vertex which is adjacent to all vertices of $AG(R)$ for finite rings. These propositions will be used to derive the other properties of $AG(R)$ graph.

Proposition 2.2. *Let R be a finite reduced ring. Then there exists a vertex $x \in Z(R)^*$ such that x is adjacent to all vertices of $AG(R)$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a finite field.*

Proof. Suppose R is a finite reduced ring then we have $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$ is a finite field for $1 \leq i \leq n$.

Suppose $x = (x_1, x_2, \ldots, x_n) \in Z(R)^*$ is a vertex which is adjacent to all the vertices of R. First we consider $n \geq 3$ and let $e_1 = (1, 0, 0, \ldots, 0) \in Z(R)^*$. Then $xe_1 = (x_1, 0, 0, \ldots, 0)$ and so $ann(xe_1) = ann(e_1)$. Thus for x and e_1 to be adjacent we must have $x_1 = 0$. Similarly taking $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the i^{th} entry, for $1 \leq i \leq n$ and continuing the same way we have $x = (0, 0, \ldots, 0)$, which is a contradiction. Hence if $n \geq 3$, there does not exist $x \in Z(R)^*$ such that x is adjacent to all vertices of $AG(R)$. So we consider $n \leq 2$. If $n = 1$ then $AG(R)$ is an empty graph. Now for $n = 2$, $R \cong \mathbb{F}_1 \times \mathbb{F}_2$ and so by [3, Thereom 3.6] $AG(R) = \Gamma(R)$. But for $\Gamma(R)$, there exists $x \in Z(R)^*$ which is adjacent to all vertices of $AG(R)$ if only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a field or R is a local ring by [2, Corrolary 2.7]. But since R is a reduced ring, we must have $R \cong \mathbb{Z}_2 \times \mathbb{F}$.
If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then clearly there is a vertex adjacent to all vertices of $AG(R)$. □

Proposition 2.3. *Let R be a finite non-reduced ring with identity. If $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local rings but not field, then there exists a vertex $x \in Z(R)^*$ such that x is adjacent to all vertices of $AG(R)$.*

Proof. Assume that R is a finite non-reduced ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local ring. Let $x = (x_1, x_2, \ldots, x_n) \in Z(R)^*$ be a vertex which is adjacent to all vertices of $AG(R)$. If atleast one of x_i is zero then for $z = (1, 1, \ldots, 1, 0, 1, \ldots, 1) \in Z(R)^*$, where zero is in the i^{th}

position, we have $ann(xz) = ann(x)$. So by [7, lemma 2.1(1)] x is not adjacent to z. Hence, if atleast one entry in x is zero then x cannot be adjacent to every vertex of $Z(R)^*$. Thus all entries of x must be non-zero. Suppose now, the k^{th} entry of x say x_k is invertible, i. e., there exists $y \in R_k$ such that $x_k y = 1$. Then for $v = (0, 0, \dots, 0, y, 0, \dots, 0) \in Z(R)^*$, $ann(xv) = ann(v)$. So x is not adjacent to some vertex of $Z(R)^*$, which is a contradiction. So we consider that each R_i is not a field. Now assume that all entries of x are non-zero and non-unit. Let $z = (z_1, z_2, \dots, z_n) \in Z(R)^*$. Then $ann(x) = ann(x_1) \times ann(x_2) \times . \times ann(x_n)$ and $ann(z) = ann(z_1) \times ann(z_2) \times \cdots \times ann(z_n)$. But as $z \in Z(R)^*$, so there exists $z_i's$, say z_k, where $z_k \in Z(R_k)^*$. As R_k is a local ring we have $AG(R_k)$ is complete and therefore $ann(x_k z_k) \neq ann(x_k) \cup ann(z_k)$. So there exists $t_j \in ann(x_k z_k) \backslash ann(x_k) \cup ann(z_k)$. Now $t = (0, 0, \dots, 0, t_j, 0, \dots, 0) \in ann(xz)$ but $t = (0, 0, \dots, 0, t_j, 0, \dots, 0) \notin ann(x) \cup ann(z)$, for if $t \in ann(x) \cup ann(z)$ then we have either $x_j t_j = 0$ or $z_j t_j = 0$ which is a contradiction. Hence there exists a vertex $x \in Z(R)^*$ such that x is adjacent to all vertices of $AG(R)$ if $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local rings but not field. □

In the next proposition we characterize a finite complete $AG(R)$ graph.

Proposition 2.4. *If $AG(R)$ is a finite complete graph then either R is a finite local ring or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. If $AG(R)$ is finite complete graph, the set of vertices of $AG(R)$ is same as $\Gamma(R)$, by [1, theorem 2.2] R must be a finite ring. So let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local ring. Let $x = (1, 0, 0, \dots, 0) \in Z(R)^*$ and $y = (1, 1, 0, \dots, 0) \in Z(R)^*$. We assume that $n \geq 3$. Then $ann(x) = ann(xy)$ shows that x is not adjacent to y, which is a contradiction. So we must have $n \leq 2$. If $n = 2$ then $R \cong R_1 \times R_2$. By proposition 2.3, we have either $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, or each R_i a local ring but not a field. First we consider that atleast one of R_i, say R_2 is not a field. Then for $t = (1, 0)$ and $w = (1, x)$, where $x \in Z(R_2)^*$, we get $ann(t) = ann(tw)$. This shows that $(1, 0)$ is not adjacent to $(1, x)$, which is a contradiction as $AG(R)$ is a complete graph. So we consider that both R_i are fields. But if both R_i are fields, there exists a vertex which is adjacent to all vertices of $AG(R)$ since $AG(R)$ is a complete graph. Hence, $R \cong \mathbb{Z}_2 \times \mathbb{F}$. As $AG(R)$ is a complete graph, we must have $\mathbb{F} \cong \mathbb{Z}_2$. Now for $n = 1$ we have R is a finite local ring. Thus, $AG(R)$ is a finite complete graph if R is a finite local ring or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

In the following proposition we characterize the finite rings for which $AG(R)$ is a regular graph.

Proposition 2.5. *If R is a finite ring with identity and $AG(R)$ is a regular graph then $R \cong \mathbb{F} \times \mathbb{F}$, i.e., $AG(R) \cong K_{t-1,t-1}$ with $|F| = t$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local ring or a field.*

Proof. Let R be a finite commutative ring with identity and $AG(R)$ be a regular graph. Since R is a finite ring, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local ring and $n \geq 1$. Now if atleast one of $R_i's$ is not a field, say R_1, then consider $e_1 = (1, 0, \dots, 0) \in Z(R)^*$ and $y = (y_1, 0, \dots, 0) \in Z(R)^*$ with $y_1 \in Z(R_1)^*$. Then clearly $deg(y) > deg(x)$, which is a contradiction. Hence if $n \geq 2$, each R_i must be field. So $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where $n \geq 2$ and $\mathbb{F}_i$'s are finite fields. If we consider e_1 as above then the vertices that are adjacent to e_1 in $AG(R)$ are those vertices y such that $e_1 y = 0$. So $deg(e_1) = |\mathbb{F}_2||\mathbb{F}_3| \cdots |\mathbb{F}_n| - 1$ and similarly if we take $e_2 = (0, 1, 0, \dots, 0) \in Z(R)^*$ then $deg(e_2) =$

$|\mathbb{F}_1||\mathbb{F}_3|\cdots|\mathbb{F}_n|-1$. As $AG(R)$ is regular, we have $deg(e_1)=deg(e_2)$ and so $|\mathbb{F}_1|=|\mathbb{F}_2|$. Thus taking each e_i for $1\leq i\leq n$, we see that all $\mathbb{F}_i$ have the same cardinality and hence $R\cong\mathbb{F}\times\mathbb{F}\times\cdots\times\mathbb{F}$. Let $|\mathbb{F}|=t$. We consider $n\geq 3$ and let $z=(1,1,0,\ldots,0)$. Then we have $\deg(e_1)=|\mathbb{F}|^{(n-1)}-1$ and $\deg(z)=(|\mathbb{F}|^{(n-2)}-1)+2(|\mathbb{F}|-1)(|\mathbb{F}|^{(n-2)}-1)$. Now if $n\geq 4$ then $deg(z)>deg(e_1)$, which is a contradiction. If $n=3$ and $|\mathbb{F}|\geq 3$ then also $deg(z)>deg(e_1)$, which is a contradiction. If $n=3$ and $|\mathbb{F}|=2$ then clearly $AG(R)$ is regular with $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Now if $n=1$, then R is a finite local ring or a field and clearly $AG(R)$ is regular. For $n=2$, we have $AG(R)=\Gamma(R)$ by [7, Theorem 3.6] and for $\Gamma(R)$ to be regular we must have $R=\mathbb{F}\times\mathbb{F}$ by [5, Theorem 8] and so $\Gamma(R)=K_{t-1,t-1}$. □

In the following proposition we find the domination number of $AG(R)$ graph.

Proposition 2.6. *If R is a finite ring then $\gamma(AG(R))\leq 2$.*

Proof. Let us consider first that R is a decomposable ring with $R\cong R_1\times R_2$. Now let us consider the sets $A=\{(x_1,0)|x_1\in R_1^*\}$, $B=\{(0,x_2)|x_2\in R_2^*\}$, $C=\{(x_1,x_2)|x_1\in Z(R_1)^*,x_2\in R_2^*\}$ and $D=\{(x_1,x_2)|x_1\in R_1^*,x_2\in Z(R_2)^*\}$. Then $Z(R)^*=A\cup B\cup C\cup D$. Next we consider two vertices $x=(1,0)\in Z(R)^*$ and $y=(0,1)\in Z(R)^*$ of $AG(R)$. Let $z=(z_1,z_2)\in Z(R)$. If $z_1\in U(R_1)$ then clearly z cannot be adjacent to x. Hence z is adjacent to x if $z_1\in Z(R_1)$ and similarly z is adjacent to y if $z_2\in Z(R_2)$. Now $xz=(z_1,0)$, $ann(x)=B\cup\{(0,0)\}$, $ann(xz)=ann(z_1,0)=B\cup\{(q,t)|q\in ann(z_1),t\in R_2\}$. If $z_2\in U(R_2)$ then $ann(z)=\{(q,0)|q\in ann(z_1)\}$ and if $z_2\in Z(R_2)^*$ then $ann(z)=\{(q_1,q_2)|q_1\in ann(z_1),q_2\in ann(z_2)\}$. Thus in all the cases we get $ann(xz)\neq ann(x)\cup ann(z)$ and so x is adjacent to z. Hence we get $Nbd(x)=B\cup C$ and similarly we get $Nbd(y)=A\cup D$. Therefore we have $Nbd(x)\cup Nbd(y)=Z(R)^*$. Now for $1\neq y_k\in U(R_2)$, we have $(0,y_k)\in Nbd(x)$ but $(0,y_k)\notin Nbd(y)$. Similarly if $x_k\in U(R_1)$ then $(x_k,0)\in Nbd(y)$ but $(x_k,0)\notin Nbd(x)$. Thus if we take $S=\{x,y\}$, then S is a dominating set of $AG(R)$. Hence for any finite commutative ring we have $\gamma(AG(R))\leq 2$. □

From propositions 2.2, 2.3 and 2.6, we have the following corollary.

Corollary 2.7. *If $R\cong R_1\times R_2\times\cdots\times R_n$, where each R_i are finite local ring but not fields or $R\cong\mathbb{Z}_2\times\mathbb{F}$, then $\gamma(AG(R))=1$.*

Next we find the criterion for the existence of a cut vertex in $AG(R)$ graph.

Proposition 2.8. *Let R be a finite ring such that $AG(R)$ has a cut vertex. Then $R\cong\mathbb{Z}_2\times\mathbb{F}$, where $\mathbb{F}$ is a finite field and $\mathbb{F}\ncong\mathbb{Z}_2$.*

Proof. Let $x\in Z(R)^*$ be a cut vertex of $AG(R)$. Clearly $AG(R)$ cannot be a complete graph and so $diam(AG(R))=2$. Now we have, $AG(R)=X\cup Y$, where $X\cap Y=\{x\}$. As x is a cut vertex and $diam(AG(R))=2$, there exist $a\in X$ and $b\in Y$ which are adjacent to x. So $a-x-b$ is a path from a to b in $AG(R)$. Now let $c\in X$, such that c is not adjacent to x in $AG(R)$ and as $diam(AG(R))=2$, so we have either c is adjacent to b or there exists a path $c-d-b$ in $AG(R)$ where $d\neq x$. In either case we get that x is not a cut vertex of $AG(R)$, which is a contradiction. Hence any vertex in $X\backslash\{x\}$ is adjacent to x. Similarly any vertex in $Y\backslash\{x\}$ is adjacent to x. Thus x is a vertex which is adjacent to all vertices of $AG(R)$. Hence by propositions 2.2 and 2.3 either $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a finite field or $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i are finite local ring but not field. If $R \cong R_1 \times R_2 \times \cdots \times R_n$ and if atleast one of R_i is such that $|Z(R_i)^*| \geq 2$ then $AG(R)$ does not have a cut vertex, which is a contradiction. Hence for each R_i we have $|Z(R_i)^*| = 1$. But when $|Z(R_i)^*| = 1$ we have either $R_i \cong \mathbb{Z}_4$ or $R_i \cong \mathbb{Z}_2[t]/(t^2)$ [1, Example 2.1(i)]. So $R \cong R_1 \times R_2 \times \cdots \times R_n$ where either $R_i \cong \mathbb{Z}_4$ or $R_i \cong \mathbb{Z}_2[t]/(t^2)$. Let $y = (y_1, y_2, \ldots, y_n)$ where $y_i = 2$ if $R_i \cong \mathbb{Z}_4$ and $y_i = t$ if $R_i \cong \mathbb{Z}_2[t]/(t^2)$. Here y is adjacent to all vertices of $AG(R)$. Now let us consider the vertices $w = (0, y_2, \ldots, y_n)$ and $z = (y_1, \ldots, y_{n-1}, 0)$. Then the vertices which not adjacent to z are the elements of the set $S = \{u = (u_1, u_2, \ldots, u_n) | u_i \in U(R_i)$ for $i = 1, 2, \ldots, n-1$ and $u_n \in Z(R_n)\}$ and the vertices which are not adjacent to w are the elements of the set $S' = \{v = (v_1, v_2, \ldots, v_n) | v_1 \in Z(R_1)$ and $v_i \in U(R_i)$ for $i = 2, \ldots, n\}$. But z is adjacent to each element of S' and similarly w is adjacent to each element of S. So the subgraph of the annihilator graph whose set of vertices is $Z(R)^* \backslash \{y\}$ is still a connected graph which shows that y is not a cut vertex of $AG(R)$. Hence $AG(R)$ does not have any cut vertex which is a contradiction. So $AG(R)$ has a cut vertex if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F} \ncong \mathbb{Z}_2$, for if $\mathbb{F} \cong \mathbb{Z}_2$ then $AG(R)$ is complete graph and a complete graph does not have a cut vertex. □

In the following two propositons we find the independence number of $AG(R)$ graph for certain classes of finite rings.

Proposition 2.9. *Let R be a finite reduced ring not a field such that $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$ are finite field, such that $|\mathbb{F}_1| \geq |\mathbb{F}_2| \geq |\mathbb{F}_3| \geq \cdots \geq |\mathbb{F}_n|$ then $\alpha(AG(R)) = |\mathbb{F}_1^*| + |\mathbb{F}_1^*||\mathbb{F}_2^*| + \cdots + |\mathbb{F}_1^*||\mathbb{F}_2^*| \cdots |\mathbb{F}_{n-1}^*|$.*

Proof. As R is a finite reduced ring, $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$. Consider the set $S_1 = \{(x_1, \ldots, x_n) | x_i = 0$ for all but one $i, 1 \leq i \leq n\}$. The independent subsets of S_1 are $S_{11} = \{(x_1, 0, \ldots, 0) | x_1 \in F_1^*\}, \ldots$, $S_{1n} = \{(0, 0, \ldots, x_n) | x_n \in F_n^*\}$. Among these independent sets, the one with maximum number of elements is S_{11} as $|\mathbb{F}_1| \geq |\mathbb{F}_i|$ $\forall_i$, $1 \leq i \leq n$. Consider the set $S_2 = \{(x_1, \ldots, x_n) | x_i = 0$ for all but 2 i, $1 \leq i \leq n\}$. The maximal independent subset of S_2 is $S_{12} = \{(x_1, x_2, 0, \ldots, 0) | x_i \in F_i^*, i = 1, 2\}$. Continuing in this way we get the maximal independent subset of S_{n-1} is $S_{1(n-1)}$. Let $S' = S_{11} \cup S_{12} \cup \ldots \cup S_{1(n-1)}$. Clearly each pair of elements in S' are nonadjacent. Also for any element $x \in Z(R)^*$ either it belong to S' or there exist an element $y \in S'$ such that x is adjacent to y. Hence we have $\alpha(AG(R)) = |\mathbb{F}_1^*| + |\mathbb{F}_1^*||\mathbb{F}_2^*| + \cdots + |\mathbb{F}_1^*||\mathbb{F}_2^*| \cdots |\mathbb{F}_{n-1}^*|$. □

Proposition 2.10. *Let R be a finite ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ where each R_i are local ring and $|U(R_1)| \geq |U(R_2)| \geq \cdots \geq |U(R_n)|$ then $\alpha(AG(R)) = |U(R_1)| + |U(R_1)||U(R_2)| + \cdots + |U(R_1)||U(R_2)| \cdots |U(R_{n-1})| + 2$.*

Proof. Let $S_1 = \{U(R_1) \times 0 \times \cdots \times 0, 0 \times U(R_2) \times 0 \times \cdots \times 0, 0 \times 0 \times \cdots \times 0 \times U(R_n)\}$. Then each element of S_1 form an independent set of $AG(R)$ and the maximal among these independent sets is $A_1 = U(R_1) \times 0 \times \cdots \times 0$. Also in the set $S_2 = \{U(R_1) \times U(R_2) \times 0 \times \cdots \times 0, U(R_1) \times 0 \times U(R_3) \times 0 \times \cdots \times 0, \ldots, 0 \times 0 \times \cdots \times 0 \times U(R_{n-1}) \times U(R_n)\}$ each element is an independent set of $AG(R)$ and the maximal among these independent sets is $A_2 = U(R_1) \times U(R_2) \times 0 \times \cdots \times 0$ since $|U(R_1)| \geq |U(R_2)| \geq$

$|U(R_i)|$ for $3 \le i \le n$. Also $A_1 \cup A_2$ is an independent set of $AG(R)$. Hence continuing similarly, we get $A_{n-1} = U(R_1) \times U(R_2) \times U(R_3) \times \cdots \times U(R_{n-1}) \times 0$ as the element of S_{n-1} that is a maximal independent set of $AG(R)$. Now if $H = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$, then H is also an independent set of $AG(R)$. Let $x = (x_1, 0, 0, \ldots, 0)$ where $x_1 \in Z(R_1)^*$ and $y = (y_1, y_2, \ldots, y_n)$ where $y_n \in Z(R_n)^*$ and $y_i \in U(R_i)$ for $1 \le i \le n-1$. Then $H' = H \cup \{x, y\}$ is a maximal independent set of $AG(R)$. For if $z = (z_1, z_2, \ldots, z_n) \in Z(R)^* \backslash H'$, then atleast one of z_i must belong to $Z(R_i)$ for some $1 \le i \le n$; if $z_n \in Z(R_n)^*$ then clearly z is adjacent to y and if $z_i \in Z(R_i)^*$ for $1 \le i \le n-1$ then clearly z is adjacent to x. Also x and y are not adjacent. So H' is disjoint and H' is maximal and hence $\alpha(AG(R))$ $=|U(R_1)| + |U(R_1)||U(R_2)| + \ldots + |U(R_1)||U(R_2)| \ldots |U(R_{n-1})| + 2$. □

We now derive the following lemma which will be needed to find the clique number of $AG(R)$ graph in the next proposition.

Lemma 2.11. *If R is a non-local ring with $R \cong R_1 \times R_2 \times . \cdots \times R_n$, where each R_i are local rings, then any two distinct elements which has the same number of non-zero entries but not identical are adjacent in $AG(R)$.*

Proof. Let $x, y \in Z(R)^*$ be non identical vertices having exactly i number of non-zero entries with $1 \le i \le n-1$. So there exist atleast one entry in x, say j^{th} with $1 \le j \le n$, which is non-zero in x but zero in y. If $xy = 0$ then clearly there exist an edge between x and y as $ann(xy) = R \ne ann(x) \cup ann(y) \subseteq Z(R)$. So we assume that $xy \ne 0$. As total number of zero entries are equal in x and y, there exists another entry, say k^{th}, which is zero in x but not in y where $1 \le k \le n$ and $k \ne j$. Then xy has less number of non-zero entries than in x and y with j^{th} and k^{th} entry zero. Now we consider $z = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ with 1 in j^{th} and k^{th} entry and 0 in the remaining entries. Then $z \in ann(xy)$ but $z \notin ann(x) \cup ann(y)$. This shows that $ann(xy) \ne ann(x) \cup ann(y)$. Hence x is adjacent to y in $AG(R)$. □

Proposition 2.12. *If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times . \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i' s$ are finite field, $\omega(AG(R)) = \binom{n}{\frac{n-1}{2}}$ if n is odd or $\binom{n}{\frac{n}{2}}$ if n is even.*

Proof. We'll prove it by induction on n. If $n = 2$, then clearly $AG(R) \cong K_{m,n}$ which is a complete bipartite graph. Hence $\omega(AG(R)) = 2$. So result is true for $n = 2$. Now let us assume that result hold for k less than n. Assume that $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times . \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$'s are finite field. Let $R' \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_{n-1}$. Then by induction hypothesis we have $\omega(AG(R')) = \binom{n-1}{\frac{n-1}{2}}$ if n is odd and $\binom{n-1}{\frac{n}{2}}$ if n is even. Let $S = \{(x_1, x_2, \ldots, x_t, 0, \ldots, 0), (x_1, x_2, \ldots, x_{t-1}, 0, x_{t+1}, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, x_{n-t}, \ldots, x_{n-1}, 0)\}$ be a set of vertices in $AG(R')$. Then clearly by lemma 2.11, S is a complete subgraph of $AG(R')$ and $|S| = \binom{n-1}{t}$ where $t = \frac{n}{2}$ when n is even and $\frac{n-1}{2}$ when n is odd. Hence S is a maximal complete subgraph of $AG(R')$. Now we extend S into S' in $AG(R)$ by adding elements $x_n \in F_n^*$ in the n^{th} co-ordinate of each element of S. Then S' is also a complete subgraph of $AG(R)$ and $|S| = |S'| = \binom{n-1}{t}$ where $t = \frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd. Now we take T to be set of elements in $V(AG(R'))$ which has $t+1$ non-zero component entries. Then T is a complete subgraph of $AG(R')$. Again we extend T to T' by adding zero element of $\mathbb{F}_n$ in the n^{th} coordinate of each element of T. Then T' is a complete subgraph of $AG(R)$ and $|T'| = |T| = \binom{n-1}{t+1}$ where $t = \frac{n}{2}$ or $\frac{n-1}{2}$. Clearly T' and S' are disjoint sets, so $|T' \cup S'| = |T'| + |S'| = \binom{n-1}{t+1} + \binom{n-1}{t} = \binom{n}{t+1}$. Here $S' \cup T'$ is a complete subgraph of $AG(R)$. If $x \notin S' \cup T'$ then $\{x\} \cup S' \cup T'$ cannot be a complete subgraph of $AG(R)$. Since x has lesser or equal or greater number of zero entries than that of elements of $S' \cup T'$.

Case 1: Suppose that x has lesser number of zero entries than that of elements of $S' \cup T'$ then we take $y \in S' \cup T'$ such that y has exactly the same position of non zero entries in x but y has more zero entries, say $x = (x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$, $y = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$. Then $ann(xy) = ann(y)$ and hence x cannot be adjacent to y.

Case 2: Suppose that x has lesser number of non-zero entries than that of elememts of $S' \cup T'$ then we take $y \in S' \cup T'$ such that y has exactly the same position of non zero entries in x but y has less number of zero entries, say $x = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$, $y = (x_1, x_2, \ldots, x_k, 0, x_{k+1}, 0, \ldots, 0)$. Then clearly $ann(xy) = ann(x)$ and hence x cannot be adjacent to y. Hence in both cases $\{x\} \cup S' \cup T'$ cannot form a complete subgraph of $AG(R)$.

Case 3: If x has the same number of zero entries as that of elements of $S' \cup T'$ then there exists an element $y \in S' \cup T'$ such that x and y have the same position of zero entries, so $ann(xy) = ann(x) = ann(y)$ which shows that x cannot be adjacent to y. Hence $\{x\} \cup S' \cup T'$ cannot be a complete subgraph of $AG(R)$.

So in order that the set of $S' \cup T'$ form a complete subgraph with $\{x\}$ we have to removed the vertices from $S' \cup T'$ which are not adjacent to x and we rename that set to be H. Then $|H \cup \{x\}| \leq \binom{n}{t+1}$. Similarly if we take $y \neq x \in Z(R)^*$ where x is adjacent to y then by similar argument we see that the complete subgraph formed by the set of vertices of $S' \cup T'$ with that of $\{x, y\}$ must be even smaller than $\binom{n}{t+1}$. Hence continuing in this way we see that the complete subgraph formed by the set of vertices of S' and that with other vertex of $AG(R)$ must have cardinality less than $\binom{n}{t+1}$. Hence the set of vertices which can form a complete subgraph with S' must have size same as that of T'. Hence $\omega(AG(R)) = \binom{n}{\frac{n-1}{2}}$ or $\binom{n}{\frac{n}{2}}$. □

For any ring $R \cong R_1 \times R_2 \times . \cdots \times R_n$ the above theorem is not true in general for if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ then $\omega(AG(R)) \neq 2$ but $\omega(AG(R)) = 6$.

Remark 2.13. *If $R \cong R_1 \times R_2 \times . \cdots \times R_n \times \mathbb{F}_1 \times \mathbb{F}_2 \times . \cdots \times \mathbb{F}_n$ where R_i are local ring not fields and $\mathbb{F}_i$ are fields then $\omega(AG(R)) \geq max$ $\{|Z(R_1)^*||R_2| \cdots |R_n||\mathbb{F}_1| \cdots |\mathbb{F}_n|, \ldots, |R_1| \ldots .|R_{n-1}||Z(R_n)^*||\mathbb{F}_1| \cdots |\mathbb{F}_n|\}$*

The corollary below follows from the above proposition and remarks.

Corollary 2.14. *If $R \cong \mathbb{F} \times R'$, where $\mathbb{F}$ is a finite field and R' is a finite local ring then $\omega(AG(R)) = |\mathbb{F}||Z(R')^*|$.*

The corollary follows from the following well known theorem.

Theorem 2.15. *A connected graph G is an Eulerian graph iff all vertices of G are of even degrees.*

Corollary 2.16. *Let R be a finite local ring with $|R| = 2^m$ for some $m \geq 3$ then $AG(R)$ is an Eulerian graph.*

Now we show that $AG(R)$ is Hamiltonian if $R \cong A \times A$ where A is a finite local ring with identity .

Proposition 2.17. *Let R be a finite ring such that $R \cong A \times A$ where A is a finite local ring with identity. Then $AG(R)$ is Hamiltonian.*

Proof. First we consider A a local ring but not a field. Let us consider the sets $A^* \times 0$, $0 \times A^*$, $A \times Z(A)^*$, $Z(A)^* \times A$. Then any non-zero zero divisors of R must belong to either one of these sets. First we show that $Z(A)^* \times A$ or $A \times Z(A)^*$ is a complete subgraph of $AG(R)$. Let $x, y \in Z(A)^* \times A$ such that $x \neq y$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$ then as A is a finite local ring so $ann(x_1y_1) \neq ann(x_1) \cup ann(y_1)$ which shows that x is adjacent to y. If $x_1 = y_1$ then $x_1^2 \neq x_1$ as A is a finite local ring and $ann(x_1^2) \neq ann(x_1)$ as $Nil(A) = Z(A)$. Hence x is adjacent to y. Therefore $Z(A)^* \times A$ and similarly $A \times Z(A)^*$ is a complete subgraph of $AG(R)$. As we can form a complete bipartite graph from the set of vertices $A^* \times 0$ and $0 \times A^*$, so there exist a path from $(0, 1)$ to $(1, 0)$ which passes through all the vertices of $A^* \times 0$ and $0 \times A^*$ exactly once and also connect $(1, 0)$ to one vertex of $Z(A)^* \times (A \backslash Z(A))$, $(0, 1)$ to one vertex of $(A \backslash Z(A)) \times Z(A)^*$ as $Z(A)^* \times Z(A)^*$ is a complete subgraph of $AG(R)$. So we get a cycle which passes through all the vertices of $AG(R)$ exactly once. Hence $AG(R)$ is a Hamiltonian graph. If A is a field then $AG(R) \cong \Gamma(R) \cong K_{|A|-1,|A|-1}$ which is clearly Hamiltonian. □

3. Planarity of $AG(R)$

In this section we characterize the finite commutative rings whose annihilator graph $AG(R)$ is planar.

Theorem 3.1. *(Kuratowski) A graph is planar if and only if it contain no sub-division heomomorphic to K_5 or $K_{3,3}$.*

Proposition 3.2. *Let R be a non-local ring then $AG(R)$ is planar if R is isomorphic to one of the following ring $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{F}$.*

Proof. **Case 1**: If $R \cong R_1 \times R_2 \times \cdots . \times R_n$ and $n \geq 4$ then as $\Gamma(R)$ is non planar by S.Akbari et al. [3], $AG(R)$ is also non-planar.

Case 2: If $R \cong R_1 \times R_2 \times R_3$ where one of $|R_i| = 4$, then $\Gamma(R)$ is non-planar by S. Akbari et al. [3] and so is $AG(R)$. So let $|R_i| \leq 3$ for $i = 1, 2, 3$. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ then the subgraph formed by the vertices $\{(2, 0, 2), (1, 2, 0), (2, 1, 0), (2, 2, 0), (0, 0, 1), (0, 0, 2)\}$ contain $K_{3,3}$ and therefore $AG(R)$ is non planar. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ then the subgraph formed by the vertices $\{(1, 2, 0), (2, 1, 0), (1, 1, 0), (0, 2, 1), (0, 1, 1), (0, 0, 1)\}$, where $X = \{(1, 2, 0), (2, 1, 0), (1, 1, 0)\}$ and $Y = \{(0, 2, 1), (0, 1, 1), (0, 0, 1)\}$, contain $K_{3,3}$ as a subgraph and therefore $AG(R)$ is non planar. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then clearly $AG(R)$ is planar. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ then the subgraph formed by the vertices $\{(0, 1, 0), (0, 1, 1), (0, 1, 2), (1, 0, 2), (1, 0, 1), (1, 0, 0)\}$, where $X = \{(0, 1, 0), (0, 1, 1), (0, 1, 2)\}$ and $Y = \{(1, 0, 2), (1, 0, 1), (1, 0, 0)\}$, contain $K_{3,3}$ as a subgraph and hence $AG(R)$ is non-planar.

Case 3: If $n = 2$ then $R \cong R_1 \times R_2$. If both $|R_1|$ and $|R_2|$ are not less than 4 then $K_{3,3}$ is a subgraph of $\Gamma(R)$ and so $AG(R)$ is non planar. So let atleast one of R_i, say $|R_1| \leq 3$. If R_2 such that $|Z(R_2)^*|$

≥ 4 then K_5 is a subgraph of $AG(R)$. Hence $AG(R)$ is non-planar. So $|Z(R_2)^*| \leq 3$.

SubCase 3.1: If $R_1 \cong \mathbb{Z}_2$ and $|Z(R_2)^*| \leq 3$. When $|Z(R_2)^*| = 3$ then $\Gamma(R_2) \cong K_{1,2}$ or K_3. If $\Gamma(R_2) \cong K_{1,2}$ then $R_2 \cong \mathbb{Z}_8$ or $\mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x, x^2-2)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ then $Z(R) = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(1,0),(1,2),(1,4),(1,6)\}$. Now let $X = \{(0,1),(0,3),(0,5),(0,7)\}$ and $Y = \{(1,4),(1,2),(1,6)\}$. As $deg_{\Gamma(R)}(x,y) = 3$ for $x \in X$ and $y \in Y$, by [7, lemma 2.1(5)], $deg_{AG(R)}(x,y) = 1$ and so $K_{4,3}$ is a subgraph of $AG(R)$ showing that $AG(R)$ is non-planar. Similarly if $R \cong \mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^3))$, $Z(\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^3))) = \{(0,0),(0,1),(0,x),(0,x^2),(0,1+x),(0,1+x^2),(0,x+x^2),(0,1+x+x^2),(1,0),(1,x),(1,x^2),(1,x+x^2)\}$, then $K_{4,3}$ is a subgraph of $AG(R)$. Hence $AG(\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^3)))$ is non-planar. Now if $R \cong \mathbb{Z}_2 \times (\mathbb{Z}_4[x]/(2x, x^2-2))$ then $Z(\mathbb{Z}_2 \times (\mathbb{Z}_4[x]/(2x, x^2-2))) = \{(0,0),(0,1),(0,2),(0,3),(0,x),(0,1+x),(0,2+x),(0,3+x),(1,0),(1,x),(1,2),(1,2+x)\}$. Now let $X = \{(0,1),(0,3),(0,1+x),(0,3+x)\}$, and $Y = \{(1,x+2),(1,2),(1,x),(1,0)\}$. As for $x \in X$ and $y \in Y$ $deg_{\Gamma(R)}(x,y) = 3$, $deg_{AG(R)}(x,y) = 1$ so $K_{4,4}$ is a subgraph of $AG(\mathbb{Z}_2 \times (\mathbb{Z}_4[x]/(2x, x^2-2)))$. Hence $AG(\mathbb{Z}_2 \times (\mathbb{Z}_4[x]/(2x, x^2-2)))$ is non-planar.

If R_2 is such that $Z(R_2) = \{0, x, y, z\}$ and $xy = yz = xz = 0$ then $K_{3,3}$ is a subgraph of $AG(R)$. Hence $AG(R)$ is non-planar. We consider $|Z(R_2)| \leq 3$.

If $|Z(R_2)^*| = 2$ then $R_2 \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$, $Z(\mathbb{Z}_2 \times \mathbb{Z}_9) = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(1,0),(1,3),(1,6)\}$. Now let $X = \{(0,1),(0,2),(0,4),(0,5),(0,7)\}$, then $Y = \{(1,0),(1,3),(1,6)\}$. As $deg_{\Gamma(R)}(x,y) = 3$ for $x \in X$ and $y \in Y$, by [7, lemma 2.1(5)] $deg_{AG(R)}(x,y) = 1$ so $K_{6,3}$ is a subgraph of $AG(\mathbb{Z}_2 \times \mathbb{Z}_9))$. Hence $AG(R)$ is non-planar. Similarly for $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $AG(R)$ is non-planar.

If $|Z(R_2)^*| = 1$ then $R_2 \cong \mathbb{Z}_4$ or $R_2 \cong \mathbb{Z}_2[x]/(x^2)$ and $AG(R)$ is clearly planar.

If $|Z(R_2)^*| = 0$ then R_2 is a field or an infinite integral domain and clearly $AG(\mathbb{Z}_2 \times R_2) \cong K_{1,n}$ or $K_{1,\infty}$ and so $AG(R)$ is planar.

SubCase 3.2: Consider $R_1 \cong \mathbb{Z}_3$. If $|Z(R_2)^*| = 3$, then by subcase 3.1 $\Gamma(R_2) \cong K_{1,2}$ or K_3. In both the cases as $AG(\mathbb{Z}_2 \times R_2)$ is a subgraph of $AG(\mathbb{Z}_3 \times R_2)$, $AG(\mathbb{Z}_3 \times R_2)$ is non-planar. If $|Z(R_2)^*| = 2$, $R_2 \cong \mathbb{Z}_9$ or $R_2 \cong \mathbb{Z}_3[x]/(x^2)$, as $AG(\mathbb{Z}_2 \times \mathbb{Z}_9)$ is a subgraph of $AG(\mathbb{Z}_3 \times \mathbb{Z}_9)$, $AG(\mathbb{Z}_3 \times \mathbb{Z}_9)$ is non-planar. Similarly, $AG(\mathbb{Z}_3 \times (\mathbb{Z}_3[x]/(x^2)))$ is non-planar as $AG(\mathbb{Z}_2 \times (\mathbb{Z}_3[x]/(x^2)))$ is a subgraph. If $|Z(R_2)^*| = 1$ then $R_2 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$, $Z(\mathbb{Z}_3 \times \mathbb{Z}_4) = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,2),(2,0),(2,2)\}$. Then clearly $AG(\mathbb{Z}_3 \times \mathbb{Z}_4)$ is planar and similarly for $\mathbb{Z}_3 \times (\mathbb{Z}_2[x]/(x^2))$, $AG(\mathbb{Z}_3 \times (\mathbb{Z}_2[x]/(x^2)))$ is planar. If $|Z(R_2)^*| = 0$ then R_2 is either a field or integral domain. $AG(\mathbb{Z}_2 \times R_2) \cong K_{2,n-1}$ or $K_{2,\infty}$ if R_2 is a field, otherwise it is a doubled star graph. In both the cases $AG(R)$ is planar. □

Proposition 3.3. *If R is a local ring such that $AG(R)$ is planar then R is isomorphic to one of the following $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x,y]/(xy, y^2-x)$, $\mathbb{Z}_9$,$\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$.*

Proof. If R is a local ring such that $|Z(R)^*| \geq 5$ then we have $AG(R)$ is a non-planar graph as K_5 is a subgraph of $AG(R)$. Therefore for a local ring R, $AG(R)$ is planar if and only if $1 \leq | Z(R)^* | \leq$

4. So the local ring for which $AG(R)$ is planar are the following: $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2[x,y]/(x,y)^2$,$\mathbb{Z}_2[x,y]/(xy, y^2-x)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$. □

Acknowledgments

The authors wish to thank the referee for careful reading the article and many useful comments.

References

[1] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, *The zero-divisor graph of a commutative ring. II*, Ideal theoretic methods in commutative algebra (Columbia, MO,1999), Lecture Notes in Pure and Appl. Math., **220**, Dekker, New York, 2001 61–72.

[2] D. F. Anderson and P. S. Livingston, The Zero-divisor graph of a commutative ring, *J. Algebra*, **217** no. 2 (1999) 434–447.

[3] S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph, *J. Algebra*, **270** no. 1 (2003) 169–180.

[4] M. F. Aitiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1969.

[5] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J.Algebra*, **274** no. 2 (2004) 847–855.

[6] M. Axtell, J. Stickles and W. Trampbachls, Zero-divisor ideals and realizable zero-divisor graphs, *Involve*, **2** no. 1 (2009) 17–27.

[7] A. Badawi, On the annihilator graph of a commutative ring, *Comm. Algebra*, **42** no. 1 (2014) 108–121.

[8] I. Beck, Coloring of commutative rings, *J. Algebra*, **116** no. 1 (1998) 208–226.

[9] T. T. Chelvam and T. Asir, Domination in the Total Graph on $\mathbb{Z}_n$, *Discrete Math. Algorithms Appl.*, **3** no. 4 (2011) 413–421.

[10] D. A. Mojdeh and A. M. Rahimi, Dominating Sets of Some Graphs Associated to Commutative Rings, *Comm. Algebra*, **40** no. 9 (2012) 3389–3396.

Sanghita Dutta

Department of Mathematics, North Eastern Hill University, Shillong - 793022, India

Email: sanghita22@gmail.com

Chanlemki Lanong

Department of Mathematics, North Eastern Hill University, Shillong - 793022, India

Email: lanongc@gmail.com